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Relationships between almost parahermitian and almost paracontact metric manifolds

Nülifer Özdemir[a,*], Murat Efe[a]

[a]*Department of Mathematics, Eskisehir Technical University, 26470 Eskisehir, Turkey*

Abstract. In this study, almost parahermitian and almost paracontact metric manifolds are considered. Almost paracontact metric manifolds are obtained from almost parahermitian manifolds with product of $\mathbb{R}$. The relations between classes of almost paracontact metric manifolds and classes of almost parahermitian manifolds are investigated. As an important outcome of our study, we obtain the decomposition of classes of $\mathbb{G}_1$, $\mathbb{G}_2$, $\mathbb{G}_3$ and $\mathbb{G}_4$ into two sub-invariant classes. Finally, explicit examples are given for each class W_i which illustrate our results.

1. Introduction

Differential manifolds having special tensor structure have been classified according to the covariant derivative of their tensor structure. For example, the classification of almost Hermitian manifolds was given in [7] by Gray-Hervella, the classification of almost contact metric manifolds was made in [2] by Chinea, the classification of almost contact manifolds B-metric was carried out in [6] by Ganchev, the calssification of Riemannian manifolds with G_2-structure was done in [4] by Fernandez. There is another classification of almost contact metric manifolds given in [12] by considering the usual almost hermitian structure on the product of almost contact metric manifold with $\mathbb{R}$. In literature, there are some recent studies concerning relations between manifolds with additional structure and their products with $\mathbb{R}$ [13–15, 18, 19].

In this study, we consider the classification of almost parahermitian and almost paracontact manifolds given by [5, 20], respectively. Almost paracontact metric manifols are obtained from almost parahermitian manifolds with product of $\mathbb{R}$ and almost paracontact structure is defined on the product manifold. Relations between classes of almost parahermitian manifolds and almost paracontact metric manifolds are obtained. Also, we partition some of the classes given in the classification of almost paracontact metric manifolds [20] into smaller invariant subclasses.

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* Corresponding author: Nülifer Özdemir
Email addresses: nozdemir@eskisehir.edu.tr (Nülifer Özdemir), eskofh26@gmail.com (Murat Efe)
ORCID iDs: https://orcid.org/0000-0003-0507-2444 (Nülifer Özdemir), https://orcid.org/0000-0001-8229-9824 (Murat Efe)

2. Preliminaries

First, we introduce almost parahermitian manifolds. An almost parahermitian manifold is an even-dimensional semi-Riemannian manifold N having an almost product structure J and a semi-Riemannian metric h such that $h(J(X), J(Y)) = -h(X, Y)$ for all $X, Y \in \chi(N)$. The structure group of the almost parahermitian manifold N is the group of matrices in the following form

$$\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix}, \quad A \in Gl(n, \mathbb{R}).$$

The 2-form defined by $F(X, Y) = h\,(J(X), Y)$ is the fundamental 2-form on N. Almost parahermitian manifolds are classified according to the Levi-Civita covariant derivative ∇F of F. It follows that

$$\alpha(X, Y, Z) := (\nabla_X F)\,(Y, Z) = h\,((\nabla_X J)\,Y, Z)\,.$$

α satisfies

$$\alpha(X, Y, Z) = -\alpha(X, Z, Y) \text{ and } \alpha(X, J(Y), J(Z)) = \alpha(X, Y, Z).$$

The tangent space $T = T_x N$ at each point $x \in N$ is a direct sum $T = \mathcal{V} \oplus \mathcal{H}$, where $\mathcal{V}$ and $\mathcal{H}$ are the n-dimensional eigenspaces of J associated with the eigenvalues $+1$ and -1. There exists a basis $\{A_1, \cdots A_n, U_1, \cdots U_n\}$, where $\{A_1, \cdots, A_n\}$ and $\{U_1, \cdots, U_n\}$ are bases of $\mathcal{V}$ and $\mathcal{H}$ respectively, in which the expressions of h and J are

$$h = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}, \quad J = \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix}.$$

Considering an adapted local basis $\{A_1, \cdots A_n, U_1, \cdots U_n\}$ with respect to the metric h which is expressed by the matrix $h = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$, we have

$$\delta F(X) = -\sum_{i=1}^{n} \{\alpha(A_i, U_i, X) + \alpha(U_i, A_i, X)\}\,.$$

The tensor α is included in the vector space

$$W = \left\{\psi \in \otimes^3 T^* : \psi(X, Y, Z) = -\psi(X, Z, Y) = \psi(X, J(Y), J(Z))\right\},$$

which splits into the direct sum of eight invariant and irreducible subspaces W_i, $i = 1, \cdots, 8$ [5].

The classification of almost parahermitian manifolds is done as follows: Consider the following eight conditions:

(1) $\mathfrak{S}_{ABC}\alpha(A, B, C) = 0$ for all $A, B, C \in \mathcal{V}$.
(2) $\nabla_A A = 0$ for all $A \in \mathcal{V}$.
(3) $\alpha(A, U, V) = \Theta(U)h(A, V) - \Theta(V)h(A, U)$ for all $A \in \mathcal{V}$, $U, V \in \mathcal{H}$.
(4) $\sum_{i=1}^{n} \alpha(A_i, U_i, U) = 0$ for all $U \in \mathcal{H}$, $\{A_i, U_i\}$ being a local adapted frame.
(5) $\mathfrak{S}_{UVW}\alpha(U, V, W) = 0$ for all $U, V, W \in \mathcal{H}$.
(6) $\nabla_U U = 0$ for all $U \in \mathcal{H}$.
(7) $\alpha(U, A, B) = \Theta(B)h(U, A) - \Theta(A)h(U, B)$ for all $U \in \mathcal{H}$, $A, B \in \mathcal{V}$.
(8) $\sum_{i=1}^{n} \alpha(U_i, A_i, A) = 0$ for all $A \in \mathcal{V}$, $\{A_i, U_i\}$ being a local adapted frame.

Here $\mathfrak{S}_{XYZ}$ stands for the cyclic sum with respect to X, Y and Z. Θ is the Lee form of almost parahermitian manifold (N, J, h) defined by

$$\Theta(X) = -\frac{1}{n-1}\delta F(J(X))$$

for every vector field X on N. If all conditions **(1)-(8)** hold, the almost parahermitian manifold N is called parakaehlerian (class $\mathcal{PK}$). The class W_i is identified by all these conditions except **(i)**-th condition. The class $W_i \oplus W_j$ satisfies all conditions except properties **(i)** and **(j)**. If $\dim N = 4$, $W_1 = W_3 = W_5 = W_7 = \{0\}$ [5].

At this step, we give fundamental definitions and properties of almost paracontact metric manifolds.

A $(2n+1)$-dimensional real differentiable manifold M is said to have an almost paracontact structure (φ, ξ, η), if it admits a tensor field φ of type $(1,1)$, a vector field ξ and a 1-form η satsifying the following conditions:

$$\eta(\xi) = 1, \quad \varphi^2 = I - \eta \otimes \xi, \tag{1}$$

where I denotes the identity map. Identities (1) imply

$$\varphi(\xi) = 0, \quad \eta \circ \varphi = 0. \tag{2}$$

If a manifold M with an almost paracontact structure (φ, ξ, η) admits a metric g such that

$$g(\varphi(X), \varphi(Y)) = -g(X,Y) + \eta(X)\eta(Y) \tag{3}$$

for all $X, Y \in \chi(M)$, then the manifold M is called an almost paracontact metric manifold.

The equation (3) gives

$$g(X,\xi) = \eta(X), \; g(\varphi(X), Y) = -g\left(X, \varphi(Y)\right). \tag{4}$$

For an almost paracontact metric manifold, the fundamental 2-form is defined as

$$\Phi(X,Y) := g\left(\varphi(X), Y\right). \tag{5}$$

Let ∇ be the Levi-Civita covariant derivative of the metric g. We denote

$$\beta(X,Y,Z) := \left(\nabla_X \Phi\right)(Y,Z) = g\left(\left(\nabla_X \varphi\right)(Y), Z\right) \tag{6}$$

for all $X, Y, Z \in \chi(M)$. The tensor β has the following properties:

$$\begin{aligned} \beta(X,Y,Z) &= -\beta(X,Z,Y), \\ \beta(X,\varphi(Y),\varphi(Z)) &= \beta(X,Y,Z) + \eta(Y)\beta(X,Z,\xi) - \eta(Z)\beta(X,Y,\xi). \end{aligned} \tag{7}$$

The 1-forms θ, θ^* and ω are defined as

$$\theta(X) = \sum_{i=1}^{2n} g^{ij}\beta(e_i, e_j, X), \tag{8}$$

$$\theta^*(X) = \sum_{i=1}^{2n} g^{ij}\beta(e_i, \varphi(e_i), X), \quad \omega(X) = \beta(\xi, \xi, X), \tag{9}$$

where $\{e_i, \cdots, e_{2n}, \xi\}$ is a basis of TM and (g^{ij}) is the inverse matrix of (g_{ij}) [20].

$\nabla\eta$, $d\eta$ and $d\Phi$ are expressed in terms of the tensor β:

$$\left(\nabla_X \eta\right)(Y) = g(\nabla_X \xi, Y) = -\beta(X, \varphi(Y), \xi), \tag{10}$$

$$2d\eta(X,Y) = \left(\nabla_X \eta\right)(Y) - \left(\nabla_Y \eta\right)(X) = -\beta(X, \varphi(Y), \xi) + \beta(Y, \varphi(X), \xi), \tag{11}$$

$$3d\Phi(X,Y,Z) = \beta(X,Y,Z) + \beta(Y,Z,X) + \beta(Z,X,Y). \tag{12}$$

Using properties (7), the spaces of covariant derivatives of the endomorphism φ are defined as

$$\mathbb{G} := \left\{\beta \in \otimes_3^0 M : \beta(X,Y,Z) = -\beta(X,Z,Y) = \beta(X,\varphi(Y),\varphi(Z)) - \eta(Y)\beta(X,Z,\xi) + \eta(Z)\beta(X,Y,\xi)\right\}.$$

The space $\mathbb{G}$ is written as the direct sum of 12 subspaces;

$$\mathbb{G} = \mathbb{G}_1 \oplus \cdots \oplus \mathbb{G}_{12}.$$

The subspaces $\mathbb{G}_i$ are orthogonal and invariant under the action of $\mathbb{U}^\pi(n) \times \{1\}$, where

$$\mathbb{U}^\pi(n) \times \{1\} = \left\{ \begin{pmatrix} \tau & 0 \\ 0 & 1 \end{pmatrix} : \tau \in \mathbb{U}^\pi(n) \right\},$$

$$\mathbb{U}^\pi(n) = \left\{ \tau = \begin{pmatrix} A & B \\ B & A \end{pmatrix} \quad : \quad \begin{array}{l} A, B \text{ are real matrices of type } n \times n \\ A^t A - B^t B = I_n,\ A^t B - B^t A = 0 \end{array} \right\},$$

here I_n is the unit matrix of size n.

Any almost paracontact metric manifold belongs to a subclass $\mathbb{G}_{i_1} \oplus \cdots \oplus \mathbb{G}_{i_k}$ for $1 \le i_1 \le \cdots \le i_k \le 12$ of $\mathbb{G}$. The defining relations of subspaces we use are as follows:

$$\begin{aligned} \mathbb{G}_1 := \{\beta \in \mathbb{G} : \beta(X,Y,Z) \quad = \quad & \tfrac{1}{2(n-1)} \Big\{ g(X, \varphi(Y))\theta_\beta(\varphi(Z)) \\ & - g(X, \varphi(Z))\theta_\beta(\varphi(Y)) \\ & - g(\varphi(X), \varphi(Y))\theta_\beta(\varphi^2(Z)) \\ & + g(\varphi(X), \varphi(Z))\theta_\beta(\varphi^2(Y) \Big\}, \end{aligned}$$

$$\mathbb{G}_2 := \Big\{ \beta \in \mathbb{G} : \beta(\varphi(X), \varphi(Y), Z) = -\beta(X,Y,Z), \quad \theta_\beta = 0 \Big\},$$

$$\mathbb{G}_3 := \{ \beta \in \mathbb{G} : \beta(\xi, Y, Z) = \beta(X, \xi, Z) = 0, \quad \beta(X,Y,Z) = -\beta(Y,X,Z) \},$$

$$\mathbb{G}_4 := \{ \beta \in \mathbb{G} : \beta(\xi, Y, Z) = \beta(X, \xi, Z) = 0, \quad \mathfrak{S}_{XYZ}\beta(X,Y,Z) = 0 \},$$

where $X, Y, Z \in \chi(M)$. Definitions of other classes can be found in [20]. Also, for any $\beta(X,Y,Z) = \sum_{i=1}^{12} \beta^i(X,Y,Z) \in \mathbb{G}$; the projections β^i of the $\beta \in \mathbb{G}$ in the subspaces $\mathbb{G}_i$ are given below [20]:

$$\begin{aligned} \beta^1(X,Y,Z) \quad = \quad & \tfrac{1}{2(n-1)} \Big\{ g(X, \varphi(Y))\theta_{\beta^1}(\varphi(Z)) - g(X, \varphi(Z))\theta_{\beta^1}(\varphi(Y)) \\ & - g(\varphi(X), \varphi(Y))\theta_{\beta^1}(\varphi^2(Z)) + g(\varphi(X), \varphi(Z))\theta_{\beta^1}(\varphi^2(Y)) \Big\}, \end{aligned}$$

$$\beta^2(X,Y,Z) = \frac{1}{2} \Big\{ \beta\left(\varphi^2(X), \varphi^2(Y), \varphi^2(Z)\right) - \beta\left(\varphi(X), \varphi^2(Y), \varphi(Z)\right) \Big\} - \beta^1(X,Y,Z),$$

$$\begin{aligned} \beta^3(X,Y,Z) \quad = \quad & \tfrac{1}{6} \Big\{ \beta\left(\varphi^2(X), \varphi^2(Y), \varphi^2(Z)\right) + \beta\left(\varphi(X), \varphi^2(Y), \varphi(Z)\right) \\ & + \beta\left(\varphi^2(Y), \varphi^2(Z), \varphi^2(X)\right) + \beta\left(\varphi(Y), \varphi^2(Z), \varphi(X)\right) \\ & + \beta\left(\varphi^2(Z), \varphi^2(X), \varphi^2(Y)\right) + \beta\left(\varphi(Z), \varphi^2(X), \varphi(Y)\right) \Big\}, \end{aligned}$$

$$\beta^4(X,Y,Z) = \frac{1}{2} \Big\{ \beta\left(\varphi^2(X), \varphi^2(Y), \varphi^2(Z)\right) + \beta\left(\varphi(X), \varphi^2(Y), \varphi(Z)\right) \Big\} - \beta^3(X,Y,Z),$$

$$\beta^5(X,Y,Z) = \frac{\theta_{\beta^5}(\xi)}{2n} \{ \eta(Y) g(\varphi(X), \varphi(Z)) - \eta(Z) g(\varphi(X), \varphi(Y)) \},$$

$$\beta^6(X,Y,Z) = -\frac{\theta^*_{\beta^6}(\xi)}{2n} \{ \eta(Y) g(X, \varphi(Z)) - \eta(Z) g(X, \varphi(Y)) \},$$

$$\begin{aligned} \beta^7(X,Y,Z) \quad = \quad & -\tfrac{1}{4}\eta(Y) \Big\{ \beta\left(\varphi^2(X), \varphi^2(Z), \xi\right) - \beta\left(\varphi(X), \varphi(Z), \xi\right) \\ & - \beta\left(\varphi^2(Z), \varphi^2(X), \xi\right) + \beta\left(\varphi(Z), \varphi(X), \xi\right) \Big\} \\ & + \tfrac{1}{4}\eta(Z) \Big\{ \beta\left(\varphi^2(X), \varphi^2(Y), \xi\right) - \beta\left(\varphi(X), \varphi(Y), \xi\right) \\ & - \beta\left(\varphi^2(Y), \varphi^2(X), \xi\right) + \beta\left(\varphi(Y), \varphi(X), \xi\right) \Big\} - \beta^6(X,Y,Z), \end{aligned}$$

$$\begin{aligned}\beta^8(X,Y,Z) &= -\tfrac{1}{4}\eta(Y)\left\{\beta\left(\varphi^2(X),\varphi^2(Z),\xi\right)-\beta\left(\varphi(X),\varphi(Z),\xi\right)\right.\\&\quad\left.+\beta\left(\varphi^2(Z),\varphi^2(X),\xi\right)-\beta\left(\varphi(Z),\varphi(X),\xi\right)\right\}\\&\quad+\tfrac{1}{4}\eta(Z)\left\{\beta\left(\varphi^2(X),\varphi^2(Y),\xi\right)-\beta\left(\varphi(X),\varphi(Y),\xi\right)\right.\\&\quad\left.+\beta\left(\varphi^2(Y),\varphi^2(X),\xi\right)-\beta\left(\varphi(Y),\varphi(X),\xi\right)\right\}-\beta^5(X,Y,Z),\end{aligned}$$

$$\begin{aligned}\beta^9(X,Y,Z) &= -\tfrac{1}{4}\eta(Y)\left\{\beta\left(\varphi^2(X),\varphi^2(Z),\xi\right)+\beta\left(\varphi(X),\varphi(Z),\xi\right)\right.\\&\quad\left.-\beta\left(\varphi^2(Z),\varphi^2(X),\xi\right)-\beta\left(\varphi(Z),\varphi(X),\xi\right)\right\}\\&\quad+\tfrac{1}{4}\eta(Z)\left\{\beta\left(\varphi^2(X),\varphi^2(Y),\xi\right)+\beta\left(\varphi(X),\varphi(Y),\xi\right)\right.\\&\quad\left.-\beta\left(\varphi^2(Y),\varphi^2(X),\xi\right)-\beta\left(\varphi(Y),\varphi(X),\xi\right)\right\},\end{aligned}$$

$$\begin{aligned}\beta^{10}(X,Y,Z) &= -\tfrac{1}{4}\eta(Y)\left\{\beta\left(\varphi^2(X),\varphi^2(Z),\xi\right)+\beta\left(\varphi(X),\varphi(Z),\xi\right)\right.\\&\quad\left.+\beta\left(\varphi^2(Z),\varphi^2(X),\xi\right)+\beta\left(\varphi(Z),\varphi(X),\xi\right)\right\}\\&\quad+\tfrac{1}{4}\eta(Z)\left\{\beta\left(\varphi^2(X),\varphi^2(Y),\xi\right)+\beta\left(\varphi(X),\varphi(Y),\xi\right)\right.\\&\quad\left.+\beta\left(\varphi^2(Y),\varphi^2(X),\xi\right)+\beta\left(\varphi(Y),\varphi(X),\xi\right)\right\},\end{aligned}$$

$$\beta^{11}(X,Y,Z)=\eta(X)\beta(\xi,\varphi^2(Y),\varphi^2(Z)),$$

$$\beta^{12}(X,Y,Z)=\eta(X)\left\{\eta(Y)\beta(\xi,\xi,\varphi^2(Z)-\eta(Z)\beta(\xi,\xi,\varphi^2(Y)\right\}.$$

3. Almost paracontact metric manifolds from almost parahermitian manifolds

Let N be an almost parahermitian manifold, $\dim N=2n$, with metric h and almost product structure J. Now, consider the product manifold $N\times\mathbb{R}$. A vector field on $N\times\mathbb{R}$ will be denoted by $\left(X,a\frac{d}{dt}\right)$, where $X\in\chi(N)$, t is the coordinate of $\mathbb{R}$ and a is a C^∞ function on $N\times\mathbb{R}$. We define an almost paracontact structure $\tilde{\varphi}$, metric $\tilde{g}$, vector field $\tilde{\xi}$ and 1-form $\tilde{\eta}$ on $N\times\mathbb{R}$ by

$$\tilde{\varphi}\left(X,a\frac{d}{dt}\right):=(J(X),0),\quad \tilde{\xi}:=\left(0,\frac{d}{dt}\right),$$

$$\tilde{g}\left(\left(X,a\frac{d}{dt}\right),\left(Y,b\frac{d}{dt}\right)\right):=h(X,Y)+ab$$

and

$$\tilde{\eta}\left(Y,b\frac{d}{dt}\right)=a.$$

The fundamental 2-form of $\left(N\times\mathbb{R},\tilde{\varphi},\tilde{\xi},\tilde{\eta},\tilde{g}\right)$ is given by

$$\tilde{\Phi}\left(\left(X,a\frac{d}{dt}\right),\left(Y,b\frac{d}{dt}\right)\right)=\tilde{g}\left(\tilde{\varphi}\left(X,a\frac{d}{dt}\right),\left(Y,b\frac{d}{dt}\right)\right).$$

Hence

$$\tilde{\Phi}\left(\left(X,a\frac{d}{dt}\right),\left(Y,b\frac{d}{dt}\right)\right)=h(J(X),Y)=F(X,Y).$$

Let $\tilde{\nabla}$ be the Levi-Civita covariant derivative of $\left(N\times\mathbb{R},\tilde{\varphi},\tilde{\xi},\tilde{\eta},\tilde{g}\right)$. Then, from the Koszul formula we have

$$\tilde{\nabla}_{(X,a\frac{d}{dt})}\left(Y,b\frac{d}{dt}\right)=\left(\nabla_XY,\left(X[b]+a\frac{db}{dt}\right)\frac{d}{dt}\right)$$

where $X, Y \in \chi(N)$ and a, b are C^∞ functions on $N \times \mathbb{R}$. In addition, the covariant derivative of the endomorphism $\tilde{\varphi}$ is calculated as

$$\left(\tilde{\nabla}_{(X,a\frac{d}{dt})}\tilde{\varphi}\right)\left(Y, b\frac{d}{dt}\right) = ((\nabla_X J)(Y), 0). \tag{13}$$

A tensor field $\tilde{\beta}$ of type $(0,3)$ is defined as follows:

$$\tilde{\beta}\left(\left(X, a\frac{d}{dt}\right), \left(Y, b\frac{d}{dt}\right), \left(Z, c\frac{d}{dt}\right)\right) = \tilde{g}\left(\left(\tilde{\nabla}_{(X,a\frac{d}{dt})}\tilde{\varphi}\right)\left(Y, b\frac{d}{dt}\right), \left(Z, c\frac{d}{dt}\right)\right) = \alpha(X, Y, Z),$$

where $\alpha(X, Y, Z) = (\nabla_X F)(Y, Z)$. The exterior derivative of the basic 2-form $\tilde{\Phi}$ can be computed as

$$d\tilde{\Phi}\left(\left(X, a\frac{d}{dt}\right), \left(Y, b\frac{d}{dt}\right), \left(Z, c\frac{d}{dt}\right)\right) = dF(X, Y, Z).$$

Let $\{A_1, , ..., A_n, U_1, ..., U_n\}$ be a local adapted frame on an open subset U of N. Then,

$$\left\{\left(\frac{1}{\sqrt{2}}(A_1 + U_1), 0\right), \cdots, \left(\frac{1}{\sqrt{2}}(A_n + U_n), 0\right),\ \varphi\left(\frac{1}{\sqrt{2}}(A_1 + U_1), 0\right), \cdots, \varphi\left(\frac{1}{\sqrt{2}}(A_n + U_n), 0\right), \left(0, \frac{d}{dt}\right)\right\}$$

is an orthonormal basis with respect to $\tilde{g}$ on the open subset $U \times \mathbb{R}$ of $N \times \mathbb{R}$. By using this frame, the followings are obtained by direct calculation:

$$\tilde{\theta}\left(X, a\frac{d}{dt}\right) = -\delta F(X), \quad \tilde{\theta}^*\left(X, a\frac{d}{dt}\right) = -\delta F(J(X)),$$

$$\tilde{\omega}\left(X, a\frac{d}{dt}\right) = 0.$$

The Riemannian curvature $\tilde{R}$ of $\left(N \times \mathbb{R}, \tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g}\right)$ is written as

$$\tilde{R}\left(\left(X, a\frac{d}{dt}\right), \left(Y, b\frac{d}{dt}\right)\right)\left(Z, c\frac{d}{dt}\right) = (R(X, Y)Z, 0) \tag{14}$$

for any vector fields $\left(X, a\frac{d}{dt}\right), \left(Y, b\frac{d}{dt}\right), \left(Z, c\frac{d}{dt}\right)$. In addition, the Ricci curvature $\tilde{Q}$ is evaluated as

$$\tilde{Q}\left(\left(X, a\frac{d}{dt}\right), \left(Y, b\frac{d}{dt}\right)\right) = Q(X, Y), \tag{15}$$

for any vector fields $\left(X, a\frac{d}{dt}\right), \left(Y, b\frac{d}{dt}\right)$. Also the scalar curvature $\tilde{s}$ is as follows

$$\tilde{s} = s. \tag{16}$$

From now on, we shall use $\tilde{X} = \left(X, a\frac{d}{dt}\right)$, $\tilde{Y} = \left(Y, b\frac{d}{dt}\right)$ and $\tilde{Z} = \left(Z, c\frac{d}{dt}\right)$ to denote smooth vector fields on $N \times \mathbb{R}$ where X, Y, Z are smooth vector fields on N and a, b, c are smooth functions on $N \times \mathbb{R}$.

Definition 3.1. *Let (M, g) be a semi-Riemannian manifold. If there exists a constant λ such that*

$$Q(X, Y) = \lambda g(X, Y) \tag{17}$$

for any vector fields X, Y on M, then M is called an Einstein manifold with Einstein constant λ [8, 11].

Definition 3.2. *Let (M, g) be a semi-Riemannian manifold. If there exists a smooth non-zero vector field v on M and a constant λ such that*

$$\frac{1}{2}\mathcal{L}_v g + Q + \lambda g = 0, \tag{18}$$

then (M, g) is called a Ricci soliton, where $\mathcal{L}$ denotes the Lie derivative [3, 9, 16].

There exist a generalization of the Ricci soliton on an almost paracontact metric manifold $(M, \varphi, \xi, \eta, g)$ as follows:

Definition 3.3. *Let $(M, \varphi, \xi, \eta, g)$ be an almost paracontact metric manifold. If there exists a smooth non-zero vector field v on M, constants λ and μ such that*

$$\frac{1}{2}\mathcal{L}_v g + Q + \lambda g + \mu\eta \otimes \eta = 0, \tag{19}$$

then $(M, \varphi, \xi, \eta, g)$ is called an η-Ricci soliton [1, 3, 16, 17].

Now, we give relations between an almost parahermitian Einstein manifold (Ricci soliton) N and the η-Ricci soliton $N \times \mathbb{R}$.

Theorem 3.4. *Almost parahermitian manifold N is Einstein with Einstein constant λ if and only if the almost paracontact metric manifold $N \times \mathbb{R}$ is $\tilde{\eta}$-Ricci soliton with vector field $\tilde{\xi}$ and constants $(-\lambda, \lambda)$.*

Proof. Let the manifold N be an Einstein manifold with constant λ, that is $Q(X, Y) = \lambda h(X, Y)$ for all $X, Y \in \chi(N)$. If we take $v = \tilde{\xi} = \left(0, \frac{d}{dt}\right)$ and $\lambda = -\tilde{\lambda} = \tilde{\mu}$, we have

$$\frac{1}{2}\mathcal{L}_{\tilde{\xi}}\tilde{g}(\tilde{X}, \tilde{Y}) + \tilde{Q}(\tilde{X}, \tilde{Y}) + \tilde{\lambda}\tilde{g}(\tilde{X}, \tilde{Y}) + \tilde{\mu}\tilde{\eta}(\tilde{X})\tilde{\eta}(\tilde{Y}) = Q(X, Y) - \lambda h(X, Y) = 0 \tag{20}$$

Hence $N \times \mathbb{R}$ is $\tilde{\eta}$-Ricci soliton with constants $(-\lambda, \lambda)$.

Conversely, if $N \times \mathbb{R}$ is an $\tilde{\eta}$-Ricci soliton with the vector field $\tilde{\xi} = \left(0, \frac{d}{dt}\right)$ and scalars $\lambda = -\tilde{\lambda} = \tilde{\mu}$, then the equation

$$Q(X, Y) - \lambda h(X, Y) = \frac{1}{2}\mathcal{L}_{\tilde{\xi}}\tilde{g}(\tilde{X}, \tilde{Y}) + \tilde{Q}(\tilde{X}, \tilde{Y}) + \tilde{\lambda}\tilde{g}(\tilde{X}, \tilde{Y}) + \tilde{\mu}\tilde{\eta}(\tilde{X})\tilde{\eta}(\tilde{Y}) = 0 \tag{21}$$

is satisfied. (21) implies $Q(X, Y) = \lambda h(X, Y)$, that is N is an Einstein manifold with the constant λ. □

Theorem 3.5. *The almost parahermitian manifold N is a Ricci soliton with non-zero vector field v and constant λ if and only if the almost paracontact metric manifold $N \times \mathbb{R}$ is an $\tilde{\eta}$-Ricci soliton with vector field $(v, 0)$ and constants $(\lambda, -\lambda)$.*

Proof. Let N be a Ricci soliton with non-zero vector field v and constant λ. Then the equation

$$\frac{1}{2}\mathcal{L}_v h(X, Y) + Q(X, Y) + \lambda h(X, Y) = 0 \tag{22}$$

is satisfied for each $X, Y \in \chi(N)$. On $N \times \mathbb{R}$, if we take $\tilde{v} = (v, 0)$ and $\tilde{\lambda} = -\tilde{\mu} = \lambda$, we get

$$\frac{1}{2}\mathcal{L}_{\tilde{v}}\tilde{g}(\tilde{X}, \tilde{Y}) + \tilde{Q}(\tilde{X}, \tilde{Y}) + \tilde{\lambda}\tilde{g}\left(\tilde{X}, \tilde{Y}\right) + \tilde{\mu}\tilde{\eta}(\tilde{X})\tilde{\eta}(\tilde{Y}) = \frac{1}{2}\mathcal{L}_v h(X, Y) + Q(X, Y) + \lambda h(X, Y) = 0.$$

Hence $N \times \mathbb{R}$ is an $\tilde{\eta}$-Ricci soliton with vector field $(v, 0)$ and constants $(\lambda, -\lambda)$.

Conversely, if $N \times \mathbb{R}$ is an $\tilde{\eta}$-Ricci soliton with the vector field $\tilde{v} = (v, 0)$ and constants $(\lambda, -\lambda)$, then we have

$$\frac{1}{2}\mathcal{L}_{\tilde{v}}\tilde{g}(\tilde{X}, \tilde{Y}) + \tilde{Q}(\tilde{X}, \tilde{Y}) + \lambda\tilde{g}(\tilde{X}, \tilde{Y}) + (-\lambda)\tilde{\eta}(\tilde{X})\tilde{\eta}(\tilde{Y}) = 0$$

From this equation (22) follows. Therefore N is a Ricci soliton with any vector field v and constants $(\lambda, -\lambda)$. □

Example 3.6. *[10] Let G be a Lie group of dimension 6 with Lie algebra $\mathfrak{g}$, where $\{e_1, \cdots, e_6\}$ is a basis for $\mathfrak{g}$ with non-zero brackets*

$$[e_1, e_4] = e_1 + e_4 \quad [e_2, e_5] = e_2 + e_5, \quad [e_3, e_6] = e_3 + e_6.$$

A left-invariant almost parahermitian structure on the Lie group G is determined by

$$J(e_1) = e_1,\ J(e_2) = e_2,\ J(e_3) = e_3,\ J(e_4) = -e_4,\ J(e_5) = -e_5,\ J(e_6) = -e_6.$$

We choose the left-invariant metric h of the Lie group as follows

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

It is easily checked that (J, h) is a left-invariant almost parahermitian structure on the Lie group G. From the Koszul formula, we obtain non-zero Levi-Civita covariant derivatives as

$$\begin{array}{llll} \nabla_{e_1} e_1 = -e_1, & \nabla_{e_1} e_4 = e_4, & \nabla_{e_2} e_4 = -e_2, & \nabla_{e_2} e_5 = e_5, \\ \nabla_{e_3} e_3 = -e_3, & \nabla_{e_3} e_6 = e_6, & \nabla_{e_4} e_1 = -e_1, & \nabla_{e_4} e_4 = e_4, \\ \nabla_{e_5} e_2 = -e_2, & \nabla_{e_5} e_5 = e_5, & \nabla_{e_6} e_3 = -e_3, & \nabla_{e_6} e_6 = e_6. \end{array}$$

The Ricci curvature is calculated as $Q(X, Y) = -2h(X, Y)$ and the scalar curvature is -12. From Theorem 3.4, the left-invariant almost paracontact metric structure on $G \times \mathbb{R}$ is an η-Ricci soliton with the vector field $\xi = \left(0, \frac{d}{dt}\right)$ and constant $(2, -2)$.

Now we investigate relations between classes of an almost parahermitian manifold N and the almost paracontact metric manifold $N \times \mathbb{R}$. Note that from (13) we have

$$\left(\tilde{\nabla}_{(X, a\frac{d}{dt})} \tilde{\varphi}\right)\left(0, \frac{d}{dt}\right) = (0, 0). \tag{23}$$

Then the projections $\tilde{\beta}^5, \cdots, \tilde{\beta}^{12}$ vanish. Hence $N \times \mathbb{R}$ is in $\mathbb{G}_1$, $\mathbb{G}_2$, $\mathbb{G}_3$, $\mathbb{G}_4$, or in one of their suitable direct sum.

We introduce the following decompositions of classes $\mathbb{G}_i$, $1 \leq i \leq 4$ to investigate relations between N and $N \times \mathbb{R}$. Define

$$f : \mathbb{G}_i \to \mathbb{G}_i,\ f(\beta)(X, Y, Z) := \frac{1}{2}\left\{\beta(\varphi(X), Y, Z) + \beta(X, Y, Z)\right\}, 1 \leq i \leq 4.$$

It can be seen that $f^2 = f$. Then $\mathbb{G}_i = \mathrm{Ker}(f) \oplus \mathrm{Im}(f)$. Let $\mathbb{G}_i^1 = \mathrm{Ker}(f)$ and $\mathbb{G}_i^2 = \mathrm{Im}(f)$, $1 \leq i \leq 4$. The classes $\mathbb{G}_i^1$ and $\mathbb{G}_i^2$ are orthogonal and invariant under the action of the structure group.

By using these decompositions, we state following theorems:

Theorem 3.7. *Almost parahermitian manifold (N, J, h) belongs to the class W_1 if and only if almost paracontact metric manifold $(N \times \mathbb{R}, \tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ belongs to the class $\mathbb{G}_3^2$.*

Proof. Let the manifold (N, J, h) be of class W_1. Then all conditions except condition **(1)** are satisfied. From conditions (**4**) and (**8**), we have $\delta F(X) = 0$ and we get

$$\tilde{\theta}\left(X, a\frac{d}{dt}\right) = 0$$

for all vector fields $\left(X, a\frac{d}{dt}\right)$. Since $\delta F = 0$ and the conditions (**3**) and (**7**) hold,

$$\alpha(A, U, V) = \alpha(U, A, B) = 0 \tag{24}$$

is obtained for all $A, B \in \mathcal{V}$, $U, V \in \mathcal{H}$. In addition, from conditions (**2**) and (**6**) we get

$$\alpha(X, X, Z) = 0. \tag{25}$$

for all $X, Z \in \chi(N)$. From the condition (**5**) and the equation (25) we have

$$\alpha(U, V, W) = 0, \tag{26}$$

for all $U, V, W \in \mathcal{H}$. Moreover, from (24) and (26)

$$\alpha(X, Y, Z) = \alpha(J(X), Y, Z) \tag{27}$$

is obtained. Hence we get

$$\tilde{\beta}(\tilde{X}, \tilde{Y}, \tilde{Z}) = \tilde{\beta}(\tilde{\varphi}(\tilde{X}), \tilde{Y}, \tilde{Z}). \tag{28}$$

Also from (25), we have

$$\tilde{\beta}(\tilde{X}, \tilde{Y}, \tilde{Z}) = -\tilde{\beta}(\tilde{Y}, \tilde{X}, \tilde{Z}). \tag{29}$$

In addition, since $\tilde{\beta}(\tilde{\xi}, \tilde{Y}, \tilde{Z}) = 0$ and the equations (28) and (29) are met, $N \times \mathbb{R}$ is of the class $\mathbb{G}_3^2$.

Conversely, if $N \times \mathbb{R}$ is of the class $\mathbb{G}_3^2$, then the equations (28) and (29) are satisfied. From (28) and (29), we obtain (27) and (25). Setting $X = Y = A$, $A \in \mathcal{V}$ in (25), $\alpha(A, A, Z) = 0$ is obtained for any vector field Z. Similarly, setting $X = Y = U$, $U \in \mathcal{H}$ in (25), we have $\alpha(U, U, Z) = 0$. Therefore the conditions **(2)** and **(6)** are satisfied. In addition, (27) yields the condition **(5)**. From the equation (25) we have $\delta F(X) = 0$. Hence the conditions **(4)** and **(8)** hold. Also from (25) and (27), we have $\alpha(U, A, B) = \alpha(A, U, V) = 0$ for all $A, B \in \mathcal{V}$, $U, V \in \mathcal{H}$. Hence the conditions (**3**) and (**7**) are satisfied. Thus N is of the class W_1. □

Example 3.8. *[10] Consider a Lie group of dimension 6 with Lie algebra, where*

$$[e_1, e_2] = 2e_3, \quad [e_1, e_6] = 2e_5, \quad [e_2, e_6] = 2e_4.$$

A left-invariant almost parahermitian structure on G is given by

$$J(e_1) = e_1,\ J(e_2) = e_2,\ J(e_3) = -e_3,\ J(e_4) = -e_4,\ J(e_5) = -e_5,\ J(e_6) = e_6.$$

The left-invariant metric h of the Lie group is chosen as

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

It can be proved that (J, h) is a left-invariant almost parahermitian structure on G. The Koszul formula gives the following non-zero Levi-Civita covariant derivatives:

$$\nabla_{e_1} e_2 = e_3, \quad \nabla_{e_2} e_1 = -e_3, \quad \nabla_{e_1} e_6 = e_5,$$
$$\nabla_{e_6} e_1 = -e_5, \quad \nabla_{e_2} e_6 = e_4, \quad \nabla_{e_6} e_2 = -e_4.$$

Also, we calculate non-zero Levi-Civita covariant derivatives of J as

$$(\nabla_{e_1} J)\, e_2 = 2e_3 = -(\nabla_{e_2} J)\, e_1, \quad (\nabla_{e_1} J)\, e_6 = 2e_5 = -(\nabla_{e_6} J)\, e_1,$$
$$(\nabla_{e_2} J)\, e_6 = 2e_4 = -(\nabla_{e_6} J)\, e_2.$$

Since $\mathcal{V} = Span\{e_1, e_2, e_6\}$ *and* $\mathcal{H} = Span\{e_3, e_4, e_5\}$,

$$\alpha(e_1, e_2, e_6) + \alpha(e_2, e_6, e_1) + \alpha(e_6, e_1, e_2) = 6 \neq 0$$

so the condition **(1)** *is not satisfied. But it can be seen that the other conditions (condition* **(2)**, *..., condition* **(8)***) are met. Hence the left-invariant parahermitian structure* (J, h) *on* G *is of the class* W_1. *Then from Theorem 3.7, left-invariant paracontact structure on* $G \times \mathbb{R}$ *is of the class* $\mathbb{G}_3^2$.

Theorem 3.9. *Almost parahermitian manifold* (N, J, h) *is in the class* W_2 *if and only if almost paracontact metric manifold* $(N \times \mathbb{R}, \tilde{\xi}, \tilde{\varphi}, \tilde{\eta}, \tilde{g})$ *is in the class* $\mathbb{G}_4^2$.

Proof. Let the manifold (N, J, h) be of class W_2. In this class we have

$$\delta F = 0, \ \alpha(A, U, V) = \alpha(U, A, B) = \alpha(U, V, W) = 0$$

for all $A, B \in \mathcal{V}$, $U, V, W \in \mathcal{H}$. Then

$$\alpha(X, Y, Z) = \alpha(J(X), Y, Z) \tag{30}$$

for all $X, Y, Z \in \chi(N)$ and we obtain

$$\tilde{\beta}(\tilde{X}, \tilde{Y}, \tilde{Z}) = \tilde{\beta}(\varphi(\tilde{X}), \tilde{Y}, \tilde{Z}). \tag{31}$$

Also from the condition **(1)**, we get

$$\mathfrak{S}_{\tilde{X}\tilde{Y}\tilde{Z}}\tilde{\beta}\left(\tilde{X}, \tilde{Y}, \tilde{Z}\right) = 0. \tag{32}$$

Thus $N \times \mathbb{R}$ is of the class $\mathbb{G}_4^2$, since $\tilde{\beta}(\tilde{\xi}, \tilde{Y}, \tilde{Z}) = \tilde{\beta}(\tilde{X}, \tilde{\xi}, \tilde{Z}) = 0$.

Conversely, if $N \times \mathbb{R}$ is of the class $\mathbb{G}_4^2$, then (31) and (32) are satisfied. From these equations, we get (30) and

$$\mathfrak{S}_{XYZ}\, (\nabla_X F)\, (Y, Z) = 0. \tag{33}$$

From (33), we get the conditions (**1**), (**5**) and we have

$$\alpha(U, A, B) = \alpha(A, U, V) = 0$$

for all $A, B \in \mathcal{V}$ and $U, V \in \mathcal{H}$ and $\delta F = 0$. Hence the conditions (**3**), (**4**), (**7**) and (**8**) are satisfied. In addition we obtain the condition (**6**) since (30) holds. Therefore N is in W_2. □

Example 3.10. *Consider a Lie group* G *of dimension* 4 *with Lie algebra* $g_{4.3}$, *where*

$$[e_1, e_4] = e_1, \quad [e_3, e_4] = e_2.$$

Suppose a left-invariant almost parahermitian structure on G *determined by*

$$J(e_1) = -e_1, \quad J(e_2) = -e_2, \quad J(e_3) = e_3, \quad J(e_4) = e_4.$$

We choose the left-invariant metric h of the Lie group as follows

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

It can be seen that (J, h) is a left-invariant almost parahermitian structure on the Lie group G. Using the Koszul formula, we get non-zero components of Levi-Civita covariant derivativess as

$$\nabla_{e_4} e_1 = -e_1, \quad \nabla_{e_4} e_3 = -e_2, \quad \nabla_{e_4} e_4 = e_4.$$

Since $\mathcal{V} = Span\{e_4, e_3\}$ and $\mathcal{H} = Span\{e_1, e_2\}$, non-zero Levi-Civita covariant derivative of J is evaluated as

$$(\nabla_{e_4} J)(e_3) = -2e_2.$$

By direct calculation, all conditions except condition **(2)** *are satisfied and the left-invariant parahermitian structure (J, h) on G is in W_2. Then from Theorem 3.9 the left-invariant paracontact structure on $G \times \mathbb{R}$ is of the class $\mathbb{G}_4^2$.*

Theorem 3.11. *Almost parahermitian manifold (N, J, h) is in W_3 if and only if $(N \times \mathbb{R}, \tilde{\xi}, \tilde{\varphi}, \tilde{\eta}, \tilde{g})$ is an almost paracontact metric manifold of class $\mathbb{G}_2^2$.*

Proof. Let (N, J, h) belongs to W_3. In this class following relations hold:

$$\alpha(A, B, C) = \alpha(U, A, B) = \alpha(U, V, W) = 0, \ \delta F = 0$$

for all $A, B, C \in \mathcal{V}$ and $U, V, W \in \mathcal{H}$. Then we have

$$\alpha(X, Y, Z) = \alpha(J(X), Y, Z) = -\alpha(J(X), J(Y), Z). \tag{34}$$

Thus we get $\tilde{\theta} = 0$ and

$$\tilde{\beta}(\tilde{X}, \tilde{Y}, \tilde{Z}) = -\tilde{\beta}(\tilde{\varphi}(\tilde{X}), \tilde{\varphi}(\tilde{Y}), \tilde{Z}) = \tilde{\beta}(\tilde{\varphi}(\tilde{X}), \tilde{Y}, \tilde{Z}). \tag{35}$$

Hence $N \times \mathbb{R}$ is of class $\mathbb{G}_2^2$.

Now, let us assume that $N \times \mathbb{R}$ is in $\mathbb{G}_2^2$, then (35) is satisfied and we have $\tilde{\theta}(\tilde{X}) = 0$. The equation (35) implies (34). Also we have $\delta F = 0$ since $\tilde{\theta}(\tilde{X}) = 0$. In addition, taking into account (34) we have

$$\alpha(A, B, Z) = \alpha(U, A, Z) = \alpha(U, V, Z) = 0,$$

so we get all conditions except the condition **(3)**. Hence the manifold N belongs to the class W_3. □

Example 3.12. *[10] Consider a Lie group G of dimension 6 with Lie algebra, where*

$$[e_4, e_5] = e_6.$$

Suppose a left-invariant almost parahermitian structure on the Lie group G defined by

$$J(e_1) = e_1, \ J(e_2) = e_2, \ J(e_3) = e_3, \ J(e_4) = -e_4, \ J(e_5) = -e_5, \ J(e_6) = -e_6.$$

We choose the left-invariant metric h of the Lie group as follows

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

(J, h) is a left-invariant almost parahermitian structure on G. From the Koszul formula, the non-zero Levi-Civita covariant derivatives are

$$\nabla_{e_3} e_4 = -\tfrac{1}{2} e_2, \quad \nabla_{e_3} e_5 = \tfrac{1}{2} e_1, \quad \nabla_{e_4} e_3 = -\tfrac{1}{2} e_2,$$
$$\nabla_{e_4} e_5 = \tfrac{1}{2} e_6, \quad \nabla_{e_5} e_3 = \tfrac{1}{2} e_1, \quad \nabla_{e_5} e_4 = -\tfrac{1}{2} e_6.$$

We calculate the non-zero Levi-Civita covariant derivatives of J as

$$(\nabla_{e_3} J)(e_4) = e_2, \quad (\nabla_{e_3} J)(e_5) = -e_1.$$

It can be proved that the condition **(3)** *is not satisfied and the other conditions (condition* **(1)**, *condition* **(2)**, *condition* **(4)** *,..., condition* **(8)***) hold. Hence $G \times \mathbb{R}$ is in $\mathbb{G}_2^2$ from Theorem 3.11.*

Theorem 3.13. *Almost parahermitian manifold (N, J, h) belongs to the class W_4 if and only if almost paracontact metric manifold $(N \times \mathbb{R}, \tilde{\xi}, \tilde{\varphi}, \tilde{\eta}, \tilde{g})$ belongs to the class $\mathbb{G}_1^2$.*

Proof. Let the manifold (N, J, h) be of class W_4. In this class we have

$$\alpha(A, B, C) = \alpha(U, V, W) = \alpha(U, A, B) = 0, \ \delta F(A) = 0$$

for all $A, B, C \in \mathcal{V}$ and $U, V, W \in \mathcal{V}$. Then

$$\tilde{\beta}(\tilde{\varphi}^2(\tilde{X}), \tilde{\varphi}^2(\tilde{Y}), \tilde{\varphi}^2(\tilde{Z})) + \tilde{\beta}(\tilde{\varphi}(\tilde{X}), \tilde{\varphi}^2(\tilde{Y}), \tilde{\varphi}(\tilde{Z})) = 0, \tag{36}$$

$$\tilde{\beta}(\tilde{\varphi}(\tilde{X}), \tilde{Y}, \tilde{Z}) = \tilde{\beta}(\tilde{X}, \tilde{Y}, \tilde{Z})$$

are fulfilled. Taking into account (36) we conclude that

$$\tilde{\beta}^3 = \tilde{\beta}^4 = 0.$$

In addition, from condition **(3)**, we obtain

$$\tilde{\beta}^1(\tilde{X}, \tilde{Y}, \tilde{Z}) = \alpha(A, U, V),$$

hence $\tilde{\beta}^2(\tilde{X}, \tilde{Y}, \tilde{Z}) = 0$. As a result $N \times \mathbb{R}$ belongs to the class $\mathbb{G}_1^2$.

Conversely, let $N \times \mathbb{R}$ be in $\mathbb{G}_1^2$. In this case we get

$$\tilde{\beta}(\tilde{X}, \tilde{Y}, \tilde{Z}) = \tilde{\beta}^1(\tilde{X}, \tilde{Y}, \tilde{Z}) \tag{37}$$

and

$$\tilde{\beta}(\tilde{X}, \tilde{Y}, \tilde{Z}) = \tilde{\beta}(\tilde{\varphi}(\tilde{X}), \tilde{Y}, \tilde{Z}). \tag{38}$$

From (38) we have

$$\alpha(U, Y, Z) = 0 \tag{39}$$

for all $U \in \mathcal{H}$, $Y, Z \in \chi(N)$. From (39), we get conditions **(5)**, **(6)**, **(7)** and **(8)**. Setting $\tilde{X} = (A, 0)$ and $\tilde{Y} = (U, 0)$ and $\tilde{Z} = (V, 0)$ in (37), we have the condition **(3)**. In addition, setting $\tilde{X} = (A, 0)$ and $\tilde{Y} = (A, 0)$ and $\tilde{Z} = (Z, 0)$ in the equation (37) we obtain

$$\alpha(A, A, Z) = 0,$$

thus we get conditions **(1)**, **(2)**. As a result, N belongs to the class W_4. □

Example 3.14. *Consider a Lie group G of dimension 4 with Lie algebra $g_{4.10}$, where*

$$[e_1, e_4] = e_1, \quad [e_3, e_4] = e_2.$$

Consider a left-invariant almost parahermitian structure on G determined by

$$J(e_1) = -e_1, \quad J(e_2) = e_2, \quad J(e_3) = e_3, \quad J(e_4) = -e_4.$$

We use the left-invariant metric h of the G given by

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

(J,h) is a left-invariant almost parahermitian structure on G. Non-zero Levi-Civita covariant derivatives are

$$\begin{array}{lll} \nabla_{e_1}e_2 = -\frac{1}{2}e_3, & \nabla_{e_1}e_3 = -\frac{1}{2}e_3, & \nabla_{e_1}e_4 = \frac{1}{2}e_1 + \frac{1}{2}e_4, \\ \nabla_{e_2}e_1 = -\frac{1}{2}e_3, & \nabla_{e_2}e_4 = \frac{1}{2}e_2, & \nabla_{e_3}e_1 = -\frac{1}{2}e_3, \\ \nabla_{e_3}e_4 = \frac{1}{2}e_2, & \nabla_{e_4}e_1 = -\frac{1}{2}e_1 + \frac{1}{2}e_4, & \nabla_{e_4}e_2 = \frac{1}{2}e_2, \\ \nabla_{e_4}e_3 = -\frac{1}{2}e_2. & & \end{array}$$

Since $\mathcal{V} = Span\{e_2, e_3\}$ and $\mathcal{H} = Span\{e_1, e_4\}$, we evaluate non-zero Levi-Civita covariant derivatives of J as

$$(\nabla_{e_2}J)(e_1) = e_3,\ (\nabla_{e_2}J)(e_4) = -e_2,\ (\nabla_{e_3}J)(e_1) = e_3,\ (\nabla_{e_3}J)(e_4) = -e_2.$$

It is seen that all conditions except condition **(4)** *are valid. The structure (J,h) on G is in W_4. Therefore from Theorem 3.13, the paracontact structure on $G \times \mathbb{R}$ is of the class $\mathbb{G}_1^2$.*

Theorem 3.15. *(N,J,h) is of the class W_5 if and only if $(N \times \mathbb{R}, \tilde{\xi}, \tilde{\varphi}, \tilde{\eta}, \tilde{g})$ is of the class $\mathbb{G}_3^1$.*

Proof. Assume that (N,J,h) is in W_5. In this class all conditions except the condition **(5)** are satisfied. Then we have the followings:

$$\alpha(A,B,C) = \alpha(A,U,V) = \alpha(U,A,B) = 0, \tag{40}$$

also

$$\alpha(X,Y,Z) = -\alpha(Y,X,Z). \tag{41}$$

for all $A,B,C \in \mathcal{V}$, $U,V \in \mathcal{H}$, $X,Y,Z \in \chi(N)$. In addition from (40) we have

$$\alpha(X,Y,Z) = -\alpha(J(X),Y,Z). \tag{42}$$

Now (42) implies

$$\tilde{\beta}(\tilde{X},\tilde{Y},\tilde{Z}) = -\tilde{\beta}(\tilde{\varphi}(\tilde{X}),\tilde{Y},\tilde{Z}). \tag{43}$$

Moreover if we use (41), we derive

$$\tilde{\beta}(\tilde{X},\tilde{Y},\tilde{Z}) = -\tilde{\beta}(\tilde{Y},\tilde{X},\tilde{Z}). \tag{44}$$

Thus $N \times \mathbb{R}$ is of the class $\mathbb{G}_3^1$.

If $N \times \mathbb{R}$ is in $\mathbb{G}_3^1$, then (43) and (44) are satisfied. Then we have (41) and (42). From (41) and (42), all conditions except the condition **(5)** are hold. Thus N is in W_5. □

Example 3.16. *Let G be a 6-dimensional Lie group with Lie algebra, whose non-zero brackets satisfy*

$$[e_1,e_2] = 2e_3, \quad [e_1,e_6] = 2e_5, \quad [e_2,e_6] = 2e_4.$$

Define a left-invariant almost parahermitian structure on G by

$$J(e_1) = -e_1,\ J(e_2) = -e_2,\ J(e_3) = e_3,\ J(e_4) = e_4,\ J(e_5) = e_5,\ J(e_6) = -e_6.$$

We choose the left-invariant metric h of the Lie group G as follows

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

It is verified that (J,h) is a left-invariant almost parahermitian structure on G. We write the non-zero Levi-Civita covariant derivatives as

$$\nabla_{e_1}e_2 = e_3, \quad \nabla_{e_2}e_1 = -e_3, \quad \nabla_{e_1}e_6 = e_5,$$
$$\nabla_{e_6}e_1 = -e_5, \quad \nabla_{e_2}e_6 = e_4, \quad \nabla_{e_6}e_2 = -e_4.$$

The non-zero components of the Levi-Civita covariant derivatives of J are

$$(\nabla_{e_1}J)\,e_2 = -2e_3 = -\left(\nabla_{e_2}J\right)e_1,$$
$$(\nabla_{e_1}J)\,e_6 = -2e_5 = -\left(\nabla_{e_6}J\right)e_1,$$
$$(\nabla_{e_2}J)\,e_6 = -2e_4 = -\left(\nabla_{e_6}J\right)e_2.$$

$\mathcal{V} = Span\{e_3, e_4, e_5\}$ *and* $\mathcal{H} = Span\{e_1, e_2, e_6\}$ *imply*

$$\alpha(e_1, e_2, e_6) + \alpha(e_2, e_6, e_1) + \alpha(e_6, e_1, e_2) = -6 \neq 0,$$

and as a results the condition **(5)** *is not satisfied. In addition it can be shown that the other conditions are met. Hence the structure (J,h) on G is in W_5. Then the left-invariant paracontact structure on $G\times\mathbb{R}$ is of the class $\mathbb{G}_3^1$ from Theorem 3.15.*

Theorem 3.17. *(N, J, h) belongs to class W_6 if and only if $(N\times\mathbb{R}, \tilde{\xi}, \tilde{\varphi}, \tilde{\eta}, \tilde{g})$ belongs to the class $\mathbb{G}_4^1$.*

Proof. Let (N, J, h) be in W_6. The equations

$$\alpha(A, B, C) = \alpha(A, U, V) = \alpha(U, A, B) = 0$$

hold in the class W_6. Then we have

$$\alpha(X, Y, Z) = -\alpha(J(X), Y, Z) \tag{45}$$

and

$$\mathfrak{S}_{XYZ}\,(\nabla_X F)\,(Y, Z) = 0. \tag{46}$$

for all $X, Y, Z \in \chi(N)$. Hence we obtain

$$\tilde{\beta}(\tilde{X}, \tilde{Y}, \tilde{Z}) = -\tilde{\beta}(\tilde{\varphi}(\tilde{X}), \tilde{Y}, \tilde{Z}) \tag{47}$$

and

$$\mathfrak{S}_{\tilde{X}\tilde{Y}\tilde{Z}}\tilde{\beta}(\tilde{X}, \tilde{Y}, \tilde{Z}) = 0. \tag{48}$$

Thus, $N\times\mathbb{R}$ is in $\mathbb{G}_4^1$.

If $N\times\mathbb{R}$ is of the class $\mathbb{G}_4^1$, then (47) and (48) are satisfied. We get (45) and (46) from (47) and (48). Then (45) and (46) yield that all conditions except the condition **(6)** hold. As a result, N is in W_6. □

Example 3.18. *Consider a Lie group G of dimension 4 with Lie algebra $g_{4.3}$, where*

$$[e_1, e_4] = e_1, \quad [e_3, e_4] = e_2.$$

A left-invariant almost parahermitian structure on G is determined by

$$J(e_1) = e_1, \quad J(e_2) = e_2, \quad J(e_3) = -e_3, \quad J(e_4) = -e_4.$$

We choose the left-invariant metric h of the Lie group as below

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

It can be seen that (J, h) is a left-invariant almost parahermitian structure on G. The non-zero Levi-Civita covariant derivatives are

$$\nabla_{e_4} e_1 = -e_1, \quad \nabla_{e_4} e_3 = -e_2, \quad \nabla_{e_4} e_4 = e_4.$$

Since $\mathcal{V} = Span\{e_1, e_2\}$ and $\mathcal{H} = Span\{e_3, e_4\}$, we evaluate non-zero Levi-Civita covariant derivative of J as

$$(\nabla_{e_4} J)(e_3) = 2e_2.$$

It is proved that all conditions except condition **(6)** *are satisfied. Hence the structure (J, h) on G is in W_6. Then Theorem 3.17 implies that $G \times \mathbb{R}$ is of the class $\mathbb{G}_4^1$.*

Theorem 3.19. *(N, J, h) is in W_7 if and only if $(N \times \mathbb{R}, \tilde{\xi}, \tilde{\varphi}, \tilde{\eta}, \tilde{g})$ is in $\mathbb{G}_2^1$.*

Proof. Let (N, J, h) belongs to W_7. We have

$$\alpha(A, B, C) = \alpha(U, V, W) = \alpha(A, U, V) = 0, \ \delta F = 0. \tag{49}$$

Taking into account (49) we get

$$\alpha(X, Y, Z) = -\alpha(J(X), Y, Z), \tag{50}$$

$$\alpha(X, Y, Z) = -\alpha(J(X), J(Y), Z). \tag{51}$$

Also we obtain $\tilde{\theta} = 0$ from $\delta F = 0$. (50) and (51) yield

$$\tilde{\beta}^3(\tilde{X}, \tilde{Y}, \tilde{Z}) = \tilde{\beta}^4(\tilde{X}, \tilde{Y}, \tilde{Z}) = 0$$

and

$$\tilde{\beta}^1(\tilde{X}, \tilde{Y}, \tilde{Z}) = 0$$

since $\tilde{\theta} = \delta F = 0$. Also, (50) and (51) imply

$$\tilde{\beta}(\tilde{X}, \tilde{Y}, \tilde{Z}) = -\tilde{\beta}(\tilde{\varphi}(\tilde{X}), \tilde{\varphi}(\tilde{Y}), \tilde{Z}) = -\tilde{\beta}(\tilde{\varphi}(\tilde{X}), \tilde{Y}, \tilde{Z}) \tag{52}$$

Thus $N \times \mathbb{R}$ is of class $\mathbb{G}_2^1$.

Conversely, if $N \times \mathbb{R}$ is in $\mathbb{G}_2^1$, then (52) is satisfied and $\tilde{\theta}_{\tilde{\beta}}(\tilde{X}) = 0$. The conditions (**4**) and (**8**) hold since $\delta F(X) = \tilde{\theta}_{\tilde{\beta}}(\tilde{X}) = 0$. From (52), we have

$$\alpha(A, B, C) = \alpha(A, U, V) = \alpha(U, V, W) = 0.$$

Thus the conditions (**1**), (**2**), (**3**), (**5**) and (**6**) are obtained. As a result the manifold N belongs to the class W_7. □

Example 3.20. *[10] Consider a Lie group G of dimension 6 with Lie algebra, where*

$$[e_1, e_2] = e_3.$$

Suppose a left-invariant almost parahermitian structure on the Lie group G determined by

$$J(e_1) = e_1, \ J(e_2) = e_2, \ J(e_3) = e_3, \ J(e_4) = -e_4, \ J(e_5) = -e_5, \ J(e_6) = -e_6.$$

We choose the left-invariant metric h of the Lie group as follows

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

(J, h) is a left-invariant almost parahermitian structure on G. Using the Koszul formula, we compute the non-zero Levi-Civita covariant derivatives as

$$\nabla_{e_1} e_2 = \tfrac{1}{2} e_3, \quad \nabla_{e_1} e_6 = -\tfrac{1}{2} e_5, \quad \nabla_{e_2} e_1 = -\tfrac{1}{2} e_3,$$
$$\nabla_{e_2} e_6 = \tfrac{1}{2} e_4, \quad \nabla_{e_6} e_1 = -\tfrac{1}{2} e_5, \quad \nabla_{e_6} e_2 = \tfrac{1}{2} e_4.$$

Also, non-zero Levi-civita covariant derivatives of J are

$$(\nabla_{e_6} J)(e_1) = -e_5, \quad (\nabla_{e_6} J)(e_2) = e_4$$

It is shown that other conditions except condition **(8)** *are valid. Hence the structure (J, h) on G is in W_7. Then Theorem 3.19 yields that the structure on $G \times \mathbb{R}$ is in $\mathbb{G}_2^1$.*

Theorem 3.21. *Almost parahermitian manifold (N, J, h) is in W_8 if and only if almost paracontact metric manifold $(N \times \mathbb{R}, \tilde{\xi}, \tilde{\varphi}, \tilde{\eta}, \tilde{g})$ belongs to the class $\mathbb{G}_1^1$.*

Proof. Let (N, J, h) be a manifold of class W_8. We have

$$\tilde{\beta}^3 = \tilde{\beta}^4 = 0$$

since $\alpha(X, Y, Z) = -\alpha(J(X), Y, Z) = -\alpha(J(X), J(Y), Z)$ in W_8. Also, from the condition **(7)**

$$\tilde{\beta}^1(\tilde{X}, \tilde{Y}, \tilde{Z}) = \tilde{\beta}(\tilde{X}, \tilde{Y}, \tilde{Z})$$

is obtained. Thus $N \times \mathbb{R}$ is of class $\mathbb{G}_1^1$.

Conversely, if $N \times \mathbb{R}$ is of class $\mathbb{G}_1^1$, then we get

$$\tilde{\beta}(\tilde{X}, \tilde{Y}, \tilde{Z}) = -\tilde{\beta}(\tilde{\varphi}(\tilde{X}), \tilde{Y}, \tilde{Z}) \tag{53}$$

and

$$\tilde{\beta}(\tilde{X}, \tilde{Y}, \tilde{Z}) = \tilde{\beta}^1(\tilde{X}, \tilde{Y}, \tilde{Z}). \tag{54}$$

(53) and (54) imply

$$\alpha(X, Y, Z) = -\alpha(J(X), Y, Z) = -\alpha(J(X), J(Y), Z).$$

Then all conditions except the condition **(8)** are valid. Therefore N is in W_8. □

Example 3.22. *Consider a Lie group G of dimension 4 with Lie algebra $g_{4.10}$, where*

$$[e_1, e_3] = e_1, \quad [e_2, e_3] = e_2, \quad [e_1, e_4] = -e_2, \quad [e_2, e_4] = e_1.$$

Suppose a left-invariant almost parahermitian structure on the Lie group G given by

$$J(e_1) = -e_1, \quad J(e_2) = -e_2, \quad J(e_3) = e_3, \quad J(e_4) = e_4.$$

The left-invariant metric h of the Lie group is chosen by

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Then (J, h) is a left-invariant almost parahermitian structure on G with non-zero components of the Levi-Civita covariant derivatives evaluated as

$$\nabla_{e_1} e_3 = e_1, \quad \nabla_{e_1} e_4 = -e_2, \quad \nabla_{e_3} e_2 = -e_2,$$
$$\nabla_{e_3} e_3 = e_3, \quad \nabla_{e_4} e_2 = -e_1, \quad \nabla_{e_4} e_4 = e_3.$$

Since $\mathcal{V} = Span\{e_4, e_3\}$ and $\mathcal{H} = Span\{e_1, e_2\}$, we write the non-zero components of Levi-Civita covariant derivative of J as

$$(\nabla_{e_1} J)(e_3) = 2e_1, \quad (\nabla_{e_1} J)(e_4) = -2e_2.$$

It is verified that all conditions except condition **(8)** *hold. Hence the structure (J, h) on G is of the class W_8. Then the left-invariant paracontact structure on $G \times \mathbb{R}$ is of the class $\mathbb{G}_1^1$ from Theorem 3.21.*

The relationship between classes are summarized in the table below:

The class to which the manifold N belongs	The class to which themanifold $N \times \mathbb{R}$ belongs
W_1	$\mathbb{G}_3^2$
W_2	$\mathbb{G}_4^2$
W_3	$\mathbb{G}_2^2$
W_4	$\mathbb{G}_1^2$
W_5	$\mathbb{G}_3^1$
W_6	$\mathbb{G}_4^1$
W_7	$\mathbb{G}_2^1$
W_8	$\mathbb{G}_1^1$

References

[1] A. M. Blaga, *η-Ricci solitons on para-Kenmotsu manifolds*, Balkan J. Geom. Appl. **20**(1) (2015), 1–13.
[2] D. Chinea, C. Gonzalez, *A Classification of Almost Contact Metric Manifolds*, Ann. Mat. Pura Appl. **156** (1990), 15–36.
[3] J. T. Cho, M. Kimura, *Ricci solitons and real hypersurfaces in a complex space form*, Tohoku Math. J. **61**(2) (2009) 205–212.
[4] M. Fernandez, A. Gray, *Riemannian Manifolds with structure group G_2*, Ann. Mat. Pura Appl. **132** (1982), 19–45.
[5] P. M. Gadea, J. M. Masque, *Classification of Almost Para-Hermitian Manifolds*, Rend. Mat. Appl. (7), **7**(11) (1991), 377–396.
[6] G. Ganchev, V. Mihova, K.Gribachev, *Almost Contact Manifolds with B-Metric*, Math. Balkanica (N.S.) **7** (1993), 261–167.
[7] A. Gray, L. M. Hervella, *The Sixteen Classes of Almost Hermitian Manifolds and Their Linear Invariants*, Ann. Mat. Pura Appl. **123** (1980), 35–58.
[8] J. M. Lee, *Riemannian Manifolds: An Introduction to Curvature*, (3rd edition), Springer, New York, 1997.
[9] M. Manev, *Ricci-like solitons on almost contact B-metric manifolds*, J. Geom. Phys. **154** (2020),103734.
[10] L. Nourmohammadifar, E. Peyghan, *Nearly Para-Kahler Geometry on Lie Groups*, Quaest. Math. **45**(11) (2022), 1713–1738.
[11] Barret O'neill, *Semi-Riemannian Geometry with Applications to Relativity*, Academic Press, London, 1983.
[12] J. A. Oubina, *A classification for Almost Contact Structure*, Preprint, 1980.
[13] N. Özdemir, N. Erdoğan, *Some Relations Between Almost Paracontact Metric Manifolds and almost Parahermitian manifolds*, Turkish J. Math. **46** (2022), 1459–1477.
[14] N. Özdemir, Ş.Aktay, M. Solgun, *Almost Hermitian Structures from Almost Contact Metric Manifolds and their Curvature Properties*, Konuralp J. Math. **12**(1) (2024), 5–12.
[15] N. Özdemir, Ş.Aktay, M. Solgun, *Some Results on Normal Almost Contact Manifolds with B-Metric*, Kragujevac J. Math. **50**(4) (2026), 597-611.
[16] D. S. Patra, *Ricci Solitons and Paracontact Geometry*, Mediterr. J. Math. **16** (2019), 137.
[17] D. S. Patra, V. Rovenski, *Almost η-Ricci solitons on Kenmotsu manifolds*, Eur. J. Math. **7** (2021), 1753–1766.
[18] M. Solgun, *On Constructing almost Complex Norden Metric Structures*, AIMS Math. **7**(10) (2022), 17942–17953.
[19] M. Solgun, Y. Karababa, *A Natural Way to Construct an Almost Complex B-metric Structure*, Math. Methods Appl. Sci. **44**(9) (2021), 7607–7613.
[20] S. Zamkovoy, G. Nakova, *The Decomposition of Almost Paracontact Metric Manifolds in Eleven Classes Revisited*, J. Geom. **109**(18) (2018), 0247–2468.