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On some common fixed point results of Górnicki-type mappings

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Abstract. In this article, we focus on a pair of Górnicki-type mappings $T_1, T_2 \in X^X$, (X,d) being a metric space, satisfying $d(T_1x, T_2y) \leq M[d(x, T_2y) + d(y, T_1x) + d(x,y)]$ for all $x, y \in X$ where $0 \leq M < 1$. The significance of such mappings lies in their broader class compared to contraction and nonexpansive mappings. Our main focus is on the common fixed point(s) of this pair of Górnicki-type mappings. Specifically, we establish conditions under which a couple of mappings share fixed points satisfying the aforementioned inequality. Additionally, we provide several non-trivial examples to validate our results.

1. Introduction

A mapping $T \in X^X$ where (X,d) is a metric space, is called a (Banach) contraction mapping if there exists a non-negative constant $\alpha < 1$ such that $d(Tx, Ty) \leq \alpha d(x,y)$ for all $x, y \in X$. The notion of contraction mappings is a fundamental concept in metric fixed point theory. It originated in 1922 through the pioneering work of Banach, who introduced these mappings. Since then, metric fixed point theory has flourished, with renowned mathematicians contributing various fixed point results. These results guarantee the existence of fixed points for mappings that satisfy different types of contraction conditions. Notable references include works by authors such as Ekeland, Reich, Browder, Nadler, and others (see [1, 3–8, 11, 12, 16–19, 23–25, 25–28, 30, 31, 33–35, 38, 38–41, 44]).

The subject has rapidly grown and become a major research area in mathematical analysis. The practical applicability of fixed-point results adds to its allure, making it an active field of study. Researchers are motivated to generalize and modify contraction mappings, leading to a rich variety of contraction-type mappings. In many of these conditions, the classical Banach contraction is enhanced by considering additional displacements, such as $d(x, Tx)$, $d(x, Ty)$ and $d(y, Tx)$, along with the original distances $d(x,y)$ and $d(Tx, Ty)$. Few of such well-known contractions are listed below:

Definition 1.1. *Let (X,d) be a metric space. A mapping $T \in X^X$ is said to be a*

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(a) Kannan contraction mapping *if there exists a constant* c *with* $0 \le c < \frac{1}{2}$ *such that*

$$d(Tx, Ty) \le c[d(x, Tx) + d(y, Ty)]$$

for all $x, y \in X$ *[25].*

(b) Chatterjea contraction mapping *if there exists a constant* c *with* $0 \le c < \frac{1}{2}$ *such that*

$$d(Tx, Ty) \le c[d(x, Ty) + d(y, Tx)]$$

for all $x, y \in X$ *[10].*

(c) Reich contraction *if*

$$d(Tx, Ty) \le \alpha_1 d(x, y) + \alpha_2 d(x, Tx) + \alpha_3 d(y, Ty)$$

for all $x, y \in X$*, where* $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}_+$ *with* $\alpha_1 + \alpha_2 + \alpha_3 < 1$ *[36].*

(d) Ćirić contraction *if*

$$d(Tx, Ty) \le \alpha \max\left\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\right\}$$

for all $x, y \in X$*, where* $\alpha \in [0, 1)$ *[11].*

Here we can note down one important observation about the constants arising in the aforementioned contractions. In each of the above mappings, the sum of the constants (i.e., α, α_i) arising above < 1 and it is known that if the sum of constants is relaxed to ≥ 1, then the mappings may not acquire fixed point(s). So, it is now legitimate to study about the fact that if the sum of the constants ≥ 1, then what sufficient condition can assure us about the existence of fixed point(s) of such mappings. From the theory of non-expansive mappings, one can get a number of such sufficient conditions if the sum of the constants is exactly equal to 1, see [9, 14, 15, 20, 21, 29, 32, 37, 42, 43]. For the case where the sum of constants is strictly greater than 1, Górnicki put in place a sufficient condition which can ensure the existence of fixed point(s) for the mapping considered in Definition 1.1 (b). He instigated such condition by obtaining the following result:

Theorem 1.2. ([22, p. 2150, Theorem 3.3]) *If* (X, d) *is a complete metric space and* $T \in X^X$*, is an asymptotically regular mapping such that there exists* $M < 1$ *satisfying*

$$d(Tx, Ty) \le M[d(x, Tx) + d(y, Ty) + d(x, y)] \tag{1}$$

for every $x, y \in X$*, then* T *has a unique fixed point* $v \in X$*.*

After this, Popescu obtained some other sufficient condition for the above type of mappings in the following result:

Theorem 1.3. ([35, p. 4, Theorem 4]) *Let* (X, d) *be a complete metric space and let* $T \in X^X$*. Suppose that* T *satisfies* (1) *and there exist non-negative real constants* a, b *with* $a < 1$ *such that for any* $x \in X$ *there exists* $u \in X$ *with* $d(u, Tu) \le a d(x, Tx)$ *and* $d(u, x) \le b d(x, Tx)$*. Then,* T *has a unique fixed point.*

According to Popescu, the mapping satisfying the conditions of the above theorem, is called a Górnicki mapping.

If we go through the literature of fixed point theory, then we can set up two scenario. The first one is that there are many fixed point results related to contraction type mappings, where the contraction conditions involve the displacements $d(x, Ty)$ and $d(y, Tx)$, and the second one is that there are again a lot of results which deal with coincidence points and/or common fixed points of two or more mappings. It is noteworthy to mention that in the context of the above two scenarios, it is legitimate to think about two important directions of research: the first one is whether the above type of results can be extended to Górnicki-type mappings where the displacements $d(x, Tx)$ and $d(y, Ty)$ can be changed to the displacements $d(x, Ty)$ and

$d(y, Tx)$, and the second one is whether the above type of results can be acquired in case of common fixed points of a pair of mappings. Recently, Debnath [13] have concentrated on the second observation and obtained the following result. Before we state that theorem, let us first recall the following definition:
A pair of mappings $T_1, T_2 \in X^X$, is said to be a Górnicki-type pair of mappings if there exists non-negative real constants α, β with $\alpha < 1$ such that for any $x \in X$, there exists $y \in X$ with

$$d(y, T_1 y) \leq \alpha d(x, T_2 x),\ d(y, x) \leq \beta d(x, T_2 x)$$

and

$$d(y, T_2 y) \leq \alpha d(x, T_1 x),\ d(y, x) \leq \beta d(x, T_1 x) \ \text{([13, p. 2, Definition 2.1])}.$$

Theorem 1.4. ([13, p. 3, Theorem 2.2]) *Let (X, d) be a complete metric space and $T_1, T_2 \in X^X$ be a Górnicki-type pair of mappings satisfying the following condition for $M \in (0, 1)$:*

$$d(T_1 x, T_2 y) \leq M[d(x, T_1 x) + d(y, T_2 y) + d(x, y)] \tag{2}$$

for all $x, y \in X$. Then, T_1 and T_2 have a unique common fixed point.

In this article, our primary goal is to achieve two significant outcomes simultaneously by proving a single result. At first, we procure a couple of common fixed point results concerning Górnicki-type mapping where the displacements $d(x, Tx)$ and $d(y, Ty)$ can be changed to $d(x, Ty)$ and $d(y, Tx)$. After this, we show that the conditions in Theorem 1.4 which are used to get the guaranty of existence of common fixed points due to Debnath can be weakened without affecting the conclusions of the result. Finally, we provide relevant examples to support our derived results.

2. Common fixed point of Górnicki-type mappings

Throughout the paper, X^X denotes the collection of all mappings $f : X \to X$ for a non-empty set X. We use the notation $Fix(T)$ to denote the set of all fixed points of a mapping T and $\mathbb{N}_0$ to denote the set $\mathbb{N} \cup \{0\}$.

At the beginning of this section, we put in place a result which deals with the existence of common fixed points of a couple of mappings satisfying the Górnicki-type condition.

Theorem 2.1. *Let (X, d) be a complete metric space and let $T_1, T_2 \in X^X$. Suppose that there exists an $M \in [0, 1)$ such that*

$$d(T_1 x, T_2 y) \leq M[d(x, T_2 y) + d(y, T_1 x) + d(x, y)] \tag{3}$$

for all $x, y \in X$. If there exist real numbers a, b with $0 \leq a < 1$, $b > 0$ such that for each $x \in X$ there exists $u \in X$ satisfying $d(u, T_1 u) \leq ad(x, T_2 x)$, $d(u, T_2 u) \leq ad(x, T_1 x)$ and $d(u, x) \leq b \max\{d(x, T_1 x), d(x, T_2 x)\}$. Then, T_1, T_2 admit a common fixed point in X.

Proof. Let $x_0 \in X$ be chosen arbitrarily. Then, by hypothesis, there exists an element $x_1 \in X$ such that

$$d(x_1, T_1 x_1) \leq ad(x_0, T_2 x_0), d(x_1, T_2 x_1) \leq ad(x_0, T_1 x_0)$$

and

$$d(x_1, x_0) \leq b \max\{d(x_0, T_1 x_0), d(x_0, T_2 x_0)\}.$$

Again, there exists $x_2 \in X$ such that

$$d(x_2, T_1 x_2) \leq ad(x_1, T_2 x_1) \leq a^2 d(x_0, T_1 x_0)$$

$$d(x_2, T_2 x_2) \leq ad(x_1, T_1 x_1) \leq a^2 d(x_0, T_2 x_0)$$

and

$$d(x_2, x_1) \leq b \max\{d(x_1, T_1 x_1), d(x_1, T_2 x_1)\}.$$

Continuing this process, we finally get a sequence $\{x_n\}$ in X such that for all $n \in \mathbb{N}$, we have

$$d(x_n, T_1x_n) \leq \begin{cases} a^n d(x_0, T_1x_0) & \text{if } n \text{ is even} \\ a^n d(x_0, T_2x_0) & \text{if } n \text{ is odd} \end{cases},$$

$$d(x_n, T_2x_n) \leq \begin{cases} a^n d(x_0, T_2x_0) & \text{if } n \text{ is even} \\ a^n d(x_0, T_1x_0) & \text{if } n \text{ is odd} \end{cases}$$

and

$$d(x_{n+1}, x_n) \leq b \max\{d(x_n T_1x_n), d(x_n, T_2x_n)\} \leq ba^n \max\{d(x_0, T_1x_0), d(x_0, T_2x_0)\}.$$

It is now an easy exercise to show that $\{x_n\}$ is a Cauchy sequence in X and hence, by completeness of X, there is an element $z \in X$ such that $d(x_n, z) \to 0$ as $n \to \infty$. Also, from the last inequality, we have $d(x_{n+1}, x_n) \to 0$ as $n \to \infty$.

We further have $d(x_n, T_1x_n) \to 0$ as $n \to \infty$ and $d(x_n, T_2x_n) \to 0$ as $n \to \infty$. Using triangle inequality, we deduce that $d(T_1x_n, z) \to 0$ as $n \to \infty$ and $d(T_2x_n, z) \to 0$ as $n \to \infty$.

Taking $x = x_n$, $y = z$ and putting in Equation (3), we get

$$d(T_1x_n, T_2z) \leq M[d(x_n, T_2z) + d(z, T_1x_n) + d(x_n, z)].$$

Now, taking limit as $n \to \infty$, we get

$$d(z, T_2z) \leq Md(z, T_2z),$$

which implies that $d(z, T_2z) = 0$, i.e., $z \in Fix(T_2)$.

Now, putting $x = z$, $y = x_n$ in Equation (3), we see that $z \in Fix(T_1)$ which proves that $z \in Fix(T_1) \cap Fix(T_2)$. Thus T_1, T_2 admit a common fixed point. □

Let us consider $T_1 = T_2 = T$ in the above theorem to establish the following result which is analogous to [35, p. 4, Theorem 4]:

Theorem 2.2. *Let (X, d) be a complete metric space and let $T \in X^X$. Suppose that there exists an $M \in [0, 1)$ such that*

$$d(Tx, Ty) \leq M[d(x, Ty) + d(y, Tx) + d(x, y)] \tag{4}$$

for all $x, y \in X$. If there exist real numbers a, b with $0 \leq a < 1$, $b > 0$ such that for each $x \in X$ there exists $u \in X$ satisfying $d(u, Tu) \leq ad(x, Tx)$ and $d(u, x) \leq bd(x, Tx)$. Then, $Fix(T)$ is a non-empty subset of X.

It's noteworthy to note that Theorem 2.1 can only guarantee the presence of the common fixed point; it cannot guarantee its uniqueness. We provide the following example to support this assertion:

Example 2.3. *Let us choose $X = \mathbb{N}_0$ and consider the metric d defined on X by $d(x, y) = 0$ or 1 according as $x = y$ or $x \neq y$. Next, we define two mappings $T_1, T_2 \in X^X$ by*

$$T_1x = x^2 \text{ and } T_2x = x^4 \text{ for all } x \in X.$$

Then, for $M = \frac{5}{6}$, one can easily check that (3) holds for all $x, y \in X$.

Now, for $x = 0$, we choose $u = 0$; for $x = 1$, we choose $u = 1$ and for $x \neq 0, 1$ we choose $u = 0$ to be sure that $d(u, T_1u) \leq ad(x, T_2x)$, $d(u, T_2u) \leq ad(x, T_1x)$ and $d(u, x) \leq b \max\{d(x, T_1x), d(x, T_2x)\}$ for $a = \frac{6}{7}$ and $b = 2$.

Thus all the assumptions of Theorem 2.1 hold here and still T_1, T_2 have more than one common fixed point viz. 0 and 1.

Now, we move on to our next proposed aim. Particularly, we weaken the assumptions of [13, p. 3, Theorem 2.2] without affecting the conclusions of the result.

Theorem 2.4. *Let (X, d) be a complete metric space and let $T_1, T_2 \in X^X$. Suppose that there exists an $M \in [0, 1)$ such that*

$$d(T_1x, T_2y) \leq M[d(x, T_1x) + d(y, T_2y) + d(x, y)] \tag{5}$$

for all $x, y \in X$. If there exist real numbers a, b with $0 \leq a < 1$, $b > 0$ such that for each $x \in X$ there exists $u \in X$ satisfying $d(u, T_1u) \leq ad(x, T_2x)$, $d(u, T_2u) \leq ad(x, T_1x)$ and $d(u, x) \leq b \max\{d(x, T_1x), d(x, T_2x)\}$. Then, T_1, T_2 admit a unique common fixed point in X.

Proof. Let $x_0 \in X$ be chosen arbitrarily. Then, by hypothesis, there exists an element $x_1 \in X$ such that

$$d(x_1, T_1x_1) \leq ad(x_0, T_2x_0), d(x_1, T_2x_1) \leq ad(x_0, T_1x_0)$$

and

$$d(x_1, x_0) \leq b \max\{d(x_0, T_1x_0), d(x_0, T_2x_0)\}.$$

Again, there exists $x_2 \in X$ such that

$$d(x_2, T_1x_2) \leq ad(x_1, T_2x_1) \leq a^2d(x_0, T_1x_0),$$

$$d(x_2, T_2x_2) \leq ad(x_1, T_1x_1) \leq a^2d(x_0, T_2x_0)$$

and

$$d(x_2, x_1) \leq b \max\{d(x_1, T_1x_1), d(x_1, T_2x_1)\}.$$

Continuing this process, we finally get a sequence $\{x_n\}$ in X such that for all $n \in \mathbb{N}$, we have

$$d(x_n, T_1x_n) \leq \begin{cases} a^n d(x_0, T_1x_0) & \text{if } n \text{ is even} \\ a^n d(x_0, T_2x_0) & \text{if } n \text{ is odd} \end{cases},$$

$$d(x_n, T_2x_n) \leq \begin{cases} a^n d(x_0, T_2x_0) & \text{if } n \text{ is even} \\ a^n d(x_0, T_1x_0) & \text{if } n \text{ is odd} \end{cases}$$

and

$$d(x_{n+1}, x_n) \leq b \max\{d(x_nT_1x_n), d(x_n, T_2x_n)\} \leq ba^n \max\{d(x_0, T_1x_0), d(x_0, T_2x_0)\}.$$

It is now an easy exercise to show that $\{x_n\}$ is a Cauchy sequence in X and hence, by completeness of X, there is an element $z \in X$ such that $d(x_n, z) \to 0$ as $n \to \infty$. We also have $d(x_n, T_1x_n) \to 0$ and $d(x_n, T_2x_n) \to 0$ as $n \to \infty$. It can also be shown that $d(T_1x_n, z) \to 0$ as $n \to \infty$ as well as $d(T_2x_n, z) \to 0$ as $n \to \infty$.

We now show that $z \in Fix(T_1) \cap Fix(T_2)$. For this, let us put $x = x_n$ and $y = z$ in Equation (5) and we have

$$d(T_1x_n, T_2z) \leq M[(d(x_n, T_1x_n) + d(z, T_2z) + d(x_n, z)].$$

Passing through limit as $n \to \infty$ and applying continuity of f and d, we get

$$d(z, T_2z) \leq Md(z, T_2z),$$

which implies that $d(z, T_2z) = 0$. This shows that $z \in Fix(T_2)$.

Similarly, putting $x = z$ and $y = x_n$ in Equation (5), we see that $z \in Fix(T_1)$ and hence, $z \in Fix(T_1) \cap Fix(T_2)$.

To prove uniqueness, let $z' \in Fix(T_1) \cap Fix(T_2)$. Then,

$$\begin{aligned} d(T_1z, T_2z') &\leq M[d(z, T_1z) + d(z', T_2z') + d(z, z')] \\ &= Md(z, z'), \end{aligned}$$

which implies that

$$d(z, z') \leq Md(z, z').$$

This yields that $d(z, z') = 0$, i.e., $z = z'$. Hence, $Fix(T_1) \cap Fix(T_2) = \{z\}$ and this completes the proof. □

Remark 2.5. *If we take $T_1 = T_2 = T$ in the above theorem, then we can obtain Theorem 1.3 due to Popescu as a consequence.*

We now present few examples in support of our established results. The first two examples are in support of Theorem 2.4.

Example 2.6. *Here we consider the metric space $\mathbb{R}^2$ equipped with the metric d defined by*

$$d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$$

for all $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$.

Let us take $X = \{(x, y) \in \mathbb{R}^2 : x^2+y^2 \leq 1\}$. Let $T_1, T_2 \in X^X$ be defined by $T_1(x, y) = \left(-\frac{x}{3}, \frac{y}{3}\right)$ and $T_2(x, y) = \left(\frac{x}{3}, -\frac{y}{3}\right)$ for all $(x, y) \in X$.

Let $x = (x_1, y_1)$ and $y = (x_2, y_2)$ be taken arbitrarily from X. Then,

$$d(T_1x, T_2y) = \frac{1}{3}\{|x_1 + x_2| + |y_1 + y_2|\},$$

$$\begin{aligned} d(x, T_1x) &= d\left((x_1, y_1), \left(-\frac{x_1}{3}, \frac{y_1}{3}\right)\right) \\ &= \frac{2}{3}|x_1| + \frac{4}{3}|y_1|, \end{aligned}$$

$$\begin{aligned} d(y, T_2y) &= d\left((x_2, y_2), \left(\frac{x_2}{3}, -\frac{y_2}{3}\right)\right) \\ &= \frac{4}{3}|x_2| + \frac{2}{3}|y_2|, \end{aligned}$$

and $d(x, y) = |x_1 - x_2| + |y_1 - y_2|$.

Now, taking $M = \frac{4}{5}$, we get

$$\begin{aligned} &M(d(x, T_1x) + d(y, T_2y) + d(x, y)) \\ &= \frac{4}{5}\left\{\frac{2}{3}|x_1| + \frac{4}{3}|y_1| + \frac{4}{3}|x_2| + \frac{2}{3}|y_2| + |x_1 - x_2| + |y_1 - y_2|\right\} \\ &= \frac{8}{15}|x_1| + \frac{16}{15}|x_2| + \frac{16}{15}|y_1| + \frac{8}{15}|y_2| + \frac{4}{15}|x_1 - x_2| + \frac{4}{15}|y_1 - y_2| \\ &\geq \frac{1}{3}\{|x_1| + |x_2| + |y_1| + |y_2|\} \\ &\geq \frac{1}{3}\{|x_1 + x_2| + |y_1 + y_2|\} \end{aligned}$$

which implies that

$$d(T_1x, T_2y) \leq M[d(x, T_1x) + d(y, T_2y) + d(x, y)]$$

for all $x, y \in X$.

Let $x = (x_1, y_1) \in X$ be arbitrary. Let us take $u = (0, 0) \in X$. Then, for $0 \leq a < 1$ and $b = 4$, we have

$$d(u, T_1u) = d((0, 0), (0, 0)) = 0 \leq ad(x, T_2x)$$

$$d(u, T_2u) = d((0, 0), (0, 0)) = 0 \leq ad(x, T_1x)$$

and

$$d(u, x) = d((0, 0), (x_1, y_1)) = |x_1| + |y_1| \leq b \max\{d(x, T_1x), d(x, T_2x)\}.$$

Thus all the conditions of Theorem 2.4 are satisfied. Therefore, we can conclude that T_1, T_2 have a unique common fixed point in X viz. $(0, 0)$.

Example 2.7. *Let us take $X = \mathbb{R}$ and consider the usual metric d on X. We define $T_1, T_2 \in X^X$ by $T_1x = -2x$ and $T_2x = -x$ for all $x \in X$.*

Let $x, y \in X$ be arbitrary. Then, taking $M = \frac{5}{6}$, we get

$$\begin{aligned} M[d(x, T_1y) + d(y, T_2x) + d(x, y)] &= \frac{5}{6}\{d(x, -2x) + d(y, -y) + d(x, y)\} \\ &= \frac{5}{6}\{3|x| + 2|y| + |x - y|\} \\ &= \frac{5}{2}|x| + \frac{5}{3}|y| + \frac{5}{6}|x - y|, \end{aligned}$$

whereas

$$\begin{aligned} d(T_1x, T_2y) &= |2x - y| \\ &\leq 2|x| + |y| \\ &\leq \frac{5}{2}|x| + \frac{5}{3}|y|. \end{aligned}$$

Therefore,

$$d(T_1x, T_2y) \leq M[d(x, T_1y) + d(y, T_2x) + d(x, y)].$$

Let $x \in X$ be arbitrary. If we take $u = 0 \in X$, then for $0 \leq a < 1$ and $b = 1$, we have

$$d(u, T_1u) = d(0, 0) = 0 \leq ad(x, T_2x)$$

$$d(u, T_2u) = d(0, 0) = 0 \leq ad(x, T_1x)$$

and

$$d(u, x) = d(0, x) = |x| \leq b \max\{d(x, T_1x), d(x, T_2x)\}.$$

Thus wee see that all the conditions of Theorem 2.4 are satisfied. Therefore, we can conclude that T_1, T_2 have a unique common fixed point in X viz. 0.

Next, we present below an example in support of Theorem 2.1.

Example 2.8. *Here we consider the metric space $X = \mathbb{R}^2$ equipped with the metric d defined by*

$$d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$$

for all $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$.

Let us define $T_1, T_2 \in X^X$ be defined by $T_1(x, y) = (0, 0)$ and $T_2(x, y) = (-x, -y)$ for all $(x, y) \in X$.

Let $x = (x_1, y_1)$ and $y = (x_2, y_2)$ be taken arbitrarily from X. Then,

$$d(T_1x, T_2y) = |x_2| + |y_2|.$$

Now,

$$\begin{aligned} d(x, T_1y) &= d((x_1, y_1), (0, 0)) \\ &= |x_1| + |y_1|, \end{aligned}$$

$$\begin{aligned} d(y, T_2x) &= d((x_2, y_2), (-x_1, -y_1)) \\ &= |x_1 + x_2| + |y_1 + y_2|, \end{aligned}$$

and $d(x, y) = |x_1 - x_2| + |y_1 - y_2|$.

Now, taking $M = \frac{4}{5}$, we have

$$
\begin{aligned}
& d(T_1x, T_2y) - M[d(x, T_1y) + d(y, T_2x) + d(x, y)] \\
& = (|x_2| + |y_2|) - \frac{4}{5}\{|x_1| + |y_1| + |x_1 + x_2| + |y_1 + y_2| + |x_1 - x_2| + |y_1 - y_2|\} \\
& = (|x_2| + |y_2|) - \frac{4}{5}(|x_1| + |y_1|) - \frac{4}{5}(|x_1 + x_2| + |x_1 - x_2|) - \frac{4}{5}(|y_1 + y_2| + |y_1 - y_2|) \\
& = (|x_2| + |y_2|) - \frac{4}{5}(|x_1| + |y_1|) - \frac{8}{5}\max\{|x_1|, |x_2|\} - \frac{8}{5}\max\{|y_1|, |y_2|\} \\
& = \frac{1}{5}\{|x_2| + |y_2| - 8\max\{|x_1|, |x_2|\} - 8\max\{|y_1|, |y_2|\} - \frac{4}{5}(|x_1| + |y_1|) \\
& \leq 0
\end{aligned}
$$

which implies that

$$d(T_1x, T_2y) \leq M[d(x, T_1y) + d(y, T_2x) + d(x, y)]$$

for all $x, y \in X$.

Let $x = (x_1, y_1) \in X$ be arbitrary. Let us take $u = (0, 0) \in X$. Then, for $0 \leq a < 1$ and $b = 1$, we have

$$d(u, T_1u) = d((0, 0), (0, 0)) = 0 \leq ad(x, T_2x)$$

$$d(u, T_2u) = d((0, 0), (0, 0)) = 0 \leq ad(x, T_1x)$$

and

$$d(u, x) = d((0, 0), (x_1, y_1)) = |x_1| + |y_1| \leq b\max\{d(x, T_1x), d(x, T_2x)\}.$$

Thus all the conditions of Theorem 2.1 are satisfied. Therefore, we can conclude that T_1, T_2 have a unique common fixed point in X viz. $(0, 0)$.

We conclude this section by the following interesting remark:
Remark. It is widely known that a mapping (a pair of mappings) satisfying non-expansive (type) condition may not posses fixed point (common fixed point) if the the underlying space(s) is (are) only complete metric space(s). However, in Example 2.7 and Example 2.8, we see that pair of mappings satisfying non-expansive condition can indeed have common fixed points when the underlying space is a complete metric space.

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References

[1] M. Albu, *A fixed point theorem of Maia-Perov type*, Studia Univ. Babeş-Bolyai Math. **23(1)** (1978), 76–79.
[2] S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fund. Math. **3** (1922), 133–181.
[3] V. Berinde, *Maia type fixed point theorems for some classes of enriched contractive mappings in Banach spaces*, Carpathian J. Math. **38(1)** (2022), 35–46.
[4] V. Berinde, A. Petruşel and I.A. Rus, *Remarks on the terminology of the mappings in fixed point iterative methods in metric spaces*, Fixed Point Theory. **24(2)** (2023), 525–540.
[5] V. Berinde, I.A. Rus, *Asymptotic regularity fixed points and successive approximations*, Filomat **34(3)** (2020), 965–981.
[6] R.M.T. Bianchini, *Su un problema di S. Reich riguardante la teoria dei punti fissi*, Boll. Un. Mat. Ital. **5** (1972), 103–108.
[7] D.W. Boyd, J.S.W. Wong, *On nonlinear contraction*, Proc. Amer. Math. Soc. **20** (1969), 458–464.
[8] A. Branciari, *A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces*, Publ. Math. Debrecen **57** (2000), 31–37.
[9] F.E. Browder, *Nonexpansive nonlinear operators in a Banach space*, Proc. Nat. Acad. Sci. USA **54(4)** (1965), 1041–1044.
[10] S.K. Chatterjea, *Fixed-point theorems*, C. R. Acad. Bulgare Sci. **25** (1972), 727–730.
[11] LB. Ćirić, *A generalization of Banach's contraction principle*, Proc. Amer. Math. Soc. **45(2)** (1974), 267–273.
[12] LB. Ćirić, *A new fixed-point theorem for contractive mappings*, Publ. Inst. Math. (Beograd) **30** (1981), 25–27.
[13] P. Debnath, *New common fixed point theorems for Górnicki-type mappings and enriched contractions*, São Paulo J. Math. Sci. **16** (2022), 1401–1408.

[14] S. Dhompongsa, W. Inthakon, A. Kaewkhao, *Edelstein's method and fixed point theorems for some generalized nonexpansive mappings*, J. Math. Anal. Appl. **350(1)** (2007), 12–17.
[15] C. Donghan, S. Jie, L. Weiyi, *Fixed point theorems and convergence theorems for a new generalized nonexpansive mapping*, Numer. Funct. Anal. Optim. **39(16)** (2018), 1742–1754.
[16] M. Edelstein, *An extension of Banach contraction principle*, Proc. Amer. Math. Soc. **37** (1961), 7–10.
[17] M. Edelstein, *On fixed and periodic points under contractive mappings*, J. Lond. Math. Soc. **37(1)** (1962), 74–79.
[18] B. Fisher, *A fixed point theorem for compact metric spaces*, Publ. Inst. Math. **25** (1978), 193–194.
[19] M. Geraghty, *On contractive mappings*, Proc. Amer. Math. Soc. **40(2)** (1973), 604–608.
[20] K. Goebel, W.A. Kirk, *A fixed point theorem for asymptotically nonexpansive mappings*, Proc. Amer. Math. Soc. **35(1)** (1972), 171–174.
[21] K. Goebel, W.A. Kirk, *Iteration processes for nonexpansive mappings*, Contemp. Math. **21** (1983), 115–123.
[22] J. Górnicki, *Fixed point theorems for Kannan type mappings*, J. Fixed Point Theory Appl. **19(3)** (2017), 2145–2152.
[23] M. Jovanović, Z. Kadelburg, S. Radenović, *Common fixed point results in metric-type spaces*, Fixed Point Theory Appl. **2010**, 2010. Article ID 978121.
[24] G. Jungck, *Compatible mappings and common fixed points*, J. Math. Math. Sci. **9(4)** (1986), 771–779.
[25] R. Kannan, *Some results on fixed points*, Bull. Calcutta Math. Soc. **60** (1968), 71–76.
[26] R. Kannan, *Some results on fixed points- II*, Amer. Math. Monthly **76** (1969), 405–408.
[27] R. Kannan, *Some results on fixed points- III*, Fund. Math. **70** (1971), 169–177.
[28] M.S. Khan, *On fixed point theorems*, Math. Japonica **23(2)** (1978/79), 201–204.
[29] S.H. Khan, T. Suzuki, *A Reich-type convergence theorem for generalized nonexpansive mappings in uniformly convex Banach spaces*, Nonlinear Anal. **80** (2013), 211–215.
[30] W.A. Kirk, P.S. Srinivasan, P. Veeramani, *Fixed points for mappings satisfying cyclical contractive conditions*, Fixed Point Theory **4** (2003), 79–89.
[31] S. B. Nadler, *Multivalued contraction mappings*, Pacific Journal of Mathematics **30(2)** (1969), 475–488.
[32] R. Pandey, R. Pant, V. Rakočević, R. Shukla, *Approximating fixed points of a general class of nonexpansive mappings in Banach spaces with applications*, Results Math. **74** (2019).
[33] H.K. Pathak, M.S. Khan, R. Tiwari, *A common fixed point theorem and its application to nonlinear integral equations*, Computers & Mathematics with Applications **53** (2007), 961–971.
[34] O. Popescu, *Some new fixed point theorems for α-Geraghty contractive type maps in metric spaces*, Fixed Point Theory Appl. **2014** (2014), 190.
[35] O. Popescu, *A new class of contractive mappings*, Acta Math. Hungar. **164** (2021), 570–579.
[36] S. Reich, *Some remarks concerning contraction mappings*, Canad. Math. Bull. **14(1)** (1971), 121–124.
[37] S. Reich, *Weak convergence theorems for nonexpansvie mappings in Banach spaces*, *J. Math. Anal. Appl.*, 67(2):274–276, 1979.
[38] S. Reich, R. Kannan, *Fixed point theorem*, Boll. Un. Math. Ital. **4** (1971), 1–11.
[39] B.E. Rhoades, *Some theorems on weakly contractive maps*, Nonlinear Anal. **47(4)** (2001), 2683–2693.
[40] A.F. Roldán-López-de-Hierro, N. Shahzad, *Common fixed point theorems under* $(\mathcal{R}, \mathcal{S})$*-contractivity conditions*, Fixed Point Theory Appl. **2016** (2016), 55.
[41] S.L. Singh, S.N. Mishra, R. Chugh, R. Kamal, *General common fixed point theory and applications*, J. Appl. Math. **2012** (2012). Article ID 902312.
[42] S. Som, A. Petruşel, H. Garai, L.K. Dey, *Some characterizations of Reich and Chatterjea type nonexpansive mappings*, J. Fixed Point Theory Appl. **21(4)** (2019). Article Number 94.
[43] T. Suzuki, *Fixed point theorems and convergence theorems for some generalized nonexpansive mappings*, J. Math. Anal. Appl. **340(2)** (2008), 1088–1095.
[44] T. Suzuki, W. Takahashi, *Fixed point theorems and characterizations of metric completeness*, Topol. Methods in Nonlinear Anal. **8** (1996), 371–382.