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Non-Newtonian Jacobsthal and Jacobsthal-Lucas numbers: A new look

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Abstract. In this study, we introduce a novel version of Jacobsthal and Jacobsthal-Lucas numbers, termed as non-Newtonian Jacobsthal and non-Newtonian Jacobsthal-Lucas numbers. We investigate various characteristics of these newly defined sequences. Additionally, we explore several formulas and identities such as Cassini's identity, d'Ocagne's identity, Binet's formula, Gelin-Cesàro's identity, Honsberger's identity, and Melham's identity associated with these new types. Furthermore, we find the generating functions for such sequences. The novel feature of this study is to generalize the notions of Jacobsthal numbers by using non-Newtonian calculus. If we take the identity function I instead of the generator α in the construction of non-Newtonian Jacobsthal numbers, then non-Newtonian Jacobsthal numbers turn into the classical Jacobsthal numbers, so our results in this paper improve and generalize the known corresponding results in the literature.

1. Introduction and Background

The world of mathematics provides an endless journey of exploration, encouraging us to delve into the intricacies of various integer sequences. One of these number sequences is Jacobsthal numbers, which have a recurrence relationship similar to Fibonacci numbers and are named after the German number theorist Ernst Jacobsthal. Jacobsthal sequence begins with 0 and 1, and each subsequent number is found by adding twice the previous number. Historically, the roots of Jacobsthal numbers date back to 1880; While Henri Brocard [8] focused on the properties of a triangle sequence, he explained the recurrence relation and Binet form of the resulting number sequence in a trigonometric context. Subsequently, in between 1919-1920, Jacobsthal [32] identified the expression

$$f_{n+1} = f_n + x f_{n-1},\ f_{-1} = 0,\ f_0 = 1\ \ (n = 0, 1, 2, ...)$$

as the closest approximation to Jacobsthal numbers. This recursive relationship generates Jacobsthal sequence when $x = 2$. However, Jacobsthal did not explicitly state that $x = 2$ for Jacobsthal sequence, instead indicating its equivalence to Fibonacci sequence when $x = 1$. The Jacobsthal number sequence, discovered by Carlitz et al. [9], remained unnamed until Horadam's article [30], where the term "Jacobsthal numbers" was first used. Horadam also made the first reference to Jacobsthal's work [32] in this article. Before

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Horadam's article [30], it is observed that the term "Jacobsthal polynomials" appeared in the article by Hoggatt and Bicknell [29]. Horadam [31] also included some features of Jacobsthal and Jacobsthal-Lucas numbers and noted the history of these numbers at the end of the article. This chronological setting is important in understanding the evolution and naming of studies on Jacobsthal numbers.

Beyond the boundaries of pure mathematics, Jacobsthal numbers have found versatile applications in various disciplines. Their utility extends to problem solving and model formulation in fields as diverse as astronomy [2], combinatorics [18, 46], graph theory [7], coding theory [38], theoretical computer science and engineering [1, 43]. This multifaceted significance underscores the continuing relevance and broad applicability of Jacobsthal numbers.

Let's give the basic facts on Jacobsthal and Jacobsthal-Lucas numbers.

Jacobsthal numbers are the terms of the integer sequence

$$\{0, 1, 1, 3, 5, 11, 21, 43, 85, 171, 341, ..., J_n, ...\}$$

defined by the recurrence relation

$$J_{n+2} = J_{n+1} + 2J_n \text{ for each } n \in \{0, 1, 2, ...\}$$

with $J_0 = 0, J_1 = 1$, it is well known as the $n-$th term of the Jacobsthal sequence (J_n) which is a numerical sequence.

Jacobsthal-Lucas numbers are the terms of the integer sequence

$$\{2, 1, 5, 7, 17, 31, 65, 127, 257, 511, 1025, ..., j_n, ...\}$$

defined by the recurrence relation

$$j_{n+2} = j_{n+1} + 2j_n \text{ for each } n \in \{0, 1, 2, ...\}$$

with $j_0 = 2, j_1 = 1$, it is well known as the $n-$th term of the Jacobsthal-Lucas sequence (j_n) which is a numerical sequence.

Generating functions for Jacobsthal and Jacobsthal-Lucas numbers are as follows:

$$\sum_{n=1}^{\infty} J_n x^{n-1} = \left(1 - x - 2x^2\right)^{-1}, \tag{1}$$

and

$$\sum_{n=1}^{\infty} j_n x^{n-1} = (1 + 4x)\left(1 - x - 2x^2\right)^{-1}. \tag{2}$$

The Binet's formulas for Jacobsthal and Jacobsthal-Lucas numbers are as follows:

$$J_n = \frac{\nu^n - \omega^n}{3} = \frac{2^n - (-1)^n}{3}, \tag{3}$$

and

$$j_n = \nu^n + \omega^n = 2^n + (-1)^n \tag{4}$$

where $\nu = 2$ and $\omega = -1$.

The Simson's (Cassini) identity for Jacobsthal and Jacobsthal-Lucas numbers are as follows:

$$J_{n+1}J_{n-1} - J_n^2 = (-1)^n 2^{n-1} \tag{5}$$

and

$$j_{n+1}j_{n-1} - j_n^2 = 9(-1)^{n-1} 2^{n-1} = -9(J_{n+1}J_{n-1} - J_n^2). \tag{6}$$

If we choose $k = 2$ in Theorem 5.3 and Theorem 5.9 of Köken (see [34]), the Catalan identities of the Jacobsthal and Jacobsthal-Lucas numbers for $n, r \geq 0$ and $n \geq r$ are as follows, respectively:

$$J_{n+r}J_{n-r} - J_n^2 = (-1)^{n+1-r}\, 2^{n-r} J_r^2, \tag{7}$$

$$j_{n+r}j_{n-r} - j_n^2 = 9\,(-2)^{n-r}\, J_r^2. \tag{8}$$

In the same way, d'Ocagne identities of Jacobsthal and Jacobsthal-Lucas numbers are as follows, respectively:

$$J_m J_{n+1} - J_{m+1} J_n = (-2)^n\, J_{m-n}, \tag{9}$$

$$j_m j_{n+1} - j_{m+1} j_n = 9\,(-1)^{n+1}\, 2^n J_{m-n}. \tag{10}$$

If we take $H_n = \frac{(-1)^{n+1}}{2^n} J_n$ in the equality (41) and $I_n = \frac{(-1)^n}{2^n} j_n$ in the equality (42) of Theorem 2.7 in Daşdemir's work (see [15]), then the Gelin-Cesàro identities of Jacobsthal and Jacobsthal-Lucas numbers for $n \geq 2$ are the followings, respectively:

$$J_n^4 - J_{n-2}J_{n-1}J_{n+1}J_{n+2} = 2^{n-2}\left((-1)^{n+1} J_n^2 + 2^{n-1}\right), \tag{11}$$

$$j_n^4 - j_{n-2}j_{n-1}j_{n+1}j_{n+2} = 9 \times 2^{n-2}\left((-1)^n j_n^2 + 9 \times 2^{n-1}\right). \tag{12}$$

In [31], the following result is proved:

$$\lim_{n\to\infty} \frac{J_{n+1}}{J_n} = \lim_{n\to\infty} \frac{j_{n+1}}{j_n} = 2. \tag{13}$$

In [31], the following result is also proved:

$$\lim_{n\to\infty} \frac{j_n}{J_n} = 3. \tag{14}$$

For $n, m \geq 1$, the following relations hold (see [34–36]):

$$J_{m+n} = J_m J_{n+1} + 2J_{m-1}J_n, \tag{15}$$

$$j_{m+n} = j_n J_{m+1} + 2j_{n-1}J_m. \tag{16}$$

For $n \geq 0$, the following relations hold (see [31]):

$$j_n J_n = J_{2n}. \tag{17}$$

$$3\,(J_{n+1} + J_n) = j_{n+1} + j_n. \tag{18}$$

$$3\,(J_{n+1} - J_n) + 4\,(-1)^{n+1} = j_{n+1} - j_n. \tag{19}$$

$$j_{n+1} - 2j_n = 6J_n - 3J_{n+1}. \tag{20}$$

$$3J_n + 2\,(-1)^n = j_n. \tag{21}$$

$$2^{n+1} = 3J_n + j_n. \tag{22}$$

$$2J_{n+1} = J_n + j_n. \tag{23}$$

$$j_n^2 + 9J_n^2 = 2j_{2n} \quad [m = n \to (33)]. \tag{24}$$

$$j_n^2 - 9J_n^2 = (-1)^n\, 2^{n+2} \quad [m = n \to 35)]. \tag{25}$$

For $n \geq 1$, the following relations hold (see [31]):

$$J_{n+1} + 2J_{n-1} = j_n. \tag{26}$$

$$j_{n+1} + 2j_{n-1} = 9J_n. \tag{27}$$

$$2j_{n+1} + j_{n-1} = 6J_{n+1} + 3J_{n-1} + 6(-1)^{n+1}. \tag{28}$$

For $n \geq 2$, the following relation holds (see [31]):

$$j_{n+2}j_{n-2} - j_n^2 = -9J_{n+2}J_{n-2} + 9J_n^2. \tag{29}$$

For $n, m \geq 0$ and $m \geq n$, the following relations hold (see [31]):

$$3(J_{m+n} + J_{m-n}) + 4(-1)^{m-n} = j_{m+n} + j_{m-n}. \tag{30}$$

$$3(J_{m+n} - J_{m-n}) = j_{m+n} - j_{m-n}. \tag{31}$$

$$J_m j_n + J_n j_m = 2J_{m+n} \quad [m = n \to (17)]. \tag{32}$$

$$j_m j_n + 9J_m J_n = 2j_{m+n}. \tag{33}$$

$$J_m j_n - J_n j_m = (-1)^n 2^{n+1} J_{m-n}. \tag{34}$$

$$j_m j_n - 9J_m J_n = (-1)^n 2^{n+1} j_{m-n}. \tag{35}$$

Calculus, a branch of analysis and an important area of mathematics is the mathematical study of change and motion. The history of calculus has been shaped by many contributions of mathematicians and thinkers, culminating in the 17th century when Gottfried Wilhelm Leibniz and Isaac Newton independently and simultaneously defined and developed the fundamental principles of differential and integral calculus. This fundamental framework, which encompasses the integral and the derivative, reflects the essence of arithmetic operations even on infinitesimally small scales. The work of Newton and Leibniz paved the way for profound advances in differential and integral calculus, with results that revolutionized mathematical thought and enriched scientific research, spanning many scientific fields. Over time, however, the evolution of mathematical thought and methods has allowed new perspectives and computational methods to emerge. The trajectory of calculus underwent a paradigm shift in the mid-20th century when Michael Grosman and Robert Katz [26] introduced modern calculus, the so-called non-Newtonian calculus. This new approach uses functions called generators to reshape arithmetic operations and create new mathematical structures, especially multiplicative arithmetic. Non-Newtonian calculus includes some special and infinite calculi such as harmonic, bigeometric, geometric and anageometric calculus. Non-Newtonian calculus also overcomes the limitations of traditional calculus by emphasizing differentiation and integration regardless of units of measure, thus offering a nuanced perspective on mathematical analysis.

Since the seminal work of Grossman and Katz [26], the importance of this field has been increasingly recognized among researchers from different disciplines. For example, this new calculus has been transferred to many different areas of mathematical analysis, such as sequence spaces, fixed point theory, integral equations and measurement [17, 20, 22, 27]. Especially after the work of Bashirov et al. [3], the significant applications of multiplicative calculus in various fields have become more evident and attracted the attention of researchers. Particularly noteworthy are the important applications of multiplicative calculus in actuarial science, demography and finance [4], economics [23–25], statistics [10], biomedical image analysis [28], logistic growth models [45], contour detection in noisy images [40], linear and nonlinear signal representation [5], physics [13], quantum theory and cryptology [12, 14, 44], multiplicative mechanics [37], geometric magnetic energy [21], exponential signal processing [42], neural networks [41] and cancer treatment [39], etc. The reader can refer to Tekin and Başar [47] for some spaces of non-Newtonian complex sequences; Boruah, Hazarika and Bashirov [6] for some new numerical methods, namely, bigeometric-Euler method, Taylor's bigeometric-series method and bigeometric-Runge-Kutta method for approximation of bigeometric-initial value problems together with examples; Uzer [48] for multiplicative complex calculus and extention of some useful theorems in additive complex calculus to multiplicative complex calculus by using the newly defined operators. These studies show how research on the mathematical foundations of non-Newtonian calculus can have practical implications for addressing real-world problems in a variety of disciplines. In this context, the broad spectrum potential of non-Newtonian calculus represents an exciting step towards providing innovative solutions by pushing the boundaries of mathematical thinking and methods.

Arithmetic is a term usually associated with positive integers, but here the term "arithmetic" refers to an integer ordered field whose universe is a subset of the set of real numbers. There are an infinite number of arithmetic systems. The generator of an arithmetic system generates real number classical arithmetic if I is the identity function and geometric arithmetic if *exp* is the function. Non-Newtonian calculus deals with different mathematical methods and systems from classical calculus. It is used in various applications ranging from atmospheric sciences to petroleum engineering, and from nonlinear dynamic systems to energy crisis. An arithmetic is a complete ordered field whose realm is a subset of $\mathbb{R}$. Non-Newtonian calculi utilize different types of arithmetic and their generators. Let α be a bijection whose domain $\mathbb{R}$ and whose range is a subset U of $\mathbb{R}$. Then, it is called a generator with range U and defines an arithmetic. The range of generator α is denoted by $\mathbb{R}_\alpha$. Also, every element of $\mathbb{R}_\alpha$ is called a non-Newtonian real number.

	$\alpha-$arithmetic
Realm	$U\,(=\mathbb{R}_\alpha)$
$\alpha-$addition	$r \dot{+} s = \alpha\left\{\alpha^{-1}(r) + \alpha^{-1}(s)\right\}$
$\alpha-$subtraction	$r \dot{-} s = \alpha\left\{\alpha^{-1}(r) - \alpha^{-1}(s)\right\}$
$\alpha-$multiplication	$r \dot{\times} s = \alpha\left\{\alpha^{-1}(r) \times \alpha^{-1}(s)\right\}$
$\alpha-$division	$r \dot{/} s = \frac{r}{s}\alpha = \alpha\left\{\frac{\alpha^{-1}(r)}{\alpha^{-1}(s)}\right\}\ \left(s \neq \dot{0}\right)$
$\alpha-$ordering	$r \dot{\leq} s \Longleftrightarrow \alpha^{-1}(r) \leq \alpha^{-1}(s)$

If $r \in \mathbb{R}_\alpha$ and $\dot{0} \dot{<} r$ (or $r \dot{<} \dot{0}$), then we say that it is a $\alpha-$positive number (or $\alpha-$negative number). Additionally, $\mathbb{R}_\alpha^+$ denotes the set of $\alpha-$positive numbers. Also, $\alpha(-r) = \alpha\left\{-\alpha^{-1}\left(\dot{r}\right)\right\} = \dot{-}\dot{r}$ for all $r \in \mathbb{R}$. On the other hand, the number $r \dot{\times} r$ is called the $\alpha-$ square of r, denoted by $r^{\dot{2}}$. If $r \in \mathbb{R}_\alpha^+ \cup \left\{\dot{0}\right\}$, then we say that $\alpha\left[\sqrt{\alpha^{-1}(r)}\right]$ is the $\alpha-$ square root of r, denoted by $\dot{\sqrt{r}}$ [11, 26].

Non-Newtonian calculus offers a new perspective in mathematical analysis by overcoming the limitations of classical calculus. In [16], the authors defined non-Newtonian Fibonacci and non-Newtonian Lucas numbers. Inspired by [16] and extensive applications of Jacobsthal numbers and non-Newtonian calculus, in this study, we will examine how non-Newtonian calculus can approach Jacobsthal and Jacobsthal-Lucas numbers, especially the recurrence relations used in constructing these number sequences. The main objectives of this study are to understand the place of these number sequences in the mathematical world, to establish a general mathematical structure, and to explore how they can be used in specific areas of application.

2. Non-Newtonian Jacobsthal and Non-Newtonian Jacobsthal-Lucas Numbers with Some Properties

In this part, we introduce the notions of a non-Newtonian Jacobsthal number and a non-Newtonian Jacobsthal-Lucas number with a new perspective on the concepts of a Jacobsthal number and a Jacobsthal-Lucas number. We also discuss the non-Newtonian versions of some well-known formulas and identities for classical counterparts.

Definition 2.1. *The non-Newtonian Jacobsthal and non-Newtonian Jacobsthal-Lucas numbers are defined*

$$\aleph\aleph J_n = \alpha(J_n)$$

and

$$\aleph\aleph j_n = \alpha(j_n),$$

respectively, where J_n and j_n are the $n-th$ Jacobsthal and Jacobsthal-Lucas numbers, respectively. The sets of non-Newtonian Jacobsthal and non-Newtonian Jacobsthal-Lucas numbers, which we denote by $\aleph\aleph J$ and $\aleph\aleph j$ are as follows, respectively:

$$\begin{aligned}\aleph\aleph J &= \{\aleph\aleph J_n : n \in \mathbb{N}\} \\ &= \left\{\dot{0}, \dot{1}, \dot{1}, \dot{3}, \dot{5}, \dot{11}, \dot{21}, \dot{43}, \dot{85}, \dot{171}, \dot{341}, ..., \alpha(J_n), ...\right\}\end{aligned}$$

and

$$\begin{aligned}\aleph\aleph j &= \{\aleph\aleph j_n : n \in \mathbb{N}\} \\ &= \left\{\dot{2}, \dot{1}, \dot{5}, \dot{7}, \dot{17}, \dot{31}, \dot{65}, \dot{127}, \dot{257}, \dot{511}, \dot{1025}, ..., \alpha(j_n), ...\right\}.\end{aligned}$$

Choosing the generator I defined by $\alpha(y) = y$ for each $y \in \mathbb{R}$, we obtain Jacobsthal and Jacobsthal-Lucas numbers with respect to classical arithmetic, respectively.

Also, if we consider the generator exp defined by $\alpha(y) = e^y$ for each $y \in \mathbb{R}$, we obtain Jacobsthal and Jacobsthal-Lucas numbers with respect to geometric arithmetic, respectively, as follows:

$$\begin{aligned}\aleph\aleph GJ &= \{\alpha(J_n) : n \in \mathbb{N}\} \\ &= \left\{e^{J_n} : n \in \mathbb{N}\right\} \\ &= \left\{e^0, e^1, e^1, e^3, e^5, e^{11}, e^{21}, e^{43}, ..., e^{J_n}, ...\right\}\end{aligned}$$

and

$$\begin{aligned}\aleph\aleph Gj &= \{\alpha(j_n) : n \in \mathbb{N}\} \\ &= \left\{e^{j_n} : n \in \mathbb{N}\right\} \\ &= \left\{e^2, e^1, e^5, e^7, e^{17}, e^{31}, e^{65}, ..., e^{j_n}, ...\right\}.\end{aligned}$$

We start by obtaining generating functions of the non-Newtonian Jacobsthal numbers and non-Newtonian Jacobsthal-Lucas numbers.

Theorem 2.2. *The generating function $g_{\aleph\aleph J} : \mathbb{R}_\alpha - \left\{\frac{\dot{1}}{\dot{2}}\alpha, \dot{-}\dot{1}\right\} \to \mathbb{R}_\alpha$ of the non-Newtonian Jacobsthal numbers is defined as*

$$g_{\aleph\aleph J}(s) = \frac{\dot{1}}{\dot{1} \dot{-} s \dot{-} \dot{2} \dot{\times} s^{\dot{2}}}\alpha$$

where $\dot{1} \dot{-} s \dot{-} \dot{2} \dot{\times} s^{\dot{2}} \neq \dot{0}$.

Proof. Let the generating function of the non-Newtonian Jacobsthal number $\aleph\aleph J_n$ has the form

$$g_{\aleph\aleph J}(s) =_\alpha \sum_{n=1}^{\infty} \left(\aleph\aleph J_n \dot{\times} s^{\dot{n} \dot{-} \dot{1}}\right),$$

where the symbol ${}_\alpha\sum_{n=1}^{\infty}$ denotes the non-Newtonian real number series which can be found in [19, 33]. After some arrangements and employing (1), we arrive at the following results:

$$\begin{aligned} g_{\aleph\aleph J}(s) &= {}_{\alpha}\sum_{n=1}^{\infty}\left(\aleph\aleph J_n \dot{\times} s^{\dot{n}\dot{-}\dot{1}}\right) \\ &= \dot{1} \dot{+} s \dot{+} {}_{\alpha}\sum_{n=3}^{\infty}\left(\aleph\aleph J_{n-1} \dot{\times} s^{\dot{n}\dot{-}\dot{1}}\right) \dot{+} \dot{2} \dot{\times} \left[{}_{\alpha}\sum_{n=3}^{\infty}\left(\aleph\aleph J_{n-2} \dot{\times} s^{\dot{n}\dot{-}\dot{1}}\right)\right] \\ s \dot{\times} g_{\aleph\aleph J}(s) &= {}_{\alpha}\sum_{n=1}^{\infty}\left(\aleph\aleph J_n \dot{\times} s^{\dot{n}}\right) = s \dot{+} {}_{\alpha}\sum_{n=3}^{\infty}\left(\aleph\aleph J_{n-1} \dot{\times} s^{\dot{n}\dot{-}\dot{1}}\right), \\ s^{\dot{2}} \dot{\times} g_{\aleph\aleph J}(s) &= {}_{\alpha}\sum_{n=1}^{\infty}\left(\aleph\aleph J_n \dot{\times} s^{\dot{n}\dot{+}\dot{1}}\right) =_{\alpha} \sum_{n=3}^{\infty}\left(\aleph\aleph J_{n-2} \dot{\times} s^{\dot{n}\dot{-}\dot{1}}\right). \end{aligned}$$

This implies that

$$\begin{aligned} & \left(\dot{1} \dot{-} s \dot{-} \dot{2} \dot{\times} s^{\dot{2}}\right) \dot{\times} g_{\aleph\aleph J}(s) \\ = \; & g_{\aleph\aleph J}(s) \dot{-} \left(s \dot{\times} g_{\aleph\aleph J}(s)\right) \dot{-} \dot{2} \dot{\times} \left(s^{\dot{2}} \dot{\times} g_{\aleph\aleph J}(s)\right) \\ = \; & \left[\dot{1} \dot{+} s \dot{+} {}_{\alpha}\sum_{n=3}^{\infty}\left(\aleph\aleph J_{n-1} \dot{\times} s^{\dot{n}\dot{-}\dot{1}}\right) \dot{+} \dot{2} \dot{\times} \left[{}_{\alpha}\sum_{n=3}^{\infty}\left(\aleph\aleph J_{n-2} \dot{\times} s^{\dot{n}\dot{-}\dot{1}}\right)\right]\right] \\ & \dot{-} \left[s \dot{+} {}_{\alpha}\sum_{n=3}^{\infty}\left(\aleph\aleph J_{n-1} \dot{\times} s^{\dot{n}\dot{-}\dot{1}}\right)\right] \dot{-} \dot{2} \dot{\times} \left[{}_{\alpha}\sum_{n=3}^{\infty}\left(\aleph\aleph J_{n-2} \dot{\times} s^{\dot{n}\dot{-}\dot{1}}\right)\right] \\ = \; & \dot{1}. \end{aligned}$$

Hereupon, we get the function $g_{\aleph\aleph J}(s) = \frac{\dot{1}}{\dot{1} \dot{-} s \dot{-} \dot{2} \dot{\times} s^{\dot{2}}}\alpha$ which finishes the proof. □

Theorem 2.3. *The generating function* $g_{\aleph\aleph j} : \mathbb{R}_{\alpha} - \left\{\frac{\dot{1}}{\dot{2}}\alpha, \dot{-}\dot{1}\right\} \to \mathbb{R}_{\alpha}$ *of the non-Newtonian Jacobsthal-Lucas numbers is defined as*

$$g_{\aleph\aleph j}(s) = \frac{\dot{1} \dot{+} \dot{4} \dot{\times} s}{\dot{1} \dot{-} s \dot{-} \dot{2} \dot{\times} s^{\dot{2}}}\alpha$$

where $\dot{1} \dot{-} s \dot{-} \dot{2} \dot{\times} s^{\dot{2}} \neq \dot{0}$.

Proof. Suppose that the generating function of the non-Newtonian Jacobsthal-Lucas number $\aleph\aleph j_n$ has the form

$$g_{\aleph\aleph j}(s) =_{\alpha} \sum_{n=1}^{\infty}\left(\aleph\aleph j_n \dot{\times} s^{\dot{n}\dot{-}\dot{1}}\right).$$

Then, considering (2) with the needed calculations, we arrive at the following equations:

$$\begin{aligned} g_{\aleph\aleph j}(s) &= {}_\alpha\sum_{n=1}^{\infty}\left(\aleph\aleph j_n \dot{\times} s^{\dot{n}\dot{-}\dot{1}}\right) \\ &= \dot{1}\dot{+}\dot{5}\dot{\times}s\dot{+}\ {}_\alpha\sum_{n=3}^{\infty}\left(\aleph\aleph j_{n-1}\dot{\times}s^{\dot{n}\dot{-}\dot{1}}\right)\dot{+}\dot{2}\dot{\times}\left[{}_\alpha\sum_{n=3}^{\infty}\left(\aleph\aleph j_{n-2}\dot{\times}s^{\dot{n}\dot{-}\dot{1}}\right)\right], \\ s\dot{\times}g_{\aleph\aleph j}(s) &= {}_\alpha\sum_{n=1}^{\infty}\left(\aleph\aleph j_n\dot{\times}s^{\dot{n}}\right) = s\dot{+}\ {}_\alpha\sum_{n=3}^{\infty}\left(\aleph\aleph j_{n-1}\dot{\times}s^{\dot{n}\dot{-}\dot{1}}\right), \\ s^{\dot{2}}\dot{\times}g_{\aleph\aleph j}(s) &= {}_\alpha\sum_{n=1}^{\infty}\left(\aleph\aleph j_n\dot{\times}s^{\dot{n}\dot{+}\dot{1}}\right) ={}_\alpha \sum_{n=3}^{\infty}\left(\aleph\aleph j_{n-2}\dot{\times}s^{\dot{n}\dot{-}\dot{1}}\right). \end{aligned}$$

So, in this stage, we get

$$\begin{aligned} &\left(\dot{1}\dot{-}s\dot{-}\dot{2}\dot{\times}s^{\dot{2}}\right)\dot{\times}g_{\aleph\aleph j}(s) \\ =\ & g_{\aleph\aleph j}(s)\dot{-}\left(s\dot{\times}g_{\aleph\aleph j}(s)\right)\dot{-}\dot{2}\dot{\times}\left(s^{\dot{2}}\dot{\times}g_{\aleph\aleph j}(s)\right) \\ =\ & \left[\dot{1}\dot{+}\dot{5}\dot{\times}s\dot{+}\ {}_\alpha\sum_{n=3}^{\infty}\left(\aleph\aleph j_{n-1}\dot{\times}s^{\dot{n}\dot{-}\dot{1}}\right)\dot{+}\dot{2}\dot{\times}\left[{}_\alpha\sum_{n=3}^{\infty}\left(\aleph\aleph j_{n-2}\dot{\times}s^{\dot{n}\dot{-}\dot{1}}\right)\right]\right] \\ &\dot{-}\left[s\dot{+}\ {}_\alpha\sum_{n=3}^{\infty}\left(\aleph\aleph j_{n-1}\dot{\times}s^{\dot{n}\dot{-}\dot{1}}\right)\right]\dot{-}\dot{2}\dot{\times}\left[{}_\alpha\sum_{n=3}^{\infty}\left(\aleph\aleph j_{n-2}\dot{\times}s^{\dot{n}\dot{-}\dot{1}}\right)\right] \\ =\ & \dot{1}\dot{+}\dot{4}\dot{\times}s. \end{aligned}$$

This implies that $g_{\aleph\aleph j}(s) = \frac{\dot{1}\dot{+}\dot{4}\dot{\times}s}{\dot{1}\dot{-}s\dot{-}\dot{2}\dot{\times}s^{\dot{2}}}\alpha$ as the desired result. □

In the next theorem, the Binet formulas for non-Newtonian Jacobsthal numbers and non-Newtonian Jacobsthal-Lucas numbers are deduced.

Theorem 2.4. *The Binet formulas for non-Newtonian Jacobsthal numbers $\aleph\aleph J_n$'s and non-Newtonian Jacobsthal-Lucas numbers $\aleph\aleph j_n$'s for $n \geq 0$ are as follows:*

$$\aleph\aleph J_n = \frac{\dot{1}}{\dot{3}}\alpha\dot{\times}\left(\dot{\nu}^{\dot{n}}\dot{-}\dot{\omega}^{\dot{n}}\right)$$

and

$$\aleph\aleph j_n = \dot{\nu}^{\dot{n}}\dot{+}\dot{\omega}^{\dot{n}},$$

where $\dot{\nu} = \dot{2}$ and $\dot{\omega} = \dot{-}\dot{1}$.

Proof. Taking into account the subtraction and division of non-Newtonian real numbers and by virtue of

Binet formula (3) for Jacobsthal numbers, we get

$$\begin{aligned}
& \frac{\dot{1}}{\dot{3}}\alpha \dot{\times} \left(\dot{\nu}^{\dot{n}} \dot{-} \dot{\omega}^{\dot{n}}\right) \\
= \; & \alpha\left\{\alpha^{-1}\left(\frac{\dot{1}}{\dot{3}}\alpha\right) \times \alpha^{-1}\left(\dot{\nu}^{\dot{n}} \dot{-} \dot{\omega}^{\dot{n}}\right)\right\} \\
= \; & \alpha\left\{\alpha^{-1}\left(\alpha\left\{\frac{\alpha^{-1}\left(\dot{1}\right)}{\alpha^{-1}\left(\dot{3}\right)}\right\}\right) \times \alpha^{-1}\left(\overbrace{\left(\dot{\nu} \dot{\times} \dot{\nu} \dot{\times} \dots \dot{\times} \dot{\nu}\right)}^{n \text{ times}} \dot{-} \overbrace{\left(\dot{\omega} \dot{\times} \dot{\omega} \dot{\times} \dots \dot{\times} \dot{\omega}\right)}^{n \text{ times}}\right)\right\} \\
= \; & \alpha\left\{\alpha^{-1}\left(\alpha\left\{\frac{\alpha^{-1}\left(\alpha\left(1\right)\right)}{\alpha^{-1}\left(\alpha\left(3\right)\right)}\right\}\right) \times \alpha^{-1}\left(\alpha\left\{\left(\alpha^{-1}\left(\dot{\nu}\right)\right)^{n} - \left(\alpha^{-1}\left(\dot{\omega}\right)\right)^{n}\right\}\right)\right\} \\
= \; & \alpha\left\{\frac{1}{3}\left(\nu^{n} - \omega^{n}\right)\right\} \\
= \; & \alpha\left(J_n\right) \\
= \; & \aleph\aleph J_n.
\end{aligned}$$

Also, using Binet formula (4) for Jacobsthal-Lucas numbers, we observe that

$$\begin{aligned}
\dot{\nu}^{\dot{n}} \dot{+} \dot{\omega}^{\dot{n}} \; = \; & \alpha\left[\alpha^{-1}\left(\overbrace{\left(\dot{\nu} \dot{\times} \dot{\nu} \dot{\times} \dots \dot{\times} \dot{\nu}\right)}^{n \text{ times}} \dot{+} \overbrace{\left(\dot{\omega} \dot{\times} \dot{\omega} \dot{\times} \dots \dot{\times} \dot{\omega}\right)}^{n \text{ times}}\right)\right] \\
= \; & \alpha\left[\alpha^{-1}\left(\alpha\left[\left(\alpha^{-1}\left(\dot{\nu}\right)\right)^{n}\right] \dot{+} \alpha\left[\left(\alpha^{-1}\left(\dot{\omega}\right)\right)^{n}\right]\right)\right] \\
= \; & \alpha\left[\alpha^{-1}\left(\alpha\left[\alpha^{-1}\alpha\left[\left(\alpha^{-1}\left(\dot{\nu}\right)\right)^{n}\right] + \alpha^{-1}\alpha\left[\left(\alpha^{-1}\left(\dot{\omega}\right)\right)^{n}\right]\right]\right)\right] \\
= \; & \alpha\left(\nu^{n} + \omega^{n}\right) \\
= \; & \alpha\left(j_n\right) \\
= \; & \aleph\aleph j_n.
\end{aligned}$$

□

The next theorem consists of Simson's (Cassini) identities for new types of numbers we discuss in this article in non-Newtonian sense.

Theorem 2.5. *Simson's (Cassini) identities for $\aleph\aleph J_n$ and $\aleph\aleph j_n$ for $n \geq 1$ are as follows*:

1) $\aleph\aleph J_{n+1} \dot{\times} \aleph\aleph J_{n-1} \dot{-} \aleph\aleph J_n^{\dot{2}} = \left(\dot{-}\dot{1}\right)^{\dot{n}} \dot{\times} \dot{2}^{\dot{n} \dot{-} \dot{1}}$.

2) $\aleph\aleph j_{n+1} \dot{\times} \aleph\aleph j_{n-1} \dot{-} \aleph\aleph j_n^{\dot{2}} = \dot{9} \dot{\times} \left(\dot{-}\dot{1}\right)^{\dot{n} \dot{-} \dot{1}} \dot{\times} \dot{2}^{\dot{n} \dot{-} \dot{1}} = \dot{-}\dot{9} \dot{\times} (\aleph\aleph J_{n+1} \dot{\times} \aleph\aleph J_{n-1} \dot{-} \aleph\aleph J_n^{\dot{2}})$.

Proof. 1) We get the result

$$
\begin{aligned}
& \aleph\aleph J_{n+1} \dot{\times} \aleph\aleph J_{n-1} \dot{-} \aleph\aleph J_n^{\dot{2}} \\
= \; & \aleph\aleph J_{n+1} \dot{\times} \aleph\aleph J_{n-1} \dot{-} \aleph\aleph J_n \dot{\times} \aleph\aleph J_n \\
= \; & \alpha\left(J_{n+1}\right) \dot{\times} \alpha\left(J_{n-1}\right) \dot{-} \alpha\left(J_n\right) \dot{\times} \alpha\left(J_n\right) \\
= \; & \alpha\left\{\alpha^{-1}\alpha\left(J_{n+1}\right) \times \alpha^{-1}\alpha\left(J_{n-1}\right)\right\} \dot{-} \alpha\left\{\alpha^{-1}\alpha\left(J_n\right) \times \alpha^{-1}\alpha\left(J_n\right)\right\} \\
= \; & \alpha\left\{\alpha^{-1}\alpha\left\{\alpha^{-1}\alpha\left(J_{n+1}\right) \times \alpha^{-1}\alpha\left(J_{n-1}\right)\right\} - \alpha^{-1}\alpha\left\{\alpha^{-1}\alpha\left(J_n\right) \times \alpha^{-1}\alpha\left(J_n\right)\right\}\right\} \\
= \; & \alpha\left(J_{n+1}J_{n-1} - J_n^2\right) \\
= \; & \alpha\left((-1)^n 2^{n-1}\right) \\
= \; & \alpha\left(\overbrace{(-1)\ldots(-1)}^{n \text{ times}} \; \overbrace{2\ldots2}^{n-1 \text{ times}}\right) \\
= \; & \alpha\left(\overbrace{\alpha^{-1}\left(\alpha(-1)\right)\ldots\alpha^{-1}\left(\alpha(-1)\right)}^{n \text{ times}}\overbrace{\alpha^{-1}\left(\alpha(2)\right)\ldots\alpha^{-1}\left(\alpha(2)\right)}^{n-1 \text{ times}}\right) \\
= \; & \overbrace{\alpha(-1) \dot{\times} \ldots \dot{\times} \alpha(-1)}^{n \text{ times}} \dot{\times} \overbrace{\alpha(2) \dot{\times} \ldots \dot{\times} \alpha(2)}^{n-1 \text{ times}} \\
= \; & \overbrace{\left(\dot{-}\dot{1}\right) \dot{\times} \ldots \dot{\times} \left(\dot{-}\dot{1}\right)}^{n \text{ times}} \dot{\times} \overbrace{\left(\dot{2}\right) \dot{\times} \ldots \dot{\times} \left(\dot{2}\right)}^{n-1 \text{ times}} \\
= \; & \left(\dot{-}\dot{1}\right)^{\dot{n}} \dot{\times} \dot{2}^{\dot{n} \dot{-} \dot{1}}
\end{aligned}
$$

by utilizing Simson's (Cassini) identity (5) and doing some necessary computations.

2) Proof follows directly by using the identity (6). □

Now, we want to find the Catalan identities for J_n and j_n in non-Newtonian sense.

Theorem 2.6. *The Catalan identities for* $\aleph\aleph J_n$ *and* $\aleph\aleph j_n$ *for* $n, r \geq 1$ *and* $n \geq r$ *are given as follows:*

1) $\aleph\aleph J_{n+r} \dot{\times} \aleph\aleph J_{n-r} \dot{-} \aleph\aleph J_n^{\dot{2}} = \left(\dot{-}\dot{1}\right)^{\dot{n} \dot{+} \dot{1} \dot{-} \dot{r}} \dot{\times} \dot{2}^{\dot{n} \dot{-} \dot{r}} \dot{\times} \aleph\aleph J_r^{\dot{2}}$.

2) $\aleph\aleph j_{n+r} \dot{\times} \aleph\aleph j_{n-r} \dot{-} \aleph\aleph j_n^{\dot{2}} = \dot{9} \dot{\times} \left(\dot{-}\dot{2}\right)^{\dot{n} \dot{-} \dot{r}} \dot{\times} \aleph\aleph J_r^{\dot{2}}$.

Proof. 1) (7) and some needed computations imply the following expression:

$$
\begin{aligned}
& \aleph\aleph J_{n-r} \dot{\times} \aleph\aleph J_{n+r} \dot{-} \aleph\aleph J_n^{\dot{2}} \\
= \; & \aleph\aleph J_{n-r} \dot{\times} \aleph\aleph J_{n+r} \dot{-} \aleph\aleph J_n \dot{\times} \aleph\aleph J_n \\
= \; & \alpha\left(J_{n-r}\right) \dot{\times} \alpha\left(J_{n+r}\right) \dot{-} \alpha\left(J_n\right) \dot{\times} \alpha\left(J_n\right) \\
= \; & \alpha\left\{\alpha^{-1}\alpha\left(J_{n-r}\right) \times \alpha^{-1}\alpha\left(J_{n+r}\right)\right\} \dot{-} \alpha\left\{\alpha^{-1}\alpha\left(J_n\right) \times \alpha^{-1}\alpha\left(J_n\right)\right\} \\
= \; & \alpha\left\{\alpha^{-1}\alpha\left\{\alpha^{-1}\alpha\left(J_{n-r}\right) \times \alpha^{-1}\alpha\left(J_{n+r}\right)\right\} - \alpha^{-1}\alpha\left\{\alpha^{-1}\alpha\left(J_n\right) \times \alpha^{-1}\alpha\left(J_n\right)\right\}\right\} \\
= \; & \alpha\left(J_{n-r}J_{n+r} - J_n^2\right) \\
= \; & \alpha\left((-1)^{n+1-r} 2^{n-r} J_r^2\right) \\
= \; & \left(\dot{-}\dot{1}\right)^{\dot{n} \dot{+} \dot{1} \dot{-} \dot{r}} \dot{\times} \dot{2}^{\dot{n} \dot{-} \dot{r}} \dot{\times} \aleph\aleph J_r^{\dot{2}}.
\end{aligned}
$$

The proof is reached.

1) After some elementary calculations, it can be computed similar to the property 2) in Theorem 2.6 taking it into account (8) and the proof is straightforward. □

In the following theorem, the d'Ocagne identities for non-Newtonian Jacobsthal and non-Newtonian Jacobsthal-Lucas numbers are given.

Theorem 2.7. *The d'Ocagne identities for* $\aleph\aleph J_n$ *and* $\aleph\aleph j_n$ *are as follows:*

1) $\aleph\aleph J_{n+1} \dot{\times} \aleph\aleph J_m \dot{-} \aleph\aleph J_n \dot{\times} \aleph\aleph J_{m+1} = \left(\dot{-}\dot{2}\right)^{\dot{n}} \dot{\times} \aleph\aleph J_{m-n}$

2) $\aleph\aleph j_{n+1} \dot{\times} \aleph\aleph j_m \dot{-} \aleph\aleph j_n \aleph\aleph j_{m+1} = \dot{9} \dot{\times} \left(\dot{-}\dot{1}\right)^{\dot{n}\dot{+}\dot{1}} \dot{\times} \dot{2}^{\dot{n}} \dot{\times} \aleph\aleph J_{m-n}$

for $n, m \geq 0$ *and* $m \geq n$.

Proof. 1) The assertion follows by applying (9) to

$$\begin{aligned}
& \aleph\aleph J_{n+1} \dot{\times} \aleph\aleph J_m \dot{-} \aleph\aleph J_n \dot{\times} \aleph\aleph J_{m+1} \\
= & \ \alpha\left(J_{n+1}\right) \dot{\times} \alpha\left(J_m\right) \dot{-} \alpha\left(J_n\right) \dot{\times} \alpha\left(J_{m+1}\right) \\
= & \ \alpha\left\{\alpha^{-1}\alpha\left(J_{n+1}\right) \times \alpha^{-1}\alpha\left(J_m\right)\right\} \dot{-} \alpha\left\{\alpha^{-1}\alpha\left(J_n\right) \times \alpha^{-1}\alpha\left(J_{m+1}\right)\right\} \\
= & \ \alpha\left\{\alpha^{-1}\alpha\left\{\alpha^{-1}\alpha\left(J_{n+1}\right) \times \alpha^{-1}\alpha\left(J_m\right)\right\} - \alpha^{-1}\alpha\left\{\alpha^{-1}\alpha\left(J_n\right) \times \alpha^{-1}\alpha\left(J_{m+1}\right)\right\}\right\} \\
= & \ \alpha\left(J_{n+1}J_m - J_nJ_{m+1}\right) \\
= & \ \alpha\left((-2)^n J_{m-n}\right) \\
= & \ \left(\dot{-}\dot{2}\right)^{\dot{n}} \dot{\times} \aleph\aleph J_{m-n}.
\end{aligned}$$

2) From (10) we reach the expected result. □

We are now ready to introduce Gelin-Cesàro identities including non-Newtonian Jacobsthal and non-Newtonian Jacobsthal-Lucas numbers.

Theorem 2.8. *For* $n \geq 2$*; the Gelin-Cesàro identities of* $\aleph\aleph J_n$ *and* $\aleph\aleph j_n$ *are given as*

1) $\aleph\aleph J_n^{\dot{4}} \dot{-} \aleph\aleph J_{n-2} \dot{\times} \aleph\aleph J_{n-1} \dot{\times} \aleph\aleph J_{n+1} \dot{\times} \aleph\aleph J_{n+2} = \dot{2}^{\dot{n}\dot{-}\dot{2}} \dot{\times} \left((\dot{-}\dot{1})^{\dot{n}\dot{+}\dot{1}} \dot{\times} \aleph\aleph J_n^{\dot{2}} \dot{+} \dot{2}^{\dot{n}\dot{-}\dot{1}}\right).$

2) $\aleph\aleph j_n^{\dot{4}} \dot{-} \aleph\aleph j_{n-2} \dot{\times} \aleph\aleph j_{n-1} \dot{\times} \aleph\aleph j_{n+1} \dot{\times} \aleph\aleph j_{n+2} = \dot{9} \dot{\times} \dot{2}^{\dot{n}\dot{-}\dot{2}} \dot{\times} \left((\dot{-}\dot{1})^{\dot{n}} \dot{\times} \aleph\aleph j_n^{\dot{2}} \dot{+} \dot{9} \dot{\times} \dot{2}^{\dot{n}\dot{-}\dot{1}}\right).$

Proof. 1) Based on the Gelin-Cesàro identity (11) of Jacobsthal numbers, we get the desired result:

$$\begin{aligned}
& \aleph\aleph J_n^{\dot{4}} \dot{-} \aleph\aleph J_{n-2} \dot{\times} \aleph\aleph J_{n-1} \dot{\times} \aleph\aleph J_{n+1} \dot{\times} \aleph\aleph J_{n+2} \\
= & \ \aleph\aleph J_n \dot{\times} \aleph\aleph J_n \dot{\times} \aleph\aleph J_n \dot{\times} \aleph\aleph J_n \dot{-} \aleph\aleph J_{n-2} \dot{\times} \aleph\aleph J_{n-1} \dot{\times} \aleph\aleph J_{n+1} \dot{\times} \aleph\aleph J_{n+2} \\
= & \ \alpha\left(J_n\right) \dot{\times} \alpha\left(J_n\right) \dot{\times} \alpha\left(J_n\right) \dot{\times} \alpha\left(J_n\right) \dot{-} \alpha\left(J_{n-2}\right) \dot{\times} \alpha\left(J_{n-1}\right) \dot{\times} \alpha\left(J_{n+1}\right) \dot{\times} \alpha\left(J_{n+2}\right) \\
= & \ \alpha\left\{\alpha^{-1}\alpha\left(J_n\right) \times \alpha^{-1}\alpha\left(J_n\right) \times \alpha^{-1}\alpha\left(J_n\right) \times \alpha^{-1}\alpha\left(J_n\right)\right\} \\
& \dot{-}\alpha\left\{\alpha^{-1}\alpha\left(J_{n-2}\right) \times \alpha^{-1}\alpha\left(J_{n-1}\right) \times \alpha^{-1}\alpha\left(J_{n+1}\right) \times \alpha^{-1}\alpha\left(J_{n+2}\right)\right\} \\
= & \ \alpha\left\{\begin{array}{c} \alpha^{-1}\alpha\left\{\alpha^{-1}\alpha\left(J_n\right) \times \alpha^{-1}\alpha\left(J_n\right) \times \alpha^{-1}\alpha\left(J_n\right) \times \alpha^{-1}\alpha\left(J_n\right)\right\} \\ -\alpha^{-1}\alpha\left\{\alpha^{-1}\alpha\left(J_{n-2}\right) \times \alpha^{-1}\alpha\left(J_{n-1}\right) \times \alpha^{-1}\alpha\left(J_{n+1}\right) \times \alpha^{-1}\alpha\left(J_{n+2}\right)\right\} \end{array}\right\} \\
= & \ \alpha\left(J_n^4 - J_{n-2}J_{n-1}J_{n+1}J_{n+2}\right) \\
= & \ \alpha\left(2^{n-2}\left((-1)^{n+1}J_n^2 + 2^{n-1}\right)\right) \\
= & \ \dot{2}^{\dot{n}\dot{-}\dot{2}} \dot{\times} \left((\dot{-}\dot{1})^{\dot{n}\dot{+}\dot{1}} \dot{\times} \aleph\aleph J_n^{\dot{2}} \dot{+} \dot{2}^{\dot{n}\dot{-}\dot{1}}\right).
\end{aligned}$$

2) The Gelin-Cesàro identity (12) explains the proof. □

Theorem 2.9. ${}_{\alpha}\lim\limits_{n\to\infty}\frac{\aleph\aleph J_{n+1}}{\aleph\aleph J_n}\alpha = {}_{\alpha}\lim\limits_{n\to\infty}\frac{\aleph\aleph j_{n+1}}{\aleph\aleph j_n}\alpha = \dot{2}.$

Proof. Let $\frac{\aleph\aleph J_n}{\aleph\aleph J_{n-1}}\alpha = T_n$. Then, by the definition of non-Newtonian Jacobsthal numbers we have

$$\frac{\aleph\aleph J_{n+1}}{\aleph\aleph J_n}\alpha = \dot{1} \dot{+} \dot{2} \dot{\times} \frac{\aleph\aleph J_{n-1}}{\aleph\aleph J_n}\alpha.$$

This can be written as $T_{n+1} = \dot{1} \dot{+} \dot{2} \dot{\times} \frac{\dot{1}}{T_n}\alpha$. So, we get ${}_{\alpha}\lim\limits_{n\to\infty}T_{n+1} = \dot{1} \dot{+} \dot{2} \dot{\times} \frac{\dot{1}}{{}_{\alpha}\lim\limits_{n\to\infty}T_n}\alpha$. Making $T = {}_{\alpha}\lim\limits_{n\to\infty}T_n = {}_{\alpha}\lim\limits_{n\to\infty}T_{n+1}$, it follows that $T = \dot{1} \dot{+} \frac{\dot{2}}{T}\alpha$ and equivalently $T^{\dot{2}} \dot{-} T \dot{-} \dot{2} = \dot{0}$. Thus, we see that $T = \dot{2}$ and $T = \dot{-}\dot{1}$. Since $(\aleph\aleph J_n)$ is strictly increasing, the limit value T can not be $\dot{-}\dot{1}$. It results that $T = {}_{\alpha}\lim\limits_{n\to\infty}\frac{\aleph\aleph J_{n+1}}{\aleph\aleph J_n}\alpha = \dot{2}$.

The similar proof holds for $\aleph\aleph j_n$. □

Theorem 2.10. ${}_{\alpha}\lim\limits_{n\to\infty}\frac{\aleph\aleph j_n}{\aleph\aleph J_n}\alpha = \dot{3}.$

Proof. Since $\aleph\aleph J_n = \frac{\dot{2}^{\dot{n}} \dot{-} \left(\dot{-}\dot{1}\right)^{\dot{n}}}{\dot{3}}\alpha$ and $\aleph\aleph j_n = \dot{2}^{\dot{n}} \dot{+} \left(\dot{-}\dot{1}\right)^{\dot{n}}$, we can write

$$\begin{aligned}
{}_{\alpha}\lim_{n\to\infty}\frac{\aleph\aleph j_n}{\aleph\aleph J_n}\alpha &= {}_{\alpha}\lim_{n\to\infty}\frac{\dot{2}^{\dot{n}} \dot{+} \left(\dot{-}\dot{1}\right)^{\dot{n}}}{\frac{\dot{2}^{\dot{n}} \dot{-} \left(\dot{-}\dot{1}\right)^{\dot{n}}}{\dot{3}}\alpha}\alpha \\
&= {}_{\alpha}\lim_{n\to\infty}\left(\dot{3} \dot{\times} \frac{\dot{2}^{\dot{n}} \dot{+} \left(\dot{-}\dot{1}\right)^{\dot{n}}}{\dot{2}^{\dot{n}} \dot{-} \left(\dot{-}\dot{1}\right)^{\dot{n}}}\alpha\right) \\
&= {}_{\alpha}\lim_{n\to\infty}\left(\dot{3} \dot{\times} \frac{\left(\dot{-}\dot{2}\right)^{\dot{n}} \dot{+} \dot{1}}{\left(\dot{-}\dot{2}\right)^{\dot{n}} \dot{-} \dot{1}}\alpha\right) \\
&= {}_{\alpha}\lim_{n\to\infty}\left(\dot{3} \dot{\times} \frac{\dot{1} \dot{+} \left(\frac{\dot{-}\dot{1}}{\dot{2}}\alpha\right)^{\dot{n}}}{\dot{1} \dot{+} \left(\frac{\dot{1}}{\dot{2}}\alpha\right)^{\dot{n}}}\alpha\right) \\
&= {}_{\alpha}\lim_{n\to\infty}\dot{3} = \dot{3}
\end{aligned}$$

as desired result. □

Let's give the Honsberger's identities of $\aleph\aleph J_n$'s and $\aleph\aleph j_n$'s.

Theorem 2.11. *The Honsberger's identities of $\aleph\aleph J_n$'s and $\aleph\aleph j_n$'s for $m, n \geq 1$ are*

1) $\aleph\aleph J_{m+n} = \aleph\aleph J_m \dot{\times} \aleph\aleph J_{n+1} \dot{+} \dot{2} \dot{\times} \aleph\aleph J_{m-1} \dot{\times} \aleph\aleph J_n$,

2) $\aleph\aleph j_{m+n} = \aleph\aleph j_n \dot{\times} \aleph\aleph J_{m+1} \dot{+} \dot{2} \dot{\times} \aleph\aleph j_{n-1} \dot{\times} \aleph\aleph J_m$.

Proof. 1) By some straightforward calculations and the Honsberger's identity (15) of $\aleph\aleph J_n$'s, one can easily reach that

$$\begin{aligned} & \aleph\aleph J_m \dot{\times} \aleph\aleph J_{n+1} \dot{+} \dot{2} \dot{\times} \aleph\aleph J_{m-1} \dot{\times} \aleph\aleph J_n \\ = \ & \alpha(J_m) \dot{\times} \alpha(J_{n+1}) \dot{+} \alpha(2) \dot{\times} \alpha(J_{m-1}) \dot{\times} \alpha(J_n) \\ = \ & \alpha\left\{\alpha^{-1}\alpha(J_m) \times \alpha^{-1}\alpha(J_{n+1})\right\} \dot{+} \alpha\left\{\alpha^{-1}\alpha(2) \times \alpha^{-1}\alpha(J_{m-1}) \times \alpha^{-1}\alpha(J_n)\right\} \\ = \ & \alpha\left\{\begin{array}{c} \alpha^{-1}\alpha\left\{\alpha^{-1}\alpha(J_m) \times \alpha^{-1}\alpha(J_{n+1})\right\} \\ +\alpha^{-1}\alpha\left\{\alpha^{-1}\alpha(2) \times \alpha^{-1}\alpha(J_{m-1}) \times \alpha^{-1}\alpha(J_n)\right\} \end{array}\right\} \\ = \ & \alpha(J_m J_{n+1} + 2J_{m-1}J_n) \\ = \ & \alpha(J_{m+n}) \\ = \ & \aleph\aleph J_{m+n}. \end{aligned}$$

2) The result is similarly revealed. □

The followings are properties satisfied by non-Newtonian Jacobsthal and non-Newtonian Jacobsthal-Lucas numbers.

Theorem 2.12. *The following equalities are satisfied for $n \geq 0$:*

1) $\aleph\aleph\, j_n \dot{\times} \aleph\aleph J_n = \aleph\aleph J_{2n}$.

2) $\aleph\aleph j_{n+1} \dot{+} \aleph\aleph j_n = \dot{3} \dot{\times} \left(\aleph\aleph J_{n+1} \dot{+} \aleph\aleph J_n\right)$.

3) $\aleph\aleph\, j_{n+1} \dot{-} \aleph\aleph j_n = \dot{3} \dot{\times} \left(\aleph\aleph J_{n+1} \dot{-} \aleph\aleph J_n\right) \dot{+} \dot{4} \dot{\times} \left(\dot{-}\dot{1}\right)^{\dot{n}\dot{+}\dot{1}}$.

4) $\aleph\aleph j_{n+1} \dot{-} \dot{2} \dot{\times} \aleph\aleph j_n = \dot{6} \dot{\times} \aleph\aleph J_n \dot{-} \dot{3} \dot{\times} \aleph\aleph J_{n+1}$.

5) $\aleph\aleph j_n = \dot{3} \dot{\times} \aleph\aleph J_n \dot{+} \dot{2} \dot{\times} \left(\dot{-}\dot{1}\right)^{\dot{n}}$.

6) $\dot{3} \dot{\times} \aleph\aleph J_n \dot{+} \aleph\aleph j_n = \dot{2}^{\dot{n}\dot{+}\dot{1}}$.

7) $\aleph\aleph J_n \dot{+} \aleph\aleph j_n = \dot{2} \dot{\times} \aleph\aleph J_{n+1}$.

8) $\aleph\aleph j_n^{\dot{2}} \dot{+} \dot{9} \dot{\times} \aleph\aleph J_n^{\dot{2}} = \dot{2} \dot{\times} \aleph\aleph j_{2n}$.

9) $\aleph\aleph j_n^{\dot{2}} \dot{-} \dot{9} \dot{\times} \aleph\aleph J_n^{\dot{2}} = \left(\dot{-}\dot{1}\right)^{\dot{n}} \dot{\times} \dot{2}^{\dot{n}\dot{+}\dot{2}}$.

Proof. 2) If we use (18), we obtain that

$$\begin{aligned} & \dot{3} \dot{\times} \left(\aleph\aleph J_{n+1} \dot{+} \aleph\aleph J_n\right) \\ = \ & \alpha(3) \dot{\times} \left(\alpha(J_{n+1}) \dot{+} \alpha(J_n)\right) \\ = \ & \alpha(3) \dot{\times} \alpha\left\{\alpha^{-1}\alpha(J_{n+1}) + \alpha^{-1}\alpha(J_n)\right\} \\ = \ & \alpha\left\{\alpha^{-1}\alpha(3) \times \left[\alpha^{-1}\alpha\left\{\alpha^{-1}\alpha(J_{n+1}) + \alpha^{-1}\alpha(J_n)\right\}\right]\right\} \\ = \ & \alpha(3(J_{n+1} + J_n)) \\ = \ & \alpha(j_{n+1} + j_n) \\ = \ & \aleph\aleph j_{n+1} \dot{+} \aleph\aleph j_n. \end{aligned}$$

The remaining relations are easily proved with the formulas (17) and (19)-(25). □

Theorem 2.13. *The following equalities hold for $n \geq 1$:*

1) $\aleph\aleph j_n = \aleph\aleph J_{n+1} \dot{+} \dot{2} \dot{\times} \aleph\aleph J_{n-1}$.

2) $\dot{9} \dot{\times} \aleph\aleph J_n = \aleph\aleph j_{n+1} \dot{+} \dot{2} \dot{\times} \aleph\aleph j_{n-1}$.

3) $\dot{2} \dot{\times} \aleph\aleph j_{n+1} \dot{+} \aleph\aleph j_{n-1} = \dot{6} \dot{\times} \aleph\aleph J_{n+1} \dot{+} \dot{3} \dot{\times} \aleph\aleph J_{n-1} \dot{+} \dot{6} \dot{\times} \left(\dot{-}\dot{1}\right)^{\dot{n}\dot{+}\dot{1}}$.

Proof. Based on the addition and multiplication properties of non-Newtonian real numbers, from (27) and (28), the proofs of 2) and 3) are clear.

1) By (26), we deduce that

$$\begin{aligned}
& \aleph\aleph J_{n+1} \dot{+} \dot{2} \dot{\times} \aleph\aleph J_{n-1} \\
= {} & \alpha\left(J_{n+1}\right) \dot{+} \alpha\left(2\right) \dot{\times} \alpha\left(J_{n-1}\right) \\
= {} & \alpha\left(J_{n+1}\right) \dot{+} \alpha\left\{\alpha^{-1}\alpha\left(2\right) \times \alpha^{-1}\alpha\left(J_{n-1}\right)\right\} \\
= {} & \alpha\left\{\alpha^{-1}\alpha\left(J_{n+1}\right) + \alpha^{-1}\alpha\left\{\alpha^{-1}\alpha\left(2\right) \times \alpha^{-1}\alpha\left(J_{n-1}\right)\right\}\right\} \\
= {} & \alpha\left(J_{n+1} + 2J_{n-1}\right) \\
= {} & \alpha\left(j_n\right) \\
= {} & \aleph\aleph j_n,
\end{aligned}$$

which is the desired result. □

Theorem 2.14. *The following equality holds for $n \geq 2$:*

$$\aleph\aleph j_{n+2} \dot{\times} \aleph\aleph j_{n-2} \dot{-} \aleph\aleph j_n^{\dot{2}} = \dot{-}\dot{9} \dot{\times} \aleph\aleph J_{n+2} \dot{\times} \aleph\aleph J_{n-2} \dot{+} \dot{9} \dot{\times} \aleph\aleph J_n^{\dot{2}}.$$

Proof. The proof is obtained with some calculations by using (29). □

Theorem 2.15. *The following equalities are satisfied for $n, m \geq 0$ and $m \geqslant n$:*

1) $\aleph\aleph j_{m+n} \dot{+} \aleph\aleph j_{m-n} = \dot{3} \dot{\times} \left(\aleph\aleph J_{m+n} \dot{+} \aleph\aleph J_{m-n}\right) \dot{+} \dot{4} \dot{\times} \left(\dot{-}\dot{1}\right)^{\dot{m}\dot{-}\dot{n}}$.

2) $\aleph\aleph\ j_{m+n} \dot{-} \aleph\aleph j_{m-n} = \dot{3} \dot{\times} \left(\aleph\aleph J_{m+n} \dot{-} \aleph\aleph J_{m-n}\right)$.

3) $\aleph\aleph J_m \dot{\times} \aleph\aleph j_n \dot{+} \aleph\aleph J_n \dot{\times} \aleph\aleph j_m = \dot{2} \dot{\times} \aleph\aleph J_{m+n}$.

4) $\aleph\aleph j_m \dot{\times} \aleph\aleph j_n \dot{+} \dot{9} \dot{\times} \aleph\aleph J_m \dot{\times} \aleph\aleph J_n = \dot{2} \dot{\times} \aleph\aleph j_{m+n}$.

5) $\aleph\aleph J_m \dot{\times} \aleph\aleph j_n \dot{-} \aleph\aleph J_n \dot{\times} \aleph\aleph j_m = \left(\dot{-}\dot{1}\right)^{\dot{n}} \dot{\times} \dot{2}^{\dot{n}\dot{+}\dot{1}} \dot{\times} \aleph\aleph J_{m-n}$.

6) $\aleph\aleph j_m \dot{\times} \aleph\aleph j_n \dot{-} \dot{9} \dot{\times} \aleph\aleph J_m \dot{\times} \aleph\aleph J_n = \left(\dot{-}\dot{1}\right)^{\dot{n}} \dot{\times} \dot{2}^{\dot{n}\dot{+}\dot{1}} \dot{\times} \aleph\aleph j_{m-n}$.

Proof. 1) From (30), we have

$$\begin{aligned}
& \dot{3} \dot{\times} \left(\aleph\aleph J_{m+n} \dot{+} \aleph\aleph J_{m-n}\right) \dot{+} \dot{4} \dot{\times} \left(\dot{-}\dot{1}\right)^{\dot{m}\dot{-}\dot{n}} \\
= {} & \alpha\left(3\right) \dot{\times} \left(\alpha\left(J_{m+n}\right) \dot{+} \alpha\left(J_{m-n}\right)\right) \dot{+} \alpha\left(4\right) \dot{\times} \overbrace{\alpha\left(-1\right) \dot{\times} \ldots \dot{\times} \alpha\left(-1\right)}^{m-n \text{ times}} \\
= {} & \alpha\left(3\right) \dot{\times} \alpha\left\{\alpha^{-1}\alpha\left(J_{m+n}\right) + \alpha^{-1}\alpha\left(J_{m-n}\right)\right\} \dot{+} \alpha\left\{\alpha^{-1}\alpha\left(4\right) \times \overbrace{\alpha^{-1}\alpha\left(-1\right) \times \ldots \times \alpha^{-1}\alpha\left(-1\right)}^{m-n \text{ times}}\right\} \\
= {} & \alpha\left\{\alpha^{-1}\alpha\left(3\right) \times \alpha^{-1}\left(\alpha\left\{\alpha^{-1}\alpha\left(J_{m+n}\right) + \alpha^{-1}\alpha\left(J_{m-n}\right)\right\}\right)\right\} \dot{+} \alpha\left\{\alpha^{-1}\alpha\left(4\right) \times \left(\alpha^{-1}\alpha\left(-1\right)\right)^{m-n}\right\} \\
= {} & \alpha\left\{\begin{array}{c} \alpha\left[\alpha^{-1}\left(\alpha^{-1}\alpha\left(3\right) \times \alpha^{-1}\alpha\left\{\alpha^{-1}\alpha\left(J_{m+n}\right) + \alpha^{-1}\alpha\left(J_{m-n}\right)\right\}\right)\right] \\ +\alpha\left[\alpha^{-1}\left(\alpha\left\{\alpha^{-1}\alpha\left(4\right) \times \left(\alpha^{-1}\alpha\left(-1\right)\right)^{m-n}\right\}\right)\right] \end{array}\right\} \\
= {} & \alpha\left(3\left(J_{n+1} - J_n\right) + 4\left(-1\right)^{m-n}\right) \\
= {} & \alpha\left(j_{m+n} + j_{m-n}\right) \\
= {} & \aleph\aleph j_{m+n} \dot{+} \aleph\aleph j_{m-n},
\end{aligned}$$

which is the desired result.

The proofs of others can be done similarly to the proof of 1) using (31)-(35). □

While examining the relevant literature, we observed that the Melham identity is also given for classical Jacobsthal and classical Jacobsthal-Lucas numbers. So, at the end of the study, we filled the gap in here and transferred the identity, we found, to the non-Newtonian version.

Theorem 2.16. *The Melham's identity of Jacobsthal numbers J_n and Jacobsthal-Lucas numbers j_n is given as*

1) $J_{n+1}J_{n+2}J_{n+6} - J_{n+3}^3 = -\frac{1}{3}(3J_n + j_n)\left(j_n^2 - 9J_n^2 + 5\right)$

2) $j_{n+1}j_{n+2}j_{n+6} - j_{n+3}^3 = 9(3J_n + j_n)\left(j_n^2 - 9J_n^2 - 5\right)$

for $n \geq 0$.

Proof. 1)Using Binet formula, (22) and (25) we compute the following expression:

$$
\begin{aligned}
& J_{n+1}J_{n+2}J_{n+6} - J_{n+3}^3 \\
= \; & \frac{2^{n+1} - (-1)^{n+1}}{3}\,\frac{2^{n+2} - (-1)^{n+2}}{3}\,\frac{2^{n+6} - (-1)^{n+6}}{3} - \left(\frac{2^{n+3} - (-1)^{n+3}}{3}\right)^3 \\
= \; & -\frac{8}{3}(-1)^n 2^{2n} - \frac{10}{3}2^n \\
= \; & -\frac{1}{3}2^{n+1}\left((-1)^n 2^{n+2} + 5\right) \\
= \; & -\frac{1}{3}(3J_n + j_n)\left(j_n^2 - 9J_n^2 + 5\right)
\end{aligned}
$$

which is Melham's identity for Jacobsthal numbers.

2)Using Binet formula, (22) and (25) we compute the following expression:

$$
\begin{aligned}
& j_{n+1}j_{n+2}j_{n+6} - j_{n+3}^3 \\
= \; & \left(2^{n+1} + (-1)^{n+1}\right)\left(2^{n+2} + (-1)^{n+2}\right)\left(2^{n+6} + (-1)^{n+6}\right) - \left(2^{n+3} + (-1)^{n+3}\right)^3 \\
= \; & 72 \times (-1)^n 2^{2n} - 90 \times 2^n \\
= \; & 9 \times 2^{n+1}((-1)^n 2^{n+2} - 5) \\
= \; & 9(3J_n + j_n)\left(j_n^2 - 9J_n^2 - 5\right)
\end{aligned}
$$

which is Melham's identity for Jacobsthal-Lucas numbers. □

Theorem 2.17. *The Melham's identity of non-Newtonian Jacobsthal numbers $\aleph\aleph J_n$ and non-Newtonian Jacobsthal-Lucas numbers $\aleph\aleph j_n$ is given as*

1) $\aleph\aleph J_{n+1} \dot{\times} \aleph\aleph J_{n+2} \dot{\times} \aleph\aleph J_{n+6} \dot{-} \aleph\aleph J_{n+3}^{\dot{3}} = \dot{-}\frac{\dot{1}}{\dot{3}} \dot{\times} \left(\dot{3} \dot{\times} \aleph\aleph J_n \dot{+} \aleph\aleph j_n\right) \dot{\times} \left(\aleph\aleph j_n^{\dot{2}} \dot{-} \dot{9} \dot{\times} \aleph\aleph J_n^{\dot{2}} \dot{+} \dot{5}\right)$

2) $\aleph\aleph j_{n+1} \dot{\times} \aleph\aleph j_{n+2} \dot{\times} \aleph\aleph j_{n+6} \dot{-} \aleph\aleph j_{n+3}^{\dot{3}} = \dot{-}\dot{9} \dot{\times} \left(\dot{3} \dot{\times} \aleph\aleph J_n \dot{+} \aleph\aleph j_n\right) \dot{\times} \left(\aleph\aleph j_n^{\dot{2}} \dot{-} \dot{9} \dot{\times} \aleph\aleph J_n^{\dot{2}} \dot{-} \dot{5}\right)$

for $n \geq 0$.

Proof. 1) By 1) in Theorem 2.16, we have the following:

$$\begin{aligned}
& \aleph\aleph J_{n+1} \dot{\times} \aleph\aleph J_{n+2} \dot{\times} \aleph\aleph J_{n+6} \dot{-} \aleph\aleph J_{n+3}^{\dot{3}} \\
= \; & \aleph\aleph J_{n+1} \dot{\times} \aleph\aleph J_{n+2} \dot{\times} \aleph\aleph J_{n+6} \dot{-} \aleph\aleph J_{n+3} \dot{\times} \aleph\aleph J_{n+3} \dot{\times} \aleph\aleph J_{n+3} \\
= \; & \alpha(J_{n+1}) \dot{\times} \alpha(J_{n+2}) \dot{\times} \alpha(J_{n+6}) \dot{-} \alpha(J_{n+3}) \dot{\times} \alpha(J_{n+3}) \dot{\times} \alpha(J_{n+3}) \\
= \; & \alpha\left\{\alpha^{-1}\alpha(J_{n+1}) \times \alpha^{-1}\alpha(J_{n+2}) \times \alpha^{-1}\alpha(J_{n+6})\right\} \dot{-} \alpha\left\{\alpha^{-1}\alpha(J_{n+3}) \times \alpha^{-1}\alpha(J_{n+3}) \times \alpha^{-1}\alpha(J_{n+3})\right\} \\
= \; & \alpha\left\{\begin{array}{c} \alpha^{-1}\alpha\left\{\alpha^{-1}\alpha(J_{n+1}) \times \alpha^{-1}\alpha(J_{n+2}) \times \alpha^{-1}\alpha(J_{n+6})\right\} \\ -\alpha^{-1}\alpha\left\{\alpha^{-1}\alpha(J_{n+3}) \times \alpha^{-1}\alpha(J_{n+3}) \times \alpha^{-1}\alpha(J_{n+3})\right\} \end{array}\right\} \\
= \; & \alpha\left(J_{n+1}J_{n+2}J_{n+6} - J_{n+3}^{3}\right) \\
= \; & \alpha\left(-\frac{1}{3}(3J_n + j_n)\left(j_n^2 - 9J_n^2 + 5\right)\right) \\
= \; & \dot{-}\frac{\dot{1}}{\dot{3}} \dot{\times} \left(\dot{3} \dot{\times} \aleph\aleph J_n \dot{+} \aleph\aleph j_n\right) \dot{\times} \left(\aleph\aleph j_n^{\dot{2}} \dot{-} \dot{9} \dot{\times} \aleph\aleph J_n^{\dot{2}} \dot{+} \dot{5}\right).
\end{aligned}$$

2) The conclusion is easily reached using 2) in Theorem 2.16. □

3. Conclusion and Future Works

This study provides insights into the mathematical properties of non-Newtonian versions of Jacobsthal and Jacobsthal-Lucas numbers within a broader context. It is a relatively new addition to the existing literature and generalizes known Jacobsthal and Jacobsthal-Lucas numbers. By presenting significant formulas and identities derived from the classical properties of these numbers, we extend the role of these sequences in mathematical literature. The findings of this study may serve as a valuable resource for researchers interested in these new types of numbers. Moreover, they can pave the way for future research endeavors in the analysis and applications of these sequences.

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