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On some matrix mean inequalities via the log-convexity property

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Abstract. Let $f:[0,1]\longrightarrow[0,+\infty)$ be a log-convex function, $0\le\mu\le\frac{1}{2}\le\tau\le 1$ and m be a positive integer. Then by using the Jensen's inequality we prove that

$$\frac{\mu^m}{\tau}\Big(f^m(0)\nabla_\tau f^m(1)-f^m(\tau)\Big)+r_m\Big(f(0)^{\frac{m}{2}}-f^{\frac{m}{2}}(\tau)\Big)^2\le(f(0)\nabla_\mu f(1))^m-f^m(\mu)$$

and

$$\frac{(1-\tau)^m}{1-\mu}\Big(f^m(0)\nabla_\mu f^m(1)-f^m(\mu)\Big)+r'_m\Big(f(1)^{\frac{m}{2}}-f^{\frac{m}{2}}(\mu)\Big)^2\le(f(0)\nabla_\tau f(1))^m-f^m(\tau).$$

Here, ∇_μ denotes the weighted arithmetic mean, and r_m, r'_m are two positive constants. Moreover, by choosing suitable log-convex functions, we derive new refinements of several classical inequalities that relate the difference between the arithmetic-power, arithmetic-harmonic, and arithmetic-geometric means for both scalars and matrices and matrices, as well as matrix norms and determinants.

1. Introduction

Let $\mathbb{M}_n$ be the algebra of all complex matrices of order $n\times n$. The positive semidefinite matrix $A\in\mathbb{M}_n$ written as $A\ge 0$, is a Hermitian matrix with $\langle Ax,x\rangle\ge 0$ for all $x\in\mathbb{C}^n$. If $A\in\mathbb{M}_n$ is a Hermitian matrix with $\langle Ax,x\rangle>0$ for all nonzero $x\in\mathbb{C}^n$, then A is called a positive definite matrix, written as $A>0$. The set of all positive matrices is denoted by $\mathbb{M}_n^+$ and the set of all positive semidefinite matrices in $\mathbb{M}_n^+$ is denoted by $\mathbb{M}_n^{++}$. The singular values of a matrix $A\in\mathbb{M}_n$ are the eigenvalues of the positive semidefinite matrix $|A|=(A^*A)^{1/2}$, denoted by $s_i(A)$ for $i=1,2,3,\dots,n$. A norm $|||\cdot|||$ on $\mathbb{M}_n$ is called unitarily invariant if $|||UAV|||=|||A|||$ for all $A\in\mathbb{M}_n$ and all unitary matrices $U,V\in\mathbb{M}_n$. The trace norm is given by $\|A\|_1=tr|A|=\sum_{k=1}^{n}s_k(A)$, where

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tr is the usual trace. This norm is unitarily invariant. Another important example of unitarily invariant norms is the Hilbert-Schmidt norm $\|\cdot\|_2$ defined by

$$\|A\|_2 = tr(AA^*) = \Big(\sum_{i,j} |a_{i,j}|^2\Big)^{\frac{1}{2}}, \quad (A = (a_{i,j})).$$

For $A, B \in \mathbb{M}_n^{++}$, $\mu \in [0,1]$ and $p \in (-1,1)$, the operators arithmetic, geometric, harmonic and power means are defined respectively, by

$$A\nabla_\mu B := (1-\mu)A + \mu B, \;\; A\sharp_\mu B := A^{1/2}\Big(A^{-1/2}BA^{-1/2}\Big)^\mu A^{1/2}, \;\; A!_\mu B = (A^{-1}\nabla_\mu B^{-1})^{-1}$$

and

$$A\sharp_{p,\mu} B = A^{1/2}\Big((1-\mu)I + \mu(A^{-1/2}BA^{-1/2})^p\Big)^{\frac{1}{p}} A^{1/2}, \quad p \in \mathbb{R}\backslash\{0\}.$$

The limit as $p \longrightarrow 0$ implies

$$A\sharp_{0,\mu} B = A\sharp_\mu B := A^{1/2}\Big(A^{-1/2}BA^{-1/2}\Big)^\mu A^{1/2}.$$

Further, the values $p = 1, -1$ give the arithmetic and harmonic means.

An important inequalities between the powers, geometric, and arithmetic means for operators is stated as follows, for $p \in [-1, 0]$ we have

$$A\sharp_{p,\mu} B \le A\sharp_\mu B \le A\nabla_\mu B. \tag{1}$$

The theory of convex and log-convex functions has been crucial in numerous fields, such as mathematical inequalities, optimization theory, functional analysis, applied mathematics, and mathematical physics. These functions play a central role in deriving important results and solving problems in these areas. Convexity helps establish key properties like monotonicity and optimality. Log-convex functions, in particular, are useful in refining inequalities and characterizing behavior in various applications. Their study has led to significant advancements in both theoretical and applied mathematics.

Recall that a function $f : I \to \mathbb{R}$ is said to be convex on the interval $I \subset \mathbb{R}$ if

$$f((1-\mu)a + \mu b) \le (1-\mu)f(a) + \mu f(b), \tag{2}$$

for all $a, b \in I$ and $\mu \in [0,1]$. If the inequality (2) is reversed, then f is said to be concave. If $\log f$ is convex, then f is called log-convex. Therefore, a log-convex function is a positive function satisfying

$$f((1-\mu)a + \mu b) \le f(a)^{1-\mu} f(b)^\mu, \tag{3}$$

for the same parameters as in (2). Specifically, if f is a log-convex function, it is also convex, As established by the well-known Young's inequality, stated in the second inequality of (4).

Using the same notation for positive numbers a and b, the harmonic-geometric-arithmetic mean inequalities state

$$a!_\mu b \le a\sharp_\mu b \le a\nabla_\mu b, \tag{4}$$

for $a, b > 0$ and $\mu \in [0,1]$, with equality if and only if $a = b$. Here the latter one of (4) is the classical Young's inequality.

F. Kittaneh and Y. Manasrah [19] derived the following noteworthy refinement of Young's inequality.

$$a\sharp_\mu b + r_0(\sqrt{a} - \sqrt{b})^2 \le a\nabla_\mu b, \tag{5}$$

where $r_0 = \min\{\mu, 1-\mu\}$.

In [4], H. Alzer et al. presented the following Young-type inequalities, which play a significant role in various mathematical contexts, particularly mathematics inequalities.

Theorem 1.1 (Alzer-Fonseca-Kovačec). *Let $a, b > 0$ and let λ, μ and τ be real numbers with $\lambda \geq 1$ and $0 \leq \mu < \tau \leq 1$. Then*

$$\Big(\frac{\mu}{\tau}\Big)^{\lambda} \leq \frac{(a\nabla_{\mu}b)^{\lambda} - (a\sharp_{\mu}b)^{\lambda}}{(a\nabla_{\tau}b)^{\lambda} - (a\sharp_{\tau}b)^{\lambda}} \leq \Big(\frac{1-\mu}{1-\tau}\Big)^{\lambda}.$$

The Alzer-Fonseca-Kovačec inequalities have, in fact, become one of the most significant extensions of Young's inequalities Another significant refinement of the power form for the case of two weighted Young's inequalities was presented in [14].

J. Liao and J. Wu [26] replicated the Alzer-Fonseca-Kovačec inequalities for the difference between arithmetic and harmonic means as follows

Theorem 1.2 ([26]). *Let $a, b > 0$ and let λ, μ and τ be real numbers with $\lambda \geq 1$ and $0 \leq \mu < \tau \leq 1$. Then*

$$\Big(\frac{\mu}{\tau}\Big)^{\lambda} \leq \frac{(a\nabla_{\mu}b)^{\lambda} - (a!_{\mu}b)^{\lambda}}{(a\nabla_{\tau}b)^{\lambda} - (a!_{\tau}b)^{\lambda}} \leq \Big(\frac{1-\mu}{1-\tau}\Big)^{\lambda}.$$

At the same time, M. Khosravi [17] derived the following inequalities that relate the difference between the arithmetic and the power means.

Theorem 1.3. *Let $a, b > 0$, $p \in (-1, 1)$ and let μ, τ be real numbers with $0 \leq \mu < \tau \leq 1$. Then*

$$\frac{\mu}{\tau} \leq \frac{(a\nabla_{\mu}b) - (a\sharp_{\mu,p}b)}{(a\nabla_{\tau}b) - (a\sharp_{\tau,p}b)} \leq \frac{1-\mu}{1-\tau}.$$

Interestingly, Theorems 1.1, 1.2 and 1.3 happened to be special cases of a more general result obtained by M. Sababheh via convexity:

Theorem 1.4 ([21]). *Let $f : [0, 1] \longrightarrow [0, +\infty)$ be convex and let λ, μ and τ be real numbers with $\lambda \geq 1$ and $0 \leq \mu < \tau \leq 1$. Then*

$$\Big(\frac{\mu}{\tau}\Big)^{\lambda} \leq \frac{((1-\mu)f(0) + \mu f(1))^{\lambda} - f^{\lambda}(\mu)}{((1-\tau)f(0) + \tau f(1))^{\lambda} - f^{\lambda}(\tau)} \leq \Big(\frac{1-\mu}{1-\tau}\Big)^{\lambda}.$$

For additional reading on the generalized refinement of Young's inequality, the reader is encouraged to consult recent papers [1, 3, 10–13, 15, 16, 19, 22, 23, 25].

The structure of this paper is as follows: In Section 2, We present Theorem 2.4, which establishes the primary inequalities for log-convex functions. In Section 3, As an application of these results, we derive a significant new refinement of inequalities involving the differences among the arithmetic-geometric, arithmetic-harmonic, and arithmetic-power means, as discussed in the introduction. In Section 4, we explores some applications of the main results from Section 2 to derive analogous inequalities for matrices. In Section 5, by employing specific log-convex functions, we refine certain inequalities between the arithmetic-power, arithmetic-harmonic, and arithmetic-geometric means for particular norms. In the final section, we expand the applications of Section 2 to derive analogous inequalities for determinants.

2. Log-convexity results

In this section, our goal is to explore new inequalities related to log-convex functions. Before presenting and proving our results, we first introduce the following theorem on Jensen's inequality.

Theorem 2.1. *Let $f : I \longrightarrow \mathbb{R}$ be convex, $\{x_1, \ldots, x_n\} \subset I$ and $\{\mu_1, \ldots, \mu_n\} \subset [0, 1]$ be such that $\sum_{k=1}^{n} \mu_k = 1$. Then,*

$$f\Big(\sum_{k=1}^{n} \mu_k x_k\Big) \leq \sum_{k=1}^{n} \mu_k f(x_k). \tag{6}$$

We also require the following two lemmas.

Lemma 2.2 ([12]). *Let m be a positive integer and let μ be a positive number, such that $0 \leq \mu \leq 1$. Then*

$$\sum_{k=1}^{m} \binom{m}{k} k\mu^k (1-\mu)^{m-k} = m\mu, \tag{7}$$

where $\binom{m}{k}$ is the binomial coefficient.

Lemma 2.3 ([14]). *Let μ and τ be a two positive numbers such that $0 \leq \mu \leq \frac{1}{2} \leq \tau \leq 1$, and m be a positive integer. Then*

$$(1-\mu)^m - (1-\tau)\frac{\mu^m}{\tau} \geq 0 \quad \textit{and} \quad \tau^m - (1-\tau)^m \frac{\mu}{1-\mu} \geq 0.$$

Throughout the rest of this paper, we denote

$$\min\left\{\frac{\mu^m}{\tau}, (1-\mu)^m - (1-\tau)\frac{\mu^m}{\tau}\right\}$$

and

$$\min\left\{\frac{(1-\tau)^m}{1-\mu}, \tau^m - (1-\tau)^m \frac{\mu}{1-\mu}\right\},$$

respectively, by r_m and r'_m.

We are now prepared to prove our main results on log-convex functions. Additionally, the significance of these results is highlighted in Remark 2.5 below.

Theorem 2.4. *Let $f : [0,1] \longrightarrow [0,+\infty)$ be log-convex and $0 \leq \mu \leq \frac{1}{2} \leq \tau \leq 1$. Then for all positive integers m,*

$$\frac{\mu^m}{\tau}\Big(f^m(0)\nabla_\tau f^m(1) - f^m(\tau)\Big) + r_m\Big(f(0)^{\frac{m}{2}} - f^{\frac{m}{2}}(\tau)\Big)^2 \leq (f(0)\nabla_\mu f(1))^m - f^m(\mu)$$

and

$$\frac{(1-\tau)^m}{1-\mu}\Big(f^m(0)\nabla_\mu f^m(1) - f^m(\mu)\Big) + r'_m\Big(f(1)^{\frac{m}{2}} - f^{\frac{m}{2}}(\mu)\Big)^2 \leq (f(0)\nabla_\tau f(1))^m - f^m(\tau).$$

Proof. 1. Suppose that $0 \leq \mu \leq \frac{1}{2} \leq \tau \leq 1$. We claim that

$$f^m(\mu) + \frac{\mu^m}{\tau}\Big(f^m(1)\nabla_\tau f^m(0) - f^m(\tau)\Big) + r_m\Big(f^{\frac{m}{2}}(0) - f^{\frac{m}{2}}(\tau)\Big)^2 \leq (f(0)\nabla_\mu f(1))^m.$$

We have the following identities

$$\begin{aligned}
&(f(0)\nabla_\mu f(1))^m - \frac{\mu^m}{\tau}\Big(f^m(0)\nabla_\tau f^m(1) - f^m(\tau)\Big) - r_m\Big(f^{\frac{m}{2}}(0) - f^{\frac{m}{2}}(\tau)\Big)^2 \\
&= \sum_{k=0}^{m} \binom{m}{k} \mu^k (1-\mu)^{m-k} f^k(1) f^{m-k}(0) - \frac{\mu^m}{\tau}\Big((1-\tau)f^m(0) + \tau f^m(1) - f^m(\tau)\Big) \\
&\quad - r_m\Big(f^m(\tau) + f^m(0) - 2f^{\frac{m}{2}}(\tau) f^{\frac{m}{2}}(0)\Big) \\
&= \sum_{k=0}^{m} \binom{m}{k} \mu^k (1-\mu)^{m-k} f^k(1) f^{m-k}(0) - \mu^m f^m(1) - \mu^m \frac{1-\tau}{\tau} f^m(0) + \frac{\mu^m}{\tau} f^m(\tau)
\end{aligned}$$

$$
\begin{aligned}
&-r_m\Big(f^m(\tau)+f^m(0)-2f^{\frac{m}{2}}(\tau)f^{\frac{m}{2}}(0)\Big)\\
&=\sum_{k=1}^{m-1}\binom{m}{k}\mu^k(1-\mu)^{m-k}f^k(1)f^{m-k}(0)+\Big((1-\mu)^m-\mu^m\frac{1-\tau}{\tau}-r_m\Big)f^m(0)\\
&\quad+\Big(\frac{\mu^m}{\tau}-r_m\Big)f^m(\tau)+2r_mf^{\frac{m}{2}}(\tau)f^{\frac{m}{2}}(0)\\
&\quad\geq\sum_{k=1}^{m-1}\binom{m}{k}\mu^k(1-\mu)^{m-k}f^m(\frac{k}{m})+\Big((1-\mu)^m-\mu^m\frac{1-\tau}{\tau}-r_m\Big)f^m(0)\\
&\quad+\Big(\frac{\mu^m}{\tau}-r_m\Big)f^m(\tau)+2r_mf^m\left(\frac{\tau}{2}\right)\qquad \textbf{(by Lemma 2.3)}\\
&=\sum_{k=0}^{m+1}\mu_kf^m(x_k),
\end{aligned}
$$

where

$$
x_k=\begin{cases}0 & \text{if } k=0,\\[2ex] \frac{k}{m} & \text{if } 1\leq k\leq m-1,\\[2ex] \tau & \text{if } k=m,\\[2ex] \frac{\tau}{2} & \text{if } k=m+1,\end{cases}
$$

and

$$
\mu_k=\begin{cases}(1-\mu)^m-\mu^m\frac{1-\tau}{\tau}-r_m & \text{if } k=0,\\[2ex] \binom{m}{k}\mu^k(1-\mu)^{m-k} & \text{if } 1\leq k\leq m-1.\\[2ex] \frac{\mu^m}{\tau}-r_m & \text{if } k=m,\\[2ex] 2r_m & \text{if } k=m+1.\end{cases}
$$

By using Lemma 2.3, we have

(a) $x_k\geq 0$ for all $k\in\{0,1,\ldots,m,m+1\}$,

(b) $\mu_k\geq 0$ for all $k\in\{0,1,\ldots,m,m+1\}$. Further $\sum_{k=0}^{m+1}\mu_k=1$.

Since, the function f^m is convex. Theorem 2.1 implies

$$
\begin{aligned}
(f(0)\nabla_\mu f(1))^m \quad - \quad & \frac{\mu^m}{\tau}\Big(f^m(0)\nabla_\tau f^m(1)-f^m(\tau)\Big)-r_m\Big(f^{\frac{m}{2}}(0)-f^{\frac{m}{2}}(\tau)\Big)^2\\
\geq \quad & \sum_{k=0}^{m+1}\mu_kf^m(x_k)\geq f^m\Big(\sum_{k=0}^{m+1}\mu_kx_k\Big)=f^m(\mu).
\end{aligned}
$$

2nd case: For the second inequality, first notice that if the function $f(x)$ is log-convex on $[0,1]$, then $f(1-x)$ is log-convex on $[0,1]$. If $0\leq\mu\leq\dfrac{1}{2}\leq\tau\leq 1$, then we have $0\leq 1-\tau\leq\dfrac{1}{2}\leq 1-\mu\leq 1$. So by changing $f(x)$, μ and τ into $f(1-x)$, $1-\tau$ and $1-\mu$, respectively in the first inequality. We obtain the desired results.

□

Remark 2.5. *Before delving into further results, we briefly discuss the relationship between Theorem 2.4 and Theorem 1.4.*
Notice that the first inequality in Theorem 1.4 can be written as

$$\left(\frac{\mu}{\tau}\right)^m \left[(f(0)\nabla_\tau f(1))^m - f^m(\tau)\right] \le \left(f(0)\nabla_\mu f(1)\right)^m - f^m(\mu); 0 \le \mu < \tau \le 1, \tag{8}$$

while the second inequality in the same theorem can be stated as

$$\left[\left(f(0)\nabla_\mu f(1)\right)^m - f^m(\mu)\right] \le \left(\frac{1-\mu}{1-\tau}\right)^m (f(0)\nabla_\tau f(1))^m - f^m(\tau); 0 \le \mu < \tau \le 1 \tag{9}$$

where $m = 1, 2, \cdots$.
As a result, the first inequality in Theorem 2.4 provides a new refinement of (8)*, while the second inequality introduces a refining term for* (9)*. Together, the three parts of Theorem 2.4 offer a significant refinement of Theorem 1.4.*

It is important to note that these refinements have been established for integer powers m and for log-convex functions. Additionally, the assumption that f is log-convex played a crucial role in the proof.

Since Theorem 1.4 generalized the results from [4, 17, 26], our findings in this section offer improved estimates compared to those in these references, highlighting the significance of our results. In the following sections, we provide explicit examples of refined inequalities for both scalars and operators.

3. Applications to scalar inequalities

In this section, by choosing suitable log-convex functions, we derive new refinements of classical inequalities involving the differences between the arithmetic-power, arithmetic-harmonic, and arithmetic-geometric means for scalars.

Let $a, b > 0$, $\lambda \in [0,1]$ and $p \in \mathbb{R} \setminus \{0\}$. It is widely known that the function

$$p \mapsto a\sharp_{p,\lambda} b = ((1-\lambda)a^p + \lambda b^p)^{\frac{1}{p}}$$

is increasing on $\mathbb{R} \setminus \{0\}$. In particular, we have

$$a\sharp_{p,\lambda} b \le a\nabla_\lambda b,$$

for every $p \in (-\infty, 0)$. Furthermore, it is known that $a\sharp_\lambda b = \lim\limits_{\substack{p\to 0\\ p\neq 0}} a\sharp_{p,\lambda} b$.

On the other hand, we can easily show that for every $p \in (-\infty, 0)$, the function $\lambda \mapsto a\sharp_{p,\lambda} b$ is log-convex on $[0,1]$. So, by applying Theorem 2.4 we obtain the following new lower bound for the difference between the arithmetic and power means.

Corollary 3.1. *Let* $a, b > 0$, $p \in (-\infty, 0)$ *and* $0 \le \mu \le \frac{1}{2} \le \tau \le 1$. *Then for all positive integers m, we have*

$$\begin{aligned}\frac{\mu^m}{\tau}\Big((a^m\nabla_\tau b^m) - (a\sharp_{p,\tau} b)^m\Big) \quad &+ \quad r_m\Big(b^{\frac{m}{2}} - (a\sharp_{p,\tau} b)^{\frac{m}{2}}\Big)^2 \\ &\le \quad \Big(a\nabla_\mu b\Big)^m - \Big(a\sharp_{p,\mu} b\Big)^m,\end{aligned} \tag{10}$$

and

$$\begin{aligned}\frac{(1-\tau)^m}{1-\mu}\Big((a^m\nabla_\mu b^m) - (a\sharp_{p,\mu} b)^m\Big) \quad &+ \quad r'_m\Big(a^{\frac{m}{2}} - (a\sharp_{p,\mu} b)^{\frac{m}{2}}\Big)^2 \\ &\le \quad \Big(a\nabla_\tau b\Big)^m - \Big(a\sharp_{p,\tau} b\Big)^m.\end{aligned} \tag{11}$$

If we take $p = -1$ in Corollary 3.1, we obtain the following corollary, that proves a generalized refinement of the difference between arithmetic and harmonic mean inequality.

Corollary 3.2. *Let $a, b > 0$ and $0 \le \mu \le \frac{1}{2} \le \tau \le 1$. Then for all positive integers m,*

$$\begin{aligned}\frac{\mu^m}{\tau}\Big((a^m\nabla_\tau b^m) - (a!_\tau b)^m\Big) \quad &+ \quad r_m\Big(b^{\frac{m}{2}} - (a!_\tau b)^{\frac{m}{2}}\Big)^2 \\ &\le \quad \Big(a\nabla_\mu b\Big)^m - \Big(a!_\mu b\Big)^m, \end{aligned} \tag{12}$$

and

$$\begin{aligned}\frac{(1-\tau)^m}{1-\mu}\Big((a^m\nabla_\mu b^m) - (a!_\mu b)^m\Big) \quad &+ \quad r'_m\Big(a^{\frac{m}{2}} - (a!_\mu b)^{\frac{m}{2}}\Big)^2 \\ &\le \quad \Big(a\nabla_\tau b\Big)^m - \Big(a!_\tau b\Big)^m. \end{aligned} \tag{13}$$

If we take the limit as $p \longrightarrow 0$ in Corollary 3.1, we obtain the following corollary, that presents a new generalized refinement of the difference between arithmetic and geometric mean inequality presented in [14].

Corollary 3.3. *Let $a, b > 0$ and $0 \le \mu \le \frac{1}{2} \le \tau \le 1$. Then for all positive integers m,*

$$\begin{aligned}\frac{\mu^m}{\tau}\Big((a^m\nabla_\tau b^m) - (a\sharp_\tau b)^m\Big) \quad &+ \quad r_m\Big(b^{\frac{m}{2}} - (a\sharp_\tau b)^{\frac{m}{2}}\Big)^2 \\ &\le \quad \Big(a\nabla_\mu b\Big)^m - \Big(a\sharp_\mu b\Big)^m, \end{aligned} \tag{14}$$

and

$$\begin{aligned}\frac{(1-\tau)^m}{1-\mu}\Big((a^m\nabla_\mu b^m) - (a\sharp_\mu b)^m\Big) \quad &+ \quad r'_m\Big(a^{\frac{m}{2}} - (a\sharp_\mu b)^{\frac{m}{2}}\Big)^2 \\ &\le \quad \Big(a\nabla_\tau b\Big)^m - \Big(a\sharp_\tau b\Big)^m. \end{aligned} \tag{15}$$

Remark 3.4. *If we set $\tau = \frac{1}{2}$ in* (14) *and $\mu = \frac{1}{2}$ in (15) respectively, then we recapture Theorem 3 from [1].*

4. Applications to some matrix inequalities

Our aim in this section is to discuss some matrix inequalities that correspond to scalar inequalities derived in the previous section.

In order to obtain matrix inequalities from the corresponding scalar inequalities, we will use the monotonicity property of operator functions described in the following lemma.

Lemma 4.1 ([24], p. 3). *Let $A \in \mathbb{M}_n$ be Hermitian. If f and g are both continuous functions with $f(t) \ge g(t)$ for $t \in Sp(A)$ (where the sign $Sp(A)$ denotes the spectrum of matrix A), then $f(A) \ge g(A)$.*

An analogue of Corollary 3.1 for matrices is the following theorem, which establishes a refinement of the difference between the arithmetic and power mean inequalities for matrices.

Theorem 4.2. *Let $A, B \in \mathbb{M}_n^{++}$, $p \in (-\infty, 0)$ and $0 \le \mu \le \frac{1}{2} \le \tau \le 1$. Then for all positive integers m,*

$$\begin{aligned}\frac{\mu^m}{\tau}\Big(A\nabla_\tau(A\sharp_m B) \quad &- \quad A\sharp_m(A\sharp_{p,\tau}B)\Big) \\ &+ \quad r_m\Big(A + A\sharp_m(A\sharp_{p,\tau}B) - 2A\sharp_{\frac{m}{2}}(A\sharp_{p,\tau}B)\Big) \\ &\le \quad A\sharp_m(A\nabla_\mu B) - A\sharp_m(A\sharp_{p,\mu}B), \end{aligned} \tag{16}$$

$$\begin{aligned}\frac{(1-\tau)^m}{1-\mu}\Big(A\nabla_\mu(A\sharp_m B) \quad &- \quad A\sharp_m(A\sharp_{p,\mu}B)\Big)\\ &+ \quad r'_m\Big(A\sharp_m B + A\sharp_m(A\sharp_{p,\mu}B) - 2(A\sharp_{\frac{m}{2}}B)A^{-1}A\sharp_{\frac{m}{2}}(A\sharp_{p,\mu}B)\Big)\\ &\le \quad A\sharp_m(A\nabla_\tau B) - A\sharp_m(A\sharp_{p,\tau}B).\end{aligned} \tag{17}$$

Proof. Let $b = 1$ in inequality (10). Then

$$\begin{aligned}\frac{\mu^m}{\tau}\Big((\tau a^m + (1-\tau)) \quad &- \quad (\tau a^p + (1-\tau))^{\frac{m}{p}}\Big)\\ &+ \quad r_m\Big(1 + (\tau a^p + (1-\tau))^{\frac{m}{p}} - 2(\tau a^p + (1-\tau))^{\frac{m}{2p}}\Big)\\ &\le \quad (\mu a + (1-\mu))^m - (\mu a^p + (1-\mu))^{\frac{m}{p}}.\end{aligned} \tag{18}$$

Since the matrix $C = A^{\frac{-1}{2}}BA^{\frac{-1}{2}}$ has a positive spectrum, Lemma 4.1 and (18) imply

$$\begin{aligned}\frac{\mu^m}{\tau}\Big((\tau C^m + (1-\tau)I) \quad &- \quad (\tau C^p + (1-\tau)I)^{\frac{m}{p}}\Big)\\ &+ \quad r_m\Big(I + (\tau C^p + (1-\tau)I)^{\frac{m}{p}} - 2(\tau C^p + (1-\tau)I)^{\frac{m}{2p}}\Big)\\ &\le \quad (\mu C + (1-\mu)I)^m - (\mu C^p + (1-\mu)I)^{\frac{m}{p}}.\end{aligned} \tag{19}$$

Finally, multiplying inequality (19) by $A^{\frac{1}{2}}$ on the left and right hand sides, we get

$$\begin{aligned}\frac{\mu^m}{\tau}\Big(A\nabla_\tau(A\sharp_m B) \quad &- \quad A\sharp_m(A\sharp_{p,\tau}B)\Big)\\ &+ \quad r_m\Big(A + A\sharp_m(A\sharp_{p,\tau}B) - 2A\sharp_{\frac{m}{2}}(A\sharp_{p,\tau}B)\Big)\\ &\le \quad A\sharp_m(A\nabla_\mu B) - A\sharp_m(A\sharp_{p,\mu}B).\end{aligned}$$

Using the same technique in (11), we get (17). This completes the proof. □

To deduce our first corollary of the above theorem we need the following simple lemma from [27].

Lemma 4.3 ([27]). *Let $A, B \in \mathbb{M}_n^{++}$ and $\mu, \tau \in [0, 1]$ are be real numbers. Then*

$$A\sharp_\tau(A\sharp_\mu B) = A\sharp_{\tau\mu}B.$$

If we take the limit as $p \longrightarrow 0$ in Theorem 4.2, we obtain the following corollary, that proves a refinement of the difference between arithmetic and geometric mean inequality for matrices.

Corollary 4.4. *Let $A, B \in \mathbb{M}_n^{++}$ and $0 \le \mu \le \frac{1}{2} \le \tau \le 1$. Then for all positive integers m,*

$$\begin{aligned}\frac{\mu^m}{\tau}\Big(A\nabla_\tau(A\sharp_m B) - A\sharp_{m\tau}B\Big) \quad &+ \quad r_m\Big(A + A\sharp_{m\tau}B - 2A\sharp_{\frac{m\tau}{2}}B\Big)\\ &\le \quad A\sharp_m(A\nabla_\mu B) - A\sharp_{m\mu}B,\end{aligned} \tag{20}$$

and

$$\begin{aligned}\frac{(1-\tau)^m}{1-\mu}\Big(A\nabla_\mu(A\sharp_m B) - A\sharp_{m\mu}B\Big) \quad &+ \quad r'_m\Big(A\sharp_m B + A\sharp_{m\mu}B - 2A\sharp_{\frac{m}{2}+\frac{m\mu}{2}}B\Big)\\ &\le \quad A\sharp_m(A\nabla_\tau B) - A\sharp_{m\tau}B.\end{aligned} \tag{21}$$

Take $p = -1$ in Theorem 4.2, we obtain the following corollary, that prove a refinement of the difference between arithmetic and harmonic mean inequality for matrices.

Corollary 4.5. *Let $A, B \in \mathbb{M}_n^{++}$ and $0 \le \mu \le \frac{1}{2} \le \tau \le 1$. Then for all positive integer m, we have*

$$\begin{aligned}\frac{\mu^m}{\tau}\Big(A\nabla_\tau(A\sharp_m B) &- A\sharp_m(A!_\tau B)\Big)\\ &+ r_m\Big(A + A\sharp_m(A!_\tau B) - 2A\sharp_{\frac{m}{2}}(A!_\tau B)\Big)\\ &\le A\sharp_m(A\nabla_\mu B) - A\sharp_m(A!_\mu B),\end{aligned} \tag{22}$$

$$\begin{aligned}\frac{(1-\tau)^m}{1-\mu}\Big(A\nabla_\tau(A\sharp_m B) &- A\sharp_m(A!_\mu B)\Big)\\ &+ r'_m\Big(A\sharp_m B + A\sharp_m(A!_\mu B) - 2(A\sharp_{\frac{m}{2}} B)A^{-1}A\sharp_{\frac{m}{2}}(A!_\mu B)\Big)\\ &\le A\sharp_m(A\nabla_\tau B) - A\sharp_m(A!_\tau B).\end{aligned} \tag{23}$$

5. Applications to some norm inequalities

In this section, by selecting some appropriate log-convex functions, we obtain new refinements of some inequalities between arithmetic-power, arithmetic-harmonic and arithmetic-geometric means for certain norms.

The matrix version of classical Young inequality $a\sharp_t b \le a\nabla_t b$ states that [7],

$$|||A^{1-t}XB^t||| \le (1-t)|||AX||| + t|||XB|||, \ 0 \le t \le 1. \tag{24}$$

It is known that when $A, B \in \mathbb{M}_n^+$ and $X \in \mathbb{M}_n$, the function $f(\mu) = |||A^{1-\mu}XB^\mu|||$ is log-convex on $[0, 1]$, (see [20]) for any unitarily invariant norm $|||.|||$ on $\mathbb{M}_n$. Then by using the Theorem 2.4 we have the following corollary which presents a new generalized refinement of Young's ineuality (24).

Corollary 5.1. *Let $A, B \in \mathbb{M}_n^+$ and $X \in \mathbb{M}_n$, $0 \le \mu \le \frac{1}{2} \le \tau \le 1$ and m, be a positive integer. Then*

$$\begin{aligned}\frac{\mu^m}{\tau}\Big(|||AX|||^m\nabla_\tau|||XB|||^m &- |||A^{1-\tau}XB^\tau|||^m\Big)\\ &+ r_m\Big(|||XB|||^{\frac{m}{2}} - |||A^{1-\tau}XB^\tau|||^{\frac{m}{2}}\Big)^2\\ &\le \Big(|||AX|||\nabla_\mu|||XB|||\Big)^m - |||A^{1-\mu}XB^\mu|||^m,\end{aligned}$$

and

$$\begin{aligned}\frac{(1-\tau)^m}{1-\mu}\Big(|||AX|||^m\nabla_\mu|||XB|||^m &- |||A^{1-\mu}XB^\mu|||^m\Big)\\ &+ r'_m\Big(|||AX|||^{\frac{m}{2}} - |||A^{1-\mu}XB^\mu|||^{\frac{m}{2}}\Big)^2\\ &\le \Big(|||AX|||\nabla_\tau|||XB|||\Big)^m - |||A^{1-\tau}XB^\tau|||^m.\end{aligned}$$

It is known that when $A, B \in \mathbb{M}_n^+$ and $X \in \mathbb{M}_n$, the function

$$f(\mu) = |||A^{1-\mu}XB^\mu|||.|||A^\mu XB^{1-\mu}|||$$

is log-convex on $[0, 1]$, (see [20]) for any unitarily invariant norm $|||.|||$ on $\mathbb{M}_n$. Then by using Theorem 2.4 we have the following corollary.

Corollary 5.2. *Let $A, B \in \mathbb{M}_n^+$ and $X \in \mathbb{M}_n$, $0 \le \mu \le \frac{1}{2} \le \tau \le 1$ and m, be a positive integer. Then*

$$\begin{aligned}\frac{\mu^m}{\tau}\Big((|||AX|||.|||XB|||)^m &- (|||A^{1-\tau}XB^{\tau}|||.|||A^{\tau}XB^{1-\tau}||)^m\Big)\\ &+r_m\Big((|||AX|||.|||XB|||)^{\frac{m}{2}} - (|||A^{1-\tau}XB^{\tau}|||.|||A^{\tau}XB^{1-\tau}||)^{\frac{m}{2}}\Big)^2\\ &\le \Big(|||AX|||.|||XB|||\Big)^m - (|||A^{1-\mu}XB^{\mu}|||.|||A^{\mu}XB^{1-\mu}||)^m,\end{aligned}$$

and

$$\begin{aligned}\frac{(1-\tau)^m}{1-\mu}\Big((|||AX|||.|||XB|||)^m &- (|||A^{1-\mu}XB^{\mu}|||.|||A^{\mu}XB^{1-\mu}||)^m\Big)\\ &+r'_m\Big((|||AX|||.|||XB|||)^{\frac{m}{2}} - (|||A^{1-\mu}XB^{\mu}|||.|||A^{\mu}XB^{1-\mu}||)^{\frac{m}{2}}\Big)^2\\ &\le \Big(|||AX|||.|||XB|||\Big)^m - (|||A^{1-\tau}XB^{\tau}|||.|||A^{\tau}XB^{1-\tau}||)^m.\end{aligned}$$

The next lemma provides a technical result which we will need in the next result.

Lemma 5.3 ([21]). *Let $A, B \in \mathbb{M}_n^{++}$ and $X \in \mathbb{M}_n$, and let $f(\mu) = \|A^{\mu}XB^{1-\mu} + A^{1-\mu}XB^{\mu}\|_2$. Then f is log-convex on* $[0, 1]$.

Using this lemma, together with Theorem 2.4, we have the following corollary.

Corollary 5.4. *Let $A, B \in \mathbb{M}_n^{++}$ and $X \in \mathbb{M}_n$, $0 \le \mu \le \frac{1}{2} \le \tau \le 1$ and m, be a positive integer. Then*

$$\begin{aligned}\frac{\mu^m}{\tau}\Big(\|AX+XB\|_2^m &- \|A^{\tau}XB^{1-\tau} + A^{1-\tau}XB^{\tau}\|_2^m\Big)\\ &+ r_m\Big(\|AX+XB\|_2^{\frac{m}{2}} - \|A^{\tau}XB^{1-\tau} + A^{1-\tau}XB^{\tau}\|_2^{\frac{m}{2}}\Big)^2\\ &\le \|AX+XB\|_2^m - \|A^{\mu}XB^{1-\mu} + A^{1-\mu}XB^{\mu}\|_2^m,\end{aligned}$$

and

$$\begin{aligned}\frac{(1-\tau)^m}{1-\mu}\Big(\|AX+XB\|_2^m &- \|A^{\mu}XB^{1-\mu} + A^{1-\mu}XB^{\mu}\|_2^m\Big)\\ &+ r'_m\Big(\|AX+XB\|_2^{\frac{m}{2}} - \|A^{\mu}XB^{1-\mu} + A^{1-\mu}XB^{\mu}\|_2^{\frac{m}{2}}\Big)^2\\ &\le \|AX+XB\|_2^m - \|A^{\tau}XB^{1-\tau} + A^{1-\tau}XB^{\tau}\|_2^m.\end{aligned}$$

It is known that when $A, B \in \mathbb{M}_n^+$ the function $f(\mu) = tr(A^{1-\mu}B^{\mu})$ is log-convex on $[0, 1]$, (see [20]). Then by using Theorem 2.4 we obtain the following corollary.

Corollary 5.5. *Let $A, B \in \mathbb{M}_n^{++}$, $0 \le \mu \le \frac{1}{2} \le \tau \le 1$ and m be a positive integer. Then*

$$\begin{aligned}\frac{\mu^m}{\tau}\Big(tr^m(A)\nabla_{\tau}tr^m(B) &- tr(A^{1-\tau}B^{\tau})^m\Big)\\ &+ r_m\Big(tr(B)^{\frac{m}{2}} - tr(A^{1-\tau}B^{\tau})^{\frac{m}{2}}\Big)^2\\ &\le tr(A\nabla_{\mu}B)^m - tr(A^{1-\mu}B^{\mu})^m,\end{aligned}$$

and

$$\begin{aligned}\frac{(1-\tau)^m}{1-\mu}\Big(tr^m(A)\nabla_{\mu}tr^m(B) &- tr(A^{1-\mu}B^{\mu})^m\Big)\\ &+ r'_m\Big(tr(A)^{\frac{m}{2}} - tr(A^{1-\mu}B^{\mu})^{\frac{m}{2}}\Big)^2\\ &\le tr(A\nabla_{\tau}B)^m - tr(A^{1-\tau}B^{\tau})^m.\end{aligned}$$

6. Application to determinant inequalities

In this section, we examine the matrix version of inequalities involving the arithmetic, power, harmonic, and geometric means for determinants, which correspond to the scalar inequalities derived in Section 2. Let A and B be two positive definite matrices. The matrix version of the classical Young inequality $a\sharp_t b \leq a\nabla_t b$ for determinant states that [7],

$$det\left(A^{1-t}B^t\right) \leq det\left((1-t)A + tB\right),\ 0 \leq t \leq 1. \tag{25}$$

Several other forms of Young's inequalities for the determinant, based on weaker scalar inequalities, can also be found in a paper of J. Liao [26].

Before giving our result, we need the following two lemmas.

Lemma 6.1 ([6], p. 26). *(Minkowski inequality) Let $a = [a_j]$, $b = [b_j]$, $j = 1, \ldots, n$ such that a_j, b_j are positive real numbers. Then*

$$\Big(\prod_{j=1}^{n} a_j\Big)^{\frac{1}{n}} + \Big(\prod_{j=1}^{n} b_j\Big)^{\frac{1}{n}} \leq \Big(\prod_{j=1}^{n}(a_j + b_j)\Big)^{\frac{1}{n}}.$$

Equality holds if and only if $a = b$.

Lemma 6.2. *Let a and b be positive real numbers with $a > b$. If $\lambda \geq 1$, then*

$$a^\lambda - b^\lambda \geq (a-b)^\lambda.$$

Proof. For $\lambda \geq 1$ the function $f(t) = t^\lambda$ is super-additive, in the sense that $f(a+b) \geq f(a) + f(b)$. Since, $a > b$, we have

$$a^\lambda = (a - b + b)^\lambda \geq (a-b)^\lambda + b^\lambda.$$

It follows that $a^\lambda - b^\lambda \geq (a-b)^\lambda$. □

The first key result of this section is stated as follows.

Theorem 6.3. *Let $A, B \in \mathbb{M}_n^{++}$, $p \in (-\infty, 0)$, $T = A^{\frac{-1}{2}}BA^{\frac{-1}{2}}$ and $0 \leq \mu \leq \dfrac{1}{2} \leq \tau \leq 1$. Then for all positive integers m,*

$$\begin{aligned} \frac{\mu^m}{\tau}\det\Big(A\nabla_\tau B \quad &- \quad (A\sharp_{p,\tau}B)\Big)^{\frac{m}{n}} \\ &+ \quad r_m \det\Big(A^{\frac{m}{2}}(I - (I\sharp_{p,\tau}T)^{\frac{m}{2}})^2 A^{\frac{m}{2}}\Big)^{\frac{1}{n}} \\ &\leq \quad \det(A\nabla_\mu B)^{\frac{m}{n}} - \det(A\sharp_{p,\mu}B)^{\frac{m}{n}}, \end{aligned} \tag{26}$$

and

$$\begin{aligned} \frac{(1-\tau)^m}{1-\mu}\det\Big(A\nabla_\mu B \quad &- \quad (A\sharp_{p,\mu}B)\Big)^{\frac{m}{n}} \\ &+ \quad r'_m \det\Big(A^{\frac{m}{2}}(T^{\frac{m}{2}} - (I\sharp_{p,\tau}T)^{\frac{m}{2}})^2 A^{\frac{m}{2}}\Big)^{\frac{1}{n}} \\ &\leq \quad \det(A\nabla_\tau B)^{\frac{m}{n}} - \det(A\sharp_{p,\tau}B)^{\frac{m}{n}}. \end{aligned} \tag{27}$$

Proof. Since the determinant of a positive definite matrix is the product of its singular values, so we have

$$\begin{aligned} \det(I\nabla_\mu T)^{\frac{m}{n}} &= \det((1-\mu)I + \mu T)^{\frac{m}{n}} \\ &= \Big[\prod_{j=1}^{n}((1-\mu) + \mu s_j(T))^m\Big]^{\frac{1}{n}}. \end{aligned}$$

From inequality (10) of Corollary 3.1 we have

$$\begin{aligned}\det(I\nabla_\mu T)^{\frac{m}{n}} &\geq \Big[\prod_{j=1}^{n}(\mu s_j^p(T)+(1-\mu))^{\frac{m}{p}}\\ &+ \frac{\mu^m}{\tau}\Big((\tau s_j^m(T)+1-\tau)-(\tau s_j^p(T)+(1-\tau))^{\frac{m}{p}}\Big)\\ &+ r_m\Big(1-(\tau s_j^p(T)+(1-\tau))^{\frac{m}{2p}}\Big)^2\Big]^{\frac{1}{n}}.\end{aligned}$$

So,

$$\begin{aligned}\det(I\nabla_\mu T)^{\frac{m}{n}} &\geq \Big[\prod_{j=1}^{n}\Big((\mu s_j^p(T)+(1-\mu))^{\frac{m}{p}}\Big)\Big]^{\frac{1}{n}}\\ &+ \frac{\mu^m}{\tau}\Big[\prod_{j=1}^{n}\Big((\tau s_j(T)+(1-\tau))-(\tau s_j^p(T)+(1-\tau))^{\frac{1}{p}}\Big)^m\Big]^{\frac{1}{n}}\\ &+ r_m\Big[\prod_{j=1}^{n}\Big(1-(\tau s_j^p(T)+(1-\tau))^{\frac{m}{2p}}\Big)^2\Big]^{\frac{1}{n}}\\ &= \det(I\sharp_{p,\mu}T)^{\frac{m}{n}}+\frac{\mu^m}{\tau}\det\Big(I\nabla_\tau T-(I\sharp_{p,\tau}T)\Big)^{\frac{m}{n}}\\ &+ r_m\det\Big((I-(I\sharp_{p,\tau}T)^{\frac{m}{2}})^2\Big)^{\frac{1}{n}},\end{aligned}$$

where the second inequality follows by the convexity of the function $f(t)=t^m$ and Lemmas 6.1 and 6.2. So, multiplying the above inequality by $(\det A^{\frac{1}{2}})^{\frac{m}{n}}$ on the left-hand side and on the right hand side, we can deduce the result. Using the same technique above we can easily get the proof of the other inequality. This complete the proof. □

If we take the limit as $p \longrightarrow 0$ in Theorem 6.3, we obtain the following corollary, that proves a refinement of the difference between arithmetic and geometric mean inequality for determinant.

Corollary 6.4. *Let $A, B \in \mathbb{M}_n^{++}$, $T = A^{\frac{-1}{2}}BA^{\frac{-1}{2}}$ and $0 \leq \mu \leq \frac{1}{2} \leq \tau \leq 1$. Then for all positive integers m,*

$$\begin{aligned}\frac{\mu^m}{\tau}\det\Big(A\nabla_\tau B &- (A\sharp_\tau B)\Big)^{\frac{m}{n}}\\ &+ r_m\det\Big(A^{\frac{m}{2}}(I-(I\sharp_\tau T)^{\frac{m}{2}})^2A^{\frac{m}{2}}\Big)^{\frac{1}{n}}\\ &\leq \det(A\nabla_\mu B)^{\frac{m}{n}}-\det(A\sharp_\mu B)^{\frac{m}{n}},\end{aligned}\tag{28}$$

and

$$\begin{aligned}\frac{(1-\tau)^m}{1-\mu}\det\Big(A\nabla_\mu B &- (A\sharp_\mu B)\Big)^{\frac{m}{n}}\\ &+ r'_m\det\Big(A^{\frac{m}{2}}(T^{\frac{m}{2}}-(I\sharp_\tau T)^{\frac{m}{2}})^2A^{\frac{m}{2}}\Big)^{\frac{1}{n}}\\ &\leq \det(A\nabla_\tau B)^{\frac{m}{n}}-\det(A\sharp_\tau B)^{\frac{m}{n}}.\end{aligned}\tag{29}$$

If we take $p=-1$ in Theorem 6.3, we obtain the following corollary, which present a new generalized refinement of the difference between arithmetic and harmonic mean inequality for determinant.

Corollary 6.5. *Let* $A, B \in \mathbb{M}_n^{++}$, $T = A^{\frac{-1}{2}} B A^{\frac{-1}{2}}$ *and* $0 \leq \mu \leq \dfrac{1}{2} \leq \tau \leq 1$. *Then for all positive integer* m, *we have*

$$\begin{aligned} \frac{\mu^m}{\tau} \det\Big(A\nabla_\tau B \quad &- \quad (A!_\tau B)\Big)^{\frac{m}{n}} \\ &+ \quad r_m \det\Big(A^{\frac{m}{2}}(I - (I!_\tau T)^{\frac{m}{2}})^2 A^{\frac{m}{2}}\Big)^{\frac{1}{n}} \\ &\leq \quad \det(A\nabla_\mu B)^{\frac{m}{n}} - \det(A!_\mu B)^{\frac{m}{n}}, \end{aligned} \tag{30}$$

$$\begin{aligned} \frac{(1-\tau)^m}{1-\mu} \det\Big(A\nabla_\mu B \quad &- \quad (A!_\mu B)\Big)^{\frac{m}{n}} \\ &+ \quad r'_m \det\Big(A^{\frac{m}{2}}(T^{\frac{m}{2}} - (I!_\tau T)^{\frac{m}{2}})^2 A^{\frac{m}{2}}\Big)^{\frac{1}{n}} \\ &\leq \quad \det(A\nabla_\tau B)^{\frac{m}{n}} - \det(A!_\tau B)^{\frac{m}{n}}. \end{aligned} \tag{31}$$

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