Filomat 39:4 (2025), 1119–1133
https://doi.org/10.2298/FIL2504119J

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

SEP elements in a ring with involution

Yan Ji[a,*], Xinran Wang[a], Junchao Wei[a]

[a] *School of Mathematical Sciences, Yangzhou University, Yangzhou, Jiangsu 225002, P.R.China*

Abstract. In this paper, we provide many new characterizations of SEP elements in terms of exponentiation, projection, and anti-Hermitian elements. Additionally, we investigate idempotent elements and one-sided $x-$idempotence to characterize SEP elements. Finally, we discuss x-commutativity and one-sided x-equality to characterize SEP elements.

1. Introduction

Throughout this article, $\mathbb{Z}^+$ is the set of positive integers and R is an associative ring with identity 1. An involution $a \mapsto a^*$ in a ring R is an anti-isomorphism of degree 2, that is, for any $a,\ b \in R$,

$$(a^*)^* = a,\ (a+b)^* = a^* + b^*,\ (ab)^* = b^*a^*.$$

In this case, R is also called a $*-$ ring.

An element $e \in R$ satisfying $e^2 = e$ is called an idempotent element. The set of all idempotent elements of R is denoted by $E(R)$. If $e \in E(R)$ and $e^* = e$, then e is a projection of R. The set of all projections of R is denoted by $PE(R)$. If $a \in R$ and $a = aa^*a$, then a is said to be partial isometry (or PI) and we use R^{PI} to denote the set of all PI elements of R.

An element $a \in R$ is called group invertible if there is $x \in R$ which is the unique solution to equations:

$$axa = a,\ xax = x,\ ax = xa$$

such an x is determined group inverse of a [6, 7, 10], written $x = a^\#$. Denote by $R^\#$ the set of all group invertible element of R.

An element $a \in R$ is Moore-Penose invertible if there exists $x \in R$ satisfying the following equations:

$$axa = a,\ xax = x,\ (ax)^* = ax,\ (xa)^* = xa$$

such an x is called the Moore-Penose inverse (or MP-inverse) of a [8, 9], which is unique and denote by $x = a^+$. The set of all Moore-Penrose invertible elements of R will be denoted by R^+.

2020 *Mathematics Subject Classification*. Primary 16B99; Secondary 16W10, 46L05.
Keywords. EP element; PI element; SEP element; $x-$idempotent; projection; $x-$commutativity; $x-$equality.
Received: 05 June 2024; Revised: 24 November 2024; Accepted: 09 December 2024
Communicated by Dijana Mosić
Research supported by PPZY2015B109(202411117168Y;XCX20240272;XCX20240259).
* Corresponding author: Yan Ji
Email addresses: 15371471030@163.com (Yan Ji), 1092078512@qq.com (Xinran Wang), jcweiyz@126.com (Junchao Wei)
ORCID iDs: https://orcid.org/0009-0003-4622-8844 (Yan Ji), https://orcid.org/0009-0004-4505-1490 (Xinran Wang), https://orcid.org/0000-0002-7310-1836 (Junchao Wei)

Let $a \in R^{\#} \cap R^{+}$. If $a^{\#} = a^{+}$, then a is called an EP element. We denote the set of all EP elements in R by R^{EP} . If $a \in R^{EP} \cap R^{PI}$, then a is said to be a strong EP element of R [2, 3, 8, 9, 13]. Let R^{SEP} denote the set of all SEP elements of R.

In recent years, many achievements have been made in the characterization of *SEP* elements. Mosić and Djordjević characterized SEP elements in $*-$rings by some equivalent conditions in [2]. In [5, 11, 13], *SEP* elements are characterized by equations. More results on *SEP* elements could be referred to [1, 3, 4].

Motivated by these results above, different methods are used in this paper to characterize *SEP* elements. We use projection; $x-$idempotent; $x-$commutativity; $x-$equality and so on to characterize *SEP* elements.

2. Using the power to characterize SEP elements

Lemma 2.1. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{SEP}$ if and only if $(a^{\#})^2 a^{+} = a^* a^{+} a^{\#}$.*

PROOF. " $\Longrightarrow$ " Since $a \in R^{SEP}$, $a^* = a^{\#} = a^{+}$. Hence, $(a^{\#})^2 a^{+} = a^* a^{+} a^{\#}$.

" $\Longleftarrow$ " Assume that $(a^{\#})^2 a^{+} = a^* a^{+} a^{\#}$. Multiplying the equality on the right by a^2, one gets $a^{\#} = a^* a^{+} a$. Hence, $a \in R^{SEP}$ by [2, Theorem 1.5.3]. □

Theorem 2.2. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{SEP}$ if and only if $((a^{\#})^2 a^{+})^k = (a^* a^{+} a^{\#})^k$ for $k = 2,\ 3$.*

PROOF. " $\Longrightarrow$ " It is an immediate result of Lemma 2.1.

" $\Longleftarrow$ " Assume that $((a^{\#})^2 a^{+})^k = (a^* a^{+} a^{\#})^k$, where $k = 2,\ 3$. Then

$$((a^{\#})^2 a^{+})^3 = (a^* a^{+} a^{\#})^3 = (a^* a^{+} a^{\#})(a^* a^{+} a^{\#})^2 = (a^* a^{+} a^{\#})((a^{\#})^2 a^{+})^2.$$

Multiplying the equality $((a^{\#})^2 a^{+})^3 = (a^* a^{+} a^{\#})((a^{\#})^2 a^{+})^2$ on the right by a^8, one yields $a^{\#} = a^* a^{+} a$. Hence, by [2, Theorem 1.5.3], $a \in R^{SEP}$. □

Corollary 2.3. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{SEP}$ if and only if $(a^{\#})^2 a^* = a^* a^{+} a^*$.*

PROOF. " $\Longrightarrow$ " Since $a \in R^{SEP}$, $(a^{\#})^2 a^{+} = a^* a^{+} a^{\#}$ by Lemma 2.1 and $a^* = a^{\#} = a^{+}$. Hence, $(a^{\#})^2 a^* = a^* a^{+} a^*$.

" $\Longleftarrow$ " Multiplying the equality $(a^{\#})^2 a^* = a^* a^{+} a^*$ on the right by $(a^{+})^*$, one gets

$$(a^{\#})^2 = a^* a^{+} a^{+} a.$$

This gives

$$(a^{\#})^2 = a^{+} a (a^* a^{+} a^{+} a) = a^{+} a (a^{\#})^2 = a^{+} a^{\#}.$$

Hence, $a \in R^{EP}$ by [2, Theorem 1.2.1]. It follows that

$$(a^{\#})^2 = a^* a^{+} a^{+} a = a^* a^{+} = a^* a^{\#}.$$

By [2, Theorem 1.5.3], $a \in R^{SEP}$. □

Theorem 2.4. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{SEP}$ if and only if $((a^{\#})^2 a^*)^k = (a^* a^{+} a^*)^k$ for $k = 2,\ 3$.*

PROOF. " $\Longrightarrow$ " It is an immediate result of Corollary 2.3.

" $\Longleftarrow$ " From the assumption, we have

$$(a^* a^{+} a^*)^3 = ((a^{\#})^2 a^*)^3 = (a^{\#})^2 a^* ((a^{\#})^2 a^*)^2 = (a^{\#})^2 a^* (a^* a^{+} a^*)^2.$$

Multiplying the equality on the right by $(a^{\#})^*$, one gets

$$a^* a^{+} a^* a^* a^{+} a^* a^* a^{+} = (a^{\#})^2 a^* a^* a^{+} a^* a^* a^{+}.$$

By [3, Lemma 2.11], one has $a^* a^{+} a^* a^* a^{+} a^* a^* = (a^{\#})^2 a^* a^* a^{+} a^* a^*$.

Multiplying the last equality on the right by $(a^{\#})^* (a^{\#})^*$, and then, by [3, Lemma 2.11], one yields

$$a^* a^{+} a^* a^* = (a^{\#})^2 a^* a^*.$$

This induces $a^* a^{+} a^* = a^* a^{+} a^* a^* (a^{\#})^* = (a^{\#})^2 a^* a^* (a^{\#})^* = (a^{\#})^2 a^*$.

By Corollary 2.3, $a \in R^{SEP}$. □

Theorem 2.5. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{SEP}$ if and only if $(a^*a^+a^2)^{k+1} = (a^*a^2a^+)^k$ for some $k \in \mathbb{Z}^+$.*

PROOF. " $\Longrightarrow$ " Assume that $a \in R^{SEP}$. Then $(a^{\#})^2a^+ = a^*a^+a^{\#}$ by Lemma 2.1. Noting that $a \in R^{EP}$. Then

$$a^*a^2a^+ = a^*a^+a^2 = (a^*a^+a^{\#})a^3 = (a^{\#})^2a^+a^3 = aa^{\#}.$$

This implies that for any $k \in \mathbb{Z}^+$, we have

$$(a^*a^+a^2)^{k+1} = aa^{\#} = (a^*a^2a^+)^k.$$

" $\Longleftarrow$ " Multiplying the equality $(a^*a^+a^2)^{k+1} = (a^*a^2a^+)^k$ on the right by a^+a, one gets $(a^*a^2a^+)^k = (a^*a^2a^+)^ka^+a$ for some $k \in \mathbb{Z}^+$.

Multiplying the last equality on the left by $(aa^{\#})^*a^{\#}(a^+)^*$, one obtains

$$(a^*a^2a^+)^{k-1} = (a^*a^2a^+)^{k-1}a^+a.$$

Repeating the process mentioned above, one arrives at $a^*a^2a^+ = a^*a^2a^+a^+a$.

This gives

$$aa^+ = a^{\#}(a^+)^*a^*a^2a^+ = a^{\#}(a^+)^*a^*a^2a^+a^+a = aa^+a^+a.$$

Hence, $a \in R^{EP}$, this induces $(a^*a)^{k+1} = (a^*a^+a^2)^{k+1} = (a^*a^2a^+)^k = (a^*a)^k$ for some $k \in \mathbb{Z}^+$, and

$$(a^*a)^k = a^+(a^+)^*(a^*a)^{k+1} = a^+(a^+)^*(a^*a)^k = (a^*a)^{k-1}.$$

This deduces $(a^*a)^2 = a^*a$. Hence, $a \in R^{PI}$ and so $a \in R^{SEP}$. □

Corollary 2.6. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{SEP}$ if and only if $((a^*a^+a^2)(a^*a^2a^+))^2 = (a^*a^+a^2)(a^*a^2a^+)(a^*a^+a^2)$.*

PROOF. " $\Longrightarrow$ " Assume that $a \in R^{SEP}$. Then $a \in R^{EP}$ and $(a^*a)^2 = a^*a$.

It follows that $((a^*a)^2)^2 = (a^*a)^2(a^*a)^2 = a^*a(a^*a)^2 = (a^*a)^2a^*a$.

Noting that $a \in R^{EP}$. Then $(a^*a^+a^2)(a^*a^2a^+) = (a^*a)(a^*a) = (a^*a)^2$, one obtains

$$((a^*a^+a^2)(a^*a^2a^+))^2 = ((a^*a)^2)^2 = (a^*a)^2a^*a = (a^*a^+a^2)(a^*a^2a^+)a^*a = (a^*a^+a^2)(a^*a^2a^+)(a^*a^+a^2).$$

" $\Longleftarrow$ " Suppose that $((a^*a^+a^2)(a^*a^2a^+))^2 = (a^*a^+a^2)(a^*a^2a^+)(a^*a^+a^2)$.

Then, multiplying the equality on the left by $a^+a^{\#}(a^+)^*a^{\#}(a^{\#})^*$, one gets

$$a^+(a^*a^+a^2)(a^*a^2a^+) = a^+a^*a^+a^2.$$

By [3, Lemma 2.11], one obtains $a^*a^+a^2a^*a^2a^+ = a^*a^+a^2$, and so

$$a^+a^2a^*a^2a^+ = (a^{\#})^*a^*a^+a^2a^*a^2a^+ = (a^{\#})^*a^*a^+a^2 = a^+a^2.$$

Hence, $a = a^{\#}aa^+a^2 = a^{\#}aa^+a^2a^*a^2a^+ = aa^*a^2a^+ = (aa^*a^2a^+)aa^+ = a(aa^+) = a^2a^+$, one yields $a \in R^{EP}$ and $a = aa^*a^2a^+ = aa^*a$, it follows that $a \in R^{PI}$. Thus, $a \in R^{SEP}$. □

Corollary 2.7. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{SEP}$ if and only if $((a^*a^+a^2)(a^*a^2a^+))^2 = (a^*a^2a^+)(a^*a^+a^2)^2$.*

PROOF. " $\Longrightarrow$ " Since $a \in R^{SEP}$, $a \in R^{EP}$ and by Corollary 2.6,

$$((a^*a^+a^2)(a^*a^2a^+))^2 = (a^*a^+a^2)(a^*a^2a^+)(a^*a^+a^2).$$

Noting that $a \in R^{EP}$. Then $(a^*a^+a^2)(a^*a^2a^+) = a^*aa^*a = (a^*a^2a^+)(a^*a^+a^2)$.

Hence, $((a^*a^+a^2)(a^*a^2a^+))^2 = (a^*a^2a^+)(a^*a^+a^2)^2$.

" $\Longleftarrow$ " Multiplying the equality $((a^*a^+a^2)(a^*a^2a^+))^2 = (a^*a^2a^+)(a^*a^+a^2)^2$ on the right by aa^+, one gets

$$(a^*a^2a^+)(a^*a^+a^2)^2 = (a^*a^2a^+)(a^*a^+a^2)^2aa^+.$$

Multiplying the last equality on the left by $a^+a^\#(a^+)^*$, one obtains

$$a^+a^*a^+a^2a^*a^+a^2 = a^+a^*a^+a^2a^*a^+a^3a^+.$$

By [3, Lemma 2.11], one has $a^*a^+a^2a^*a^+a^2 = a^*a^+a^2a^*a^+a^3a^+$.

Multiplying the equality mentioned above on the left by $a^\#a(a^\#)^*a^+aa^\#(a^\#)^*$, one yields $a = a^2a^+$.

Hence, $a \in R^{EP}$, it follows $a^*a^2a^+ = a^*a = a^*a^+a^2$, and

$$((a^*a^+a^2)(a^*a^2a^+))^2 = (a^*a^2a^+)(a^*a^+a^2)^2 = (a^*a)^3 = (a^*a^+a^2)(a^*a^2a^+)(a^*a^+a^2).$$

By Corollary 2.6, $a \in R^{SEP}$. □

Similarly, we can show the following Corollary 2.8.

Corollary 2.8. *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if $((a^*a^+a^2)(a^*a^2a^+))^2 = (a^*a^2a^+)^2(a^*a^+a^2)$.*

3. Using the projections to characterize SEP elements

Theorem 3.1. *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if $a^*a^2a^+$ is a projection.*

PROOF. " $\Longrightarrow$ " Since $a \in R^{SEP}$, $(a^\#)^2a^+ = a^*a^+a^\#$ by Lemma 2.1 and $a^\# = a^+$. This gives

$$a^*a^2a^+ = a^*a^+a^2 = (a^*a^+a^\#)a^3 = (a^\#)^2a^+a^3 = a^\#a = a^+a.$$

Hence, $a^*a^2a^+$ is a projection.

" $\Longleftarrow$ " Assume that $a^*a^2a^+$ is a projection. Then

$$a^*a^2a^+ = (a^*a^2a^+)(a^*a^2a^+)^* = a^*a^2a^+a^*a.$$

Multiplying the equality on the left by $a^+a^\#(a^+)^*$, one gets $a^+ = a^+a^*a$. Hence, $a \in R^{SEP}$ by [2, Theorem 1.5.3]. □

It is well known that $e \in R$ is projection if and only if e^* is projection. Noting that $(a^*a^2a^+)^* = aa^+a^*a$. Hence, Theorem 3.1 implies the following corollary.

Corollary 3.2. *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if aa^+a^*a is a projection.*

Lemma 3.3. *Let $e \in R^+$. If e is a projection, then e^+ is projection.*

Lemma 3.4. *Let $a \in R^\# \cap R^+$. Then $(aa^+a^*a)^+ = a^+(a^\#)^*$.*

PROOF. It is a routine verfication. □

By Corollary 3.2, Lemma 3.3 and Lemma 3.4, we have the following corollary.

Corollary 3.5. *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if $a^+(a^\#)^*$ is a projection.*

Noting that $a^+(a^\#)^* = a^+(a^\#)^*aa^+$. Then we have $a^+(a^+)^*aa^+ = a^+(a^\#)^*aa^+$ whenever $a \in R^{EP}$. This implies the following corollary.

Corollary 3.6. *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if $a^+(a^+)^*aa^+$ is a projection.*

PROOF. " $\Longrightarrow$ " Assume that $a \in R^{SEP}$. Then $a^+(a^\#)^*aa^+ = a^+(a^\#)^*$ is a projection by Corollary 3.5. Noting that $a \in R^{EP}$. Then $a^+(a^+)^*aa^+ = a^+(a^\#)^*aa^+$ is a projection.

" $\Longleftarrow$ " From the condition, we have

$$a^+(a^+)^*aa^+ = a^+(a^+)^*aa^+(a^+(a^+)^*aa^+)^* = a^+(a^+)^*aa^+a^+(a^+)^*.$$

This gives

$$a^+a^2a^+ = a^*(a^+)^*aa^+ = a^*aa^+(a^+)^*aa^+ = a^*aa^+(a^+)^*aa^+a^+(a^+)^* = a^+a^2a^+a^+(a^+)^*,$$

and

$$a^+ = a^+aa^\# a^+a^2a^+ = a^+aa^\# a^+a^2a^+a^+(a^+)^* = a^+a^+(a^+)^* = a^+a^+(a^+)^*aa^\# = a^+aa^\#.$$

Hence, $a \in R^{EP}$, it follows $a^+(a^\#)^* = a^+(a^+)^* = a^+(a^+)^*aa^\# = a^+(a^+)^*aa^+$ is a projection. By Corollary 3.5, $a \in R^{SEP}$. □

If $a = a^*$, then a is called a Hermitian element [2]. We generally write R^{Her} to denote the set of all Hermitian elements of R.The following lemma is evident.

Lemma 3.7. *Let $a, b \in R^{Her}$. If ab is projection. Then $ab = ba$.*

PROOF. Since $a, b \in R^{Her}$, $a^* = a, b^* = b$. Then $(ab)^* = b^*a^* = ba$. Assume that ab is projection. Then $(ab)^* = ab$. Hence, $ab = ba$. □

By Corollary 3.6 and Lemma 3.7, we get the following corollary.

Corollary 3.8. *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if $aa^+a^+(a^+)^*$ is a projection.*

Noting that $(aa^+a^+(a^+)^*)^+ = aa^\# a^*a(aa^\#)^*aa^\#$. Then Corollary 3.8 and Lemma 3.3 induce the following theorem.

Theorem 3.9. *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if $aa^\# a^*a(aa^\#)^*aa^\#$ is a projection.*

Lemma 3.10. *Let $x, y \in R$ are projections. If $yxy = y$, then xyx is projection.*

PROOF. Since $x,\ y$ are projections,

$$x = x^2 = x^*,\ y = y^2 = y^*.$$

Clearly,

$$(xyx)^2 = xyx^2yx = xyxyx = x(yxy)x = xyx;$$

$$(xyx)^* = x^*y^*x^* = xyx.$$

Hence xyx is projection. □

In Theorem 3.9, choose $x = a^+a,\ y = aa^\# a^*a(aa^\#)^*aa^\#$. Then $xyx = a^*a(aa^\#)^*aa^\#$. If $a \in R^{EP}$, then $xyx = a^*a(aa^\#)^*$ and $yxy = y^2$. Hence, Theorem 3.9 implies.

Theorem 3.11. *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if $a^*a(aa^\#)^*$ is a projection.*

PROOF. " $\Longrightarrow$ " Since $a \in R^{SEP}$, $a \in R^{EP}$ and $aa^\# a^*a(aa^\#)^*aa^\# = y$ is a projection by Theorem 3.9. Noting that $x = a^+a$ is a projection and $yxy = y^2 = y$. By Lemma 3.10, xyx is a projection. Clearly,

$$xyx = a^*a(aa^\#)^*aa^\# = a^*a(aa^\#)^*aa^+ = a^*a(aa^\#)^*.$$

Hence, $a^*a(aa^\#)^*$ is a projection.

" $\Longleftarrow$ " Using the assumption, one gets

$$a^*a(aa^\#)^* = (a^*a(aa^\#)^*)^*a^*a(aa^\#)^* = aa^\# a^*aa^*a(aa^\#)^*.$$

This gives $a^* = a^*aa^+ = a^*a(aa^\#)^*a^+ = aa^\# a^*aa^*a(aa^\#)^*a^+ = aa^\# a^*aa^*$, and

$$a^+ = a^*(a^+)^*a^+ = aa^\# a^*aa^*(a^+)^*a^+ = aa^\# a^*.$$

Hence, $a \in R^{SEP}$ by [2, Theorem 1.5.3]. □

4. Using anti-Hermitians to characterize SEP elements

An element $a \in R$ is called weakly projection if $a^* = -a^2$. Denote the set of all weakly projections of R by R^{Wp}. Theorem 3.11 inspires us to give the following theorem.

Theorem 4.1. *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if $a^*a - (aa^\#)^* \in R^{Wp}$.*

PROOF. " $\Longrightarrow$ " Assume that $a \in R^{SEP}$. Then $a^*a = aa^\#$ by [2, Theorem 1.5.3] and $aa^\# = (aa^\#)^*$ by [2, Theorem 1.1.3]. Hence, $a^*a - (aa^\#)^* = 0 \in R^{Wp}$.
" $\Longleftarrow$ " The condition "$a^*a - (aa^\#)^* \in R^{Wp}$" gives

$$a^*a - aa^\# = (a^*a - (aa^\#)^*)^* = -(a^*a - (aa^\#)^*)^2 = -(a^*a)^2 + a^*a(aa^\#)^* + a^*a - (aa^\#)^*.$$

i.e.,

$$aa^\# = (a^*a)^2 - a^*a(aa^\#)^* + (aa^\#)^*. \tag{1}$$

Multiplying (4.1) on the left by a^+a, one gets $aa^\# = a^+a$. Hence, $a \in R^{EP}$, it follows from (4.1) that

$$(a^*a)^2 = a^*a.$$

Hence, $a \in R^{PI}$ by [12, Theorem 3.1]. Thus, $a \in R^{SEP}$. □
Clearly, $a \in R^{Wp}$ if and only if $a^* \in R^{Wp}$, then Theorem 4.1 implies

Corollary 4.2. *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if $a^*a - aa^\# \in R^{Wp}$.*

It is well known that if $a \in R^+$, then $a \in R^{PI}$ implies $a^*a = a^+(a^+)^*$. Hence Corollary 4.2 implies the following corollary.

Corollary 4.3. *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if $a^+(a^+)^* - aa^\# \in R^{Wp}$.*

PROOF. " $\Longrightarrow$ " It is an immediate result of Corollary 4.2.
" $\Longleftarrow$ " With the hypothesis, one gets

$$(a^+(a^+)^* - aa^\#)^* = -(a^+(a^+)^* - aa^\#)^2.$$

e.g.,

$$a^+(a^+)^* - (aa^\#)^* = -a^+(a^+)^*a^+(a^+)^* + a^+(a^+)^* + a^\#(a^+)^* - aa^\#,$$

it follows that

$$(aa^\#)^* = a^+(a^+)^*a^+(a^+)^* - a^\#(a^+)^* + aa^\#. \tag{2}$$

Multiplying (4.2) on the right by $aa^\#$, one arrives at $(aa^\#)^* = (aa^\#)^*aa^\#$. By [2, Theorem 1.1.3], $a \in R^{EP}$, it follows from (4.2) that

$$a^+(a^+)^*a^+(a^+)^* = a^\#(a^+)^*.$$

This gives $(a^+)^* = aa^\#(a^+)^* = aa^+(a^+)^*a^+(a^+)^* = (a^+)^*a^+(a^+)^*$. Therefore

$$a^+a = a^*(a^+)^* = a^*(a^+)^*a^+(a^+)^* = a^+(a^+)^*,$$

and

$$a^* = a^+aa^* = a^+(a^+)^*a^* = a^+.$$

Hence, $a \in R^{SEP}$. □
Clearly, if $a \in R^\# \cap R^+$, then $a \in R^{EP}$ if and only if $a^\#(a^\#)^* = a^+(a^+)^*$. Hence, from Corollary 4.3, we have.

Theorem 4.4. *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if $a^\#(a^\#)^* - aa^\# \in R^{Wp}$.*

PROOF. “ $\Longrightarrow$ ” This is a direct result of Corollary 4.3.
“ $\Longleftarrow$ ” Using the assumption, one yields

$$(a^{\#}(a^{\#})^* - aa^{\#})^* = -(a^{\#}(a^{\#})^* - aa^{\#})^2.$$

that is,

$$-(aa^{\#})^* = -a^{\#}(a^{\#})^*a^{\#}(a^{\#})^* + a^{\#}(a^{\#})^*aa^{\#} - aa^{\#}.$$

Multiplying the equality on the left by $a^{\#}a$, one gets $(aa^{\#})^* = aa^{\#}(aa^{\#})^*$. Hence, $a \in R^{EP}$ by [2, Theorem 1.13]. By Corollary 4.3, it follows that

$$a^+(a^+)^* - aa^{\#} = a^{\#}(a^{\#})^* - aa^{\#} \in R^{Wp}.$$

Hence, $a \in R^{SEP}$. □

Theorem 4.5. *Let $a \in R^{\#} \cap R^+$. Then $a \in R^{SEP}$ if and only if $a^*a^+a^2 - aa^{\#} \in R^{Wp}$.*

PROOF. “ $\Longrightarrow$ ” Since $a \in R^{SEP}$, $a^*a^+a^2 = a^*a$ and By Corollary 4.2, we yield $a^*a^+a^2 - aa^{\#} = a^*a - aa^{\#} \in R^{Wp}$.
“ $\Longleftarrow$ ” It follows from $a^*a^+a^2 - aa^{\#} \in R^{Wp}$ that

$$\begin{aligned} a^*a^+a^2 - (aa^{\#})^* &= (a^*a^+a^2 - aa^{\#})^* = -(a^*a^+a^2 - aa^{\#})^2 \\ &= -a^*a^+a^2a^*a^+a^2 + a^*a^+a^2 + aa^{\#}a^*a^+a^2 - aa^{\#}. \end{aligned}$$

e.g.,

$$(aa^{\#})^* = a^*a^+a^2a^*a^+a^2 - aa^{\#}a^*a^+a^2 + aa^{\#} \tag{3}$$

Multiplying (4.3) on the right by $aa^{\#}$, one has

$$(aa^{\#})^* = (aa^{\#})^*aa^{\#}.$$

Hence, $a \in R^{EP}$, this induces

$$a^*a - aa^{\#} = a^*a^+a^2 - aa^{\#} \in R^{Wp}.$$

By Corollary 4.2, $a \in R^{SEP}$. □

5. Using idempotents to characterize SEP elements

Observing Theorem 4.5, we can get the following theorem.

Theorem 5.1. *Let $a \in R^{\#} \cap R^+$. Then $a \in R^{SEP}$ if and only if $aa^{\#} - a^*a^2a^+ \in E(R)$.*

PROOF. “ $\Longrightarrow$ ” Assume that $a \in R^{SEP}$. Then $a^*a^2a^+ = a^{\#}a^2a^{\#} = aa^{\#}$. Hence $aa^{\#} - a^*a^2a^+ = 0 \in E(R)$.
“ $\Longleftarrow$ ” From $aa^{\#} - a^*a^2a^+ \in E(R)$, one gets

$$aa^{\#} - a^*a^2a^+ = (aa^{\#} - a^*a^2a^+)^2 = aa^{\#} - aa^{\#}a^*a^2a^+ - a^*a + (a^*a^2a^+)^2.$$

i.e.,

$$a^*a^2a^+ = aa^{\#}a^*a^2a^+ + a^*a - (a^*a^2a^+)^2.$$

Multiplying the last equality on the right by $aa^{\#}a^+$, one obtains

$$aa^{\#}a^* = a^*a^2a^+a^*.$$

This gives

$$aa^{\#} = aa^{\#}a^+a = aa^{\#}a^*(a^+)^* = a^*a^2a^+a^*(a^+)^* = a^*a^2a^+a^+a,$$

and

$$a^+a = a^+a(aa^{\#}) = a^+a(a^*a^2a^+a^+a) = a^*a^2a^+a^+a = aa^{\#}.$$

Hence, $a \in R^{EP}$, this induces

$$aa^{\#} = a^*a^2a^+a^+a = a^*a.$$

Thus, $a \in R^{SEP}$. □

Corollary 5.2. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{SEP}$ if and only if $aa^{+} - a^{*}a^{2}a^{+} \in E(R)$.*

PROOF. " $\Longrightarrow$ " Assume that $a \in R^{SEP}$. Then $a^{\#} = a^{+}$ and $aa^{\#} - a^{*}a^{2}a^{+} \in E(R)$ by Theorem 5.1, it follows that $aa^{+} - a^{*}a^{2}a^{+} \in E(R)$.

" $\Longleftarrow$ " The condition $aa^{+} - a^{*}a^{2}a^{+} \in E(R)$ implies

$$aa^{+} - a^{*}a^{2}a^{+} = (aa^{+} - a^{*}a^{2}a^{+})^{2} = aa^{+} - aa^{+}a^{*}a^{2}a^{+} - a^{*}a^{2}a^{+} + (a^{*}a^{2}a^{+})^{2}.$$

i.e.,

$$aa^{+}a^{*}a^{2}a^{+} = (a^{*}a^{2}a^{+})^{2}.$$

Multiplying the last equality on the right by $a^{\#}aa^{+}(a^{\#})^{*}a$, one gets

$$a = a^{*}a^{2}.$$

By [2, Theorem 1.5.3], $a \in R^{SEP}$. □

Since $e \in E(R)$ if and only if $e^{*} \in E(R)$, Corollary 5.2 leads to the following result.

Corollary 5.3. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{SEP}$ if and only if $aa^{+} - aa^{+}a^{*}a \in E(R)$.*

Noting that $a \in R^{SEP}$ if and only if $a^{*} \in R^{SEP}$. Then instead a by a^{*} in Corollary 5.3, we have

Corollary 5.4. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{SEP}$ if and only if $a^{+}a - a^{+}a^{2}a^{*} \in E(R)$.*

6. Using one-sided $x-$idempotency to characterize SEP elements

Let $a,\ x \in R$. Then a is called left (right) $x-$idempotent if $a^{2} = xa$ ($a^{2} = ax$).

Example 6.1. *Let R be a ring and $a \in R$. Then*

(1) a is left and right $a-$idempotent.

(2) The following conditions are equivalent:

(a) a is idempotent;

(b) a is left $1-$idempotent;

(c) a is right 1-idempotent;

(d) a is left $2a-1$-idempotent;

(e) a is right $2a-1$-idempotent.

(3) The following conditions are equivalent:

(a) $a^{2} = 1$;

(b) $1-a$ is left 2-idempotent;

(c) $1-a$ is right 2-idempotent;

(d) $1+a$ is left 2-idempotent;

(e) $1+a$ is right 2-idempotent.

(4) a is nilpotent if and only if a is left $a^{k}+a$-idempotent for some $k \in \mathbb{Z}^{+}$.

Theorem 6.2. *Let $a \in R^{\#} \cap R^{+}$. Then the following conditions are equivalent:*

(1) $a \in R^{SEP}$;

(2) $(a^{\#})^{2}a^{+}$ is left $a^{*}a^{+}a^{\#}$-idempotent;

(3) $(a^{\#})^{2}a^{+}$ is right $a^{*}a^{+}a^{\#}$-idempotent.

PROOF. (1) $\Longrightarrow$(2) The assumption of $a \in R^{SEP}$ implies $(a^{\#})^{2}a^{+} = a^{*}a^{+}a^{\#}$ by Lemma 2.1. Hence, $(a^{\#})^{2}a^{+}$ is left $a^{*}a^{+}a^{\#}$-idempotent.

(2) $\Longrightarrow$(3) From the hypothesis, one gets

$$((a^{\#})^{2}a^{+})^{2} = a^{*}a^{+}a^{\#}(a^{\#})^{2}a^{+}.$$

This gives

$$a^{\#} = (a^{\#})^2a^+a^2 = (a^{\#})^2a^+(a^{\#})^2a^+a^5 = a^*a^+a^{\#}(a^{\#})^2a^+a^5 = a^*a^+a.$$

By [2, Theorem 1.5.3], $a \in R^{EP}$, and

$$((a^{\#})^2a^+)(a^*a^+a^{\#}) = ((a^{\#})^2a^+)(a^*a^+a)(a^{\#})^2 = ((a^{\#})^2a^+)a^{\#}(a^{\#})^2 = ((a^{\#})^2a^+)((a^{\#})^2a^+).$$

Hence, $(a^{\#})^2a^+$ is right $a^*a^+a^{\#}$-idempotent.

(3) $\Longrightarrow$(1) By the condition, we have $((a^{\#})^2a^+)^2 = ((a^{\#})^2a^+)(a^*a^+a^{\#})$, and

$$a^+(a^{\#})^2a^+ = a^+a^3(a^{\#})^2a^+(a^{\#})^2a^+ = a^+a^3((a^{\#})^2a^+)(a^*a^+a^{\#})$$

$$= a^+a^*a^+a^{\#} = (a^+a^*a^+a^{\#})a^{\#}a = (a^+(a^{\#})^2a^+)a^{\#}a = a^+(a^{\#})^3.$$

So,

$$aa^+ = a^4a^+(a^{\#})^2a^+ = a^4a^+(a^{\#})^3 = aa^{\#}.$$

Hence, $a \in R^{EP}$, it follows that

$$a^* = aa^+a^* = a(a^+a^*a^+a^{\#})a^2 = a(a^+(a^{\#})^2a^+)a^2 = a^{\#}.$$

Thus, $a \in R^{SEP}$. □

Lemma 6.3. *Let $a,\ x \in R$. If a is left x idempotent, then*
(1) xa is right a^2-idempotent.
(2) ax is left a^2-idempotent.

PROOF. Since a is left x idempotent, $a^2 = xa$. Then
(1) $(xa)^2 = (xa)(xa) = (xa)a^2$. Hence, xa is right a^2-idempotent.
(2) $(ax)^2 = (ax)(ax) = a(xa)x = aa^2x = a^2(ax)$. Hence, ax is left a^2-idempotent. □

Theorem 6.4. *Let $a \in R^{\#} \cap R^+$. Then the following conditions are equivalent:*
(1) $a \in R^{SEP}$;
(2) $a^*a^+(a^{\#})^3a^+$ is right $(a^+)^2a^{\#}(a^+)^2a^{\#}$-idempotent;
(3) $a^*a^+(a^{\#})^4$ is right $(a^{\#})^2a^+(a^{\#})^2a^+$-idempotent.

PROOF. (1) $\Longrightarrow$ (2) Assume that $a \in R^{SEP}$. Then $(a^{\#})^2a^+$ is left $a^*a^+a^{\#}$-idempotent by Theorem 6.2. From Lemma 6.3, we have $(a^*a^+a^{\#})((a^{\#})^2a^+)$ is right $((a^{\#})^2a^+)^2$-idempotent. Noting that $a^{\#} = a^+$. Then $a^*a^+(a^{\#})^3a^+$ is right $(a^+)^2a^{\#}(a^+)^2a^{\#}$-idempotent.

(2) $\Longrightarrow$ (3) From the assumption, we have

$$(a^*a^+(a^{\#})^3a^+)^2 = (a^*a^+(a^{\#})^3a^+)((a^+)^2a^{\#}(a^+)^2a^{\#}).$$

Multiplying the equality on the right by a^+a, we get

$$(a^*a^+(a^{\#})^3a^+)^2 = (a^*a^+(a^{\#})^3a^+)^2a^+a.$$

Multiplying the last equality on the left by $a^+a^5(a^{\#})^*$, we obtain

$$a^+a^*a^+(a^{\#})^3a^+ = a^+a^*a^+(a^{\#})^3a^+a^+a.$$

By [3, Lemma 2.10], we yield

$$a^*a^+(a^{\#})^3a^+ = a^*a^+(a^{\#})^3a^+a^+a.$$

Hence,

$$a^+ = (a^+a^5(a^{\#})^*)(a^*a^+(a^{\#})^3a^+) = (a^+a^5(a^{\#})^*)(a^*a^+(a^{\#})^3a^+a^+a) = a^+a^+a,$$

it follows that $a \in R^{EP}$ and so $a^{\#} = a^+$.

Thus, by (2), we have $a^*a^+(a^{\#})^4$ is right $(a^{\#})^2a^+(a^{\#})^2a^+$-idempotent.

(3) $\Longrightarrow$ (1) Using the hypothesis, we have

$$(a^*a^+(a^\#)^4)^2 = (a^*a^+(a^\#)^4)((a^\#)^2a^+(a^\#)^2a^+).$$

Multiplying the last equality on the left by $a^4(a^\#)^*$, we obtain

$$a^\#a^*a^+(a^\#)^4 = (a^\#)^3a^+(a^\#)^2a^+ = (a^\#)^6a^+.$$

This gives

$$(a^\#)^6a^+ = a^\#a^*a^+(a^\#)^4 = (a^\#a^*a^+(a^\#)^4)aa^\# = (a^\#)^6a^+aa^\# = (a^\#)^7.$$

Hence, $a \in R^{EP}$ by [2, Theorem 1.2.2], which implies

$$a^\#a^*(a^\#)^5 = a^\#a^*a^+(a^\#)^4 = (a^\#)^6a^+ = (a^\#)^7.$$

It follows that

$$a = a(a^\#)^7a^7 = a(a^\#a^*(a^\#)^5a^7) = a^*a^2.$$

Hence, $a \in R^{SEP}$ by [2, Theorem 1.5.3]. □

Theorem 6.5. *Let $a \in R^\# \cap R^+$. Then the following conditions are equivalent:*
(1) $a \in R^{SEP}$;
(2) $a^*a^+a^\# - (a^\#)^2a^+$ is right $a^*a^+a^\#$-idempotent;
(3) $a^*a^+a^\# - (a^\#)^2a^+$ is left $a^*a^+a^\#$-idempotent.

PROOF. (1) $\Longrightarrow$ (2) Since $a \in R^{SEP}$, $(a^\#)^2a^+$ is left $a^*a^+a^\#$-idempotent by Theorem 6.2, that is $((a^\#)^2a^+)^2 = (a^*a^+a^\#)((a^\#)^2a^+)$, it follows that

$$[a^*a^+a^\# - (a^\#)^2a^+]^2 = (a^*a^+a^\#)^2 - (a^*a^+a^\#)((a^\#)^2a^+) - ((a^\#)^2a^+)(a^*a^+a^\#) + ((a^\#)^2a^+)^2$$

$$= (a^*a^+a^\#)^2 - ((a^\#)^2a^+)(a^*a^+a^\#) = (a^*a^+a^\# - (a^\#)^2a^+)(a^*a^+a^\#).$$

Hence, $a^*a^+a^\# - (a^\#)^2a^+$ is right $a^*a^+a^\#$-idempotent.

(2) $\Longrightarrow$ (3) From the assumption, we have

$$(a^*a^+a^\# - (a^\#)^2a^+)^2 = (a^*a^+a^\# - (a^\#)^2a^+)(a^*a^+a^\#).$$

By computing, we obtain

$$((a^\#)^2a^+)^2 = (a^*a^+a^\#)((a^\#)^2a^+).$$

By Theorem 6.2, $a \in R^{SEP}$. Again, by Theorem 6.2, $(a^\#)^2a^+$ is right $a^*a^+a^\#$-idempotent. Then it is easy to show that $a^*a^+a^\# - (a^\#)^2a^+$ is left $a^*a^+a^\#$-idempotent.

(3) $\Longrightarrow$ (1) From (3), we have

$$(a^*a^+a^\# - (a^\#)^2a^+)^2 = (a^*a^+a^\#)(a^*a^+a^\# - (a^\#)^2a^+).$$

This induces

$$((a^\#)^2a^+))^2 = (a^\#)^2a^+)(a^*a^+a^\#).$$

By Theorem 6.2, $a \in R^{SEP}$. □

7. Using $x-$commutativity to characterize SEP elements

Let R be a ring and $a, b, x \in R$. Then a, b are called $x-$commutativity if $xa = bx$.
Clearly, (1) a, b are always $0-$commutativity for any $a, b \in R$;
(2) a, a are $a-$commutativity for each $a \in R$;
(3) $e \in E(R)$ if and only if $e,\ 1$ are $e-$commutativity;
(4) $e \in E(R)$ if and only if $e,\ 2e-1$ are $e-$commutativity;
(5) $a \in N(R)$ if and only if $xa,\ a^k + ax$ are $a-$commutativity for any $x \in R$ and some $k \in \mathbb{Z}^+$;
(6) a is left $b-$idempotent if and only if $a,\ b$ are $a-$commutativity;
(7) a is right $b-$idempotent if and only if $b,\ a$ are $a-$commutativity.

Theorem 7.1. *Let $a \in R^{\#} \cap R^{+}$. Then the following conditions are equivalent:*
(1) $a \in R^{SEP}$;
(2) $(a^{\#})^2 a^{+},\ a^* a^+ a^{\#}$ are $a-$commutativity;
(3) $(a^{\#})^2 a^{+},\ a^* a^+ a^{\#}$ are $a^{\#}-$commutativity.

PROOF. (1) $\Longrightarrow$ (2) Since $a \in R^{SEP}$, $(a^{\#})^2 a^+ = a^* a^+ a^{\#}$ by Lemma 2.1. Noting that $a \in R^{EP}$. Then $a((a^{\#})^2 a^+) = ((a^{\#})^2 a^+)a$. Hence $(a^{\#})^2 a^+,\ a^* a^+ a^{\#}$ are $a-$commutativity.

(2) $\Longrightarrow$ (3) From (2), one gets

$$a^{\#}((a^{\#})^2 a^+) = a^{\#}a^{\#}(a(a^{\#})^2 a^+) = a^{\#}a^{\#}(a^* a^+ a^{\#} a)$$

$$= a^{\#}a^{\#}(a^* a^+ a^{\#} a^2)a^{\#} = a^{\#}a^{\#}(a(a^{\#})^2 a^+)aa^{\#} = a^{\#}(a^{\#})^2 a^+ aa^{\#} = (a^{\#})^4.$$

and $(a^* a^+ a^{\#})a^{\#} = (a^* a^+ a^{\#})aa^{\#}a^{\#} = a(a^{\#})^2 a^+ a^{\#} a^{\#} = (a^{\#})^4$.

Hence $(a^{\#})^2 a^+,\ a^* a^+ a^{\#}$ are $a^{\#}$-commutativity.

(3) $\Longrightarrow$ (1) Using the equality, one gets

$$a^{\#}((a^{\#})^2 a^+) = (a^* a^+ a^{\#})a^{\#} = (a^* a^+ a^{\#} a^{\#})a^+ a = (a^{\#})^3 a^+ a^+ a.$$

This gives

$$aa^+ = a^4 (a^{\#})^3 a^+ = a^4 (a^{\#})^3 a^+ a^+ a = aa^+ a^+ a.$$

Hence, $a \in R^{EP}$, it follows that

$$a = (a^{\#})^3 a^+ a^5 = (a^* a^+ a^{\#} a^{\#})a^5 = a^* a^2.$$

Thus, $a \in R^{SEP}$ by [2, Theorem 1.5.3]. □

Lemma 7.2. *Let $a, b, c \in R$ and $b,\ c$ are $a-$commutativity. Then $b,\ c$ are $ab-$commutativity.*

PROOF. Since $b,\ c$ are $a-$commutativity, $ab = ca$. This implies $(ab)b = (ca)b = c(ab)$. Hence, $b,\ c$ are $ab-$commutativity. □

Corollary 7.3. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{SEP}$ if and only if $(a^{\#})^2 a^+,\ a^* a^+ a^{\#}$ are $a^{\#}a^{+}-$commutativity.*

PROOF. " $\Longrightarrow$ " Assume that $a \in R^{SEP}$. Then $(a^{\#})^2 a^+,\ a^* a^+ a^{\#}$ are $a-$commutativity by Theorem 7.1. By Lemma 7.2, one gets $(a^{\#})^2 a^+,\ a^* a^+ a^{\#}$ are $a^{\#}a^{+}-$commutativity.

" $\Longleftarrow$ " From the assumption, we have

$$(a^{\#}a^+)(a^{\#})^2 a^+ = a^* a^+ a^{\#}(a^{\#}a^+).$$

It follows that

$$a^{\#} = a^{\#}a^+ a = a^{\#}a^+((a^{\#})^2 a^+ a^4) = a^* a^+ a^{\#}(a^{\#}a^+)a^4 = a^* a^+ a.$$

Hence, $a \in R^{SEP}$ by [2, Theorem 1.5.3]. □

Lemma 7.4. *Let $x, y, z \in R$ and $y,\ z$ are $x-$commutativity. Then $xz,\ xy$ are $z-$commutativity.*

PROOF. Since y, z are $x-$commutativity, $xy = zx$. One gets $z(xz) = (zx)z = (xy)z$. Hence, xz, xy are $z-$commutativity. □

Corollary 7.5. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{SEP}$ if and only if $aa^*a^+a^{\#}$, $a^{\#}a^+$ are $a^*a^+a^{\#}-$commutativity.*

PROOF. " $\Longrightarrow$ " Assume that $a \in R^{SEP}$. Then $(a^{\#})^2a^+$, $a^*a^+a^{\#}$ are $a-$commutativity by Theorem 7.1. By Lemma 7.4, one gets $a(a^*a^+a^{\#})$, $a((a^{\#})^2a^+)$ are $a^*a^+a^{\#}-$commutativity, that is, $aa^*a^+a^{\#}$, $a^{\#}a^+$ are $a^*a^+a^{\#}-$commutativity.

" $\Longleftarrow$ " Using the assumption, we get

$$(a^*a^+a^{\#})(aa^*a^+a^{\#}) = (a^{\#}a^+)(a^*a^+a^{\#}).$$

Multiplying the equality on the right by a^2a^+, we have

$$a^*a^+a^{\#}aa^*a^+ = a^{\#}a^+a^*a^+.$$

By [3, Lemma 2.10], we obtain

$$a^*a^+a^{\#}aa^* = a^{\#}a^+a^*,$$

and

$$a^*a^+a^{\#}a = a^*a^+a^{\#}aa^*(a^+)^* = a^{\#}a^+a^*(a^+)^* = a^{\#}a^+a^+a.$$

Multiplying the last equality on the left by a^+a, we yield

$$a^{\#}a^+a^+a = a^+aa^{\#}a^+a^+a,$$

and

$$a^{\#}a^+a^+ = a^{\#}a^+a^+aa^+ = a^+aa^{\#}a^+a^+aa^+ = a^+aa^{\#}a^+a^+.$$

Again, by [3, Lemma 2.10], we have

$$a^{\#}a^+ = a^+aa^{\#}a^+.$$

It follows that

$$a = a^{\#}a^+a^3 = a^+aa^{\#}a^+a^3 = a^+a^2.$$

Hence, $a \in R^{EP}$ by [2, Theorem 1.2.1]. Now we have

$$a^* = (a^*a^+a^{\#}a)a = (a^{\#}a^+a^+a)a = a^{\#}.$$

Thus, $a \in R^{SEP}$. □

Theorem 7.6. *Let $a \in R^{\#} \cap R^{+}$. Then the following conditions are equivalent:*
(1) $a \in R^{SEP}$;
(2) $a^*a^+a^{\#}$, $(a^{\#})^2a^+$ are a^+-commutativity;
(3) $a^*a^+a^{\#}$, $(a^{\#})^2a^+$ are $(a^{\#})^*-$commutativity.

Proof. (1) $\Longrightarrow$ (2) Assume that $a \in R^{SEP}$. Then $(a^{\#})^2a^+$, $a^*a^+a^{\#}$ are $a^{\#}-$commutativity by Theorem 7.1 and $(a^{\#})^2a^+ = a^*a^+a^{\#}$ by Lemma 2.1. Noting that $a^+ = a^{\#}$. Hence, $a^*a^+a^{\#}$, $(a^{\#})^2a^+$ are a^+-commutativity.

(2) $\Longrightarrow$ (3) From the assumption, one gets

$$a^+a^*a^+a^{\#} = (a^{\#})^2a^+a^+ = ((a^{\#})^2a^+a^+)aa^+ = a^+a^*a^+a^{\#}aa^+.$$

By [3, Lemma 2.10], one yields $a^*a^+a^{\#} = a^*a^+a^{\#}aa^+$, so

$$a^{\#} = aa^+a^{\#} = a(a^{\#})^*(a^*a^+a^{\#}) = a(a^{\#})^*(a^*a^+a^{\#}aa^+) = a^{\#}aa^+.$$

Hence, $a \in R^{EP}$, this induces $a^{\#}a^*a^{\#}a^{\#} = a^+a^*a^+a^{\#} = (a^{\#})^2a^+a^+ = (a^{\#})^4$ and $a^* = aa^{\#}a^*a^{\#}a = a(a^{\#}a^*a^{\#}a^{\#})a^2 = a(a^{\#})^4a^2 = a^{\#}$. Hence,

$$(a^{\#})^*(a^*a^+a^{\#}) = a(a^*a^+a^{\#}) = aa^{\#}a^+a^{\#} = a^{\#}a^{\#} = (a^{\#})^2a^+a = (a^{\#})^2a^+(a^{\#})^*.$$

(3) $\Longrightarrow$ (1) According to the hypothesis, one obtains $(a^{\#})^*(a^*a^+a^{\#}) = (a^{\#})^2a^+(a^{\#})^*$, e.g.,

$$a^+a^{\#} = (a^{\#})^2a^+(a^{\#})^*.$$

and

$$a = a^3a^+a^{\#} = a^3(a^{\#})^2a^+(a^{\#})^* = aa^+(a^{\#})^*.$$

So, $a^* = a^{\#}aa^+$. Hence, $a \in R^{SEP}$ by [2, Theorem 1.5.3].

Theorem 7.7. *Let $a \in R^{\#} \cap R^+$. Then the following conditions are equivalent:*
(1) $a \in R^{SEP}$;
(2) $(a^+)^2a^{\#}, a^*a^+a^{\#}$ are a^*-commutativity;
(3) $a^*a^+a^{\#}$, $(a^{\#})^2a^+$ are $(a^{\#})^*-$commutativity.

PROOF. (1) $\Longrightarrow$ (2) Suppose that $a \in R^{SEP}$. Then, by Theorem 7.1, $(a^{\#})^2a^+$, $a^*a^+a^{\#}$ are $a^{\#}-$commutativity. Noting that $a^{\#} = a^+ = a^*$. Hence, $(a^+)^2a^{\#}, a^*a^+a^{\#}$ are a^*-commutativity.
(2) $\Longrightarrow$ (3) Using the hypothesis, one gets $a^*(a^+)^2a^{\#} = a^*a^+a^{\#}a^*$, this gives

$$(a^+)^2a^{\#} = (a^{\#})^*a^*(a^+)^2a^{\#} = (a^{\#})^*a^*a^+a^{\#}a^* = a^+a^{\#}a^* = a^+a^{\#}a^*aa^+ = (a^+)^2a^{\#}aa^+.$$

By [3, Lemma 2.10], one has $a^+a^{\#} = a^+a^{\#}aa^+$, and $a = a^3a^+a^{\#} = a^3a^+a^{\#}aa^+ = a^2a^+$. Hence, $a \in R^{EP}$, this leads to

$$(a^+)^2a^{\#}(a^{\#})^* = a^+a^{\#}a^*(a^{\#})^* = a^+a^+a^*(a^{\#})^* = a^+a^+ = (a^{\#})^*a^*a^+a^+ = (a^{\#})^*a^*a^+a^{\#}.$$

Hence, $a^*a^+a^{\#}$, $(a^{\#})^2a^+$ are $(a^{\#})^*-$commutativity.
(3) $\Longrightarrow$ (1) With the assumption, one gets

$$a^+a^{\#} = (a^{\#})^*a^*a^+a^{\#} = (a^+)^2a^{\#}(a^{\#})^* = ((a^+)^2a^{\#}(a^{\#})^*)aa^+ = a^+a^{\#}aa^+.$$

Hence, $a \in R^{EP}$ by (2) $\Longrightarrow$ (3), it follows that

$$a = a^3a^+a^{\#} = a^3(a^+)^2a^{\#}(a^{\#})^* = a^3a^+a^{\#}a^{\#}(a^+)^* = (a^+)^*.$$

This induces $a \in R^{PI}$. Thus, $a \in R^{SEP}$. □

8. Using one-sided $x-$equality to characterize SEP elements

Let $x,\ y,\ z \in R$. Then $y,\ z$ are called left (right) $x-$equality if $xy = xz$ ($yx = zx$).
Clearly, (1) y is right $z-$idempotent if and only if $y,\ z$ are left $y-$equality;
(2) $e \in R$ is idempotent if and only if $e,\ 1$ are left $e-$equality;
(3) $e \in R$ is idempotent if and only if $e,\ 2e - 1$ are left $e-$equality.

Theorem 8.1. *Let $a \in R^{\#} \cap R^+$. Then $a \in R^{SEP}$ if and only if $(a^{\#})^2a^+$, $a^*a^+a^{\#}$ are left $x-$equality for some $x \in \chi_a = \{a,\ a^{\#},\ a^+,\ a^*,\ (a^+)^*,\ (a^{\#})^*\}$.*

PROOF. "$\Longrightarrow$" Since $a \in R^{SEP}$, $(a^{\#})^2a^+ = a^*a^+a^{\#}$ by Lemma 2.1.
Hence, $(a^{\#})^2a^+, a^*a^+a^{\#}$ are left $x-$equality for any $x \in \chi_a$.
"$\Longleftarrow$" If there exists some $x_0 \in \chi_a$, such that $(a^{\#})^2a^+, a^*a^+a^{\#}$ are left $x-$equality. Then

$$x_0(a^{\#})^2a^+ = x_0a^*a^+a^{\#} = (x_0a^*a^+a^{\#})a^+a = x_0(a^{\#})^2a^+a^+a.$$

Noting that if $x_0 \in \tau_a = \{a,\ a^{\#},\ (a^+)^*\}$, then $x_0^{\#}x_0 = aa^{\#}$. It follows that

$$(a^{\#})^2a^+ = a^{\#}a(a^{\#})^2a^+ = x_0^{\#}x_0(a^{\#})^2a^+ = x_0^{\#}x_0(a^{\#})^2a^+a^+a = a^{\#}a(a^{\#})^2a^+a^+a = (a^{\#})^2a^+a^+a,$$

this gives $aa^+ = a^3(a^{\#})^2a^+ = a^3(a^{\#})^2a^+a^+a = aa^+a^+a$. Hence, $a \in R^{EP}$.

Also, if $x_0 \in \gamma_a = \{a^+,\ a^*,\ (a^\#)^*\}$, then $x_0^\# x_0 = (aa^\#)^*$. Hence,

$$(aa^\#)^*(a^\#)^2a^+ = x_0^\# x_0(a^\#)^2a^+ = x_0^\# x_0(a^\#)^2a^+a^+a = (aa^\#)^*(a^\#)^2a^+a^+a,$$

and

$$aa^+ = a^4a^+(a^\#)^2a^+ = a^4a^+(aa^\#)^*(a^\#)^2a^+ = a^4a^+(aa^\#)^*(a^\#)^2a^+a^+a = aa^+a^+a.$$

Hence, $a \in R^{EP}$.

In any casewe have $a \in R^{EP}$ and $x_0^\# x_0 = aa^\#$ for $x \in \chi_a$. Thus, we have

$$(a^\#)^2a^+ = (aa^\#)(a^\#)^2a^+ = (x_0^\# x_0)(a^\#)^2a^+ = x_0^\# x_0 a^*a^+a^\# = aa^\#a^*a^+a^\# = a^*a^+a^\#.$$

By Lemma 2.1, $a \in R^{SEP}$. □

Theorem 8.2. *Let $a \in R^\# \cap R^+$. Then the following conditions are equivalent:*
(1) $a \in R^{SEP}$;
(2) $(a^\#)^2a^+,\ a^*a^+a^\#$ are left $(a^\#)^2a^+-$equality;
(3) $(a^\#)^2a^+,\ a^*a^+a^\#$ are left $a^*a^+a^\#-$equality.

PROOF. (1) ⟹ (2) It follows from $a \in R^{SEP}$ and Lemma 2.1 that $(a^\#)^2a^+ = a^*a^+a^\#$. Hence, $(a^\#)^2a^+, a^*a^+a^\#$ are left $(a^\#)^2a^+-$equality.

(2) ⟹ (3) From the assumption, one gets $(a^\#)^2a^+$ is right $a^*a^+a^\#$-idempotent. By Theorem 6.2, $a \in R^{SEP}$, it follows from Lemma 2.1 that $(a^\#)^2a^+ = a^*a^+a^\#$ and so $(a^\#)^2a^+,\ a^*a^+a^\#$ are left $a^*a^+a^\#-$equality.

(3) ⟹ (1) From the hypothesis, we have

$$(a^*a^+a^\#)(a^\#)^2a^+ = (a^*a^+a^\#)^2 = (a^*a^+a^\#)^2a^+a = (a^*a^+a^\#)(a^\#)^2a^+a^+a.$$

Multiplying the equality on the left by $a^5(a^\#)^*$, one has

$$aa^+ = aa^+a^+a.$$

Hence, $a \in R^{EP}$, this gives

$$a^*(a^\#)^5 = (a^*a^+a^\#)(a^\#)^2a^+ = (a^*a^+a^\#)^2 = (a^*a^\#a^\#)^2.$$

Multiplying the last equality by $a^2(a^+)^*$ on the left, one gets

$$(a^\#)^3 = a^*(a^\#)^2.$$

It follows that

$$a = (a^\#)^3a^4 = a^*(a^\#)^2a^4 = a^*a^2.$$

Thus, $a \in R^{SEP}$ by [2, Theorem 1.5.3]. □

Theorem 8.3. *Let $a \in R^\# \cap R^+$. Then the following conditions are equivalent:*
(1) $a \in R^{SEP}$;
(2) $(a^\#)^2a^+a^+,\ a^+a^*a^+a^\#$ are left a^+-equality;
(3) $(a^\#)^2a^+a^+,\ a^+a^*a^+a^\#$ are right a^+-equality.

PROOF. (1) ⟹ (2) Since $a \in R^{SEP}$, $a^\# = a^+ = a^*$, this gives

$$a^+a^*a^+a^\# = a^\#a^\#a^+a^+ = (a^\#)^2a^+a^+.$$

Hence, $(a^\#)^2a^+a^+,\ a^+a^*a^+a^\#$ are left a^+-equality.

(2) ⟹ (3) Using the assumption, one gets

$$a^+(a^\#)^2a^+a^+ = a^+a^+a^*a^+a^\# = (a^+a^+a^*a^+a^\#)a^+a = a^+(a^\#)^2a^+a^+a^+a$$

$$a^+a^+ = a^+aa^+a^+ = a^+a^4a^+(a^\#)^2a^+a^+ = a^+a^4a^+(a^\#)^2a^+a^+a^+a = a^+a^+a^+a.$$

By [3, Lemma 2.11], $a^+ = a^+a^+a$. Hence, $a \in R^{EP}$, it follows that

$$(a^\#)^5 = a^+(a^\#)^2a^+a^+ = a^+a^+a^*a^+a^\# = (a^\#)^2a^*(a^\#)^2$$

$$a = a^3(a^\#)^5a^3 = a^3(a^\#)^2a^*(a^\#)^2a^3 = aa^*a.$$

Thus, $a \in R^{SEP}$, which implies $(a^\#)^2a^+a^+ = a^+a^*a^+a^\#$. Therefore, $(a^\#)^2a^+a^+$, $a^+a^*a^+a^\#$ are right a^+-equality.

(3) $\Longrightarrow$ (1) With the assumption, one has

$$(a^\#)^2a^+a^+a^+ = a^+a^*a^+a^\#a^+ = a^+a(a^+a^*a^+a^\#a^+) = a^+a((a^\#)^2a^+a^+a^+) = a^+a^\#a^+a^+a^+.$$

By [3, Lemmma 2.11], one gets $(a^\#)^2a^+ = a^+a^\#a^+$, and

$$a = (a^\#)^2a^+a^4 = a^+a^\#a^+a^4 = a^+a^2.$$

Hence, $a \in R^{EP}$ and so

$$(a^\#)^5 = (a^\#)^2a^+a^+a^+ = a^+a^*a^+a^\#a^+ = a^\#a^*(a^\#)^3$$

and

$$a = a^2(a^\#)^5a^4 = a^2a^\#a^*(a^\#)^3a^4 = aa^*a.$$

Thus, $a \in R^{SEP}$. □

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