Filomat 39:4 (2025), 1135–1147
https://doi.org/10.2298/FIL2504135K

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Rough ideal convergence of sequences in gradual normed linear spaces

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Abstract. For a non-empty set X, an ideal $\mathcal{I}$ represents a family of subsets of X that is closed under taking finite unions and subsets of its elements. Considering $X = \mathbb{N}$, in the present study, we set forth with the new concept of rough $\mathcal{I}$ and $\mathcal{I}^*-$convergence in gradual normed linear spaces (GNLS). We produce significant results that present several fundamental features of the notions utilizing $\mathcal{I}^r(\mathcal{G})$ and $\mathcal{I}^{*,r}(\mathcal{G})-$limit set. In the end, we investigate their interrelationships and establish a necessary and sufficient condition for the equivalency of the two notions.

1. Introduction

The concept of fuzzy sets (FS), introduced by Zadeh [50], has gained wide-ranging applications across various fields of engineering and science. The concept "fuzzy number (FN)" is significant in the work of FS theory. FNs are essentially the generalisation of intervals, not numbers. Indeed, FNs do not supply a couple of algebraic features of the well-known numbers. For this reason the concept "FN":is debatable to some researchers owing to its not similar behaviour. The concept "fuzzy intervals" is commonly utilized by several researchers in place of FNs. In order to succeed the confusion of the authors, Fortin et al. [19] put forward to the concept of gradual real numbers (GRNs) as elements of fuzzy intervals. GRNs are known by their respective assignment function whose domain is the interval $(0, 1]$. So, each $\mathbb{R}$ numbers can be thought of as a gradual number with a constant assignment function. Furthermore, the GRNs supply all the algebraic features of the $\mathbb{R}$ numbers and have been used in optimization problems and computation.

Sadeqi and Azari [38] were the first to introduce the concept of GNLS. They worked significant features from both the topological and algebraic points of view. Improvement in this way has taken place owing to Ettefagh et al. [16, 17], Choudhury and Debnath [11], and many others. For an comprehensive research on GRNs, one can refer to [1, 14, 29, 43].

On the other side, the idea of convergence of a real sequence was extended to statistical convergence by Fast [18] as follows:

A real valued sequence $w = (w_k)$ is called statistically convergent to the number w provided that for all $\gamma > 0$,

2020 *Mathematics Subject Classification*. Primary 03E72; Secondary 40A35, 40A05.
Keywords. Ideal, rough ideal convergence, $\mathcal{I}^r(\mathcal{G})-$limit set, gradual number, gradual normed linear space.
Received: 11 June 2024; Revised: 14 November 2024; Accepted: 01 December 2024
Communicated by Dijana Mosić
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$$\lim_m \frac{1}{m} \left|\{k \leq m : |w_k - w| \geq \gamma\}\right| = 0.$$

We denote $S-\lim w_k = w$ or $w_k \to w(S)$. The main thought behind statistical convergence was the notion of natural density. The natural density of a set $S \subseteq \mathbb{N}$ is demonstrated and determined by

$$\delta(S) = \lim_n \frac{1}{n} |\{k \in S : k \leq n\}|.$$

Clearly, $\delta(\mathbb{N} \setminus S) + \delta(S) = 1$ and $S \subseteq T$ implies $\delta(S) \leq \delta(T)$ for $S, T \subseteq \mathbb{N}$. It is obvious that when S is a finite set then $\delta(S) = 0$.

Statistical convergence became remarkable after the works of Connor [12], Fridy [20], Šalát [40], Tripathy [44–47], and many others (see, for details [3, 4, 25, 30, 31]). Many improvements of statistical convergence can be found in number theory and mathematical analysis.

Statistical convergence was extended to two kinds of convergence namely $\mathcal{I}$ and $\mathcal{I}^*$−convergence by Kostyrko et al. [28], where the ideal $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is nonempty and closed under choosing finite unions and subsets of its elements. Several advancements in the direction of ideal convergence can be found from the works of Demirci [13], Nabiev et al. [33], and many others (see, for details [21–24, 27, 32, 34, 39, 41, 42, 48, 49]).

The concept of rough convergence was first put forward by Phu [36] for finite dimensional normed spaces. But a similar idea was independently studied by Burgin [10] in the fuzzy setting. In [36], Phu mainly proved that the limit set $LIM^r w$ is closed, bounded and convex and examined the fundamental features of this interesting concept. It ought to be mentioned that the idea of rough convergence occurs quite naturally in numerical analysis and has significant applications there. Phu [37], further investigated the notion of rough convergence in an infinite dimensional normed space setting. Combining the conceptions of rough convergence and statistical convergence, Aytar [7] developed rough statistical convergence. But Akçay and Aytar [2] were the first who introduced and investigated the notion of rough convergence of a sequence of fuzzy numbers. For extensive study in this topic, one may refer to the works of [5, 6, 8, 9, 15, 26, 35], where many more references can be found.

2. Preliminaries

In the present section, we give some existing definitions and results which are crucial for our findings.

The gradual numbers and the gradual operations between the elements of $\mathcal{G}(\mathbb{R})$ was determined by Fortin et al. [19] as follows:

Definition 2.1. *A GRN $\tilde{s}$ is determined by an assignment function $\mathcal{F}_{\tilde{s}} : (0, 1] \to \mathbb{R}$. A GRN $\tilde{s}$ is called to be non-negative provided that for each $\rho \in (0, 1]$, $\mathcal{F}_{\tilde{s}}(\rho) \geq 0$. The set of all GRNs and non-negative GRNs are demonstrated by $\mathcal{G}(\mathbb{R})$ and $\mathcal{G}^*(\mathbb{R})$, respectively.*

Definition 2.2. *Presume that $*$ be any operation in $\mathbb{R}$ and presume $\tilde{u}_1, \tilde{u}_2 \in \mathcal{G}(\mathbb{R})$ with assignment functions $\mathcal{F}_{\tilde{u}_1}$ and $\mathcal{F}_{\tilde{u}_2}$ respectively. At that time, $\tilde{u}_1 * \tilde{u}_2 \in G(\mathbb{R})$ is determined with the assignment function $\mathcal{F}_{\tilde{u}_1 * \tilde{u}_2}$ denoted by $\mathcal{F}_{\tilde{u}_1 * \tilde{u}_2}(\rho) = \mathcal{F}_{\tilde{u}_1}(\rho) * \mathcal{F}_{\tilde{u}_2}(\rho)$, $\forall \rho \in (0, 1]$.*

Especially, the gradual addition $\tilde{u}_1 + \tilde{u}_2$ and the gradual scalar multiplication $p\tilde{u}$ ($p \in \mathbb{R}$) are given as follows:

$$\mathcal{F}_{\tilde{u}_1 + \tilde{u}_2}(\rho) = \mathcal{F}_{\tilde{u}_1}(\rho) + \mathcal{F}_{\tilde{u}_2}(\rho) \quad \text{and} \quad \mathcal{F}_{p\tilde{u}}(\rho) = p\mathcal{F}_{\tilde{u}}(\rho), \ \forall \rho \in (0, 1].$$

Utilizing the gradual numbers, Sadeqi and Azari [38] developed the GNLS and determined the notion of gradual convergence as follows:

Definition 2.3. *Take Y as a real vector space. Afterwards, the function $||\cdot||_{\mathcal{G}} : Y \to \mathcal{G}^*(\mathbb{R})$ is named to be a gradual norm (GN) on Y, provided that for each $\rho \in (0, 1]$, subsequent situations are correct for any $w, v \in Y$:*

1. $\mathcal{F}_{||w||_{\mathcal{G}}}(\rho) = \mathcal{F}_{\tilde{0}}(\rho)$ *iff* $w = 0$;
2. $\mathcal{F}_{||\mu w||_{\mathcal{G}}}(\rho) = |\mu| \mathcal{F}_{||w||_{\mathcal{G}}}(\rho)$ *for any* $\mu \in \mathbb{R}$;
3. $\mathcal{F}_{||w+v||_{\mathcal{G}}}(\rho) \leq \mathcal{F}_{||w||_{\mathcal{G}}}(\rho) + \mathcal{F}_{||v||_{\mathcal{G}}}(\rho)$.

Here, $(Y, \|\cdot\|_{\mathcal{G}})$ is named GNLS.

Example 2.4. *Take $Y = \mathbb{R}^n$ and for $w = (w_1, w_2, ..., w_m) \in \mathbb{R}^n, \gamma \in (0, 1]$, determine $\|\cdot\|_{\mathcal{G}}$ as*

$$\mathcal{F}_{\|w\|_{\mathcal{G}}}(\gamma) = e^{\gamma} \sum_{j=1}^{n} |w_j|.$$

Here, $\|\cdot\|_{\mathcal{G}}$ is a GN on $\mathbb{R}^n$, also $(\mathbb{R}^n, \|\cdot\|_{\mathcal{G}})$ denotes a GNLS.

Definition 2.5. *Take $(w_k) \in (Y, \|\cdot\|_{\mathcal{G}})$. Then, (w_k) is called to be rough gradually convergent (briefly $r(\mathcal{G})-$ convergent) to $w_0 \in Y$, provided that for each $\rho \in (0, 1]$ and $\gamma > 0$, there is an $N(= N_\gamma(\rho)) \in \mathbb{N}$ so that $\mathcal{F}_{\|w_k - w_0\|_{\mathcal{G}}}(\rho) < r + \gamma$, : $\forall k \geq N$.*

Symbolically we write, $w_k \xrightarrow{r-\|\cdot\|_{\mathcal{G}}} w_0$.

On the other hand, Ettefagh et al. [17] determined the gradual boundedness of a sequence in a GNLS and investigated its relationship with gradual convergence.

Definition 2.6. *Suppose that $(Y, \|\cdot\|_{\mathcal{G}})$ be a GNLS. In that time, a sequence (w_k) in Y is named to be gradual bounded provided that for each $\rho \in (0, 1]$, there is an $M = M(\rho) > 0$ so that*
$\mathcal{F}_{\|w_k\|_{\mathcal{G}}}(\rho) < M$, $\forall k \in \mathbb{N}$.

Definition 2.7. *Let $Y \neq \emptyset$. $\mathcal{I} \subset 2^Y$ is called an ideal on Y provided that*

1. *for each $U, V \in \mathcal{I}$ implies $U \cup V \in \mathcal{I}$;*
2. *for each $U \in \mathcal{I}$ and $V \subset U$ implies $V \in \mathcal{I}$.*

Definition 2.8. *Let $Y \neq \emptyset$. $\mathcal{F} \subset 2^Y$ is named a filter on Y provided that*

1. *for all $U, V \in \mathcal{F}$ implies $U \cap V \in \mathcal{F}$;*
2. *for all $U \in \mathcal{F}$ and $V \supset U$ implies $V \in \mathcal{F}$.*

An ideal $\mathcal{I}$ is known as non-trivial provided that $Y \notin \mathcal{I}$ and $\mathcal{I} \neq \emptyset$. A non-trivial ideal $\mathcal{I} \subset P(Y)$ is known as an admissible ideal in Y iff $\mathcal{I} \supset \{\{w\} : w \in Y\}$. Afterwards, the filter $\mathcal{F} = \mathcal{F}(\mathcal{I}) = \{Y - S : S \in \mathcal{I}\}$ is named the filter connected with the ideal.

Utilizing the notion of ideals, Kostyrko et al. [28] determined the notion of $\mathcal{I}$ and $\mathcal{I}^*-$convergence as follows:

Definition 2.9. *A sequence $w = (w_k)$ is said to be $\mathcal{I}$-convergent to w provided that for all $\gamma > 0$,*

$$\{k \in \mathbb{N} : |w_k - w| \geq \gamma\} \in \mathcal{I}.$$

In this situation, we denote as $\mathcal{I}\text{-}\lim w_k = w$.

Definition 2.10. *A sequence $w = (w_k)$ is called to be $\mathcal{I}^*-$convergent to l, provided that there is a set $M = \{s_1 < s_2 < ... < s_k < ...\} \in \mathcal{F}(\mathcal{I})$ so that $\lim_{k \in M} w_k = l$.*

Also, in [28], Kostyrko et al. gave the definition of (AP) condition for admissible ideal, and examined the relation between $\mathcal{I}$ and $\mathcal{I}^*-$convergence under (AP) condition.

The notion of rough convergence was investigated by Phu [36] and subsequently generalised by Aytar [7] and Pal et al. [35] as follows:

Definition 2.11. *Presume r be a non-negative real number. A sequence $w = (w_k)$ in a normed linear space $(Y, \|\cdot\|)$ is called to be rough convergent to $w_0 \in Y$ with roughness degree r, provided that for each $\gamma > 0$, there exists $N = (N_\gamma)$ so that for all $k \geq N$,*

$$\|w_k - w_0\| < r + \gamma.$$

Symbolically, it is demonstrated as $w_k \xrightarrow{r-\|\cdot\|} w_0$.

Definition 2.12. *A sequence $w = (w_k)$ in a normed linear space $(Y, \|\cdot\|)$ is called rough statistical convergent to $w_0 \in Y$ with roughness degree r, provided that for all $\gamma > 0$,*

$$\delta\left(\{k \in \mathbb{N} : \|w_k - w_0\| \geq r + \gamma\}\right) = 0.$$

Symbolically, it is indicated as $w_k \xrightarrow{st^r - \|\cdot\|} w_0$.

Definition 2.13. *A sequence $w = (w_k)$ is called rough $\mathcal{I}-$convergent to $w_0 \in Y$ with roughness degree r, provided that for each $\gamma > 0$,*

$$\{k \in \mathbb{N} : \|w_k - w_0\| \geq r + \gamma\} \in \mathcal{I}.$$

Symbolically, it is demonstrated as $w_k \xrightarrow{\mathcal{I}^r - \|\cdot\|} w_0$.

It is obvious from the above definition that when we contemplate $\mathcal{I} = \mathcal{I}_f = \{P \subseteq \mathbb{N} : card(P) < \infty\}$, then it reduces to Definition 2.11, whereas when we contemplate $\mathcal{I} = \mathcal{I}_d = \{P \subseteq \mathbb{N} : \delta(P) = 0\}$, then it turns to Definition 2.12. Thus, rough $\mathcal{I}-$convergence presents a general framework to study the features of several types of rough convergence.

In another direction, Choudhury and Debnath [11] put forward the notion of $\mathcal{I}$ and $\mathcal{I}^*-$convergence of sequences in GNLS as follows:

Definition 2.14. *Presume $\mathcal{I}$ be an admissible ideal in $\mathbb{N}$ and take $(w_k) \in (Y, \|\cdot\|_{\mathcal{G}})$. Then, (w_k) is named to be gradually $\mathcal{I}-$convergent to $w_0 \in Y$ provided that for each $\rho \in (0, 1]$ and $\gamma > 0$,*

$$\left\{k \in \mathbb{N} : \mathcal{F}_{\|w_k - w_0\|_{\mathcal{G}}}(\rho) \geq \gamma\right\} \in \mathcal{I}.$$

Symbolically, $w_k \xrightarrow{\mathcal{I} - \|\cdot\|_{\mathcal{G}}} w_0$.

Definition 2.15. *The sequence (w_k) is said to be gradually $\mathcal{I}-$bounded if for each $\rho \in (0, 1]$, there is an $M(= M(\rho)) > 0$ so that*

$$\left\{k \in \mathbb{N} : \mathcal{F}_{\|w_k\|_{\mathcal{G}}}(\rho) > M\right\} \in \mathcal{I}.$$

Definition 2.16. *The sequence (w_k) is named to be gradually $\mathcal{I}^*-$convergent to $w_0 \in Y$ provided that there is a set $M = \{s_1 < s_2 < ... < s_k < ...\} \in \mathcal{F}(\mathcal{I})$ so that the sequence (w_{s_k}) is gradual convergent to w_0. In that case, $w_k \xrightarrow{\mathcal{I}^* - \|\cdot\|_{\mathcal{G}}} w_0$.*

Motivating from the above mentioned works, in this study we present the notions of rough $\mathcal{I}$ and rough $\mathcal{I}^*-$convergence as a continuation of the work of Choudhury and Debnath [11]. Throughout our discussion, $\mathcal{I}$ will indicate a non-trivial admissible ideal in $\mathbb{N}$ and r stands for a non-negative real number.

3. Rough $\mathcal{I}-$convergence in GNLS

In this section, we present our findings regarding rough $\mathcal{I}-$convergence of sequences in GNLS. We begin with the subsequent definition:

Definition 3.1. *Suppose $\mathcal{I}$ be an admissible ideal in $\mathbb{N}$ and take $(w_k) \in (Y, \|\cdot\|_{\mathcal{G}})$. Then, (w_k) is called gradually rough $\mathcal{I}-$convergent (briefly $\mathcal{I}^r(\mathcal{G})-$convergent) to $w_0 \in Y$, provided that for all $\rho \in (0, 1]$ and $\gamma > 0$,*

$$\left\{k \in \mathbb{N} : \mathcal{F}_{\|w_k - w_0\|_{\mathcal{G}}}(\rho) \geq r + \gamma\right\} \in \mathcal{I}.$$

Symbolically we write, $w_k \xrightarrow{\mathcal{I}^r - \|\cdot\|_{\mathcal{G}}} w_0$.

Here r is called the degree of roughness. For $r = 0$, Definition 3.1 reduces to Definition 2.14. But our main intention is to deal with the case $r > 0$. There are some reasons for such interest. Since a gradually ideal convergent sequence (q_k) with $q_k \xrightarrow{\mathcal{I}-\|\cdot\|_{\mathcal{G}}} w_0$ often cannot be calculated or measured accurately, one has to deal with a statistically approximated sequence (w_k) supplying

$$\left\{k \in \mathbb{N} : \mathcal{F}_{\|w_k - q_k\|_{\mathcal{G}}}(\rho) > r\right\} \in \mathcal{I}.$$

Then, no one can guarantee the gradually $\mathcal{I}-$convergence of (w_k), but since for any $\gamma > 0$, the following inclusion

$$\left\{k \in \mathbb{N} : \mathcal{F}_{\|w_k - w_0\|_{\mathcal{G}}}(\rho) \geq r + \gamma\right\} \subseteq \left\{k \in \mathbb{N} : \mathcal{F}_{\|q_k - w_0\|_{\mathcal{G}}}(\rho) \geq \gamma\right\}$$

holds, one can certainly assure the $\mathcal{I}^r(\mathcal{G})-$convergence of (w_k). We surve the subsequent example to illustrate the above fact more preciously.

Example 3.2. *Assume $Y = \mathbb{R}^n$ and let $\|\cdot\|_{\mathcal{G}}$ be the norm determined in Example 2.4 and $\mathcal{I}_d = \{P \subseteq \mathbb{N} : \delta(P) = 0\}$. Examine the sequence (q_k) in Y determined as*

$$q_k = \begin{cases} (0, 0, ..., 0, 0.5), & \text{when } k \text{ is a perfect square} \\ \left(0, 0, ..., 0, 0.5 + 2 \cdot \frac{(-1)^k}{k}\right), & \text{otherwise.} \end{cases}$$

Then, we have

$$\mathcal{F}_{\|q_k - (0,0,...,0,0.5)\|_{\mathcal{G}}}(\rho) = \begin{cases} 0, & \text{when } k \text{ is a perfect square} \\ \frac{2e^{\rho}}{k}, & \text{otherwise.} \end{cases}$$

So, for any $\gamma > 0$, the following inclusion

$$\left\{k \in \mathbb{N} : \mathcal{F}_{\|q_k - (0,0,...,0,0.5)\|_{\mathcal{G}}}(\rho) \geq \gamma\right\} \subseteq \{1, 4, 9, ...\}$$

supplies and eventually $q_k \xrightarrow{\mathcal{I}_d-\|\cdot\|_{\mathcal{G}}} (0, 0, ..., 0, 0.5)$. But for sufficiently large k, it is not possible to compute q_k exactly by computer however it is rounded to the nearest one. So, in the interest of simplicity, we approximate q_k by $w_k = (0, 0, ..., 0, t)$ at the non-perfect square positions where t is the integer satisfying $t - 0.5 < q_k < t + 0.5$. At that time, the sequence (w_k) does not gradually $\mathcal{I}_d-$converge anymore. On the other hand according to definition $w_k \xrightarrow{\mathcal{I}_d^r-\|\cdot\|_{\mathcal{G}}} (0, 0, ..., 0, 0.5)$ for $r = 0.5$.

It is obvious that for $r > 0$, the $\mathcal{I}^r(\mathcal{G})-$limit of a sequence is not necessarily unique. As a result, our main interest is to deal with the case $r > 0$. Therefore, we construct $\mathcal{I}^r(\mathcal{G})-$limit set of a sequence $w = (w_k)$, demonstrated and determined as follows:

$$\mathcal{I} - LIM_w^r(\mathcal{G}) = \left\{w_0 \in Y : w_k \xrightarrow{\mathcal{I}^r-\|\cdot\|_{\mathcal{G}}} w_0\right\}.$$

Theorem 3.3. *Let (w_k) and (q_k) be two sequences in $(Y, \|\cdot\|_{\mathcal{G}})$ so that $w_k \xrightarrow{\mathcal{I}^{r_1}-\|\cdot\|_{\mathcal{G}}} w_0$ and $q_k \xrightarrow{\mathcal{I}^{r_2}-\|\cdot\|_{\mathcal{G}}} q_0$. Then, (i) $w_k + q_k \xrightarrow{\mathcal{I}^{(r_1+r_2)}-\|\cdot\|_{\mathcal{G}}} w_0 + q_0$ and (ii) $\mu w_k \xrightarrow{\mathcal{I}^{|\mu| r_1}-\|\cdot\|_{\mathcal{G}}} \mu w_0$ for any $\mu \in \mathbb{R}$.*

Proof. (i) Since, $w_k \xrightarrow{\mathcal{I}^{r_1}-\|\cdot\|_{\mathcal{G}}} w_0$ and $q_k \xrightarrow{\mathcal{I}^{r_2}-\|\cdot\|_{\mathcal{G}}} q_0$, so for any $\rho \in (0, 1]$ and $\gamma > 0$, $P, Q \in \mathcal{I}$, where

$$P = \left\{k \in \mathbb{N} : \mathcal{F}_{\|w_k - w_0\|_{\mathcal{G}}}(\rho) \geq r_1 + \frac{\gamma}{2}\right\} \text{ and } Q = \left\{k \in \mathbb{N} : \mathcal{F}_{\|q_k - q_0\|_{\mathcal{G}}}(\rho) \geq r_2 + \frac{\gamma}{2}\right\}.$$

As the inclusion

$$(\mathbb{N}\setminus P)\cap(\mathbb{N}\setminus Q)\subseteq\left\{k\in\mathbb{N}:\mathcal{F}_{||(w_k+q_k)-(w_0+q_0)||_{\mathcal{G}}}(\rho)<r_1+r_2+\gamma\right\}$$

supplies, so we have to obtain

$$\left\{k\in\mathbb{N}:\mathcal{F}_{||(w_k+q_k)-(w_0+q_0)||_{\mathcal{G}}}(\rho)\geq r_1+r_2+\gamma\right\}\in\mathcal{I}$$

and as a result, $w_k+q_k\xrightarrow{\mathcal{I}^{(r_1+r_2)}-||\cdot||_{\mathcal{G}}}w_0+q_0$.

(ii) When $\mu=0$, then there is nothing to prove. So let us presume that $\mu\neq 0$. Now as the situations

$$\mathcal{F}_{||w_k-w_0||_{\mathcal{G}}}(\rho)\leq r_1 \text{ and } \mathcal{F}_{||\mu w_k-\mu w_0||_{\mathcal{G}}}(\rho)\leq|\mu|r_1$$

are equivalent in gradual normed algebras, so the result follows. □

Now for $r_1=r_2=0$, the above theorem reduces to the following result of [11]:

Corollary 3.4. *Let (w_k) and (q_k) be two sequences in $(Y,||\cdot||_{\mathcal{G}})$ so that $w_k\xrightarrow{\mathcal{I}-||\cdot||_{\mathcal{G}}}w_0$ and $q_k\xrightarrow{\mathcal{I}-||\cdot||_{\mathcal{G}}}q_0$. Then, (i) $w_k+q_k\xrightarrow{\mathcal{I}-||\cdot||_{\mathcal{G}}}w_0+q_0$ and (ii) $\mu w_k\xrightarrow{\mathcal{I}-||\cdot||_{\mathcal{G}}}\mu w_0$ for any $\mu\in\mathbb{R}$.*

Theorem 3.5. *Take $(w_k)\in(Y,||\cdot||_{\mathcal{G}})$. Then,*
$diam(\mathcal{I}-LIM^r_w(\mathcal{G}))=\sup\left\{\mathcal{F}_{||q-t||_{\mathcal{G}}}(\rho):q,t\in\mathcal{I}-LIM^r_w(\mathcal{G}),\rho\in[0,1)\right\}\leq 2r.$
In general, $diam(\mathcal{I}-LIM^r_w(\mathcal{G}))$ has no smaller bound.

Proof. If possible, let us suppose that $diam(\mathcal{I}-LIM^r_w(\mathcal{G}))>2r$. Afterwards, there are $q_0,t_0\in\mathcal{I}-LIM^r_w(\mathcal{G})$ and $\rho_0\in[0,1)$ so that $\mathcal{F}_{||q_0-t_0||_{\mathcal{G}}}(\rho_0)>2r$. Select $\gamma>0$ in such a manner that

$$\gamma<\frac{\mathcal{F}_{||q_0-t_0||_{\mathcal{G}}}(\rho_0)}{2}-r. \tag{1}$$

Since, $q_0,t_0\in\mathcal{I}-LIM^r_w(\mathcal{G})$, so for any $\rho\in(0,1]$ and $\gamma>0$, $A,B\in\mathcal{I}$, where

$$A=\left\{k\in\mathbb{N}:\mathcal{F}_{||w_k-q_0||_{\mathcal{G}}}(\rho)\geq r+\gamma\right\}\text{ and }B=\left\{k\in\mathbb{N}:\mathcal{F}_{||w_k-t_0||_{\mathcal{G}}}(\rho)\geq r+\gamma\right\}.$$

As a result, $(\mathbb{N}\setminus A)\cap(\mathbb{N}\setminus B)\in F(\mathcal{I})$ is non-empty. Take $p\in(\mathbb{N}\setminus A)\cap(\mathbb{N}\setminus B)$. At that time, we acquire

$$\mathcal{F}_{||q_0-t_0||_{\mathcal{G}}}(\rho_0)\leq\mathcal{F}_{||w_p-q_0||_{\mathcal{G}}}(\rho_0)+\mathcal{F}_{||w_p-t_0||_{\mathcal{G}}}(\rho_0)<2(r+\gamma),$$

which contradicts (1).

For the second part, presume (w_k) be a sequence in $(Y,||\cdot||_{\mathcal{G}})$ so that $w_k\xrightarrow{\mathcal{I}-||\cdot||_{\mathcal{G}}}w_0$. As a result, for any $\rho\in(0,1]$ and $\gamma>0$,

$$\left\{k\in\mathbb{N}:\mathcal{F}_{||w_k-w_0||_{\mathcal{G}}}(\rho)\geq\gamma\right\}\in\mathcal{I}.$$

Now for each $q_0\in(w_0+\bar{N}(r,\rho))=\left\{w\in Y:\mathcal{F}_{||w_0-w||_{\mathcal{G}}}(\rho)\leq r\right\}$, the following inequation

$$\mathcal{F}_{||w_k-q_0||_{\mathcal{G}}}(\rho)\leq\mathcal{F}_{||w_k-w_0||_{\mathcal{G}}}(\rho)+\mathcal{F}_{||w_0-q_0||_{\mathcal{G}}}(\rho)<r+\gamma,$$

supplies whenever $k\notin\left\{k\in\mathbb{N}:\mathcal{F}_{||w_k-w_0||_{\mathcal{G}}}(\rho)\geq\gamma\right\}$. This shows that $q_0\in\mathcal{I}-LIM^r_w(\mathcal{G})$ and subsequently

$$\mathcal{I}-LIM^r_w(\mathcal{G})=(w_0+\bar{N}(r,\rho))$$

supplies. Since, $diam(w_0+\bar{N}(r,\rho))=2r$, so in general upper bound $2r$ of the diameter of the set $\mathcal{I}-LIM^r_w(\mathcal{G})$ cannot be decreased anymore. □

Taking $r = 0$ in the above theorem, we can get the subsequent result of [11]:

Corollary 3.6. *Let $w = (w_k)$ be a sequence in $(Y, \|\cdot\|_{\mathcal{G}})$ so that $w_k \xrightarrow{\mathcal{I}-\|\cdot\|_{\mathcal{G}}} w_0$. Then, w_0 is uniquely determined.*

Theorem 3.7. *A sequence $w = (w_k)$ in $(Y, \|\cdot\|_{\mathcal{G}})$ is gradually $\mathcal{I}-$bounded iff there exists some $r \geq 0$ so that $\mathcal{I} - LIM_w^r(\mathcal{G}) \neq \emptyset$.*

Proof. Let $w = (w_k)$ be gradually $\mathcal{I}-$bounded. Then, for each $\rho \in (0, 1]$, there exists $M(= M(\rho)) > 0$ so that

$$A \in \mathcal{I}, \text{ where } A = \left\{k \in \mathbb{N} : \mathcal{F}_{\|w_k\|_{\mathcal{G}}}(\rho) > M\right\}.$$

Suppose

$$B = \sup\left\{\mathcal{F}_{\|w_k\|_{\mathcal{G}}}(\rho) : k \in \mathbb{N} \setminus A, \rho \in [0, 1)\right\}.$$

Then, the set $\mathcal{I} - LIM_w^B(\mathcal{G})$ includes the zero vector of X and eventually

$$\mathcal{I} - LIM_w^B(\mathcal{G}) \neq \emptyset.$$

Conversely presume that $\mathcal{I} - LIM_w^r(\mathcal{G}) \neq \emptyset$ for some $r \geq 0$. At that time, for $w_0 \in \mathcal{I} - LIM_w^M(\mathcal{G})$,

$$\left\{k \in \mathbb{N} : \mathcal{F}_{\|w_k - w_0\|_{\mathcal{G}}}(\rho) \geq r + \gamma\right\} \in \mathcal{I}$$

supplies for any $\rho \in (0, 1]$ and $\gamma > 0$. This implies that x is gradually $\mathcal{I}-$bounded. □

Theorem 3.8. *Let $(w_k) \in (Y, \|\cdot\|_{\mathcal{G}})$. When $q_0 \in \mathcal{I} - LIM_w^{r_0}(\mathcal{G})$ and $q_1 \in \mathcal{I} - LIM_w^{r_1}(\mathcal{G})$, then*

$$q_\lambda = (1 - \lambda)q_0 + \lambda q_1 \in \mathcal{I} - LIM_w^{(1-\lambda)r_0+\lambda r_1}(\mathcal{G}), \text{ whenever } \lambda \in [0, 1].$$

Proof. Since, $q_0 \in \mathcal{I} - LIM_w^{r_0}(\mathcal{G})$ and $q_1 \in \mathcal{I} - LIM_w^{r_1}(\mathcal{G})$, so for each $\rho \in (0, 1]$ and $\gamma > 0$, $A, B \in \mathcal{I}$, where

$$A = \left\{k \in \mathbb{N} : \mathcal{F}_{\|w_k - q_0\|_{\mathcal{G}}}(\rho) \geq r_0 + \gamma\right\} \text{ and } B = \left\{k \in \mathbb{N} : \mathcal{F}_{\|w_k - q_1\|_{\mathcal{G}}}(\rho) \geq r_1 + \gamma\right\}.$$

Subsequently, for any $k \in (\mathbb{N} \setminus A) \cap (\mathbb{N} \setminus B)$,

$$\begin{aligned}\mathcal{F}_{\|w_k - q_\lambda\|_{\mathcal{G}}}(\rho) &\leq (1 - \lambda)\mathcal{F}_{\|w_k - q_0\|_{\mathcal{G}}}(\rho) + \lambda\mathcal{F}_{\|w_k - q_1\|_{\mathcal{G}}}(\rho) \\ &< (1 - \lambda)(r_0 + \gamma) + \lambda(r_1 + \gamma) \\ &= (1 - \lambda)r_0 + \lambda r_1 + \gamma.\end{aligned}$$

This demonstrates that,

$$\left\{k \in \mathbb{N} : \mathcal{F}_{\|w_k - q_\lambda\|_{\mathcal{G}}}(\rho) \geq (1 - \lambda)r_0 + \lambda r_1 + \gamma\right\} \subseteq A \cup B.$$

Now since the set in the right-hand side of the above inclusion belongs to $\mathcal{I}$, hence the set in the left-hand side belongs to $\mathcal{I}$. Hence, $q_\lambda \in \mathcal{I} - LIM_w^{(1-\lambda)r_0+\lambda r_1}(\mathcal{G})$. □

Corollary 3.9. *Let $(w_k) \in (Y, \|\cdot\|_{\mathcal{G}})$. Then, the set $\mathcal{I} - LIM_w^r(\mathcal{G})$ is convex.*

Theorem 3.10. *Let $(w_k) \in (Y, \|\cdot\|_{\mathcal{G}})$. Then, the set $\mathcal{I} - LIM_w^r(\mathcal{G})$ is gradually closed.*

Proof. Let $q = (q_k) \in \mathcal{I} - LIM_w^r(\mathcal{G})$. It supplies

$$q_k \xrightarrow{\|\cdot\|_{\mathcal{G}}} q_0.$$

Then, for each $\rho \in (0, 1]$ and $\gamma > 0$, there is an $N(= N_\gamma(\rho)) \in \mathbb{N}$ so that for all $k \geq N$,

$$\mathcal{F}_{\|q_k - q_0\|_{\mathcal{G}}}(\rho) < \frac{\gamma}{2}.$$

Select $k_0 \in \mathbb{N}$ such that $k_0 \geq N$. Then, $\mathcal{F}_{||y_{k_0}-q_0||_{\mathcal{G}}}(\rho) < \frac{\gamma}{2}$. On the other hand, since $(q_k) \subseteq \mathcal{I} - LIM^r_w(\mathcal{G})$, we must have

$$\left\{k \in \mathbb{N} : \mathcal{F}_{||w_k-y_{k_0}||_{\mathcal{G}}}(\rho) \geq r + \frac{\gamma}{2}\right\} \in \mathcal{I}. \tag{2}$$

Suppose $p \notin \left\{k \in \mathbb{N} : \mathcal{F}_{||w_k-y_{k_0}||_{\mathcal{G}}}(\rho) \geq r + \frac{\gamma}{2}\right\}$. Then, $\mathcal{F}_{||x_p-y_{k_0}||_{\mathcal{G}}}(\rho) < r + \frac{\gamma}{2}$ and eventually

$$\mathcal{F}_{||x_p-q_0||_{\mathcal{G}}}(\rho) \leq \mathcal{F}_{||x_p-y_{k_0}||_{\mathcal{G}}}(\rho) + \mathcal{F}_{||y_{k_0}-q_0||_{\mathcal{G}}}(\rho) < r + \gamma.$$

This gives that $p \notin \left\{k \in \mathbb{N} : \mathcal{F}_{||w_k-q_0||_{\mathcal{G}}}(\rho) \geq r + \gamma\right\}$ and subsequently from (2) we acquire

$$\left\{k \in \mathbb{N} : \mathcal{F}_{||w_k-q_0||_{\mathcal{G}}}(\rho) \geq r + \gamma\right\} \in \mathcal{I}.$$

Hence, $q_0 \in \mathcal{I} - LIM^r_w(\mathcal{G})$ and the proof ends. □

Theorem 3.11. *Let $r_1 \geq 0$ and $r_2 \geq 0$. A sequence $w = (w_k)$ in a GNLS $(Y, ||\cdot||_{\mathcal{G}})$ is $\mathcal{I}^{(r_1+r_2)}(\mathcal{G})$−convergent to w_0 if and only if there is a sequence $q = (q_k)$ such that*

$$q_k \xrightarrow{\mathcal{I}^{r_1}-||\cdot||_{\mathcal{G}}} w_0 \text{ and } \mathcal{F}_{||w_k-q_k||_{\mathcal{G}}}(\rho) \leq r_2$$

for all $k \in \mathbb{N}$.

Proof. Let us assume that $q_k \xrightarrow{\mathcal{I}^{r_1}-||\cdot||_{\mathcal{G}}} w_0$. Then, according to definition for any $\rho \in (0, 1]$ and $\gamma > 0$,

$$P \in \mathcal{I}, \text{ where } P = \left\{k \in \mathbb{N} : \mathcal{F}_{||q_k-w_0||_{\mathcal{G}}}(\rho) \geq r_1 + \gamma\right\}.$$

Now since $\mathcal{F}_{||w_k-q_k||_{\mathcal{G}}}(\rho) \leq r_2$ supplies for all $k \in \mathbb{N}$, so for all $k \notin P$,

$$\mathcal{F}_{||w_k-w_0||_{\mathcal{G}}}(\rho) \leq \mathcal{F}_{||w_k-q_k||_{\mathcal{G}}}(\rho) + \mathcal{F}_{||q_k-w_0||_{\mathcal{G}}}(\rho) < r_1 + r_2 + \gamma.$$

This implies that

$$\left\{k \in \mathbb{N} : \mathcal{F}_{||w_k-w_0||_{\mathcal{G}}}(\rho) \geq r_1 + r_2 + \gamma\right\} \subseteq P$$

and eventually by the property of ideal,

$$\left\{k \in \mathbb{N} : \mathcal{F}_{||w_k-w_0||_{\mathcal{G}}}(\rho) \geq r_1 + r_2 + \gamma\right\} \in \mathcal{I}.$$

Hence, $w_k \xrightarrow{\mathcal{I}^{(r_1+r_2)}-||\cdot||_{\mathcal{G}}} w_0$.

For the converse part, let us suppose that

$$w_k \xrightarrow{\mathcal{I}^{(r_1+r_2)}-||\cdot||_{\mathcal{G}}} w_0. \tag{3}$$

Define $q = (q_k)$ by

$$q_k = \begin{cases} w_0, & if\ \mathcal{F}_{||w_k-w_0||_{\mathcal{G}}}(\rho) \leq r_2 \\ w_k + r_2 \frac{w_0-w_k}{\mathcal{F}_{||w_k-w_0||_{\mathcal{G}}}(\rho)}, & otherwise. \end{cases}$$

Then, it is easy to observe that $\mathcal{F}_{||w_k-q_k||_{\mathcal{G}}}(\rho) \leq r_2$ for all $k \in \mathbb{N}$.

Moreover,

$$\mathcal{F}_{\left\|q_k-w_0\right\|_{\mathcal{G}}}(\rho)=\begin{cases}0, & if\ \mathcal{F}_{\|w_k-w_0\|_{\mathcal{G}}}(\rho)\leq r_2\\ \mathcal{F}_{\|w_k-w_0\|_{\mathcal{G}}}(\rho)-r_2, & otherwise.\end{cases}$$

By (3), for each $\rho\in(0,1]$ and $\gamma>0$,

$$\left\{k\in\mathbb{N}:\mathcal{F}_{\|w_k-w_0\|_{\mathcal{G}}}(\rho)\geq r_1+r_2+\gamma\right\}\in\mathcal{I}.$$

Now as the inclusion

$$\left\{k\in\mathbb{N}:\mathcal{F}_{\|w_k-w_0\|_{\mathcal{G}}}(\rho)\geq r_1+r_2+\gamma\right\}\supseteq\left\{k\in\mathbb{N}:\mathcal{F}_{\left\|q_k-w_0\right\|_{\mathcal{G}}}(\rho)\geq r_1+\gamma\right\}$$

supplies, so we have to obtain

$$\left\{k\in\mathbb{N}:\mathcal{F}_{\left\|q_k-w_0\right\|_{\mathcal{G}}}(\rho)\geq r_1+\gamma\right\}\in\mathcal{I}.$$

Hence, $q_k\xrightarrow{\mathcal{I}^{r_1}-\|\cdot\|_{\mathcal{G}}}w_0$ and the proof ends. □

Corollary 3.12. *A sequence $(w_k)\in(Y,\|\cdot\|_{\mathcal{G}})$ is $\mathcal{I}^r(\mathcal{G})-$convergent to $w_0\in Y$ with roughness degree $r\geq 0$ iff there is a sequence $q=(q_k)$ in Y so that $w_k\xrightarrow{\mathcal{I}-\|\cdot\|_{\mathcal{G}}}w_0$ and $\mathcal{F}_{\left\|w_k-q_k\right\|}\leq r$ for all $k\in\mathbb{N}$.*

Theorem 3.13. *Presume $(Y,\|\cdot\|)$ be a normed linear space and suppose $f:(0,1]\to\mathbb{R}^+$ be a non-zero function. In [16], it was demonstrated that the map $\mathcal{F}_{\|x\|_{\mathcal{G}}}:(0,1]\to\mathbb{R}^+$ determined by*
$\mathcal{F}_{\|x\|_{\mathcal{G}}}(\rho)=f(\rho)\|x\|, x\in Y$
defines an GN on Y. For any sequence (w_k) in Y,
(i) If $w_k\xrightarrow{\mathcal{I}^r-\|\cdot\|}w_0$, then $w_k\xrightarrow{\mathcal{I}^r-\|\cdot\|_{\mathcal{G}}}w_0$.
(ii) If $w_k\xrightarrow{\mathcal{I}^r-\|\cdot\|_{\mathcal{G}}}w_0$ and there exists a $\rho_0\in(0,1]$ such that $f(\rho_0)=1$, then $w_k\xrightarrow{\mathcal{I}^{r'}-\|\cdot\|}w_0$ for some $r'\geq 0$.

Proof. (i) Since $w_k\xrightarrow{\mathcal{I}^r-\|\cdot\|}w_0$, so for any $\gamma>0$ and $\rho_0\in(0,1]$,

$$\left\{k\in\mathbb{N}:\|w_k-w_0\|\geq r+\tfrac{\gamma}{f(\rho_0)}\right\}\in\mathcal{I}.$$

So, the following inequation

$$\mathcal{F}_{\|w_k-w_0\|_{\mathcal{G}}}(\rho_0)=f(\rho_0)\|w_k-w_0\|<rf(\rho_0)+\gamma$$

supplies for any $k\notin\left\{k\in\mathbb{N}:\|w_k-w_0\|\geq r+\frac{\gamma}{f(\rho_0)}\right\}$ and the result follows by taking $r'=rf(\rho_0)$.
(ii) Since $w_k\xrightarrow{\mathcal{I}^r-\|\cdot\|_{\mathcal{G}}}w_0$, so for any $\gamma>0$ and $\rho\in(0,1]$,

$$\{k\in\mathbb{N}:\mathcal{F}_{\|w_k-w_0\|_{\mathcal{G}}}(\rho)\geq r+\gamma\}\in\mathcal{I}.$$

Especially, for $\rho=\rho_0$, the following inequation

$$\|w_k-w_0\|=f(\rho_0)\|w_k-w_0\|=\mathcal{F}_{\|w_k-w_0\|_{\mathcal{G}}}(\rho_0)<r+\gamma$$

supplies for any $k\notin\left\{k\in\mathbb{N}:\mathcal{F}_{\|w_k-w_0\|_{\mathcal{G}}}(\rho)\geq r+\gamma\right\}$ and the rest follows from the hereditary property of an ideal. □

4. Rough $\mathcal{I}^*$−convergence in GNLS

The main reason behind the study of rough $\mathcal{I}^*$−convergence can be described in a similar way of rough $\mathcal{I}$−convergence. So in this section, our main objective is to investigate the relationship between rough $\mathcal{I}$ and rough $\mathcal{I}^*$−convergence of a sequence in a GNLS for a given roughness degree $r \geq 0$. We begin with the following definition:

Definition 4.1. *Suppose that $\mathcal{I}$ be an admissible ideal and take $(w_k) \in (Y, \|\cdot\|_{\mathcal{G}})$. Then, (w_k) is named to be gradually rough $\mathcal{I}^*$−convergent (in short $\mathcal{I}^{*,r}(\mathcal{G})$− convergent) to $w_0 \in Y$ provided that there is a set $M = \{s_1 < s_2 < ... < s_k < ...\} \in \mathcal{F}(\mathcal{I})$ such that the subsequence (w_{s_k}) is $r(\mathcal{G})$−convergent to w_0. We denote $w_k \xrightarrow{\mathcal{I}^{*,r}-\|\cdot\|_{\mathcal{G}}} w_0$.*

Here r is contemplated as the degree of roughness. For $r = 0$, the above definition reduces to Definition 2.16. But our main goal is to deal with the case $r > 0$. Because for $r > 0$, like $\mathcal{I}^r(\mathcal{G})$−limit, the $\mathcal{I}^{*,r}(\mathcal{G})$−limit of a sequence is also not necessarily unique. Hence, we establish $\mathcal{I}^{*,r}(\mathcal{G})$−limit set of a sequence $w = (w_k)$, demonstrated and determined as follows:

$$\mathcal{I}^* - LIM^r_w(\mathcal{G}) = \left\{ w_0 \in Y : w_k \xrightarrow{\mathcal{I}^{*,r}-\|\cdot\|_{\mathcal{G}}} w_0 \right\}.$$

Theorem 4.2. *Let $(w_k) \in (Y, \|\cdot\|_{\mathcal{G}})$. Then, $w_k \xrightarrow{\mathcal{I}^{*,r}-\|\cdot\|_{\mathcal{G}}} w_0$ implies $w_k \xrightarrow{\mathcal{I}^{r}-\|\cdot\|_{\mathcal{G}}} w_0$.*

Proof. Since $w_k \xrightarrow{\mathcal{I}^{*,r}-\|\cdot\|_{G}} w_0$, so there is an $M = \{s_1 < s_2 < ... < s_k < ...\} \in \mathcal{F}(\mathcal{I})$ so that for each $\gamma > 0$ and $\rho \in (0, 1]$, there is an $N(= N_\gamma(\rho)) \in \mathbb{N}$ so that for all $k > N$,

$$\mathcal{F}_{\|w_{s_k} - w\|_{\mathcal{G}}}(\rho) < r + \gamma.$$

As $\mathcal{I}$ is admissible, we must get

$$\left\{ k \in \mathbb{N} : \mathcal{F}_{\|w_k - w\|_{\mathcal{G}}}(\rho) \geq r + \gamma \right\} \subseteq (\mathbb{N} \setminus M) \cup \{s_1, s_2, ..., s_N\} \in \mathcal{I}.$$

Hence, $w_k \xrightarrow{\mathcal{I}^{r}-\|\cdot\|_{\mathcal{G}}} w_0$. □

Corollary 4.3. *[11] Let $(w_k) \in (Y, \|\cdot\|_{\mathcal{G}})$. Then, $w_k \xrightarrow{\mathcal{I}^{*}-\|\cdot\|_{\mathcal{G}}} w_0$ implies $w_k \xrightarrow{\mathcal{I}-\|\cdot\|_{\mathcal{G}}} w_0$.*

The converse of Theorem 4.2 is not necessarily true. The following example indicates this fact.

Example 4.4. *Let $Y = \mathbb{R}$ and $\|\cdot\|_G$ be the norm determined by $\mathcal{F}_{\|w\|_{\mathcal{G}}}(\rho) = e^\rho |w|$. Examine the ideal $\mathcal{I}$ including of whole subsets of $\mathbb{N}$ that intersects finitely many V_l's where $V_l = \{2^{l-1}(2s - 1) : s \in \mathbb{N}\}, l \in \mathbb{N}$ is the decomposition of $\mathbb{N}$ into disjoint subsets i.e., $\mathbb{N} = \bigcup_{l=1}^{\infty} V_l$ and $V_i \cap V_j = \emptyset$ for $i \neq j$. Contemplate the sequence $w = (w_k)$ in $\mathbb{R}$ determined by $w_k = \frac{1}{l^l}$, whenever $k \in V_l$.*

Let r be an arbitrary non-negative real number and suppose $\gamma > 0$ be given. Select $m \in \mathbb{N}$ so that $\gamma > \frac{e^\rho}{m^m}$. Then, the following inclusion

$$\left\{ k \in \mathbb{N} : \mathcal{F}_{\|w_k - w_0\|_{\mathcal{G}}}(\rho) \geq r + \gamma \right\} \subseteq \bigcup_{l=1}^{m} V_l$$

supplies for any $w_0 \in [-r, r]$. Hence, $[-r, r] \subseteq \mathcal{I} - LIM^r_w(\mathcal{G})$.

Now we claim that $-r \notin \mathcal{I}^* - LIM^r_w(\mathcal{G})$. To demonstrate the claim, we assume the contrary. For any $H \in \mathcal{I}$ there is an $q \in \mathbb{N}$ so that $H \subseteq \bigcup_{j=1}^{q} V_j$ and then $V_{q+1} \subseteq \mathbb{N} \setminus H$. Let $M = \{s_1 < s_2 < ... < s_k < ...\}$ indicate the set $\mathbb{N} \setminus H$, afterwards $M \in \mathcal{F}(\mathcal{I})$ and $w_{s_k} = \frac{1}{(q+1)^{q+1}}$ whenever $s_k \in V_{q+1}$. So, for $\gamma = \frac{1}{(q+2)^{q+1}}$ and $s_k \in V_{q+1}$, $\mathcal{F}_{\|w_{s_k} + r\|_G}(\rho) \geq r + \gamma$ supplies for infinitely many k's, which is a contradiction.

Theorem 4.5. *Let $\mathcal{I}$ be an admissible ideal in $\mathbb{N}$ which satisfies the condition (AP). Afterwards, for a sequence $(w_k) \in (Y, \|\cdot\|_{\mathcal{G}})$, $w_k \xrightarrow{\mathcal{I}^r-\|\cdot\|_{\mathcal{G}}} w_0$ implies $w_k \xrightarrow{\mathcal{I}^{*,r}-\|\cdot\|_{\mathcal{G}}} w_0$.*

Proof. Since $w_k \xrightarrow{\mathcal{I}^r-\|\cdot\|_{\mathcal{G}}} w_0$, so for each $\rho \in (0,1]$ and $\eta > 0$,

$$P(\rho,\eta) = \left\{k \in \mathbb{N} : \mathcal{F}_{\|w_k-w_0\|_G}(\rho) \geq r+\eta\right\} \in \mathcal{I}.$$

We construct a countable family of mutually disjoint sets $\{P_l(\rho)\}_{l\in\mathbb{N}}$ in $\mathcal{I}$ by considering

$$P_1(\rho) = \left\{k \in \mathbb{N} : \mathcal{F}_{\|w_k-w_0\|_{\mathcal{G}}}(\rho) \geq r+1\right\}$$

and

$$P_l(\rho) = \left\{k \in \mathbb{N} : \frac{1}{l} \leq \mathcal{F}_{\|w_k-w_0\|_{\mathcal{G}}}(\rho) - r < \frac{1}{l-1}\right\} = P\left(\rho, \frac{1}{l}\right) \setminus P\left(\rho, \frac{1}{l-1}\right), \text{ for } l \geq 2.$$

As $\mathcal{I}$ supplies the condition (AP), so there exists another countable family of subsets $\{Q_l(\rho)\}_{l\in\mathbb{N}}$ of $\mathbb{N}$ so that

$$P_t(\rho)\triangle Q_t(\rho) \text{ is finite } \forall t \in \mathbb{N} \text{ and } Q(\rho) = \bigcup_{t=1}^{\infty} Q_t(\rho) \in \mathcal{I}. \tag{4}$$

Let $\gamma > 0$ be given. Select $l \in \mathbb{N}$ such that $\frac{1}{l+1} < \gamma$. Then, the following inclusion

$$\left\{k \in \mathbb{N} : \mathcal{F}_{\|w_k-w_0\|_{\mathcal{G}}}(\rho) \geq r+\gamma\right\} \subseteq \left\{k \in \mathbb{N} : \mathcal{F}_{\|w_k-w_0\|_{\mathcal{G}}}(\rho) \geq r + \frac{1}{l+1}\right\} = \bigcup_{t=1}^{l+1} P_t(\rho) \in \mathcal{I}$$

supplies and eventually by virtue of (4), there is an integer $k' \in \mathbb{N}$, so that

$$\bigcup_{t=1}^{l+1} Q_t(\rho) \cap (k',\infty) = \bigcup_{t=1}^{l+1} P_t(\rho) \cap (k',\infty).$$

Select $k \in \mathbb{N} \setminus Q(\rho) \in \mathcal{F}(\mathcal{I})$ so that $k > k'$. At that time, we have to obtain $k \notin \bigcup_{t=1}^{l+1} P_t(\rho)$ and consequently

$$\mathcal{F}_{\|w_k-w_0\|_{\mathcal{G}}}(\rho) < r + \tfrac{1}{l+1} < r+\gamma.$$

Hence, $w_k \xrightarrow{\mathcal{I}^{*,r}-\|\cdot\|_{\mathcal{G}}} w_0$, and the proof ends. □

We end up by stating the following theorems without proof as those are quite similar for the case of $\mathcal{I}^r(\mathcal{G})$−convergence.

Theorem 4.6. *Suppose (w_k) and (q_k) be two sequences in $(Y, \|\cdot\|_{\mathcal{G}})$ so that $w_k \xrightarrow{\mathcal{I}^{*,r_1}-\|\cdot\|_{\mathcal{G}}} w_0$ and $q_k \xrightarrow{\mathcal{I}^{*,r_2}-\|\cdot\|_{\mathcal{G}}} q_0$. Then, (i) $w_k + q_k \xrightarrow{\mathcal{I}^{*,(r_1+r_2)}-\|\cdot\|_{\mathcal{G}}} w_0 + q_0$ and (ii) $\mu w_k \xrightarrow{\mathcal{I}^{*,|\mu| r_1}-\|\cdot\|_{\mathcal{G}}} \mu w_0$ for any $\mu \in \mathbb{R}$.*

Theorem 4.7. *Let $(w_k) \in (Y, \|\cdot\|_{\mathcal{G}})$. Then, the set $\mathcal{I}^* - LIM_w^r(\mathcal{G})$ is convex.*

Theorem 4.8. *Let $(w_k) \in (Y, \|\cdot\|_{\mathcal{G}})$. Then, the set $\mathcal{I}^* - LIM_w^r(\mathcal{G})$ is gradually closed.*

5. Conclusion

In this work, we generalize the notion of $\mathcal{I}$ and $\mathcal{I}^*-$convergence to rough $\mathcal{I}$ and $\mathcal{I}^*-$convergence in an GNLS. We mainly examine several properties of the limit sets $\mathcal{I}-LIM_w^r(\mathcal{G})$ and $\mathcal{I}^*-LIM_w^r(\mathcal{G})$. Moreover, for a given roughness degree $r \geq 0$, we establish Theorem 4.2 and Theorem 4.5 which reveals the relationship between $\mathcal{I}^r(\mathcal{G})-$convergence and $\mathcal{I}^{*,r}(\mathcal{G})-$convergence of a sequence $w = (w_k)$ in a GNLS. In future, one may investigate the notion of rough ideal Cauchy sequences and the obtained results may be helpful to study the interrelationships between the notions.

6. Acknowledgements

The authors thank the anonymous referees for their constructive suggestions to improve the quality of the paper.

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