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On the monoid of all order-decreasing partial transformations

Ping Zhao[a,b], Huabi Hu[a,*]

[a]School of Biology and Engineering, Guizhou Medical University, Guiyang 50004, Guizhou Province, China
[b]School of Mathematical Sciences, Guizhou Normal University, Guiyang 550001, Guizhou Province, China

Abstract. A partial transformation α on an n-element set $\mathbf{n} = \{1, \ldots, n\}$ is called order-decreasing if $x\alpha \leq x$ for all $x \in \mathrm{dom}(\alpha)$. The set of all partial order-decreasing transformations on $\mathbf{n}$ forms a monoid $\mathcal{PD}_n$. In this paper, we determine the maximal subsemigroups as well as the maximal idempotent generated subsemigroups of $\mathcal{PD}_n$. Furthermore, we investigate the abundance of the ideals of $\mathcal{PD}_n$, and characterize the structure of the left (right) abundant principal ideal of $\mathcal{PD}_n$.

1. Introduction and preliminaries

Fix a positive integer n. We write $\mathbf{n}$ for the finite set $\{1, \ldots, n\}$. We denote by $\mathcal{PT}_n$ the monoid of all partial transformations of $\mathbf{n}$ and by $\mathcal{T}_n$ the monoid of all full transformations of $\mathbf{n}$. We say that a transformation $\alpha \in \mathcal{PT}_n$ is *order-preserving* [*order-reversing*] if $x \leq y$ implies $x\alpha \leq y\alpha$ [$x\alpha \geq y\alpha$], for all $x, y \in \mathrm{dom}(\alpha)$, and α is *decreasing* [*increasing* or *extensive*] if $x\alpha \leq x$ [$x\alpha \geq x$], for all $x \in \mathrm{dom}(\alpha)$. Denote by $\mathcal{O}_n$ the monoid of all order-preserving full transformations, by $\mathcal{PO}_n$ the monoid of all order-preserving partial transformations and by $\mathcal{POE}_n$ the of all order-preserving and extensive partial transformations. We also denote by $\mathcal{D}_n$ the monoid of all order-decreasing full transformations and $\mathcal{PD}_n$ the monoid of all order-decreasing partial transformations.

Let $c = (c_1, c_2, \ldots, c_t)$ be a sequence of t ($t \geq 0$) elements from the set $\mathbf{n}$. We say that c is *cyclic* if there exists no more than one index $i \in \{1, \ldots, t\}$ such that $c_i > c_{i+1}$, where c_{t+1} denotes c_1. Let $\alpha \in \mathcal{PT}_n$ and suppose that $\mathrm{dom}(\alpha) = \{a_1, \ldots, a_t\}$, with $t \geq 0$ and $a_1 < \cdots < a_t$. We say that α is orientation-preserving if the sequence of its image $(a_1\alpha, \ldots, a_t\alpha)$ is cyclic. We denote by $\mathcal{POP}_n$ the submonoid of $\mathcal{PT}_n$ of all partial orientation-preserving transformations and by $\mathcal{OP}_n$ the submonoid $\mathcal{POP}_n \cap \mathcal{T}_n$ of $\mathcal{PT}_n$ of all full orientation-preserving transformations. We also denote by $\mathcal{OPE}_n$ the monoid of orientation-preserving and extensive full transformations and by $\mathcal{POPE}_n$ of all orientation-preserving and extensive partial transformations.

Algebraic, combinatorial, and rank properties of various kinds of transformation semigroups have been studied over a long period and many interesting results have emerged. In particular, Dimitrova and Koppitz

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* Corresponding author: Huabi Hu
Email addresses: pingzhao731108@163.com (Ping Zhao), huhuabi1978@163.com (Huabi Hu)
ORCID iDs: https://orcid.org/0000-0003-2639-9841 (Ping Zhao), https://orcid.org/0000-0002-7152-2629 (Huabi Hu)

[1] (2008) characterized the maximal subsemigroups of the ideals of $\mathcal{O}_n$ as well as of the ideals of $\mathcal{OD}_n$ the monoid of all order-preserving or order-reversing full transformations. Further, Dimitrova and Koppitz [2] (2011) classified the maximal regular subsemigroups of the ideals of $\mathcal{O}_n$. Dimitrova, Fernandes and Koppitz [4] (2011) characterized completely the maximal subsemigroups of the ideals of $\mathcal{OP}_n$. Dimitrova and Koppitz [3] (2012) described the maximal subsemigroups as well as the maximal idempotent generated subsemigroups of $\mathcal{POE}_n$. Zhao et al.[14] (2022) completely determined the maximal subsemigroups as well as the maximal idempotent generated subsemigroups of the ideals of the monoid $\mathcal{POE}_n$. Li, Zhang and Luo [9] (2022) characterized the maximal subsemigroups as well as the maximal idempotent generated subsemigroups of the monoid $\mathcal{OPE}_n$. Recently, Zhao and Hu [15] (2023) completely determined the maximal subsemigroups as well as the maximal idempotent generated subsemigroups of the monoid $\mathcal{POPE}_n$.

In 1986, Pin [10] proved that a finite monoid is $\mathcal{R}$-trivial if and only if it can be embedded in $\mathcal{D}_n$ for some n. In 1992, Umar [11] showed that both the rank and the idempotent rank of the singular subsemigroup of $\mathcal{D}_n$ of all singular order-decreasing full transformations are equal to $\frac{n(n-1)}{2}$. In 2004, Laradji and Umar [8] studied algebraic, combinatorial and rank properties of certain Rees quotient semigroups of $\mathcal{D}_n$. Yağci [13] (2023) investigated the maximum nilpotent subsemigroup of $\mathcal{D}_n$ and determined the minimum generating set as well as the cardinality of the maximum nilpotent subsemigroup of $\mathcal{D}_n$. Recently, Zhao and Hu [16] characterized the maximal subsemigroups as well as the maximal idempotent generated subsemigroups of the monoid $\mathcal{D}_n$.

Regarding the monoid $\mathcal{PD}_n$, Umar [12] studied combinatorial and rank properties of certain Rees quotient semigroups of $\mathcal{PD}_n$. They showed that the ideals $\mathcal{PD}_{n,r} = \{\alpha \in \mathcal{PD}_n : |\operatorname{im}(\alpha)| \leq r\}$ $(1 \leq r \leq n)$ of $\mathcal{PD}_n$ are abundant (see [12, Corollary 2.4.3 and Theorem 2.2.5]). However, the results about algebraic properties of the monoid $\mathcal{PD}_n$ are very few. The main aim of this paper is to study the monoid $\mathcal{PD}_n$. We notice that each ideal of $\mathcal{PD}_n$ is not always the form $\mathcal{PD}_{n,r}$, for $1 \leq r \leq n$, and $\mathcal{PD}_n$ is the principal ideal $\mathcal{PD}_n 1_n \mathcal{PD}_n$ generated by 1_n (the identity transformation on $\mathbf{n}$). In this paper, we determine the maximal subsemigroups as well as the maximal idempotent generated subsemigroups of $\mathcal{PD}_n$ in Sect.2. In Sect.3, we characterize the abundance of the ideals of $\mathcal{PD}_n$. Moreover, we characterize the structure of the left (right) abundant principal ideal of $\mathcal{PD}_n$.

Given a subset A of a semigroup S and $u \in S$, we denote by $E(A)$ the set of idempotents of S belonging to A and by L_u^S and R_u^S the $\mathscr{L}$-class and $\mathscr{R}$-class of u, respectively. For general background on Semigroup Theory, we refer the reader to Howie's book [6].

We denote by $\theta_{\mathbf{n}}$ the empty transformation on $\mathbf{n}$. Let $\alpha \in \mathcal{PT}_n \setminus \{\theta_{\mathbf{n}}\}$, we will write

$$\alpha = \begin{pmatrix} A_1 & \cdots & A_m \\ a_1 & \cdots & a_m \end{pmatrix}$$

to indicate that $\operatorname{dom}(\alpha) = A_1 \sqcup \cdots \sqcup A_m$, $\operatorname{im}(\alpha) = \{a_1, \ldots, a_m\}$ and $A_i\alpha = a_i$ for each $i \in \{1, \ldots, m\}$ (the symbol "$\sqcup$" denotes disjoint union). Usually this notation will imply that $a_1, \cdots, a_m$ are distinct, but occasionally this will not be the case, and we will always specify this. As usual, we denote the *kernel* of $\alpha \in \mathcal{PT}_n \setminus \{\theta_{\mathbf{n}}\}$ by

$$\ker(\alpha) = \{(x, y) \in \operatorname{dom}(\alpha) \times \operatorname{dom}(\alpha) : x\alpha = y\alpha\}.$$

We will sometimes write $\ker(\alpha) = (A_1|\ldots|A_m)$ to indicate that $\ker(\alpha)$ has equivalence classes $A_1, \ldots, A_m$, and this notation will always imply that A_i are pairwise disjoint and non-empty.

Let $\alpha \in \mathcal{PD}_n$ with $|\operatorname{im}(\alpha)| = r \geq 2$. Then α can be expressed as

$$\alpha = \left(\begin{array}{c|c|c|c} A_1 & A_2 & \cdots & A_r \\ a_1 & a_2 & \cdots & a_r \end{array}\right),$$

where $a_1 < \cdots < a_r$ and $a_i \leq \min A_i$, for $1 \leq i \leq r$. Notice that if $1 \in A_1$, then $a_1 = 1$. Notice that if $\alpha \in E(\mathcal{PD}_n)$, then $a_i = minA_i$ for $1 \leq i \leq r$.

2. Maximal (idempotent generated) subsemigroups of $\mathcal{PD}_n$

We shall say that a proper subsemigroup S of $\mathcal{PD}_n$ is *maximal subsemigroup* (idempotent generated subsemigroups) if any subsemigroup (idempotent generated subsemigroups) of $\mathcal{PD}_n$ properly containing S must be $\mathcal{PD}_n$. In this section, we describe all maximal subsemigroups and maximal idempotent generated subsemigroups of $\mathcal{PD}_n$. For $1 \leq r \leq n$, put

$$J_r = \{\alpha \in \mathcal{PD}_n : |\operatorname{im}(\alpha)| = r\}, \ \ E_r = E(J_r) \ \ and \ \ \mathcal{D}_{n,r} = \{\alpha \in \mathcal{D}_n : |\operatorname{im}(\alpha)| \leq r\}.$$

Then the sets $\mathcal{PD}_{n,r}$ and $\mathcal{D}_{n,r}$ are the two-sided ideals of $\mathcal{PD}_n$ and $\mathcal{D}_n$, respectively. Clearly, $\mathcal{PD}_{n,r} = J_0 \cup J_1 \cup \cdots \cup J_r$, where J_0 consists of the empty transformation $\theta_{\mathbf{n}}$.

Lemma 2.1. *Let $0 \leq m \leq n-2$. Then $E_m \subseteq \langle E_{m+1}\rangle$.*

Proof. Let $\varepsilon \in E_m$ be arbitrary. To prove that $\varepsilon \in \langle E_{m+1}\rangle$, we distinguish two cases:

Case 1. $m = 0$. Clearly, $\varepsilon = \theta_{\mathbf{n}}$. Put

$$\eta = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \ and \ \ \xi = \begin{pmatrix} 2 \\ 2 \end{pmatrix}.$$

Then $\eta, \xi \in E_1$ and $\varepsilon = \eta\xi$. Thus $\varepsilon = \eta\xi \in \langle E_1\rangle$.

Case 2. $m \geq 1$. We can suppose that

$$\varepsilon = \left(\begin{array}{c|c|c} A_1 & \cdots & A_m \\ a_1 & \cdots & a_m \end{array}\right),$$

where $a_i = minA_i$, for $1 \leq i \leq m$. Notice that $\operatorname{dom}(\varepsilon) = A_1 \sqcup \cdots \sqcup A_m$. Clearly, $|\operatorname{dom}(\varepsilon)| \geq m$. We distinguish two subcases:

Subcase 2.1. $|\operatorname{dom}(\varepsilon)| = n$. Since $m \leq n-2$, there exist $1 \leq p \leq m$ such that $|A_p| \geq 2$. Let $x_p = min(A_p \backslash \{a_p\})$. Take $y \in \mathbf{n}\backslash\{a_1, \ldots, a_m, x_p\}$. Put

$$\eta = \left(\begin{array}{c|c|c|cc|c|c|c} A_1 & \cdots & A_{p-1} & a_p & A_p\backslash\{a_p\} & A_{p+1} & \cdots & A_m \\ a_1 & \cdots & a_{p-1} & a_p & x_p & a_{p+1} & \cdots & a_m \end{array}\right),$$

$$\xi = \left(\begin{array}{c|c|c|c|c|c|c|c} a_1 & \cdots & a_{p-1} & \{a_p, x_p\} & a_{p+1} & \cdots & a_m & y \\ a_1 & \cdots & a_{p-1} & a_p & a_{p+1} & \cdots & a_m & y \end{array}\right).$$

Then $\eta, \xi \in E_{m+1}$ and $\varepsilon = \eta\xi$. Thus $\varepsilon = \eta\xi \in \langle E_{m+1}\rangle$.

Subcase 2.2. $|\operatorname{dom}(\varepsilon)| \leq n-1$. Take $x \in \mathbf{n}\backslash \operatorname{dom}(\varepsilon)$ and $y \in \mathbf{n}\backslash\{a_1, \ldots, a_m, x\}$. Put

$$\eta = \left(\begin{array}{c|c|c|c} A_1 & \cdots & A_m & x \\ a_1 & \cdots & a_m & x \end{array}\right) and \ \xi = \left(\begin{array}{c|c|c|c} a_1 & \cdots & a_m & y \\ a_1 & \cdots & a_m & y \end{array}\right).$$

Then $\eta, \xi \in E_{m+1}$ and $\varepsilon = \eta\xi$. Thus $\varepsilon = \eta\xi \in \langle E_{m+1}\rangle$. $\square$

Lemma 2.2. *Let $0 \leq m \leq n$. Then $J_m \subseteq \langle E_m\rangle$.*

Proof. Notice that $J_0 = E_0 = \{\theta_{\mathbf{n}}\}$ and $J_n = E_n = \{1_n\}$. Then $J_m = \langle E_m \rangle$, for $m = 0, n$. Suppose that $1 \leq m \leq n-1$. Let

$$\alpha = \left(\begin{array}{c|c|c} B_1 & \cdots & B_m \\ b_1 & \cdots & b_m \end{array} \right) \in J_m,$$

where $b_i \leq minB_i$, for $1 \leq i \leq m$. Let $q_i = minB_i$, for $1 \leq i \leq m$. Then $b_i \leq q_i$, for $1 \leq i \leq m$. We denote by $\mathcal{S}_r$ the symmetric group on $\{1, \ldots, m\}$. Then there exists $\sigma \in \mathcal{S}_m$ such that $q_{1\sigma} < q_{2\sigma} < \cdots < q_{m\sigma}$. Thus

$$b_{k\sigma} \leq q_{k\sigma} < \cdots < q_{m\sigma}, \ for \ 1 \leq k \leq m.$$

Put

$$\tau = \begin{pmatrix} B_{1\sigma} & B_{2\sigma} & \cdots & B_{m\sigma} \\ q_{1\sigma} & q_{2\sigma} & \cdots & q_{m\sigma} \end{pmatrix}, \ \tau_1 = \begin{pmatrix} \{b_{1\sigma}, q_{1\sigma}\} & q_{2\sigma} & \cdots & q_{m\sigma} \\ b_{1\sigma} & q_{2\sigma} & \cdots & q_{m\sigma} \end{pmatrix}$$

and

$$\tau_i = \begin{pmatrix} b_{1\sigma} & \cdots & b_{(i-1)\sigma} & \{b_{i\sigma}, q_{i\sigma}\} & q_{(i+1)\sigma} & \cdots & q_{m\sigma} \\ b_{1\sigma} & \cdots & b_{(i-1)\sigma} & b_{i\sigma} & q_{(i+1)\sigma} & \cdots & q_{m\sigma} \end{pmatrix}$$

for $2 \leq i \leq m$. Clearly, $\tau, \tau_1, \ldots, \tau_m \in E_m$. It is easy to verify that

$$\alpha = \tau\tau_1 \ldots \tau_m.$$

Thus $\alpha \in \langle E_m \rangle$. □

Notice that $\mathcal{PD}_{n,r} = J_0 \cup J_1 \cup \cdots \cup J_r$, for $1 \leq r \leq n-1$. As an immediate consequence of Lemmas 2.1 and 2.2, we have the following result:

Lemma 2.3. *Let* $1 \leq r \leq n-1$. *Then* $\mathcal{PD}_{n,r} = \langle E_r \rangle$.

Let $S \in \{\mathcal{T}_n, \mathcal{D}_n\}$. Put

$$J_r^S = \{\alpha \in S : |\operatorname{im}(\alpha)| = r\} \ and \ E_r^S = E(J_r^S).$$

Then $J_r^{\mathcal{D}_n} \subseteq J_r$ and $E_r^{\mathcal{D}_n} \subseteq E_r$. Now, recall that Umar [8, Theorem 1.3] proved:

Lemma 2.4. *Let* $1 \leq r \leq n-1$. *Then* $\mathcal{D}_{n,r} = \langle E_r^{\mathcal{D}_n} \rangle$.

Notice that each idempotent ε of $E_{n-1}^{\mathcal{T}_n}$ has a form $\binom{a}{b}$ for some $a, b \in \mathbf{n}$, $a \neq b$, which maps a to b and x to itself for $x \neq a$. Then

$$E_{n-1}^{\mathcal{D}_n} = \{\begin{pmatrix} i \\ j \end{pmatrix} : i, j \in \mathbf{n} \ with \ i > j\}.$$

For $1 \leq i \leq n$, we denote by δ_i the identity mapping on $X_n \backslash \{i\}$. Put

$$F_{n-1} = \{\delta_i : 1 \leq i \leq n\}.$$

Then $E_{n-1} = E_{n-1}^{\mathcal{D}_n} \sqcup F_{n-1}$.

Let S be a semigroup. We say that an element $\alpha \in S$ is *undecomposable* in S if there are no $\beta, \gamma \in S \backslash \{\alpha\}$ such that $\alpha = \beta\gamma$. Given a subset U of S, we say that U is a *undecomposable subset* of S if each element of U is *undecomposable* in S. Let A be a subset of $\mathbf{n}$. We denote by 1_A the identity mapping on A. Clearly, 1_n is undecomposable in $\mathcal{PD}_n$. In fact, we have the following lemma:

Lemma 2.5. *The elements of the idempotent set* E_{n-1} *are undecomposable in* $\mathcal{PD}_n$.

Proof. Let $\varepsilon \in E_{n-1}$ be arbitrary. Suppose that there exist $\beta, \gamma \in \mathcal{PD}_n \backslash \{\varepsilon\}$ such that $\varepsilon = \beta\gamma$. Notice that

$$E_{n-1} = E^{\mathcal{D}_n}_{n-1} \sqcup F_{n-1}.$$

We distinguish two cases:

Case 1. $\varepsilon \in E^{\mathcal{D}_n}_{n-1}$. Then there exist $i, j \in \mathbf{n}$ with $i > j$ such that $\varepsilon = \binom{i}{j}$. Assume that there exist $\beta, \gamma \in \mathcal{PD}_n \backslash \{\varepsilon\}$ such that $\varepsilon = \beta\gamma$. Clearly, $\mathrm{dom}(\beta) = \mathbf{n}$. Let $x \in \mathbf{n}\backslash\{i\}$. Then $x = x\varepsilon = (x\beta)\gamma \leq x\beta \leq x$. It follows that

$$x\beta = x\gamma = x, \ for \ x \in \mathbf{n}\backslash\{i\}. \tag{2.1}$$

If $i\beta = i$, then $\beta = 1_n$ and so $\gamma = \beta\gamma = \varepsilon$, a contradiction. If $i\beta \neq i$, then, by (2.1), $(i\beta)\gamma = i\beta$ and so $i\beta = i\beta\gamma = i\varepsilon = j$. Thus, by (2.1), $\beta = \binom{i}{j} = \varepsilon$, a contradiction.

Case 2. $\varepsilon \in F_{n-1}$. Then $\varepsilon = \delta_i$ for some $1 \leq i \leq n$. Let $x \in \mathbf{n}\backslash\{i\}$. Then $x = x\varepsilon = (x\beta)\gamma \leq x\beta \leq x$. It follows that

$$x\beta = x\gamma = x, \ for \ x \in \mathbf{n}\backslash\{i\}. \tag{2.2}$$

If $i \notin \mathrm{dom}(\gamma)$, then, by (2.2), $\gamma = \delta_i$, a contradiction. If $i \in \mathrm{dom}(\gamma)$, then $\mathrm{dom}(\gamma) = \mathbf{n}$. It follows from $\delta_i = \varepsilon = \beta\gamma$ and (2.2) that $i \notin \mathrm{dom}(\beta)$. Then, by (2.2), $\beta = \delta_i$, a contradiction. □

We can now present one of the main results of this section.

Theorem 2.6. *Let $n \geq 3$. Then each maximal subsemigroup S of $\mathcal{PD}_n$ must be one of the following forms:*

$$S = \mathcal{PD}_{n,n-1} \ or \ S = \mathcal{PD}_n \backslash \{\varepsilon\}, \ for \ some \ \varepsilon \in E_{n-1}.$$

Proof. Notice that 1_n is undecomposable in $\mathcal{PD}_n$. Let $\varepsilon \in E_{n-1} \cup \{1_n\}$ be arbitrary. Then, by Lemma 2.5, we obtain the set $\mathcal{PD}_n \backslash \{\varepsilon\}$ is a maximal subsemigroup of $\mathcal{PD}_n$. Let S be a maximal subsemigroup of $\mathcal{PD}_n$. Notice that $\mathcal{PD}_n = \mathcal{PD}_{n,n-1} \cup \{1_n\}$. If $1_n \notin S$, then $S \subseteq \mathcal{PD}_{n,n-1} \subset \mathcal{PD}_n$. Thus, by the maximality of S, $S = \mathcal{PD}_{n,n-1}$. If $1_n \in S$, then, by Lemma 2.3 and $S \subset \mathcal{PD}_n$, $E_{n-1} \nsubseteq S$. Then there exists $\varepsilon \in E_{n-1}$ such that $\varepsilon \notin S$. Thus $S \subseteq \mathcal{PD}_n \backslash \{\varepsilon\} \subset \mathcal{PD}_n$. Hence, by the maximality of S, $S = \mathcal{PD}_n \backslash \{\varepsilon\}$. □

For $i, j \in \mathbf{n}$ with $i > j$, put

$$G_{(i,j)} = \{\alpha \in J^{\mathcal{D}_n}_{n-1} : i\alpha \neq j\}.$$

Notice that $E^{\mathcal{D}_n}_{n-1} \backslash \{\binom{i}{j}\} \subseteq G_{(i,j)}$. Recall that Zhao and Hu [16, Lemma 2.6]) proved the following result:

Lemma 2.7. *Let $n \geq 3$. Then $G_{(i,j)} = \langle E^{\mathcal{D}_n}_{n-1} \backslash \{\binom{i}{j}\} \rangle \cap J^{\mathcal{D}_n}_{n-1}$, for $i, j \in \boldsymbol{n}$ with $i > j$.*

A product $\varepsilon_1\varepsilon_2 \ldots \varepsilon_m$ of idempotents in $\mathcal{PT}_n$ will be called *irreducible* if $\varepsilon_i\varepsilon_{i+1} \neq \varepsilon_i$, $\varepsilon_i\varepsilon_{i+1} \neq \varepsilon_{i+1}$ $(i = 1, \ldots, m-1)$. Now, recall that Howie [7, Lemma 4] proved:

Lemma 2.8. *Let $\alpha \in J^{\mathcal{T}_n}_{n-1}$. If $\alpha = \binom{i_1}{j_1}\binom{i_2}{j_2} \cdots \binom{i_m}{j_m}$ is irreducible. Then $i_{r-1} = j_r$ and $j_{r-1} \neq i_r$, for $2 \leq r \leq m$.*

Notice that $J^{\mathcal{D}_n}_{n-1} \subseteq J^{\mathcal{T}_n}_{n-1}$. What is clear is that if a is expressible as a product of idempotents then the product can be 'pruned down' until it is irreducible (see [7, page 2]). Thus, by Lemma 2.8, we immediately deduce the following result:

Lemma 2.9. *Let $I \subseteq E^{\mathcal{D}_n}_{n-1}$. If $\alpha \in \langle I \rangle \cap J^{\mathcal{D}_n}_{n-1}$, then α can be written as*

$$\alpha = \begin{pmatrix} x_1 \\ x_0 \end{pmatrix}\begin{pmatrix} x_2 \\ x_1 \end{pmatrix} \cdots \begin{pmatrix} x_m \\ x_{m-1} \end{pmatrix},$$

where all $\binom{x_{k+1}}{x_k} \in I$, for $0 \leq k \leq m-1$.

For $i, j \in \mathbf{n}$ with $i > j$, put

$$G^{\triangle}_{(i,j)} = \{\alpha \in J_{n-1} : i\alpha \neq j\}, \;\; \Delta_i = \{\alpha \in J_{n-1} : i \notin \mathrm{dom}(\alpha)\}$$

and

$$PG_{(i,j)} = G^{\triangle}_{(i,j)} \sqcup \Delta_i.$$

Clearly, $G_{(i,j)} \subseteq G^{\triangle}_{(i,j)}$ and $F_{n-1} \subseteq PG_{(i,j)}$. Let $\alpha \in \mathcal{PD}_n$, we put

$$Shift(\alpha) = \{i \in \mathrm{dom}(\alpha) : i\alpha \neq i\}.$$

Lemma 2.10. *Let $n \geq 3$. Then $PG_{(i,j)} = \langle E_{n-1} \backslash \{\binom{i}{j}\}\rangle \cap J_{n-1}$, for $i, j \in \mathbf{n}$ with $i > j$.*

Proof. Let $\alpha \in PG_{(i,j)}$ be arbitrary. Notice that $PG_{(i,j)} = G^{\triangle}_{(i,j)} \sqcup \Delta_i$. To prove that $\alpha \in \langle E_{n-1} \backslash \{\binom{i}{j}\}\rangle \cap J_{n-1}$, we distinguish two cases.

Case 1. $\alpha \in G^{\triangle}_{(i,j)}$. Then $i\alpha \neq j$ and $\alpha \in J_{n-1}$. If $|\mathrm{dom}(\alpha)| = n$, then $\alpha \in G_{(i,j)}$. Thus, by Lemma 2.7,

$$\alpha \in G_{(i,j)} = \langle E^{\mathcal{D}_n}_{n-1} \backslash \{\begin{pmatrix} i \\ j \end{pmatrix}\}\rangle \cap J^{\mathcal{D}_n}_{n-1} \subseteq \langle E_{n-1} \backslash \{\begin{pmatrix} i \\ j \end{pmatrix}\}\rangle \cap J_{n-1}.$$

If $|\mathrm{dom}(\alpha)| = n-1$, then $\mathrm{dom}(\alpha) = \mathbf{n}\backslash\{k\}$ for some $k \in \mathbf{n}\backslash\{i\}$. (*i*) If $1 \in \mathrm{im}(\alpha)$, we define α^* by

$$x\alpha^* = \begin{cases} 1, & x = k, \\ x\alpha, & x \neq k. \end{cases}$$

Then $\alpha^* \in G_{(i,j)}$ and $\alpha = \delta_k \alpha^*$. Thus, by Lemma 2.7,

$$\alpha = \delta_k \alpha^* \in \delta_k \cdot G_{(i,j)} = \delta_k \cdot [\langle E^{\mathcal{D}_n}_{n-1} \backslash \{\begin{pmatrix} i \\ j \end{pmatrix}\}\rangle \cap J_{n-1}] \subseteq \langle E_{n-1} \backslash \{\begin{pmatrix} i \\ j \end{pmatrix}\}\rangle \cap J_{n-1}.$$

(*ii*) If $1 \notin \mathrm{im}(\alpha)$, then $k = 1$ otherwise $1\alpha = 1$. Thus, by $\alpha \in J_{n-1}$, $\mathrm{dom}(\alpha) = \mathrm{im}(\alpha) = \mathbf{n}\backslash\{1\}$. It follows from $\alpha \in \mathcal{PD}_n$ that

$$\alpha = \begin{pmatrix} 2 & 3 & \cdots & n \\ 2 & 3 & \cdots & n \end{pmatrix} = \delta_1 \in \langle E_{n-1} \backslash \{\begin{pmatrix} i \\ j \end{pmatrix}\}\rangle \cap J_{n-1}.$$

Case 2. $\alpha \in \Delta_i$. Then $\mathrm{dom}(\alpha) = \mathbf{n}\backslash\{i\}$. Notice that $i > j \geq 1$. If $i \geq 3$, then there exists $s \in \{1, 2\}$ such that $s \neq j$. Now, we define α^* by

$$x\alpha^* = \begin{cases} s, & x = i, \\ x\alpha, & x \neq i. \end{cases}$$

Then $\alpha^* \in G_{(i,j)}$ and $\alpha = \delta_i \alpha^*$. Thus, by Lemma 2.7,

$$\alpha = \delta_i \alpha^* \in \delta_i \cdot G_{(i,j)} = \delta_i \cdot [\langle E^{\mathcal{D}_n}_{n-1} \backslash \{\begin{pmatrix} i \\ j \end{pmatrix}\}\rangle \cap J_{n-1}] \subseteq \langle E_{n-1} \backslash \{\begin{pmatrix} i \\ j \end{pmatrix}\}\rangle \cap J_{n-1}.$$

Notice that if $i = 2$, then $j = 1$ since $i > j$. (*i*) If $i = 2$ and $2 \notin \mathrm{im}(\alpha)$, then $\mathrm{dom}(\alpha) = \mathrm{im}(\alpha) = \mathbf{n}\backslash\{2\}$. It follows from $\alpha \in \mathcal{PD}_n$ that

$$\alpha = \begin{pmatrix} 1 & 3 & \cdots & n \\ 1 & 3 & \cdots & n \end{pmatrix} = \delta_2 \in \langle E_{n-1} \backslash \{\begin{pmatrix} i \\ j \end{pmatrix}\}\rangle \cap J_{n-1}.$$

(*ii*) If $i = 2$ and $2 \in \mathrm{im}(\alpha)$, then we define α^* by

$$x\alpha^* = \begin{cases} 2, & x = 2, \\ x\alpha, & x \neq 2. \end{cases}$$

Then $\alpha^* \in G_{(i,j)}$ and $\alpha = \delta_2\alpha^*$. Thus, by Lemma 2.7,

$$\alpha = \delta_2\alpha^* \in \delta_2 \cdot G_{(i,j)} = \delta_2 \cdot [\langle E^{\mathcal{D}_n}_{n-1}\backslash\{\binom{i}{j}\}\rangle \cap J_{n-1}] \subseteq \langle E_{n-1}\backslash\{\binom{i}{j}\}\rangle \cap J_{n-1}.$$

It remains to prove that $\langle E_{n-1}\backslash\{\binom{i}{j}\}\rangle \cap J_{n-1} \subseteq PG_{(i,j)}$. Let $\alpha \in \langle E_{n-1}\backslash\{\binom{i}{j}\}\rangle \cap J_{n-1}$ be arbitrary. To prove that $\alpha \in PG_{(i,j)}$, we distinguish two cases.

Case 1. $\alpha \in J^{\mathcal{D}_n}_{n-1}$. Then, by Lemma 2.9, α can be written as

$$\alpha = \binom{y_1}{y_0}\binom{y_2}{y_1}\cdots\binom{y_{t-1}}{y_{t-2}}\binom{y_t}{y_{t-1}}$$

where $Shift(\alpha) = \{y_1, y_2, \ldots, y_t\}$ and $y_1 < y_2 < \cdots < y_t$ such that $\binom{y_{k+1}}{y_k} \neq \binom{i}{j}$ for all $0 \leq k \leq t-1$. If $i \notin Shift(\alpha)$, then $i\alpha = i > j$. If $i = y_{k+1} \in Shift(\alpha)$ for some $k \in \{0, 1, \ldots, t-1\}$, then $j \neq y_k$ and so

$$i\alpha = i\binom{y_1}{y_0}\binom{y_2}{y_1}\cdots\binom{y_{k+1}}{y_k}\cdots\binom{y_t}{y_{t-1}} = y_k \neq j.$$

Thus $\alpha \in G_{(i,j)} \subseteq PG_{(i,j)}$.

Case 2. $\alpha \in J_{n-1}\backslash J^{\mathcal{D}_n}_{n-1}$. Notice that $E_{n-1} = E^{\mathcal{D}_n}_{n-1} \cup F_{n-1}$ and $\alpha \in \langle E_{n-1}\backslash\{\binom{i}{j}\}\rangle \cap J_{n-1}$. Then, by Lemma 2.8, α can be written as

$$\alpha = \delta_k \text{ for some } 1 \leq k \leq n$$

or

$$\alpha = \delta_s\binom{y_1}{y_0}\binom{y_2}{y_1}\cdots\binom{y_{t-1}}{y_{t-2}}\binom{y_t}{y_{t-1}},$$

where $Shift(\alpha) = \{y_1, y_2, \ldots, y_t\}\backslash\{s\}$ and $y_1 < y_2 < \cdots < y_t$ such that $\binom{y_{k+1}}{y_k} \neq \binom{i}{j}$ for all $0 \leq k \leq t-1$, and $1 \leq s \leq n$. If $\alpha = \delta_k$ for some $1 \leq k \leq n$, then clearly $\alpha = \delta_k \in F_{n-1} \subseteq PG_{(i,j)}$. Notice that $\mathrm{dom}(\alpha) = \mathbf{n}\backslash\{s\}$. If $\alpha = \delta_s\binom{y_1}{y_0}\binom{y_2}{y_1}\cdots\binom{y_{t-1}}{y_{t-2}}\binom{y_t}{y_{t-1}}$, to prove that $\alpha \in PG_{(i,j)}$, we distinguish two subcases.

Subcase 2.1. $s = i$. Then clearly $\alpha \in \Delta_i \subseteq PG_{(i,j)}$.

Subcase 2.2. $s \neq i$. Then $i \in \mathrm{dom}(\alpha)$. If $i \notin Shift(\alpha)$, then $i\alpha = i > j$. If $i = y_{k+1} \in Shift(\alpha)$ for some $k \in \{0, 1, \ldots, t-1\}$, then $j \neq y_k$ and so

$$i\alpha = i\delta_s\binom{y_1}{y_0}\binom{y_2}{y_1}\cdots\binom{y_{t-1}}{y_{t-2}}\binom{y_t}{y_{t-1}} = i\binom{y_1}{y_0}\binom{y_2}{y_1}\cdots\binom{y_{k+1}}{y_k}\cdots\binom{y_t}{y_{t-1}} = y_k \neq j.$$

Thus $\alpha \in G^{\triangle}_{(i,j)} \subseteq PG_{(i,j)}$. □

For $1 \leq i \leq n$, put

$$\Omega_i = \{\alpha \in J_{n-1} : i \in \mathrm{dom}(\alpha)\}.$$

Lemma 2.11. *Let $n \geq 3$. Then $\Omega_i = \langle E_{n-1}\backslash\{\delta_i\}\rangle \cap J_{n-1}$, for $1 \leq i \leq n$.*

Proof. By Lemma 2.4, we have $\mathcal{D}_{n,n-1} = \langle E^{\mathcal{D}_n}_{n-1}\rangle$. Notice that $J^{\mathcal{D}_n}_{n-1} \subseteq J_{n-1}$. Then $J^{\mathcal{D}_n}_{n-1} \subseteq \mathcal{D}_{n,n-1} \cap J_{n-1} = \langle E^{\mathcal{D}_n}_{n-1}\rangle \cap J_{n-1}$. Let $\alpha \in \Omega_i$ be arbitrary. To prove that $\alpha \in \langle E_{n-1}\backslash\{\delta_i\}\rangle \cap J_{n-1}$, we distinguish two cases.

Case 1. $\alpha \in J^{\mathcal{D}_n}_{n-1}$. Then

$$\alpha \in J^{\mathcal{D}_n}_{n-1} \subseteq \langle E^{\mathcal{D}_n}_{n-1}\rangle \cap J_{n-1} \subseteq \langle E_{n-1}\backslash\{\delta_i\}\rangle \cap J_{n-1}.$$

Case 2. $\alpha \in J_{n-1} \backslash J_{n-1}^{\mathcal{D}_n}$. Then $\mathrm{dom}(\alpha) = \mathbf{n} \backslash \{k\}$, for some $k \in \mathbf{n} \backslash \{i\}$. We distinguish two subcases.
Subcase 2.1. $1 \in \mathrm{im}(\alpha)$. We define α^* by

$$x\alpha^* = \begin{cases} 1, & x = k, \\ x\alpha, & x \neq k. \end{cases}$$

Then $\alpha^* \in J_{n-1}^{\mathcal{D}_n}$ and $\alpha = \delta_k \alpha^*$. Thus

$$\alpha = \delta_k \alpha^* \in \delta_k \cdot J_{n-1}^{\mathcal{D}_n} \subseteq \delta_k \cdot \langle E_{n-1}^{\mathcal{D}_n} \rangle \cap J_{n-1} \subseteq \langle E_{n-1} \backslash \{\delta_i\} \rangle \cap J_{n-1}.$$

Subcase 2.2. $1 \notin \mathrm{im}(\alpha)$. Then $k = 1$ otherwise $1\alpha = 1$. Thus, by $\alpha \in J_{n-1}$, $\mathrm{dom}(\alpha) = \mathrm{im}(\alpha) = \mathbf{n} \backslash \{1\}$. Notice that $i \neq k = 1$. It follows from $\alpha \in \mathcal{PD}_n$ that

$$\alpha = \begin{pmatrix} 2 & 3 & \cdots & n \\ 2 & 3 & \cdots & n \end{pmatrix} = \delta_1 \in \langle E_{n-1} \backslash \{\delta_i\} \rangle \cap J_{n-1}.$$

It remains to prove that $\langle E_{n-1} \backslash \{\delta_i\} \rangle \cap J_{n-1} \subseteq \Omega_i$. Let $\alpha \in \langle E_{n-1} \backslash \{\delta_i\} \rangle \cap J_{n-1}$ be arbitrary. To prove that $\alpha \in \Omega_i$, we distinguish two subcases.

Case 1. $\alpha \in J_{n-1}^{\mathcal{D}_n}$. Then clearly $\alpha \in J_{n-1}^{\mathcal{D}_n} \subseteq \Omega_i$.

Case 2. $\alpha \in J_{n-1} \backslash J_{n-1}^{\mathcal{D}_n}$. It is obvious that, for all $\beta \in J_{n-1}$ and $\delta_i \in F_{n-1}$, if $\beta\delta_i \in J_{n-1}$, then clearly $\beta\delta_i = \beta$. Notice that $E_{n-1} = E_{n-1}^{\mathcal{D}_n} \cup F_{n-1}$ and $\alpha \in \langle E_{n-1} \backslash \{\delta_i\} \rangle \cap J_{n-1}$. Then α can be written as $\alpha = \delta_k$ for some $k \in \mathbf{n} \backslash \{i\}$ or

$$\alpha = \delta_s \varepsilon_1 \cdots \varepsilon_m,$$

where $s \in \mathbf{n} \backslash \{i\}$ and $\varepsilon_1, \ldots, \varepsilon_m \in E_{n-1}^{\mathcal{D}_n}$. Then clearly $i \in \mathrm{dom}(\alpha)$. Thus $\alpha \in \Omega_i$. □

We are now ready to prove the main result of this section.

Theorem 2.12. *Let $n \geq 3$. Then each maximal idempotent generated subsemigroup S of $\mathcal{PD}_n$ must be one of the following forms:*

(1) $S = \mathcal{PD}_{n,n-1}$.

(2) $S = \mathcal{PD}_{n,n-2} \cup PG_{(i,j)} \cup \{1_n\}$, *for* $1 \leq j < i \leq n$.

(3) $S = \mathcal{PD}_{n,n-2} \cup \Omega_i \cup \{1_n\}$, *for* $1 \leq i \leq n$.

Proof. Notice that $\mathcal{PD}_n = \mathcal{PD}_{n,n-1} \cup \{1_n\}$ and $\mathcal{PD}_{n,r} = \langle E_r \rangle = \langle E(\mathcal{PD}_{n,r}) \rangle$, for $1 \leq r \leq n-1$ (by Lemma 2.3). It is clear that $\mathcal{PD}_{n,n-1}$ is a maximal idempotent generated subsemigroup of $\mathcal{PD}_n$. Put

$$M_{i,j} = \mathcal{PD}_{n,n-2} \cup PG_{(i,j)} \cup \{1_n\},\ 1 \leq j < i \leq n,$$

$$N_i = \mathcal{PD}_{n,n-2} \cup \Omega_i \cup \{1_n\},\ 1 \leq i \leq n.$$

Then, by Lemmas 2.10 and 2.11,

$$M_{i,j} = \mathcal{PD}_{n,n-2} \cup [\langle E_{n-1} \backslash \{\begin{pmatrix} i \\ j \end{pmatrix}\} \rangle \cap J_{n-1}] \cup \{1_n\}$$

$$= \mathcal{PD}_{n,n-2} \cup \langle E_{n-1} \backslash \{\begin{pmatrix} i \\ j \end{pmatrix}\} \rangle \cup \{1_n\}$$

$$= \langle E(\mathcal{PD}_{n,n-2}) \cup [E_{n-1} \backslash \{\begin{pmatrix} i \\ j \end{pmatrix}\}] \cup \{1_n\} \rangle$$

$$= \langle E(\mathcal{PD}_n)\backslash\{\begin{pmatrix} i \\ j \end{pmatrix}\}\rangle,$$

$$\begin{aligned} N_i &= \mathcal{PD}_{n,n-2} \cup [\langle E_{n-1}\backslash\{\delta_i\}\rangle) \cap J_{n-1}] \cup \{1_n\} \\ &= \mathcal{PD}_{n,n-2} \cup \langle E_{n-1}\backslash\{\delta_i\}\rangle \cup \{1_n\} \\ &= \langle E(\mathcal{PD}_{n,n-2}) \cup [E_{n-1}\backslash\{\delta_i\}] \cup \{1_n\}\rangle \\ &= \langle E(\mathcal{PD}_n)\backslash\{\delta_i\}\rangle. \end{aligned}$$

Thus clearly $M_{i,j}$ and N_i are maximal idempotent generated subsemigroups of $\mathcal{PD}_n$. Let S be a maximal idempotent generated subsemigroup of $\mathcal{PD}_n$. Notice that $\mathcal{PD}_n = \mathcal{PD}_{n,n-1} \cup \{1_n\}$. If $1_n \notin S$, then $S \subseteq \mathcal{PD}_{n,n-1} \subset \mathcal{PD}_n$. Thus, by the maximality of S, $S = \mathcal{PD}_{n,n-1}$. If $1_n \in S$, then, by Lemma 2.3 and $S \subset \mathcal{PD}_n$, $E_{n-1} \nsubseteq S$. Then $\binom{i}{j} \notin S$ for some $i, j \in \mathbf{n}$ with $i > j$ or $\delta_i \notin S$ for some $1 \leq i \leq n$. Thus $S \subseteq \langle E(\mathcal{PD}_n)\backslash\{\binom{i}{j}\}\rangle = M_{i,j} \subset \mathcal{PD}_n$ or $S \subseteq \langle E(\mathcal{PD}_n)\backslash\{\delta_i\}\rangle = N_i \subset \mathcal{PD}_n$. Hence, by the maximality of S, $S = M_{i,j}$ or $S = N_i$. □

Notice that $|E_{n-1}| = \frac{n(n+1)}{2}$. By Theorems 2.6 and 2.12, we have the following result:

Corollary 2.13. *Let $n \geq 3$. Then the semigroup $\mathcal{PD}_n$ contains exactly $\frac{n(n+1)}{2} + 1$ maximal (idempotent generated) subsemigroups.*

3. Abundance for the (principal) ideals of $\mathcal{PD}_n$

A subset I of a semigroup S is an *ideal* if it is closed under multiplication by arbitrary elements of S: for all $x \in S$ and $y \in I$, we have $xy, yx \in I$. The *principal ideal* generated by an element a of the semigroup S is the set $SaS = \{xay : x, y \in S\}$. Notice that $\mathcal{PD}_n$ is the principal ideal $\mathcal{PD}_n 1_n \mathcal{PD}_n$ generated by 1_n.

In 1992, Umar [12] showed that the ideals $\mathcal{PD}_{n,r}$ $(1 \leq r \leq n)$ of $\mathcal{PD}_n$ are abundant. In this section, we give necessary and sufficient conditions for the ideals of $\mathcal{PD}_n$ to be abundant. Moreover, we characterize the structure of the left (right) abundant principal ideal of $\mathcal{PD}_n$.

On a semigroup S the relation $\mathscr{L}^*$ is defined by the rule that $(a, b) \in \mathscr{L}^*$ if and only if the elements a, b of S are related by Green's relation $\mathscr{L}$ in some oversemigroup of S. The relation $\mathscr{R}^*$ is defined dually. A semigroup S is called *left abundant* if each of its $\mathscr{L}^*$-classes contains an idempotent. Dually, a semigroup S is called right abundant if each of its $\mathscr{R}^*$-classes contains an idempotent. A semigroup S is abundant if it is both left and right abundant (see [5]). Given a semigroup S, we denote by L_u^{*S} and R_u^{*S} the $\mathscr{L}^*$-class and $\mathscr{R}^*$-class, respectively, of an element $u \in S$.

The following lemma and its dual give a characterization of $\mathscr{L}^*$ and $\mathscr{R}^*$ [5, Lemma 1.1].

Lemma 3.1. *Let S be a semigroup and let $a, b \in S$. Then the following conditions are equivalent:*

(1) $(a, b) \in \mathscr{L}^*$.

(2) *for all $x, y \in S^1$, $ax = ay$ if and only if $bx = by$.*

Now, recall that Umar [12, Corollary 2.4.3, Theorem 2.2.5 and Lemma 2.2.6] proved:

Lemma 3.2. *Let $1 \leq r \leq n$, and let $\alpha, \beta \in \mathcal{PD}_{n,r}$. Then*

(1) $(\alpha, \beta) \in \mathscr{L}^*$ *if and only if* $\mathrm{im}(\alpha) = \mathrm{im}(\beta)$.

(2) $(\alpha, \beta) \in \mathscr{R}^*$ *if and only if* $\ker(\alpha) = \ker(\beta)$.

(3) *the semigroup $\mathcal{PD}_{n,r}$ is abundant.*

Notice that the idempotents in E_r are exactly of the following form:

$$\varepsilon = \left(\begin{array}{c|c|c|c} A_1 & A_2 & \cdots & A_r \\ c_1 & c_2 & \cdots & c_r \end{array} \right),$$

where $c_1 < c_2 < \cdots < c_r$ and $c_i = minA_i$, for $1 \le i \le r$. Notice that $\ker(\varepsilon) = (A_1|A_2|\cdots|A_r)$. Thus, we have:

Lemma 3.3. *Let $1 \le r \le n$ and $\varepsilon, \eta \in E_r$. Then $\ker(\varepsilon) = \ker(\eta)$ if and only if $\varepsilon = \eta$.*

It is well known that the Green relations on $\mathcal{PT}_n$ can be characterized as $\alpha\mathscr{L}\beta \Leftrightarrow \mathrm{im}(\alpha) = \mathrm{im}(\beta)$, $\alpha\mathscr{R}\beta \Leftrightarrow \ker(\alpha) = \ker(\beta)$ and $\alpha\mathscr{J}\beta \Leftrightarrow |\mathrm{im}(\alpha)| = |\mathrm{im}(\beta)|$. Using Lemma 3.1, we can prove the following lemma:

Lemma 3.4. *Let S be a subsemigroup of $\mathcal{PT}_n$, and let $m = max\{|\mathrm{im}(\alpha)| : \alpha \in S\}$. If $E(L_\alpha^{\mathcal{PT}_n}) \cap S = \emptyset$ for some $\alpha \in S$ with $|\mathrm{im}(\alpha)| = m$, then S is not left abundant.*

Proof. Assume that S is left abundant. Then there exists an idempotent in L_α^{*S}, say ε. It follows from Lemma 3.1 that

$$\alpha\varepsilon = \alpha$$

since $\varepsilon \cdot \varepsilon = \varepsilon \cdot 1_n$ and so $\mathrm{im}(\alpha) \subseteq \mathrm{im}(\varepsilon)$ which implies that $m = |\mathrm{im}(\alpha)| \le |\mathrm{im}(\varepsilon)|$. By the maximality of m, we have $|\mathrm{im}(\varepsilon)| = |\mathrm{im}(\alpha)| = m$ and so $\mathrm{im}(\varepsilon) = \mathrm{im}(\alpha)$. Thus $(\alpha, \varepsilon) \in \mathscr{L}^{\mathcal{PT}_n}$ and $\varepsilon \in E(L_\alpha^{\mathcal{PT}_n}) \cap S$, a contradiction. □

Lemma 3.5. *Let S be a subsemigroup of $\mathcal{PT}_n$, and let $m = max\{|\mathrm{im}(\alpha)| : \alpha \in S\}$. If $E(R_\alpha^{\mathcal{PT}_n}) \cap S = \emptyset$ for some $\alpha \in S$ with $|\mathrm{im}(\alpha)| = m$, then S is not right abundant.*

Proof. Assume that S is right abundant. Then there exists an idempotent in R_α^{*S}, say ε. It follows from Lemma 3.1 that

$$\varepsilon\alpha = \alpha$$

since $\varepsilon \cdot \varepsilon = 1_n \cdot \varepsilon$. Thus $\mathrm{dom}(\alpha) \subseteq \mathrm{dom}(\varepsilon)$ and $\ker(\varepsilon) \subseteq \ker(\alpha)$ which implies that $m = |\mathrm{im}(\alpha)| = |\mathrm{dom}(\alpha)/\ker(\alpha)| \le |\mathrm{dom}(\varepsilon)/\ker(\varepsilon)| = |\mathrm{im}(\varepsilon)|$. By the maximality of m, we have $|\mathrm{im}(\varepsilon)| = |\mathrm{im}(\alpha)| = m$ and so $\ker(\varepsilon) = \ker(\alpha)$. Thus $(\alpha, \varepsilon) \in \mathscr{R}^{\mathcal{PT}_n}$ and $\varepsilon \in E(R_\alpha^{\mathcal{PT}_n}) \cap S$, a contradiction. □

Now, it is easy to prove the following result:

Theorem 3.6. *Let $\mathcal{I}$ be an ideal of $\mathcal{PD}_n$. Then $\mathcal{I}$ is abundant if and only if there exists $r \in \{0, 1, \ldots, n\}$ such that $\mathcal{I} = \mathcal{PD}_{n,r}$.*

Proof. Notice that $\mathcal{PD}_n = \mathcal{PD}_{n,n-1} \cup \{1_n\}$ and $\mathcal{PD}_n$ is abundant (by Lemma 3.2). Suppose that $\mathcal{I}$ is abundant. If $1_n \in \mathcal{I}$. Then clearly $\alpha = \alpha \cdot 1_n \in \mathcal{I}$, for all $\alpha \in \mathcal{PD}_n$. Thus $\mathcal{I} = \mathcal{PD}_n = \mathcal{PD}_{n,n}$. If $1_n \notin \mathcal{I}$, we put

$$r = \max\{|\mathrm{im}(\alpha)| : \alpha \in \mathcal{I}\}.$$

Then clearly $0 \le r \le n-1$ and $\mathcal{I} \subseteq \mathcal{PD}_{n,r}$. Notice that $\mathcal{PD}_{n,0} = \{\theta_{\mathbf{n}}\}$. If $r = 0$, then clearly $\mathcal{I} = \mathcal{PD}_{n,0}$. If $r \ge 1$, there exists $\alpha \in \mathcal{I}$ with $|\mathrm{im}(\alpha)| = r$. Suppose that

$$\alpha = \left(\begin{array}{c|c|c|c} A_1 & A_2 & \cdots & A_r \\ a_1 & a_2 & \cdots & a_r \end{array} \right),$$

where $a_1 < \cdots < a_r$ and $a_i \le \min A_i$, for $1 \le i \le r$. Notice that $a_i \le n - r + i$, for $1 \le i \le r$. Put

$$\beta = \left(\begin{array}{c|c|c|c} n-r+1 & n-r+2 & \cdots & n \\ \min A_1 & \min A_2 & \cdots & \min A_r \end{array}\right),$$

$$\xi = \left(\begin{array}{c|c|c|c} n-r+1 & \cdots & n-1 & n \\ a_1 & \cdots & a_{r-1} & a_r \end{array}\right).$$

Then $\xi = \beta\alpha \in \mathcal{I}$ since $\mathcal{I}$ is an ideal of $\mathcal{PD}_n$. Notice that clearly $|\operatorname{im}(\xi)| = |\operatorname{im}(\alpha)| = r$. By Lemma 3.5 and $\mathcal{I}$ is abundant, we have $E(R_\xi^{\mathcal{PD}_n}) \cap \mathcal{I} \neq \emptyset$. Then there exists $\eta \in E(\mathcal{I})$ such that $\ker(\eta) = \ker(\xi)$. Notice that

$$\lambda_r = \left(\begin{array}{c|c|c|c} n-r+1 & n-r+2 & \cdots & n \\ n-r+1 & n-r+2 & \cdots & n \end{array}\right) \in E_r.$$

Then $\ker(\eta) = \ker(\xi) = \ker(\lambda_r)$ and so $\eta = \lambda_r$ by Lemma 3.3. Now, let

$$\varepsilon = \left(\begin{array}{c|c|c|c} A_1 & A_2 & \cdots & A_r \\ c_1 & c_2 & \cdots & c_r \end{array}\right) \in E_r,$$

where $c_1 < c_2 < \cdots < c_r$ and $c_i = \min A_i$, for $1 \le i \le r$. Notice that $c_i \le n - r + i$, for $1 \le i \le r$. Put

$$\gamma = \left(\begin{array}{c|c|c|c} n-r+1 & \cdots & n-1 & r \\ c_1 & \cdots & c_{r-1} & c_r \end{array}\right).$$

Since $\eta \in \mathcal{I}$ and $\mathcal{I}$ is an ideal of $\mathcal{PD}_n$, we have $\gamma = \lambda_r\gamma = \eta\gamma \in \mathcal{I}$. By Lemma 3.4 and $\mathcal{I}$ is abundant, we have $E(L_\gamma^{\mathcal{PD}_n}) \cap \mathcal{I} \neq \emptyset$. Then there exists $\delta \in E(\mathcal{I})$ such that $\operatorname{im}(\delta) = \operatorname{im}(\gamma)$. Suppose that

$$\delta = \left(\begin{array}{c|c|c|c} B_1 & B_2 & \cdots & B_r \\ c_1 & c_2 & \cdots & c_r \end{array}\right)$$

Since $\delta \in E(\mathcal{I})$, we have $c_i = \min B_i$, for $1 \le i \le r$. It is obvious that $\varepsilon = \varepsilon\delta$ and so $\varepsilon \in \mathcal{I}$ (since $\mathcal{I}$ is an ideal of $\mathcal{PD}_n$ and $\delta \in \mathcal{I}$). Then we have proved that $E_r \subseteq \mathcal{I}$. Thus, by Lemma 2.3, $\mathcal{I} = \mathcal{PD}_{n,r}$.

Conversely, if $\mathcal{I} = \mathcal{PD}_{n,0}$, then clearly $\mathcal{I}$ is abundant. If there exists $1 \le r \le n$ such that $\mathcal{I} = \mathcal{PD}_{n,r}$, then, by Lemma 3.2, $\mathcal{I}$ is abundant. □

For any $\alpha \in \mathcal{PD}_n$, we denote by $\triangle_\alpha$ the principal ideal

$$\mathcal{PD}_n\alpha\mathcal{PD}_n$$

generated by α. Notice that if $\alpha = 1_n$, then $\Delta_\alpha = \mathcal{PD}_n$; if $|\operatorname{im}(\alpha)| = 1$, then $\alpha = \binom{\mathbf{n}}{1}$. Notice that if $\alpha = 1_n$, then $\Delta_\alpha = \mathcal{PD}_n$. Let $\beta \in \Delta_\alpha$ be arbitrary. Then there exist $\gamma, \delta \in \mathcal{PD}_n$ such that $\beta = \gamma\alpha\delta$. Clearly, $|\operatorname{im}(\beta)| \le |\operatorname{im}(\alpha)|$. Notice that $\alpha = 1_n\alpha 1_n \in \Delta_\alpha$. Thus $|\operatorname{im}(\alpha)| = \max\{|\operatorname{im}(\beta)| : \beta \in \Delta_\alpha\}$.

Lemma 3.7. *Let $\alpha \in \mathcal{PD}_n$ and α is not an idempotent. Then $E(L_\alpha^{\mathcal{PT}_n}) \cap \Delta_\alpha \neq \emptyset$ and $E(R_\alpha^{\mathcal{PT}_n}) \cap \Delta_\alpha \neq \emptyset$.*

Proof. Suppose that $|\operatorname{im}(\alpha)| = r$. Then $r \ge 1$ since α is not an idempotent. Thus we can suppose that

$$\alpha = \left(\begin{array}{c|c|c|c} A_1 & A_2 & \cdots & A_r \\ a_1 & a_2 & \cdots & a_r \end{array}\right),$$

where $a_1 < a_2 < \cdots < a_r$ and $a_i \le \min A_i$, for $1 \le i \le r$. Let $c_i = \min A_i$, for $1 \le i \le r$. Since α is not an idempotent, there exist $m \in \{1, \ldots, r\}$ such that $a_m < c_m$. Clearly, $a_m \notin A_m$. Then $a_m\alpha \neq a_m$ (if $a_m \in \operatorname{dom}(\alpha)$).

Assume that $E(L_\alpha^{\mathcal{PT}_n}) \cap \Delta_\alpha \neq \emptyset$. Let $\varepsilon \in E(L_\alpha^{\mathcal{PT}_n}) \cap \Delta_\alpha$. Then there exist $\beta, \gamma \in \mathcal{PD}_n$ such that $\varepsilon = \beta\alpha\gamma$ and $\mathrm{im}(\varepsilon) = \mathrm{im}(\alpha) = \{a_1, \ldots, a_r\}$. Since ε is an idempotent, we have $a_i = a_i\varepsilon$, for $1 \leq i \leq r$. Then

$$a_m = a_m\varepsilon = (a_m\beta\alpha)\gamma \leq (a_m\beta)\alpha \leq a_m\beta \leq a_m.$$

It follows that $a_m = a_m\beta = (a_m\beta)\alpha$ and so $a_m\alpha = a_m$, a contradiction. Thus $E(L_\alpha^{\mathcal{PT}_n}) \cap \Delta_\alpha = \emptyset$.

Assume that $E(R_\alpha^{\mathcal{PT}_n}) \cap \Delta_\alpha \neq \emptyset$. Let $\varepsilon \in E(R_\alpha^{\mathcal{PT}_n}) \cap \Delta_\alpha$. Then there exist $\beta, \gamma \in \mathcal{PD}_n$ such that $\varepsilon = \beta\alpha\gamma$ and $\ker(\varepsilon) = \ker(\alpha) = (A_1|\cdots|A_r)$. Notice that $c_i = \min A_i$, for $1 \leq i \leq r$. Since ε is an idempotent, we have $c_i\varepsilon = c_i$ for $1 \leq i \leq r$. Then

$$c_m = c_m\varepsilon = (c_m\beta\alpha)\gamma \leq (c_m\beta)\alpha \leq c_m\beta \leq c_m.$$

It follows that $c_m = c_m\beta = (c_m\beta)\alpha$ and so $c_m = c_m\alpha = a_m$, a contradiction. Thus $E(R_\alpha^{\mathcal{PT}_n}) \cap \Delta_\alpha = \emptyset$. □

Using Lemmas 3.4, 3.5 and 3.7, we can prove the following result:

Lemma 3.8. *Let $\alpha \in \mathcal{PD}_n$ and α is not an idempotent. Then $\triangle_\alpha$ is neither left abundant nor right abundant.*

Proof. By Lemma 3.7, we have

$$E(L_\alpha^{\mathcal{PT}_n}) \cap \Delta_\alpha \neq \emptyset \text{ and } E(R_\alpha^{\mathcal{PT}_n}) \cap \Delta_\alpha \neq \emptyset.$$

Then, by Lemmas 3.4 and 3.5, $\triangle_\alpha$ is neither left abundant nor right abundant. □

Let $x, y \in \mathbf{n}$ with $x < y$. The set $[x, y] = \{z \in \mathbf{n} : x \leq z \leq y\}$ of $\mathbf{n}$ is called a *closed convex set*. Similarly, we can define the convex sets of other kinds, such as $(x, y]$, (x, y) and $[x, y)$.

For $1 \leq r \leq n$, put

$$E_r^\Delta = \{\varepsilon \in E_r : \mathrm{im}(\varepsilon) = [1, r]\}.$$

Then clearly $E_n^\Delta = \{1_n\}$. Let $\alpha \in \mathcal{PD}_n$. It is easy to see that $\alpha \in E_n^\Delta (\alpha = 1_n)$ if and only if $\Delta_\alpha = \mathcal{POE}_n = \{\alpha \in \mathcal{POE}_n : \mathrm{im}(\alpha) \subseteq [1, n]\}$. In fact, we have the following result:

Theorem 3.9. *Let $1 \leq r \leq n-1$. Let $\alpha \in \mathcal{PD}_n$ with $|\mathrm{im}(\alpha)| = r$. Then the following statements are equivalent:*

(1) *Δ_α is left abundant.*

(2) *$\alpha \in E_r^\Delta$.*

(3) *$\Delta_\alpha = \{\beta \in \mathcal{PD}_n : \mathrm{im}(\beta) \subseteq [1, r]\}$.*

Proof. (1) $\Longrightarrow$ (2) Suppose that Δ_α is left abundant. Then, by Lemma 3.8, α is an idempotent. We can suppose that

$$\alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_r \\ c_1 & c_2 & \cdots & c_r \end{pmatrix},$$

where $c_1 < c_2 < \cdots < c_r$ and $c_i = \min A_i$ for $1 \leq i \leq r$. Let $c_0 = 0$. Assume that $\alpha \notin E_r^\Delta$. Then $\mathrm{im}(\alpha) \neq [1, r]$. Then there exists $m \in \{1, \cdots, r\}$ such that $c_m - c_{m-1} \geq 2$. Clearly, $c_m - 1 \notin \mathrm{im}(\alpha)$. Put

$$\xi = \begin{cases} \begin{pmatrix} A_1 & A_2 & \cdots & A_r \\ c_1 - 1 & c_2 & \cdots & c_r \end{pmatrix}, & m = 1, \\ \begin{pmatrix} A_1 & \cdots & A_{m-1} & A_m & A_{m+1} & \cdots & A_r \\ c_1 & \cdots & c_{m-1} & c_m - 1 & c_{m+1} & \cdots & c_r \end{pmatrix}, & 2 \leq m \leq r-1, \\ \begin{pmatrix} A_1 & \cdots & A_{r-1} & A_r \\ c_1 & \cdots & c_{r-1} & c_r - 1 \end{pmatrix}, & m = r. \end{cases}$$

Then $\xi = \alpha\xi = 1_n\alpha\xi \in \Delta_\alpha$ and $\xi^2 \neq \xi$. Notice that clearly $|\mathrm{im}(\xi)| = |\mathrm{im}(\alpha)|$. Assume that $E(L_\xi^{\mathcal{PT}_n}) \cap \Delta_\alpha \neq \emptyset$. Let $\varepsilon \in E(L_\xi^{\mathcal{PT}_n}) \cap \Delta_\alpha$. Then there exist $\beta, \gamma \in \mathcal{PD}_n$ such that $\varepsilon = \beta\alpha\gamma$ and $\mathrm{im}(\varepsilon) = \mathrm{im}(\xi)$. Notice that $c_m - 1 \in \mathrm{im}(\xi)$ and $\Delta_\alpha \subseteq \mathcal{PD}_n$. Since ε is an idempotent, we have $c_m - 1 = (c_m - 1)\varepsilon$. Then

$$c_m - 1 = (c_m - 1)\varepsilon = [(c_m - 1)\beta\alpha]\gamma \leq [(c_m - 1)\beta]\alpha \leq (c_m - 1)\beta \leq c_m - 1$$

and so $(c_m - 1)\beta = c_m - 1$. Thus

$$c_m - 1 = (c_m - 1)\varepsilon = [(c_m - 1)\beta\alpha]\gamma \leq [(c_m - 1)\beta]\alpha = (c_m - 1)\alpha \leq c_m - 1$$

and so $(c_m - 1)\alpha = c_m - 1$. Hence, $c_m - 1 \in \mathrm{im}(\alpha)$, a contradiction. We have proved that $E(L_\xi^{\mathcal{PT}_n}) \cap \Delta_\alpha = \emptyset$ and so Δ_α is not left abundant by Lemma 3.4, a contradiction.

(2) $\Longrightarrow$ (3) Let $M = \{\beta \in \mathcal{PD}_n : \mathrm{im}(\beta) \subseteq [1, r]\}$. Suppose that $\alpha \in E_r^\Delta$. Then $\mathrm{im}(\alpha) = [1, r]$. Let $\xi \in \Delta_\alpha$ be arbitrary. Then there exist $\beta, \gamma \in \mathcal{PD}_n$ such that $\xi = \beta\alpha\gamma$. Clearly, $\mathrm{im}(\xi) \subseteq \mathrm{im}(\alpha)\gamma = [1, r]\gamma$. It follows from $\gamma \in \mathcal{PD}_n$ that $r\gamma \leq r$ and so $\mathrm{im}(\xi) \subseteq [1, r] = \mathrm{im}(\alpha)$. Then $\xi \in M$. Thus $\Delta_\alpha \subseteq M$. Conversely, let $\beta \in M$ be arbitrary. Then $\mathrm{im}(\beta) \subseteq [1, r]$. Since $\alpha \in E_r^\Delta \subseteq E_r$ and $\mathrm{im}(\alpha) = [1, r]$, we have $x\alpha = x$, for $1 \leq x \leq r$. Then $\beta = \beta\alpha = \beta\alpha 1_n \in \Delta_\alpha$. Thus $M \subseteq \Delta_\alpha$. Hence, we have proved that $M = \Delta_\alpha$.

(3) $\Longrightarrow$ (1) Suppose that $\Delta_\alpha = \{\beta \in \mathcal{PD}_n : \mathrm{im}(\beta) \subseteq [1, r]\}$. Notice that $\alpha = 1_n\alpha 1_n \in \Delta_\alpha$ and $|\mathrm{im}(\alpha)| = r$. Then $\mathrm{im}(\alpha) = [1, r]$. Let $\beta \in \Delta_\alpha$ be arbitrary. Then $\mathrm{im}(\beta) \subseteq [1, r] = \mathrm{im}(\alpha)$. Put $\varepsilon = 1_{\mathrm{im}(\beta)}$. Then clearly $\varepsilon \in E(\Delta_\alpha)$ and $\mathrm{im}(\varepsilon) = \mathrm{im}(\beta)$. Thus $(\varepsilon, \beta) \in \mathscr{L}^{\mathcal{PT}_n}$. Hence, $\varepsilon \in \mathscr{L}_\beta^*(\Delta_\alpha) \cap E(\Delta_\alpha)$. □

For $1 \leq r \leq n$, put

$$\lambda_r = \left(\begin{array}{c|c|c|c} n-r+1 & n-r+2 & \cdots & n \\ n-r+1 & n-r+2 & \cdots & n \end{array}\right).$$

Then clearly $\lambda_r \in E_r$ and $\lambda_n = 1_n$. Let $\alpha \in \mathcal{PD}_n$. It is easy to see that $\alpha = \lambda_n (= 1_n)$ if and only if $\Delta_\alpha = \mathcal{PD}_n = \{\alpha \in \mathcal{PD}_n : \mathrm{dom}(\alpha) \subseteq [1, n]\}$. In fact, we have the following result:

Theorem 3.10. *Let $1 \leq r \leq n - 1$. Let $\alpha \in \mathcal{PD}_n$ with $|\mathrm{im}(\alpha)| = r$. Then the following statements are equivalent:*

(1) *Δ_α is right abundant.*

(2) *$\alpha = \lambda_r$.*

(3) *$\Delta_\alpha = \{\beta \in \mathcal{PD}_n : \mathrm{dom}(\beta) \subseteq [n - r + 1, n]\}$.*

Proof. (1) $\Longrightarrow$ (2) Suppose that Δ_α is right abundant. Then, by Lemma 3.8, α is an idempotent. Suppose that

$$\alpha = \left(\begin{array}{c|c|c|c} A_1 & A_2 & \cdots & A_r \\ c_1 & c_2 & \cdots & c_r \end{array}\right),$$

where $c_1 < c_2 < \cdots < c_r$ and $c_i = \min A_i$, for $1 \leq i \leq r$. Notice that $c_i \leq n - r + i$, for $1 \leq i \leq r$. Put

$$\xi = \left(\begin{array}{c|c|c|c} n-r+1 & n-r+2 & \cdots & n \\ c_1 & c_2 & \cdots & c_r \end{array}\right).$$

Then $\ker(\xi) = \ker(\lambda_r)$ and $\xi = \xi\alpha = \xi\alpha 1_n \in \Delta_\alpha$. Since Δ_α is right abundant, then $\mathscr{R}_\xi^*(\Delta_\alpha) \cap E(\Delta_\alpha) \neq \emptyset$. Then there exists $\eta \in E(\Delta_\alpha)$ such that $\ker(\eta) = \ker(\xi)(= \ker(\lambda_r))$. Thus, by Lemma 3.3, $\eta = \lambda_r$. Since $\eta \in \Delta_\alpha$, there exist $\beta, \gamma \in \mathcal{PD}_n$ such that $\lambda_r = \eta = \beta\alpha\gamma$. Clearly, $\mathrm{im}(\lambda_r) \subseteq \mathrm{im}(\alpha)\gamma$ and $|\mathrm{im}(\alpha)\gamma| \leq |\mathrm{im}(\alpha)|$. Notice that $|\mathrm{im}(\lambda_r)| = |\mathrm{im}(\alpha)|$. Then $\mathrm{im}(\lambda_r) = \mathrm{im}(\alpha)\gamma$. It follows from $\gamma \in \mathcal{PD}_n$ and $\mathrm{im}(\lambda_r) = [n - r + 1, n]$ that $\mathrm{im}(\alpha) = [n - r + 1, n]$. Let $x \in \mathrm{dom}(\alpha)$ be arbitrary. Then $x \geq x\alpha \geq \min \mathrm{im}(\alpha) = n - r + 1$. Thus $\mathrm{dom}(\alpha) \subseteq [n - r + 1, n] = \mathrm{im}(\alpha)$. It follows from $|\mathrm{dom}(\alpha)| \geq |\mathrm{im}(\alpha)| = r$ that $\mathrm{dom}(\alpha) = \mathrm{im}(\alpha) = [n - r + 1, n]$. Thus, by $\alpha \in \mathcal{PD}_n$, $\alpha = \lambda_r$.

(2) $\Longrightarrow$ (3) Let $M = \{\beta \in \mathcal{PD}_n : \text{dom}(\beta) \subseteq [n-r+1, n]\}$. Suppose that $\alpha = \lambda_r$. Let $\xi \in \Delta_\alpha$ be arbitrary. Then there exist $\beta, \gamma \in \mathcal{PD}_n$ such that $\xi = \beta\alpha\gamma$. Assume that there exists $1 \leq j \leq n-r$ such that $j \in \text{dom}(\xi)$. Then $j\xi = j\beta\alpha\gamma$ and so $j\beta \in \text{dom}(\alpha) = \text{dom}(\lambda_r) = [n-r+1, n]$. Since $\beta \in \mathcal{PD}_n$, we have $j\beta \leq j \leq n-r$, a contradiction. Then $\text{dom}(\xi) \subseteq [n-r+1, n]$. Thus $\Delta_\alpha \subseteq M$. Conversely, let $\beta \in M$ be arbitrary. Then $\text{dom}(\beta) \subseteq [n-r+1, n]$. Since $\alpha = \lambda_r$, we have $x\alpha = x$, for $n-r+1 \leq x \leq n$. It follows from $\beta \in \mathcal{PD}_n$ that $\beta = \beta\alpha = \beta\alpha 1_n \in \Delta_\alpha$. Thus $M \subseteq \Delta_\alpha$. Hence, we have proved that $M = \Delta_\alpha$.

(3) $\Longrightarrow$ (1) Suppose that $\Delta_\alpha = \{\beta \in \mathcal{PD}_n : \text{dom}(\beta) \subseteq [n-r+1, n]\}$. Notice that $\alpha = 1_n\alpha 1_n \in \Delta_\alpha$ and $|\text{im}(\alpha)| = r$. Then $\text{dom}(\alpha) = [n-r+1, n]$. Let $\xi \in \Delta_\alpha$ be arbitrary. Then $\text{dom}(\xi) \subseteq [n-r+1, n]$. Take $\varepsilon \in E(\mathcal{PD}_n)$ such that $\ker(\varepsilon) = \ker(\xi)$. Then clearly $(\varepsilon, \xi) \in \mathscr{R}^{\mathcal{PT}_n}$ and $\text{dom}(\varepsilon) = \text{dom}(\xi) \subseteq [n-r+1, n]$. Then $\varepsilon \in \{\beta \in \mathcal{PD}_n : \text{dom}(\beta) \subseteq [n-r+1, n]\} = \Delta_\alpha$ and so $\varepsilon \in E(\Delta_\alpha)$. Thus $\varepsilon \in \mathscr{R}^*_\xi(\Delta_\alpha) \cap E(\Delta_\alpha)$. □

Theorem 3.11. *Let $\alpha \in \mathcal{PD}_n$. Then Δ_α is abundant if and only if $\alpha = \theta_{\mathbf{n}}$ or $\alpha = 1_n$.*

Proof. Suppose that Δ_α is abundant. Then, by Lemma 3.8, α is an idempotent. We claim that $|\text{im}(\alpha)| = r \in \{0, n\}$. Notice that $\alpha = 1_n\alpha 1_n \in \Delta_\alpha$. Assume that $1 \leq r \leq n-1$, then, by Theorems 3.9 and 3.10, $\text{im}(\alpha) = [1, r]$ and $\text{dom}(\alpha) = [n-r+1, n]$. Since α is an idempotent, we have $x\alpha = x$, for $x \in \text{im}(\alpha)$. It follows that $\text{im}(\alpha) \subseteq \text{dom}(\alpha)$ and so $[1, r] = \text{im}(\alpha) \subseteq \text{dom}(\alpha) = [n-r+1, n]$, a contradiction. Thus $r \in \{0, n\}$. If $r = 0$, then clearly $\alpha = \theta_{\mathbf{n}}$. If $r = n$, then clearly $\alpha = 1_n$.

Conversely, if $\alpha = 1_n$, then $\Delta_\alpha = \mathcal{PD}_n$. Thus, by Lemma 3.2, $\Delta_\alpha = \mathcal{PD}_n$ is abundant. On the other hand, if $\alpha = \theta_{\mathbf{n}}$, then clearly $\Delta_\alpha = \{\theta_{\mathbf{n}}\}$ is abundant. □

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Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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