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Infante-webb spectrum of nonlinear 3×3 upper triangular block operator matrices

Sami Baraket[a], Anis Ben Ghorbal[a], Aref Jeribi[a,b,]

[a]*Department of Mathematics and Statistics, College of science, Imam Mohammad Ibn Saud Islamic University (IMSIU) Riyadh, Saudi Arabia*
[b]*Department of Mathematics, University of Sfax, Faculty of Sciences of Sfax Soukra Road Km 3.5, B. P. 1171, 3000, Sfax, Tunisia*

Abstract. The paper examines the Infante-Webb spectrum within the framework of nonlinear block operator matrices, focusing on the relationship between the spectra of 3×3 matrices and their constituent components. The focus of the study is on the Infante-Webb spectrum of 3×3 nonlinear block operator matrices. This study establishes the connection between the Infante-Webb spectrum of specific 3×3 nonlinear block operator matrices and the spectrum of their individual entries.

1. Introduction

The Infante-Webb spectrum, a component of nonlinear operator theory, is essential for comprehending and resolving nonlinear differential and integral problems. The expansion of semilinear pairs and the inclusion of positively homogeneous operators provide essential insights that contribute to diverse domains like mathematics, physics, biology, and engineering. The value of transdisciplinary applications is emphasized in various scientific disciplines. The Infante-Webb spectrum, first proposed by Infante and Webb in 2002, is a collection of nonlinear operators that exhibit intriguing topological characteristics, notably in the case of positively homogeneous operators. The IW-spectrum, sometimes referred to as the Infante and Cremins spectrum, encompasses semilinear couples (L, F) as described by Infante and Cremins [5]. Additionally, various spectra for nonlinear operators are investigated in references [6] to [13]. The investigation of Kachurovskij, Furi-Martelli-Vignoli, and Feng spectra in continuous nonlinear block operator matrices enhances the depth of the research field.

The paper is structured as follows. To prove the primary results of the remaining sections, we will require the preliminary and auxiliary qualities that are contained in Section 2. Infante-Webb spectra of 3×3 nonlinear block operator matrices are examined in Section 3.

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* Corresponding author: Aref Jeribi
Email addresses: SMBaraket@imamu.edu.sa (Sami Baraket), assghorbal@imamu.edu.sa (Anis Ben Ghorbal), aref.jeribi@fss.rnu.tn (Aref Jeribi)
ORCID iDs: https://orcid.org/0000-0001-8307-6593 (Sami Baraket), https://orcid.org/0000-0001-5501-327X (Anis Ben Ghorbal), https://orcid.org/0000-0001-6715-5996 (Aref Jeribi)

2. Preliminary and auxiliary properties

Let X be an infinite dimensional complex Banach space and $\mathcal{L}(X)$ be denote the set of all bounded linear operators from X into X. The product space $X \times X \times X$ is equipped with the norm

$$\|(x, y, z)\| = \max\{\|x\|, \|y\|, \|z\|\}.$$

Further, we assume that X and $X \times X \times X$ have a fixed projection scheme $\Gamma_1 = \{X_n, P_n\}$ and $\widetilde{\Gamma}_1 = \left\{X_n \times X_n \times X_n, \begin{pmatrix} P_n & 0 & 0 \\ 0 & P_n & 0 \\ 0 & 0 & P_n \end{pmatrix}\right\}$ respectively, where $\{X_n\}$ is a sequence of finite dimensional subspaces of X and $P_n : X \longrightarrow X_n$ is a linear projection with $P_n x \to x$ for every $x \in X$ and $\|P_n\| = 1$. Let $F : X \longrightarrow X$ be an operator. The operator F is called finitely continuous at x if for any finite dimensional subspace X_0 of X and every sequence $\{x_n\}$ in X_0 with $x_n \to x$, we have $F(x_n) \rightharpoonup F(x)$ (weak convergence). The operator F is said to be positively homogeneous if $F(tx) = tF(x)$ for every $x \in X$ and $t \in \mathbb{R}_+$. The operator F is called to be bounded if $F(Q)$ is bounded whenever $Q \subset X$ is bounded. Now, consider the following notation that will be used in the sequel [15]

$$[F]_B = \sup_{x \neq 0} \frac{\|F(x)\|}{\|x\|},$$

$$[F]_b = \inf_{x \neq 0} \frac{\|F(x)\|}{\|x\|}.$$

We write $F \in \mathcal{B}(X)$ if $[F]_B < \infty$ and call the operator F linearly bounded. It is easy to see that if F is a bounded linear operator, then $[F]_B = \|F\|$. We may also consider the following

$$d_R(F) = \liminf_{n \to \infty} \inf \left\{ \frac{\|P_n F(x)\|}{\|x\|} \ : \ x \in X_n, \ \ \|x\| \geq R \right\} \quad (R > 0),$$

and

$$d(F) = \sup_{R > 0} d_R(F).$$

Obviously, if F is positively homogeneous, then $d(F) = \liminf_{n \to \infty} [P_n F]_b$.

Lemma 1. *[4] Let* $q_R(G) = \limsup_{\|x\| \geq R} \frac{\|G(x)\|}{\|x\|}$*, then* $d_R(F - G) \geq d_R(F) - q_R(G)$.

Definition 1. *Given an operator* $F : X \longrightarrow X$*, we call that* F *is A-proper with respect to* Γ_1*, if*

$$P_n F_{|X_n} : X_n \longrightarrow X_n$$

is continuous for each $n \in \mathbb{N}$*, and if* $\{x_{n_j}$ *such that* $x_{n_j} \in X_{n_j}\}$ *is any bounded sequence such that*

$$\|P_{n_j} F(x_{n_j}) - P_{n_j} y\| \to 0 \quad (j \to \infty),$$

for some $y \in X$*, then there exist a subsequence* $\{x_{n_{j(k)}}\}$ *of* $\{x_{n_j}\}$ *and* $x \in X$*, such that* $x_{n_{j(k)}} \to x$ *and* $F(x) = y$.

Definition 2. *Given an operator* $F : X \longrightarrow X$*, we say that* F *is A-proper stable, if there exists some* $\varepsilon > 0$ *such that* $F + \mu I$ *is A-proper for all* $\mu \in \mathbb{C}$ *with* $|\mu| < \varepsilon$.

Note that F is A-proper stable if and only if

$$\tau(F) = \sup\{\varepsilon > 0 \ : \ F - \mu I \text{ is } A\text{-proper for every } \mu \in \mathbb{C} \text{ with } |\mu| < \varepsilon\} > 0.$$

Lemma 2. *[16] Let* X*,* Y *be Banach spaces and* D *be a subset of* X*,* $\Gamma = \{X_n, V_n; E_n, W_n\}$ *be an admissible scheme for* (X, Y)*. If* $T : D \subset X \longrightarrow Y$ *is A-proper with respect to* Γ*, and* $C : D \subset X \longrightarrow Y$ *is continuous and compact, then* $T + C$ *is A-proper with respect to* Γ.

Definition 3. *Let $F : X \longrightarrow Y$ be a continuous operator. The operator F is called stably solvable if, given any continuous compact operator $G : X \longrightarrow Y$ with $[G]_Q = 0$, the equation $F(x) = G(x)$ has a solution $x \in X$.*

Definition 4. *Let $F : X \longrightarrow Y$ be a continuous operator. We say that $F : X \longrightarrow X$ is A-stably solvable, if there exists $n_0 \in \mathbb{N}$ such that the operator $P_n F_{|X_n}$ is stably solvable for all $n \geq n_0$.*

Definition 5. *A finitely continuous operator $F : X \longrightarrow X$ is called IW-regular if F is A-stably solvable, F is A-proper stable and $d(F) > 0$. The set*

$$\rho_{IW}(F) = \{\lambda \in \mathbb{C} \ : \ F - \lambda I \text{ is IW-regular}\}$$

is called the IW-resolvent set of F, and its complement $\sigma_{IW}(F) = \mathbb{C} \setminus \rho_{IW}(F)$ is called the IW-spectrum of F. Note that if we put

$$\sigma_{\delta,A}(F) = \{\lambda \in \mathbb{C} : F - \lambda I \text{ is not A-stably solvable}\},$$

$$\sigma_{\tau}(F) = \{\lambda \in \mathbb{C} \ : \ F - \lambda I \text{ is not A-proper stable}\},$$

$$\sigma_{d}(F) = \{\lambda \in \mathbb{C} \ : \ d(F - \lambda I) = 0\}.$$

Then,

$$\sigma_{IW}(F) = \sigma_{\delta,A}(F) \bigcup \sigma_{\tau}(F) \bigcup \sigma_{d}(F).$$

In the case of a bounded linear operator $L : X \longrightarrow X$, we have

$$\sigma_{IW}(L) = \sigma(L) \bigcup \pi_A(L),$$

where $\pi_A(L) = \{\lambda \in \mathbb{C} \ : \ L - \lambda I \text{ is not } A\text{-proper}\}$ and $\sigma(L)$ is the usual spectrum of L. Consider the following essential spectra by

$$\sigma_{r,1}(A) = \{\lambda \in \mathbb{C} \ : \ A - \lambda I \text{ is injective, } \overline{R(A - \lambda I)} \neq X \text{ and } R(A - \lambda I) \text{ is closed}\},$$

$$\sigma_{p,1}(D) = \{\lambda \in \mathbb{C} \ : \ D - \lambda I \text{ is not injective and } R(D - \lambda I) = X\}.$$

3. Main results

We begin by establishing the connection between the IW-spectrum of 3×3 diagonal nonlinear operator matrices and that of their entries, as previously discussed in [1–3].

Theorem 3.1. *Let*

$$\mathrm{L}_0 = \begin{pmatrix} A & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & K \end{pmatrix} : X \times X \times X \longrightarrow X \times X \times X$$

be a diagonal nonlinear operator matrix with L_0 *is finitely continuous and* $\mathrm{L}_0(0) = 0$. *Then,*

$$\sigma_{\tau}(\mathrm{L}_0) = \sigma_{\tau}(A) \bigcup \sigma_{\tau}(E) \bigcup \sigma_{\tau}(K).$$

Proof. To prove that $\sigma_{\tau}(\mathrm{L}_0) = \sigma_{\tau}(A) \cup \sigma_{\tau}(E) \cup \sigma_{\tau}(K)$, it suffices to prove that L_0 is A-proper with respect to $\widetilde{\Gamma}_1$ if and only if both A, E and K are A-proper with respect to Γ_1. Assume that L_0 is A-proper with respect to $\widetilde{\Gamma}_1$. It is easy to see that $\begin{pmatrix} P_n & 0 & 0 \\ 0 & P_n & 0 \\ 0 & 0 & P_n \end{pmatrix} \mathrm{L}_{0|X_n \times X_n \times X_n}$ is continuous if and only if both of $P_n A_{|X_n}$, $P_n E_{|X_n}$ and $P_n K_{|X_n}$ are continuous for each $n \in \mathbb{N}$. Now let $\{x_{n_j}$ such that $x_{n_j} \in X_{n_j}\}$, $\{y_{n_j}$ such that $y_{n_j} \in X_{n_j}\}$, and $\{w_{n_j}$ such that $w_{n_j} \in X_{n_j}\}$ be bounded sequences such that

$$\|P_{n_j} A(x_{n_j}) - P_{n_j} y_1\| \to 0 \ \ (j \to \infty),$$

for some $y_1 \in X$,

$$\|P_{n_j}E(y_{n_j}) - P_{n_j}y_2\| \to 0 \quad (j \to \infty),$$

for some $y_2 \in X$, and

$$\|P_{n_j}K(w_{n_j}) - P_{n_j}y_3\| \to 0 \quad (j \to \infty),$$

for some $y_3 \in$ X. Set $z_{n_j} = (x_{n_j}, y_{n_j}, w_{n_j})^T \in X_{n_j} \times X_{n_j} \times X_{n_j}$, $y = (y_1, y_2, y_3)^T \in X \times X \times X$. Then

$$\begin{aligned}
&\left\| \begin{pmatrix} P_{n_j} & 0 & 0 \\ 0 & P_{n_j} & 0 \\ 0 & 0 & P_{n_j} \end{pmatrix} \mathrm{L}_0(z_{n_j}) - \begin{pmatrix} P_{n_j} & 0 & 0 \\ 0 & P_{n_j} & 0 \\ 0 & 0 & P_{n_j} \end{pmatrix} y \right\| \\
&= \left\| \begin{pmatrix} P_{n_j}Ax_{n_j} - P_{n_j}y_1 \\ P_{n_j}Ex_{n_j} - P_{n_j}y_2 \\ P_{n_j}Kx_{n_j} - P_{n_j}y_3 \end{pmatrix} \right\| \\
&= \max\{\|P_{n_j}A(x_{n_j}) - P_{n_j}y_1\|, \|P_{n_j}E(y_{n_j}) - P_{n_j}y_2\|, \|P_{n_j}K(x_{n_j}) - P_{n_j}y_3\|\} \to 0 \quad (j \to \infty).
\end{aligned}$$

Hence, by the A-properness of L_0, there exist a subsequence $\{z_{n_j(k)}\} = \{(x_{n_j(k)}, y_{n_j(k)}, w_{n_j(k)})^T\}$ of $\{z_{n_j}\}$ and $x = (x_1, x_2, x_3)^T \in X \times X \times X$, such that $z_{n_j(k)} \to x$ and $\mathrm{L}_0(x) = y$, i.e., $x_{n_j(k)} \to x_1$, $y_{n_j(k)} \to x_2$, $w_{n_j(k)} \to x_3$ and $A(x_1) = y_1$, $E(x_2) = y_2$, $K(x_3) = y_3$. This shows that A, E, and K are A-proper with respect to Γ_1. Conversely, suppose that A, E, and K are A-proper with respect to Γ_1. Let $\{z_{n_j}$ such that $z_{n_j} = (x_{n_j}, y_{n_j}, w_{n_j})^T \in X_{n_j} \times X_{n_j} \times X_{n_j}\}$ be a bounded sequence such that

$$\left\| \begin{pmatrix} P_{n_j} & 0 & 0 \\ 0 & P_{n_j} & 0 \\ 0 & 0 & P_{n_j} \end{pmatrix} \mathrm{L}_0(z_{n_j}) - \begin{pmatrix} P_{n_j} & 0 & 0 \\ 0 & P_{n_j} & 0 \\ 0 & 0 & P_{n_j} \end{pmatrix} y \right\| \to 0 \quad (j \to \infty)$$

for some $y = (y_1, y_2, y_3)^T \in X \times X \times X$, i.e.,

$$\max\{\|P_{n_j}A(x_{n_j}) - P_{n_j}y_1\|, \|P_{n_j}E(y_{n_j}) - P_{n_j}y_2\|, \|P_{n_j}K(w_{n_j}) - P_{n_j}y_3\|\} \to 0 \quad (j \to \infty)$$

Then

$$\|P_{n_j}A(x_{n_j}) - P_{n_j}y_1\| \to 0 \quad (j \to \infty)$$

$$\|P_{n_j}E(y_{n_j}) - P_{n_j}y_2\| \to 0 \quad (j \to \infty)$$

and

$$\|P_{n_j}K(x_{n_j}) - P_{n_j}y_3\|\} \to 0 \quad (j \to \infty).$$

Therefore, by the A-properness of A, there exist a subsequence $\{x_{n_j(k)}\}$ of $\{x_{n_j}\}$ and $x_1 \in X$, such that $x_{n_j(k)} \to x_1$ and $A(x_1) = y_1$. Taking a subsequence $\{y_{n_j(k)}\}$ of $\{y_{n_j}\}$, then

$$\|P_{n_j(k)}E(y_{n_j(k)}) - P_{n_j(k)}y_2\| \to 0 \quad (k \to \infty).$$

Taking a subsequence $\{w_{n_j(k)}\}$ of $\{w_{n_j}\}$, then

$$\|P_{n_j(k)}K(w_{n_j(k)}) - P_{n_j(k)}y_3\| \to 0 \quad (k \to \infty).$$

Also, by the A-properness of E (resp. K), there exist a subsequence $\{y_{{n_j(k)}_m}\}$ of $\{y_{n_j(k)}\}$ (resp. $\{w_{{n_j(k)}_m}\}$ of $\{w_{n_j(k)}\}$) and $x_2 \in X$ (resp. $x_3 \in X$) such that $y_{{n_j(k)}_m} \to x_2$ (resp. $w_{{n_j(k)}_m} \to x_3$) and $E(x_2) = y_2$ (resp. $K(x_3) = y_3$). Let $x = (x_1, x_2, x_3)^T \in X \times X \times X$ and $z_{{n_j(k)}_m} = (x_{{n_j(k)}_m}, y_{{n_j(k)}_m}, w_{{n_j(k)}_m})^T$, where $\{x_{{n_j(k)}_m}\}$ is a subsequence of $\{x_{n_j(k)}\}$. Then

$$z_{{n_j(k)}_m} \to x \ \text{ and } \ F(x) = y.$$

This proves that L_0 is A-proper with respect to $\widetilde{\Gamma_1}$. □

Theorem 3.2. *Let*

$$\mathrm{L}_0 = \begin{pmatrix} A & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & K \end{pmatrix} : X \times X \times X \longrightarrow X \times X \times X$$

be a diagonal nonlinear operator matrix with L_0 *is finitely continuous and* $\mathrm{L}_0(0) = 0$*. Then,*

$$\sigma_d(\mathrm{L}_0) = \sigma_d(A) \bigcup \sigma_d(E) \bigcup \sigma_d(K).$$

Proof. Let $\lambda \notin \sigma_d(A) \bigcup \sigma_d(E) \bigcup \sigma_d(K)$. Then there exist $R_1, R_2, R_3 > 0$ such that $d_{R_1}(A - \lambda I) > 0$, $d_{R_2}(E - \lambda I) > 0$ and $d_{R_3}(K - \lambda I) > 0$. Let $z = (x, y, w)^T \in X_n \times X_n \times X_n$ be any vector. Without loss of generality we may assume that $\|x\| \geq \max(\|y\|, \|w\|)$. Hence, $\|z\| = \max\{\|x\|, \|y\|, \|w\|\} = \|x\|$, and thus

$$\begin{aligned} d_{R_1}(\mathrm{L}_0 - \lambda I) &= \liminf_{n\to\infty} \inf_{\|z\|\geq R_1} \frac{\left\| \begin{pmatrix} P_n & 0 & 0 \\ 0 & P_n & 0 \\ 0 & 0 & P_n \end{pmatrix} (\mathrm{L}_0 - \lambda I)(z) \right\|}{\|z\|} \\ &= \liminf_{n\to\infty} \inf_{\|z\|\geq R_1} \frac{\max(\|P_n(A - \lambda I)(x)\|, \|P_n(E - \lambda I)(y)\|, \|P_n(K - \lambda I)(w)\|)}{\max(\|x\|, \|y\|, \|w\|)} \\ &\geq \liminf_{n\to\infty} \inf_{\|z\|\geq R_1} \frac{\|P_n(A - \lambda I)(x)\|}{\|x\|} \\ &= d_{R_1}(A - \lambda I) > 0. \end{aligned}$$

Therefore $\lambda \notin \sigma_d(F)$. Conversely, let $\lambda \notin \sigma_d(F)$. Then there exists $R > 0$ such that

$d_R(\mathrm{L}_0 - \lambda I)$

$$\begin{aligned} &= \liminf_{n\to\infty} \inf_{\|z\|\geq R} \frac{\left\| \begin{pmatrix} P_n & 0 & 0 \\ 0 & P_n & 0 \\ 0 & 0 & P_n \end{pmatrix} (\mathrm{L}_0 - \lambda I)(z) \right\|}{\|z\|} \\ &= \liminf_{n\to\infty} \inf_{\max(\|x\|,\|y\|,\|w\|)\geq R} \frac{\max(\|P_n(A - \lambda I)(x)\|, \|P_n(E - \lambda I)(y)\|, \|P_n(K - \lambda I)(w)\|}{\max(\|x\|, \|y\|, \|w\|)} > 0. \end{aligned}$$

for all $z = (x, y, w)^T \in X_n \times X_n \times X_n$. In particular, if $x = 0$, and $y = 0$, then

$$\begin{aligned} d_R(K - \lambda I) &= \liminf_{n\to\infty} \inf_{\|w\|\geq R} \frac{\|P_n(K - \lambda I)(w)\|}{\|w\|} \\ &= d_R(\mathrm{L}_0 - \lambda I). \end{aligned}$$

If $x = 0$, and $w = 0$, then

$$\begin{aligned} d_R(E - \lambda I) &= \liminf_{n\to\infty} \inf_{\|y\|\geq R} \frac{\|P_n(E - \lambda I)(y)\|}{\|y\|} \\ &= d_R(\mathrm{L}_0 - \lambda I). \end{aligned}$$

Also, if $y = 0$ and $z = 0$, then

$$\begin{aligned} d_R(A - \lambda I) &= \liminf_{n\to\infty} \inf_{\|x\|\geq R} \frac{\|P_n(A - \lambda I)(x)\|}{\|x\|} \\ &= d_R(\mathrm{L}_0 - \lambda I). \end{aligned}$$

Thus $\lambda \notin \sigma_d(A) \bigcup \sigma_d(E) \bigcup \sigma_d(K)$. □

Theorem 3.3. *Let*

$$\mathrm{L}_0 = \begin{pmatrix} A & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & K \end{pmatrix} : X \times X \times X \longrightarrow X \times X \times X$$

be a diagonal nonlinear operator matrix with L_0 *is finitely continuous and* $\mathrm{L}_0(0) = 0$*. Then,*

$$\sigma_{\delta,A}(\mathrm{L}_0) \supset \sigma_{\delta,A}(A) \bigcup \sigma_{\delta,A}(E) \bigcup \sigma_{\delta,A}(K).$$

Proof. It suffices to show that if L_0 is A-stably solvable then so are A, E and K. Suppose that L_0 is A-stably solvable. Then there exists $n_0 \in \mathbb{N}$, such that $\begin{pmatrix} P_n & 0 & 0 \\ 0 & P_n & 0 \\ 0 & 0 & P_n \end{pmatrix} \mathrm{L}_0$ is stably solvable for $n \geq n_0$. Let $H_1 : X_n \longrightarrow X_n$, $H_2 : X_n \longrightarrow X_n$ and $H_3 : X_n \longrightarrow X_n$ are continuous operators with $[H_1]_Q = 0$, $[H_2]_Q = 0$, and $[H_3]_Q = 0$. Define an operator $H : X_n \times X_n \times X_n \longrightarrow X_n \times X_n \times X_n$ by

$$H = \begin{pmatrix} H_1 & 0 & 0 \\ 0 & H_2 & 0 \\ 0 & 0 & H_3 \end{pmatrix}$$

Clearly, H is continuous and $[H]_Q = 0$. Then

$$\begin{pmatrix} P_n & 0 & 0 \\ 0 & P_n & 0 \\ 0 & 0 & P_n \end{pmatrix} \mathrm{L}_0 \begin{pmatrix} x \\ y \\ z \end{pmatrix} = H \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

has a solution $(x, y, z)^T \in X_n \times X_n \times X_n$, i.e., $P_n A(x) = H_1(x)$, $P_n E(y) = H_2(y)$, and $P_n K(z) = H_3(z)$. This proves that both of A, E and K are A-stably solvable. □

Theorem 3.4. *Let*

$$\mathrm{L}_0 = \begin{pmatrix} A & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & K \end{pmatrix} : X \times X \times X \longrightarrow X \times X \times X$$

be a diagonal nonlinear operator matrix with L_0 *is finitely continuous and* $\mathrm{L}_0(0) = 0$*. Then,*

$$\sigma_{IW}(\mathrm{L}_0) \supset \sigma_{IW}(A) \bigcup \sigma_{IW}(E) \bigcup \sigma_{IW}(K).$$

Proof. It follows immediately from Theorems 3.1, 3.2, 3.3 and the definition of the IW-spectrum. □

Corollary 1. *Let* $\mathrm{L}_0 = \begin{pmatrix} A & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & K \end{pmatrix} \in \mathcal{L}(X \times X \times X)$*. Then*

$$\sigma_{IW}(\mathrm{L}_0) = \sigma_{IW}(A) \bigcup \sigma_{IW}(E) \bigcup \sigma_{IW}(K).$$

Proof. By virtue of Theorem 3.1, we can learn that $\pi_A(\mathrm{L}_0) = \pi_A(A) \bigcup \pi_A(E) \bigcup \pi_A(K)$, and $\sigma(\mathrm{L}_0) = \sigma(A) \bigcup \sigma(E) \bigcup \sigma(K)$ is clearly valid. Then $\sigma_{IW}(\mathrm{L}_0) = \sigma(\mathrm{L}_0) \bigcup \pi_A(\mathrm{L}_0) = \sigma(A) \bigcup \sigma(E) \bigcup \sigma(K) \bigcup \pi_A(A) \bigcup \pi_A(E) \bigcup \pi_A(K) =$
$= \sigma_{IW}(A) \bigcup \sigma_{IW}(E) \bigcup \sigma_{IW}(K)$. □

Theorem 3.5. *Let*

$$\mathrm{L} = \begin{pmatrix} A & B & C \\ 0 & E & F \\ 0 & 0 & K \end{pmatrix} : X \times X \times X \longrightarrow X \times X \times X$$

be an upper triangular nonlinear operator matrix with A*,* B*,* C*,* E*,* F*,* K *are finitely continuous and* $B(0) = C(0) = E(0) = F(0) = K(0) = 0$*. If* B*,* C*, and* F *are continuous, then* $\sigma_\tau(A) \subset \sigma_\tau(\mathrm{L}) \subset \sigma_\tau(A) \bigcup \sigma_\tau(E) \bigcup \sigma_\tau(K)$*.*

Proof. First we show that $\sigma_\tau(A) \subset \sigma_\tau(\mathrm{L})$. We only need to show that if L is A-proper with respect to $\widetilde{\Gamma}_1$ then A is A-proper with respect to Γ_1. Assume that L is A-proper with respect to $\widetilde{\Gamma}_1$. Then $\begin{pmatrix} P_n & 0 & 0 \\ 0 & P_n & 0 \\ 0 & 0 & P_n \end{pmatrix} \mathrm{L}_{|X_n \times X_n \times X_n}$ is continuous for each $n \in \mathbb{N}$, which obviously implies that $P_n A_{|X_n}$ is continuous for each $n \in \mathbb{N}$. Now let $\{x_{n_j}$ such that $x_{n_j} \in X_{n_j}\}$ be a bounded sequence such that

$$\|P_{n_j} A(x_{n_j}) - P_{n_j} y_1\| \to 0 \quad (j \to \infty),$$

for some $y_1 \in X$. Set $z_{n_j} = (x_{n_j}, 0, 0)^T \in X_{n_j} \times X_{n_j} \times X_{n_j}$, $y = (y_1, 0, 0)^T \in X \times X \times X$. Then

$$\begin{aligned} \left\| \begin{pmatrix} P_{n_j} & 0 & 0 \\ 0 & P_{n_j} & 0 \\ 0 & 0 & P_{n_j} \end{pmatrix} \mathrm{L}(z_{n_j}) - \begin{pmatrix} P_{n_j} & 0 & 0 \\ 0 & P_{n_j} & 0 \\ 0 & 0 & P_{n_j} \end{pmatrix} y \right\| &= \left\| \begin{pmatrix} P_{n_j} A x_{n_j} - P_{n_j} y_1 \\ 0 \\ 0 \end{pmatrix} \right\| \\ &= \|P_{n_j} A x_{n_j} - P_{n_j} y_1\| \to 0 \quad (j \to \infty). \end{aligned}$$

Hence, by the A-properness of L, there exist a subsequence $\{z_{n_{j(k)}}\} = \{(x_{n_{j(k)}}, 0, 0)^T\}$ of $\{z_{n_j}\}$ and $x = (x_1, 0, 0)^T \in X \times X \times X$, such that $z_{n_{j(k)}} \to x$ and $\mathrm{L}(x) = y$, i.e., $x_{n_{j(k)}} \to x_1$ and $A(x_1) = y_1$, which shows that A is A-proper with respect to Γ_1. Next we show that $\sigma_\tau(\mathrm{L}) \subset \sigma_\tau(A) \bigcup \sigma_\tau(E) \bigcup \sigma_\tau(K)$. We only need to prove that if A, E, and K are A-proper with respect to Γ_1, then L is A-proper with respect to $\widetilde{\Gamma}_1$. Suppose that A, E, and K are A-proper with respect to Γ_1. Then $P_n A_{|X_n}$, $P_n E_{|X_n}$ and $P_n K_{|X_n}$ are continuous for each $n \in \mathbb{N}$, which can deduce that $\begin{pmatrix} P_n & 0 & 0 \\ 0 & P_n & 0 \\ 0 & 0 & P_n \end{pmatrix} \mathrm{L}_{|X_n \times X_n \times X_n}$ is continuous for each $n \in \mathbb{N}$. Let $\{z_{n_j}$ such that $z_{n_j} = (x_{n_j}, y_{n_j}, w_{n_j})^T \in X_{n_j} \times X_{n_j} \times X_{n_j}\}$ be a bounded sequence such that

$$\left\| \begin{pmatrix} P_{n_j} & 0 & 0 \\ 0 & P_{n_j} & 0 \\ 0 & 0 & P_{n_j} \end{pmatrix} \mathrm{L}_0(z_{n_j}) - \begin{pmatrix} P_{n_j} & 0 & 0 \\ 0 & P_{n_j} & 0 \\ 0 & 0 & P_{n_j} \end{pmatrix} y \right\| \to 0 \quad (j \to \infty)$$

for some $y = (y_1, y_2, y_3)^T \in X \times X \times X$, i.e.,

$$\max\{\|P_{n_j} A(x_{n_j}) + P_{n_j} B(y_{n_j}) + P_{n_j} C(w_{n_j}) - P_{n_j} y_1\|, \\ \|P_{n_j} E(y_{n_j}) + P_{n_j} F(w_{n_j}) - P_{n_j} y_2\|, \|P_{n_j} K(w_{n_j}) - P_{n_j} y_3\|\} \to 0 \quad (j \to \infty).$$

Then

$$\|P_{n_j} A(x_{n_j}) + P_{n_j} B(y_{n_j}) + P_{n_j} C(w_{n_j}) - P_{n_j} y_1\| \to 0 \quad (j \to \infty)$$
$$\|P_{n_j} E(y_{n_j}) + P_{n_j} F(w_{n_j}) - P_{n_j} y_2\| \to 0 \quad (j \to \infty)$$

and

$$\|P_{n_j} K(w_{n_j}) - P_{n_j} y_3\| \to 0 \quad (j \to \infty).$$

Thus, by the A-properness of E (resp. K), there exist a subsequence $\{y_{n_{j(k)}}\}$ of $\{y_{n_j}\}$ (resp. $\{w_{n_{j(k)}}\}$ of $\{w_{n_j}\}$) and $x_2 \in X$ (resp. $x_3 \in X$), such that $y_{n_{j(k)}} \to x_2$ and $E(x_2) = y_2$ (resp. $w_{n_{j(k)}} \to x_3$ and $K(x_3) = y_3$). Taking a subsequence $\{x_{n_{j(k)}}\}$ of $\{x_{n_j}\}$, then

$$\|P_{n_{j(k)}} A(x_{n_{j(k)}}) + P_{n_{j(k)}} B(y_{n_{j(k)}}) + P_{n_{j(k)}} C(w_{n_{j(k)}}) - P_{n_{j(k)}} y_1\| \to 0 \quad (k \to \infty),$$

this and the continuity of B and C imply that

$$\begin{aligned} &\|P_{n_{j(k)}} A(x_{n_{j(k)}}) - P_{n_{j(k)}} (y_1 - B(x_2) - C(x_3))\| \\ &\leq \|P_{n_{j(k)}} A(x_{n_{j(k)}}) + P_{n_{j(k)}} B(y_{n_{j(k)}}) + P_{n_{j(k)}} C(w_{n_{j(k)}}) - P_{n_{j(k)}} y_1\| \\ &+ \|P_{n_{j(k)}} (B(y_{n_{j(k)}}) - B(x_2))\| + \|P_{n_{j(k)}} (C(w_{n_{j(k)}}) - C(x_3))\| \to 0 \quad (k \to \infty). \end{aligned}$$

Hence, by the A-properness of A, there exist a subsequence $\{x_{n_j(k)_m}\}$ of $\{x_{n_j(k)}\}$ and $x_1 \in X$, such that $x_{n_j(k)_m} \to x_1$ and $A(x_1) = y_1 - B(x_2) - C(x_3)$. Let $x = (x_1, x_2, x_3)^T \in X \times X \times X$ and $\{z_{n_j(k)_m}\} = \{(x_{n_j(k)_m}, y_{n_j(k)_m}, w_{n_j(k)_m})^T\}$, where $\{y_{n_j(k)_m}\}$ is a subsequence of $\{y_{n_j(k)}\}$ and $\{w_{n_j(k)_m}\}$ is a subsequence of $\{w_{n_j(k)}\}$. Then $z_{n_j(k)_m} \to x$ and $\mathrm{L}(x) = y$, which shows that L is A-proper with respect to $\widetilde{\Gamma}_1$. □

Theorem 3.6. *Let*

$$\mathrm{L} = \begin{pmatrix} A & B & C \\ 0 & E & F \\ 0 & 0 & K \end{pmatrix} : X \times X \times X \longrightarrow X \times X \times X$$

be an upper triangular nonlinear operator matrix with A, B, C, E, F, K are finitely continuous and $B(0) = C(0) = E(0) = F(0) = K(0) = 0$. If A, B, C, E, F and K are positively homogeneous, then $\sigma_d(A) \subset \sigma_d(\mathrm{L}) \subset \sigma_d(A) \bigcup \sigma_d(E) \bigcup \sigma_d(K)$.

Proof. It is easy to see that $\sigma_d(A) \subset \sigma_d(\mathrm{L})$, and so we only need to prove that $\sigma_d(\mathrm{L}) \subset \sigma_d(A) \bigcup \sigma_d(E) \bigcup \sigma_d(K)$. Let $\lambda \in \sigma_d(\mathrm{L})$. Then $\liminf_{n\to\infty}[\widetilde{P}_n\mathrm{L} - \lambda\widetilde{P}_n]_b = 0$, where $\widetilde{P}_n = \begin{pmatrix} P_n & 0 & 0 \\ 0 & P_n & 0 \\ 0 & 0 & P_n \end{pmatrix}$. Evidently, the factorization formula

$$\widetilde{P}_n\mathrm{L} - \lambda\widetilde{P}_n = U_nR_nV_nW_nZ_n,$$

where

$$U_n = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & P_nK - \lambda P_n \end{pmatrix},$$

$$R_n = \begin{pmatrix} I & 0 & P_nC \\ 0 & I & P_nF \\ 0 & 0 & I \end{pmatrix},$$

$$V_n = \begin{pmatrix} I & 0 & 0 \\ 0 & P_nE - \lambda P_n & 0 \\ 0 & 0 & I \end{pmatrix},$$

$$W_n = \begin{pmatrix} I & P_nB & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix},$$

and

$$Z_n = \begin{pmatrix} P_nA - \lambda P_n & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}.$$

Note that $\liminf_{n\to\infty}[R_n]_b > 0$ and $\liminf_{n\to\infty}[W_n]_b > 0$. Since $[\widetilde{P}_n\mathrm{L} - \lambda\widetilde{P}_n]_b = [U_nR_nV_nW_nZ_n]_b \geq [U_n]_b[R_n]_b[V_n]_b[W_n]_b[Z_n]_b$, it follows that either $\liminf_{n\to\infty}[U_n]_b = 0$ or $\liminf_{n\to\infty}[V_n]_b = 0$ or $\liminf_{n\to\infty}[Z_n]_b = 0$, and hence $\liminf[P_nA - \lambda P_n]_b = 0$ or $\liminf_{n\to\infty}[P_nE - \lambda P_n]_b = 0$ or $\liminf_{n\to\infty}[P_nK - \lambda P_n]_b = 0$, therefore $\lambda \in \sigma_d(A) \bigcup \sigma_d(E) \bigcup \sigma_d(K)$. □

According to Theorem 3.5, we can give a characterization of the IW-spectrum for bounded linear upper triangular operator matrices, see the following theorem:

Theorem 3.7. *Let*

$$\mathrm{L} = \begin{pmatrix} A & B & C \\ 0 & E & F \\ 0 & 0 & K \end{pmatrix} \in \mathcal{L}(X \times X \times X).$$

If $A(X_n) = X_n$, $B(X_n) \subset X_n$, $C(X_n) \subset X_n$ and $F(X_n) \subset X_n$ for all $n \in \mathbb{N}$, then

$$(\sigma_{IW}(A) \setminus (\sigma(E) \bigcup \sigma(K)) \bigcup (\sigma_{IW}(E) \setminus (\sigma(A) \bigcup \sigma(K)) \bigcup (\sigma_{IW}(K) \setminus (\sigma(A) \bigcup \sigma(E)) \subset \sigma_{IW}(\mathrm{L}) \subset \sigma_{IW}(A) \bigcup \sigma_{IW}(E) \bigcup \sigma_{IW}(K).$$

Proof. From the proof of Theorem 3.5, we obtain that $\pi_A(\mathrm{L}) \subset \pi_A(A) \bigcup \pi_A(E) \bigcup \pi_A(K)$, since $\sigma(\mathrm{L}) \subset \sigma(A) \bigcup \sigma(E) \bigcup \sigma(K)$, then $\sigma_{IW}(\mathrm{L}) \subset \sigma_{IW}(A) \bigcup \sigma_{IW}(E) \bigcup \sigma_{IW}(K)$. Next, we prove that $(\sigma_{IW}(A) \setminus (\sigma(E) \bigcup \sigma(K)) \bigcup (\sigma_{IW}(E) \setminus (\sigma(A) \bigcup \sigma(K)) \bigcup (\sigma_{IW}(K) \setminus (\sigma(A) \bigcup \sigma(E)) \subset \sigma_{IW}(\mathrm{L})$. Let $\lambda \in \sigma_{IW}(A) \setminus (\sigma(E) \bigcup \sigma(K))$, assume that $\lambda \notin \sigma_{IW}(\mathrm{L})$. Then it is not hard to see that $\lambda \in \rho(A)$, and again from Theorem 3.5, we obtain that $\lambda \notin \pi_A(A)$. Therefore $\lambda \in \rho_{IW}(A)$, which contradicts $\lambda \in \sigma_{IW}(A)$, thus $\sigma_{IW}(A) \setminus (\sigma(E) \bigcup \sigma(K)) \subset \sigma_{IW}(\mathrm{L})$. Also, let $\lambda \in \sigma_{IW}(K) \setminus (\sigma(A) \bigcup \sigma(E))$, assume that $\lambda \notin \sigma_{IW}(\mathrm{L})$. Then we easily obtain that $\lambda \in \rho(E)$, and $\lambda \notin \pi_A(E)$. In fact, let $\{y_{n_j}$ such that $y_{n_j} \in X_{n_j}\}$ be a bounded sequence such that

$$\|P_{n_j}(E - \lambda I)(y_{n_j}) - P_{n_j} y_2\| \to 0 \quad (j \to \infty),$$

for some $y_2 \in X$. Set $z_{n_j} = (-(A - \lambda I)^{-1} B y_{n_j}, y_{n_j}, 0)^T \in X_{n_j} \times X_{n_j} \times X_{n_j}$, $y = (0, y_2, 0)^T \in X \times X$. Then

$$\begin{aligned}
&\left\| \begin{pmatrix} P_{n_j} & 0 & 0 \\ 0 & P_{n_j} & 0 \\ 0 & 0 & P_{n_j} \end{pmatrix} (\mathrm{L} - \lambda I)(z_{n_j}) - \begin{pmatrix} P_{n_j} & 0 & 0 \\ 0 & P_{n_j} & 0 \\ 0 & 0 & P_{n_j} \end{pmatrix} y \right\| \\
&= \left\| \begin{pmatrix} 0 \\ P_{n_j}(E - \lambda) y_{n_j} - P_{n_j} y_2 \\ 0 \end{pmatrix} \right\| \\
&= \|P_{n_j}(E - \lambda I)(y_{n_j}) - P_{n_j} y_2\| \to 0 \quad (j \to \infty).
\end{aligned}$$

Hence, by the A-properness of $\mathrm{L} - \lambda I$, there exist a subsequence $\{z_{n_{j(k)}}\} = \{(-(A - \lambda I)^{-1} B y_{n_{j(k)}}, y_{n_{j(k)}}, 0\}$ of $\{z_{n_j}\}$ and $x = (-(A - \lambda I)^{-1} B x_2, x_2, 0)^T \in X \times X \times X$, such that $z_{n_{j(k)}} \to x$ and $\mathrm{L}(x) = y$, i.e., $y_{n_{j(k)}} \to x_2$ and $(E - \lambda)(x_2) = y_2$, which shows that $E - \lambda I$ is A-proper with respect to Γ_1, i.e., $\lambda \notin \pi_A(E)$. Thus $\lambda \in \rho_{IW}(E)$, which contradicts $\lambda \in \sigma_{IW}(E)$, and hence $\sigma_{IW}(E) \setminus (\sigma(A) \bigcup \sigma(E)) \subset \sigma_{IW}(\mathrm{L})$. The same reasoning for the set $\sigma_{IW}(K) \setminus (\sigma(A) \bigcup \sigma(E))$. The proof is completed. □

Corollary 2. *Let*

$$\mathrm{L} = \begin{pmatrix} A & B & C \\ 0 & E & F \\ 0 & 0 & K \end{pmatrix} \in \mathcal{L}(X \times X \times X)$$

with $A(X_n) = X_n$ *and* $B(X_n) \subset X_n$, $C(X_n) \subset X_n$, $F(X_n) \subset X_n$ *for all* $n \in \mathbb{N}$. *If* $\sigma(A) \bigcap \sigma(E) \bigcap \sigma(K) = \emptyset$, *then*

$$\sigma_{IW}(L) = \sigma_{IW}(A) \bigcup \sigma_{IW}(E) \bigcup \sigma_{IW}(K).$$

Proof. Since $(\sigma_{IW}(A) \setminus (\sigma(E) \bigcup \sigma(K)) \bigcup (\sigma_{IW}(E) \setminus (\sigma(A) \bigcup \sigma(K)) \bigcup (\sigma_{IW}(K) \setminus (\sigma(A) \bigcup \sigma(E)) = (\sigma_{IW}(A) \bigcup \sigma_{IW}(E) \bigcup \sigma_{IW}(K)) \setminus (\sigma(A) \bigcap \sigma(E) \bigcap \sigma(K))$, it follows from Theorem 3.7 that $\sigma_{IW}(L) = \sigma_{IW}(A) \bigcup \sigma_{IW}(E) \bigcup \sigma_{IW}(K)$. □

In Corollary 2, the conditions for $\sigma_{IW}(\mathrm{L}) = \sigma_{IW}(A) \bigcup \sigma_{IW}(E) \bigcup \sigma_{IW}(K)$ are relatively strong. In the following, according to the fact that A-properness is invariant under compact perturbation, we will give a weaker condition.

Corollary 3. *Let*

$$\mathrm{L} = \begin{pmatrix} A & B & C \\ 0 & E & F \\ 0 & 0 & K \end{pmatrix} \in \mathcal{L}(X \times X \times X)$$

with B *is compact. If* $\sigma_{r,1}(A) \bigcap \sigma_{p,1}(E) \bigcap \sigma_{p,1}(K) = \emptyset$, *then*

$$\sigma_{IW}(L) = \sigma_{IW}(A) \bigcup \sigma_{IW}(E) \bigcup \sigma_{IW}(K).$$

Proof. From Lemma 2, we infer that $\pi_A(\mathrm{L}) = \pi_A(A) \bigcup \pi_A(E) \bigcup \pi_A(K)$. Since $\sigma(A) \bigcup \sigma(E) \bigcup \sigma(K) = \sigma(\mathrm{L}) \bigcup (\sigma_{r,1}(A) \bigcap \sigma_{p,1}(E) \bigcap \sigma_{p,1}(K))$ (see [14]), we have the desired result immediately. □

By using Lemma 2 and 1, we have the following

Lemma 3. *Let M, N are finitely continuous.*

(*i*) *If N is continuous and compact, then* $\sigma_\tau(M+N) = \sigma_\tau(M)$.
(*ii*) *If* $q_R(N) = 0$ *for some* $R > 0$, *then* $\sigma_d(M+N) = \sigma_d(M)$.
(*iii*) *If N is continuous and* $[N]_Q = 0$, *then* $\sigma_{\delta,A}(M+N) = \sigma_{\delta,A}(M)$.
(*iv*) *If N is continuous, compact and* $q_R(N) = 0$ *for some* $R > 0$, *then* $\sigma_{IW}(M+N) = \sigma_{IW}(M)$.

Theorem 3.8. *Let*

$$\mathrm{L} = \begin{pmatrix} A & B & C \\ 0 & E & F \\ 0 & 0 & K \end{pmatrix} : X \times X \times X \longrightarrow X \times X \times X$$

be an upper triangular nonlinear operator matrix with A, B, C, E, F, K are finitely continuous and $A(0) = E(0) = K(0) = 0$.

(*i*) *If B, C and F are continuous and compact, then* $\sigma_\tau(\mathrm{L}) = \sigma_\tau(A) \bigcup \sigma_\tau(E) \bigcup \sigma_\tau(K)$.
(*ii*) *If* $q_R(B) = 0$ *for some* $R > 0$, *then* $\sigma_d(\mathrm{L}) = \sigma_d(A) \bigcup \sigma_d(E) \bigcup \sigma_d(K)$.
(*iii*) *If B is continuous and* $[B]_Q = 0$, *then* $\sigma_{\delta,A}(\mathrm{L}) \supset \sigma_{\delta,A}(A) \bigcup \sigma_{\delta,A}(E) \bigcup \sigma_{\delta,A}(K)$.
(*iv*) *If B is continuous, compact and* $q_R(B) = 0$ *for some* $R > 0$, *then*

$$\sigma_{IW}(\mathrm{L}) \supset \sigma_{IW}(A) \bigcup \sigma_{IW}(E) \bigcup \sigma_{IW}(K).$$

Proof. Let $M = \begin{pmatrix} A & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & K \end{pmatrix}$, $N = \begin{pmatrix} 0 & B & C \\ 0 & 0 & F \\ 0 & 0 & 0 \end{pmatrix}$. Then N is continuous, compact and $q_R(N) = 0$ for some $R > 0$. Thus, by Lemma 3, we know that $\sigma_*(\mathrm{L}) = \sigma_*(M)$, where $\sigma_* \in \{\sigma_\tau, \sigma_d, \sigma_{\delta,A}, \sigma_{IW}\}$, and by applying Theorems 3.1, 3.2, 3.3, 3.4 the desired results can be obtained immediately. □

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References

[1] Jeribi, A: Spectral theory and applications of linear operators and block operator matrices. Springer, Cham, 2015. xvi+599 pp. ISBN: 978-3-319-17565-2; 978-3-319-17566-9
[2] Jeribi, A: Denseness, bases and frames in Banach spaces and applications. (English) Zbl 06849137 Berlin : De Gruyter (ISBN 978-3-11-048488-5/hbk ; 978-3-11-049386-3/ebook). xv, 406 p. (2018).
[3] Jeribi, A: Perturbation theory for linear operators: Denseness and bases with applications. Springer-Verlag (ISBN 978-981-16-2527-5), Singapore (2021).
[4] G. Infante, J. R. L. Webb. A finite dimensional approach to nonlinear spectral theory, Non- linear Anal, 2002, 51: 171-188.
[5] G. Infante, C. T. Cremins. A semilinear A-spectrum, Discrete Contin Dyn Syst Series S, 2008, 1: 235-242.
[6] J. Appell, E D Pascale, A Vignoli. Nonlinear spectral theory, Berlin: Walter de Gruyter, 2004.
[7] W. Feng. A new spectral theory for nonlinear operators and its applications, Abstract Appl Anal, 1997, 2: 163-183.
[8] M. Furi, M Martelli, A. Vignoli. Contributions to the spectral theory for nonlinear operators in Banach spaces. Ann Matem Pura Appl, 1978, 118: 229-294.
[9] J. Appell, E. De, A. Vignoli. A comparison of different spectra for nonlinear operators, Nonlinear Anal, 2000, 40: 73-90.
[10] J. Appell, E. Giorgieri, M. Vath. On a class of maps related to the Furi-Martelli-Vignoli spectrum, Ann Matem Pura Appl, 2001, 179: 215-228.
[11] J. Ize, A. Vignoli. Equivariant nonlinear spectrum, J Fixed Point Theor Appl, 2013, 13: 51-62.
[12] P. Santucci, M. Vath. Grasping the phantom-a new approach to nonlinear spectral theory, Ann Matem Pura Appl, 2001, 180: 255-284.

[13] L. S. Wang, Z. B. Xu. Quantitative properties of nonlinear Lipschitz countious operators. IV. Spectral theory(in Chinese), Acta Math Sinica, 1995, 38, 628-631.
[14] A. Chen, Q. M. Bai, D. Y. Wu. Spectra of 2×2 upper triangular operator matrices with unbounded entries(in Chinese), Sci Sin Math, 2016, 46, 157-168.
[15] J. Appell, E. D. Pascale, A. Vignoli. Nonlinear spectral theory, Berlin: Walter de Gruyter, 2004.
[16] Petryshyn W V. Approximation Solvability of Nonlinear Functional and Differential Equa- tions, New York: Marcel Dekker, 1993.