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Sharp constants for Hausdorff-type operators on power-weighted local Morrey-type spaces

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Abstract. In this paper, we give the necessary and sufficient conditions for the boundedness of two Hausdorff-type operators on power-weighted local Morrey-type spaces. Meanwhile, the corresponding sharp constants are also obtained. As applications, the sharp estimates for the fractional Hardy operator and its adjoint operator, the weighted Hardy–Littlewood average operator and the weighted Cesàro operator on power-weighted local Morrey-type spaces are established.

1. Introduction

In this paper, we consider the following two Hausdorff-type operators:

$$\mathcal{H}_{\Phi}(f)(x) = \int_{\mathbb{R}^n} \frac{\Phi(y)}{|y|^n} f\left(\frac{x}{|y|}\right) dy \tag{1}$$

and

$$\widetilde{\mathcal{H}_{\Phi,\beta}}(f)(x) = \int_{\mathbb{R}^n} \frac{\Phi(\frac{x}{|y|})}{|y|^{n-\beta}} f(y) dy, \tag{2}$$

where $x \in \mathbb{R}^n$, Φ are nonnegative measurable functions on $\mathbb{R}^n$ and $0 \leq \beta < n$.

The operator $\mathcal{H}_{\Phi}$, named as the n-dimensional Hausdorff operator, was initially introduced by Andersen in [4]. Another n-dimensional Hausdorff operator $\widetilde{\mathcal{H}_{\Phi,0}}$ was introduced by Chen et al. [10]. $\widetilde{\mathcal{H}_{\Phi,\beta}}$, as the fractional version of $\widetilde{\mathcal{H}_{\Phi,0}}$, was defined by Lin and Sun in [21]. The boundedness and sharp estimates for the above mentioned Hausdorff-type operators and some related operators have been intensively studied; see [5, 11, 16, 23–25, 27, 28] and the references therein.

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Recently, the necessary and sufficient conditions for the boundedness of $\mathcal{H}_\Phi$ given by (1) on local Morrey-type spaces have been obtained by Burenkov and Liflyand in [6] (see also [8]). In 2023, An et al. [3] further established the boundedness and sharp constants for Hausdorff-type operators of different forms including $\widetilde{\mathcal{H}_{\Phi,0}}$ on local Morrey-type spaces. Here, the local Morrey-type space was introduced in [7]; see also [1, 17, 18]. Let $0 < p, q \le \infty$ and $0 \le \lambda < \infty$. The local Morrey-type space $LM^\lambda_{p,q}(\mathbb{R}^n)$ is the set of all measurable functions f on $\mathbb{R}^n$ for which

$$\|f\|_{LM^\lambda_{p,q}} = \left(\int_0^\infty \left(\frac{\|f\|_{L^p(B(0,r))}}{r^\lambda}\right)^q \frac{dr}{r}\right)^{\frac{1}{q}} < \infty$$

if $q < \infty$, and

$$\|f\|_{LM^\lambda_{p,\infty}} = \sup_{r>0} \frac{\|f\|_{L^p(B(0,r))}}{r^\lambda} < \infty$$

if $q = \infty$. When $p = \infty$, we have to make some ordinary modifications. Here and in what follows, $B(0, r)$ is the open ball with center at the origin and radius r. The space $LM^\lambda_{p,\infty}(\mathbb{R}^n)$ is just the central Morrey space (see [2]). If $\lambda = 0$, then $LM^0_{p,\infty}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$.

As is well known, the weighted theory is an important branch in harmonic analysis. Power weight, as the simplest weight, plays a key role in the weighted estimates for some average operators. Inspired by the definition of the local Morrey-type space, we provide the corresponding power-weighted version as follows. Here and hereafter, we do not consider the case $p = \infty$ since it does not make much sense. Let $0 < p < \infty$, $0 < q \le \infty$, $\alpha \in \mathbb{R}$ and $0 \le \lambda < \infty$. The power-weighted local Morrey-type space $LM^{\lambda,\alpha}_{p,q}(\mathbb{R}^n)$ is the set of all measurable functions f on $\mathbb{R}^n$ satisfying

$$\|f\|_{LM^{\lambda,\alpha}_{p,q}} = \left(\int_0^\infty \left(\frac{1}{r^\lambda}\left(\int_{B(0,r)} |f(y)|^p |y|^\alpha dy\right)^{\frac{1}{p}}\right)^q \frac{dr}{r}\right)^{\frac{1}{q}} < \infty$$

when $q < \infty$, and

$$\|f\|_{LM^{\lambda,\alpha}_{p,\infty}} = \sup_{r>0} \frac{1}{r^\lambda}\left(\int_{B(0,r)} |f(y)|^p |y|^\alpha dy\right)^{\frac{1}{p}} < \infty$$

when $q = \infty$. Similar to the proof of [7, Lemma 1] (see also [18]), we find that the space $LM^{\lambda,\alpha}_{p,q}(\mathbb{R}^n)$ is not trivial, in the sense that $LM^{\lambda,\alpha}_{p,q}(\mathbb{R}^n) \neq \Theta$, if and only if

$$\lambda > 0 \text{ if } q < \infty \quad \text{and} \quad \lambda \ge 0 \text{ if } q = \infty, \tag{3}$$

where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

This paper is organized as follows. In Section 2, we obtain the necessary and sufficient conditions for the Hausdorff-type operators given by (1) and (2) on power-weighted local Morrey-type spaces, and calculate the operator norms by constructing suitable radial testing functions. As applications, we establish the sharp estimates for the fractional Hardy operator and its adjoint operator, the weighted Hardy–Littlewood average operator and the weighted Cesàro operator on power-weighted local Morrey-type spaces in Section 3.

Throughout this paper, ω_n denotes the area of $\mathbb{S}^{n-1}$ (the unit sphere in $\mathbb{R}^n$ centered at the origin), and ν_n is the volume of the unit ball in $\mathbb{R}^n$. For a measurable set E, $|E|$ and χ_E represent the Lebesgue measure and the characteristic function of E, respectively. Given $1 \le p < \infty$, p' denotes the conjugate index of p, that is, $1/p + 1/p' = 1$ for $1 < p < \infty$, and $1' = \infty$.

2. Main results

Theorem 2.1. *Assume that Φ is a nonnegative, measurable and radial function. Let $1 \le p < \infty$, $1 \le q \le \infty$ and let* (3) *be satisfied. Let $0 \le \beta < n$, $\alpha, \gamma \in \mathbb{R}$ satisfy $\beta = \frac{\alpha-\gamma}{p}$. Then $\widetilde{\mathcal{H}_{\Phi,\beta}}$ is bounded from $LM^{\lambda,\alpha}_{p,q}(\mathbb{R}^n)$ to $LM^{\lambda,\gamma}_{p,q}(\mathbb{R}^n)$, that* *is,*

$$\|\widetilde{\mathcal{H}_{\Phi,\beta}}(f)\|_{LM_{p,q}^{\lambda,\gamma}} \leq C\|f\|_{LM_{p,q}^{\lambda,\alpha}}, \quad \forall f \in LM_{p,q}^{\lambda,\alpha}(\mathbb{R}^n)$$

for some nonnegative constant C if and only if

$$C_{\Phi,1} = \omega_n \int_0^\infty \frac{\Phi(t)}{t^{\lambda-\frac{\gamma+n}{p}+1}} dt < \infty.$$

Moreover, if $C_{\Phi,1} < \infty$, *then*

$$\|\widetilde{\mathcal{H}_{\Phi,\beta}}\|_{LM_{p,q}^{\lambda,\alpha}(\mathbb{R}^n)\to LM_{p,q}^{\lambda,\gamma}(\mathbb{R}^n)} = C_{\Phi,1}.$$

Proof. We first consider the "if" part. By using polar coordinates,

$$\begin{aligned}
\widetilde{\mathcal{H}_{\Phi,\beta}}(f)(x) &= \int_0^\infty \int_{\mathbb{S}^{n-1}} \frac{\Phi\left(\frac{|x|}{r}\right)}{r^{1-\beta}} f(ry')d\sigma(y')dr \\
&= \int_0^\infty \int_{\mathbb{S}^{n-1}} f\left(\frac{|x|}{t}y'\right)d\sigma(y')|x|^\beta \frac{\Phi(t)}{t^{\beta+1}}dt,
\end{aligned}$$

where $y' \in \mathbb{S}^{n-1}$ and $d\sigma(y')$ is the induced Lebesgue measure on $\mathbb{S}^{n-1}$.

Noting that $1 \leq p < \infty$ and $\beta = \frac{\alpha-\gamma}{p}$, by applying Minkowski's inequality and Hölder's inequality, we have

$$\begin{aligned}
\left(\int_{B(0,r)} \left|\widetilde{\mathcal{H}_{\Phi,\beta}}(f)(x)\right|^p |x|^\gamma dx\right)^{1/p} &= \left(\int_{B(0,r)} \left|\int_0^\infty \int_{\mathbb{S}^{n-1}} f\left(\frac{|x|}{t}y'\right)d\sigma(y')\frac{\Phi(t)}{t^{\beta+1}}dt\right|^p |x|^{p\beta+\gamma}dx\right)^{1/p} \\
&\leq \int_0^\infty \left(\int_{\mathbb{S}^{n-1}} \left(\int_{B(0,r)} \left|f\left(\frac{|x|}{t}y'\right)\right|^p |x|^{p\beta+\gamma}dx\right)^{1/p} d\sigma(y')\right)\frac{\Phi(t)}{t^{\beta+1}}dt \\
&\leq \omega_n^{1/p'} \int_0^\infty \left(\int_{\mathbb{S}^{n-1}} \int_{B(0,r)} \left|f\left(\frac{|x|}{t}y'\right)\right|^p |x|^{p\beta+\gamma}dxd\sigma(y')\right)^{1/p} \frac{\Phi(t)}{t^{\beta+1}}dt \\
&= \omega_n \int_0^\infty \left(\int_{B(0,\frac{r}{t})} \left|f(y)\right|^p |y|^\alpha dy\right)^{1/p} \Phi(t)t^{\frac{\gamma+n}{p}}\frac{dt}{t}.
\end{aligned}$$

We now devide q into two cases: $1 \leq q < \infty$ and $q = \infty$.

If $1 \leq q < \infty$, then it follows from the Minkowski's inequality that for any $\lambda > 0$,

$$\begin{aligned}
\|\widetilde{\mathcal{H}_{\Phi,\beta}}(f)\|_{LM_{p,q}^{\lambda,\gamma}} &\leq \omega_n \left(\int_0^\infty r^{-q\lambda}\left(\int_0^\infty \left(\int_{B(0,\frac{r}{t})} \left|f(y)\right|^p |y|^\alpha dy\right)^{1/p} \Phi(t)t^{\frac{\gamma+n}{p}}\frac{dt}{t}\right)^q \frac{dr}{r}\right)^{1/q} \\
&\leq \omega_n \int_0^\infty \left(\int_0^\infty r^{-q\lambda}\left(\int_{B(0,\frac{r}{t})} \left|f(y)\right|^p |y|^\alpha dy\right)^{q/p} \frac{dr}{r}\right)^{1/q} \Phi(t)t^{\frac{\gamma+n}{p}}\frac{dt}{t} \\
&\leq \omega_n \int_0^\infty \left(\int_0^\infty \frac{1}{\left(\frac{r}{t}\right)^{q\lambda}}\left(\int_{B(0,\frac{r}{t})} \left|f(y)\right|^p |y|^\alpha dy\right)^{q/p} \frac{dr}{r}\right)^{1/q} \Phi(t)t^{\frac{\gamma+n}{p}-\lambda}\frac{dt}{t} \\
&\leq \omega_n \int_0^\infty \frac{\Phi(t)}{t^{\lambda-\frac{\gamma+n}{p}+1}}dt\|f\|_{LM_{p,q}^{\lambda,\alpha}}.
\end{aligned}$$

If $q = \infty$, then for any $\lambda \geq 0$, there holds

$$\begin{aligned}\frac{1}{r^\lambda}\int_0^\infty\left(\int_{B(0,\frac{r}{t})}\left|f(y)\right|^p|y|^\alpha dy\right)^{1/p}\Phi(t)t^{\frac{\gamma+n}{p}}\frac{dt}{t} &= \int_0^\infty\frac{1}{\left(\frac{r}{t}\right)^\lambda}\left(\int_{B(0,\frac{r}{t})}\left|f(y)\right|^p|y|^\alpha dy\right)^{1/p}\Phi(t)t^{\frac{\gamma+n}{p}-\lambda}\frac{dt}{t}\\ &\leq \int_0^\infty\frac{\Phi(t)}{t^{\lambda-\frac{\gamma+n}{p}+1}}dt\|f\|_{LM^{\lambda,\alpha}_{p,\infty}}.\end{aligned}$$

By taking the supremum over $r > 0$, we obtain

$$\|\widetilde{\mathcal{H}_{\Phi,\beta}}(f)\|_{LM^{\lambda,\gamma}_{p,\infty}} \leq \omega_n\int_0^\infty\frac{\Phi(t)}{t^{\lambda-\frac{\gamma+n}{p}+1}}dt\|f\|_{LM^{\lambda,\alpha}_{p,\infty}}.$$

Now we turn to the "only if" part. To show $C_{\Phi,1} < \infty$, we also consider q for two cases: $1 \leq q \leq \infty$ and $q = \infty$.

Case 1: $1 \leq q < \infty$. For any $\lambda > 0$ and sufficiently small $\epsilon > 0$, we choose a real number δ depending on ϵ (to be chosen later) satisfying $0 < \delta + \frac{\alpha+n}{p} < \lambda$. Take

$$f_\delta(x) = |x|^\delta\chi_{\{|x|>1\}}(x). \tag{4}$$

By an estimate similar to [3, P. 1140], we obtain

$$\begin{aligned}\|f_\delta\|^q_{LM^{\lambda,\alpha}_{p,q}} &= \int_1^\infty r^{-q\lambda-1}\left(\int_{1<|y|<r}|y|^{p\delta+\alpha}dy\right)^{q/p}dr\\ &= \left(\frac{\omega_n}{n+p\delta+\alpha}\right)^{q/p}\frac{1}{n+p\delta+\alpha}B\left(\frac{q}{p}+1,\frac{q\lambda}{n+p\delta+\alpha}-\frac{q}{p}\right) < \infty,\end{aligned} \tag{5}$$

where we have used the definition of the Beta function given by

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt = \int_0^\infty\frac{t^{a-1}}{(1+t)^{a+b}}dt$$

for $a, b > 0$. This implies $f_\delta \in LM^{\lambda,\alpha}_{p,q}(\mathbb{R}^n)$. By polar coordinates, $\widetilde{\mathcal{H}_{\Phi,\beta}}(f_\delta)$ can be represented as

$$\begin{aligned}\widetilde{\mathcal{H}_{\Phi,\beta}}(f_\delta)(x) &= \int_0^\infty\int_{\mathbb{S}^{n-1}}f_\delta\left(\frac{|x|}{t}y'\right)d\sigma(y')|x|^\beta\frac{\Phi(t)}{t^{\beta+1}}dt\\ &= \omega_n|x|^{\delta+\beta}\int_0^{|x|}\Phi(t)t^{-\delta-\beta-1}dt.\end{aligned}$$

Noting that $\beta = \frac{\alpha-\gamma}{p}$, we have

$$\begin{aligned}\|\widetilde{\mathcal{H}_{\Phi,\beta}}(f_\delta)\|^q_{LM^{\lambda,\gamma}_{p,q}} &= \omega_n^q\int_0^\infty r^{-q\lambda-1}\left(\int_{|x|<r}\left(\int_0^{|x|}\Phi(t)t^{-\delta-\beta-1}dt\right)^p|x|^{p(\delta+\beta)+\gamma}dx\right)^{q/p}dr\\ &\geq \omega_n^q\int_{\epsilon^{-1}}^\infty r^{-q\lambda-1}\left(\int_{\epsilon^{-1}<|x|<r}\left(\int_0^{\epsilon^{-1}}\Phi(t)t^{-\delta-\beta-1}dt\right)^p|x|^{p(\delta+\beta)+\gamma}dx\right)^{q/p}dr\\ &= \left(\omega_n\int_0^{\epsilon^{-1}}\Phi(t)t^{-\delta-\beta-1}dt\right)^q\int_{\epsilon^{-1}}^\infty r^{-q\lambda-1}\left(\int_{\epsilon^{-1}<|x|<r}|x|^{p\delta+\alpha}dx\right)^{q/p}dr.\end{aligned}$$

Similar to the estimate of (5), we have

$$\int_{\epsilon^{-1}}^{\infty} r^{-q\lambda-1}\left(\int_{\epsilon^{-1}<|x|<r}|x|^{p\delta+\alpha}dx\right)^{q/p}dr=\epsilon^{\left(\lambda-\delta-\frac{\alpha+n}{p}\right)q}\|f_\delta\|^q_{LM^{\lambda,\alpha}_{p,q}}.$$

Thus,

$$\|\widetilde{\mathcal{H}_{\Phi,\beta}}(f_\delta)\|^q_{LM^{\lambda,\gamma}_{p,q}}\geq\left(\omega_n\int_0^{\epsilon^{-1}}\Phi(t)t^{-\delta-\beta-1}dt\right)^q\epsilon^{\left(\lambda-\delta-\frac{\alpha+n}{p}\right)q}\|f_\delta\|^q_{LM^{\lambda,\alpha}_{p,q}}.$$

Case 2: $q=\infty$. We consider two cases: $\lambda>0$ and $\lambda=0$.

Let $\lambda>0$ and $1\leq p<\infty$. Take f_δ as in (4). By the condition $0<\delta+\frac{\alpha+n}{p}<\lambda$, it is easy to verify that

$$\begin{aligned}\|f_\delta\|_{LM^{\lambda,\alpha}_{p,\infty}}&=\sup_{r>1}r^{-\lambda}\left(\int_{1<|y|<r}|y|^{p\delta+\alpha}dy\right)^{1/p}\\&=\left(\frac{\omega_n}{n+p\delta+\alpha}\right)^{1/p}\sup_{r>1}r^{-\lambda}\left(r^{p\delta+\alpha+n}-1\right)^{1/p}\\&=\left(\frac{\omega_n}{n+p\delta+\alpha}\right)^{1/p}\left(\frac{n+p\delta+\alpha}{\lambda p-n-p\delta-\alpha}\right)^{1/p}\left(\frac{\lambda p-n-p\delta-\alpha}{\lambda p}\right)^{\frac{\lambda}{n+p\delta+\alpha}}<\infty,\end{aligned}$$

which yields $f_\delta\in LM^{\lambda,\alpha}_{p,\infty}(\mathbb{R}^n)$. Thus, by using $\beta=\frac{\alpha-\gamma}{p}$, we get

$$\begin{aligned}\|\widetilde{\mathcal{H}_{\Phi,\beta}}(f_\delta)\|_{LM^{\lambda,\gamma}_{p,\infty}}&=\omega_n\sup_{r>0}r^{-\lambda}\left(\int_{|x|<r}\left|\int_0^{|x|}\Phi(t)t^{-\delta-\beta-1}dt\right|^p|x|^{p(\delta+\beta)+\gamma}dx\right)^{1/p}\\&\geq\omega_n\sup_{r>\epsilon^{-1}}r^{-\lambda}\left(\int_{\epsilon^{-1}<|x|<r}\left|\int_0^{\epsilon^{-1}}\Phi(t)t^{-\delta-\beta-1}dt\right|^p|x|^{p\delta+\alpha}dx\right)^{1/p}\\&=\omega_n\int_0^{\epsilon^{-1}}\Phi(t)t^{-\delta-\beta-1}dt\sup_{r>\epsilon^{-1}}r^{-\lambda}\left(\int_{\epsilon^{-1}<|x|<r}|x|^{p\delta+\alpha}dx\right)^{1/p}\\&=\omega_n\int_0^{\epsilon^{-1}}\Phi(t)t^{-\delta-\beta-1}dt\cdot\epsilon^{\lambda-\delta-\frac{\alpha+n}{p}}\|f_\delta\|_{LM^{\lambda,\alpha}_{p,\infty}}.\end{aligned}$$

If $\lambda=0$, notice that $LM^{0,\alpha}_{p,\infty}(\mathbb{R}^n)=L^p_\alpha(\mathbb{R}^n)$, where

$$L^p_\alpha(\mathbb{R}^n)=\left\{f:\|f\|_{L^p_\alpha}=\left(\int_{\mathbb{R}^n}|f(x)|^p|x|^\alpha dx\right)^{1/p}<\infty\right\}.$$

For suffficiently small $\epsilon>0$, we choose

$$f_\epsilon(x)=|x|^{-\frac{\alpha+n}{p}-\epsilon}\chi_{\{|x|>1\}}(x)\quad\text{for }1\leq p<\infty.\tag{6}$$

Then

$$\|f_\epsilon\|_{LM^{0,\alpha}_{p,\infty}}=\|f_\epsilon\|_{L^p_\alpha}=\left(\frac{\omega_n}{p\epsilon}\right)^{1/p}<\infty,$$

which means $f_\epsilon\in LM^{0,\alpha}_{p,\infty}(\mathbb{R}^n)$. Moreover, by an argument similar to the estimates for the case $\lambda>0$, we have

$$\|\widetilde{\mathcal{H}_{\Phi,\beta}}(f_\epsilon)\|_{LM^{0,\gamma}_{p,\infty}}\geq\omega_n\int_0^{\epsilon^{-1}}\Phi(t)t^{\frac{n+\gamma}{p}-1}dt\cdot\epsilon^\epsilon\|f_\epsilon\|_{LM^{0,\alpha}_{p,\infty}}.$$

Combining Cases 1 and 2, taking $\delta = \lambda - \frac{\alpha+n}{p} - \epsilon$, we obtain

$$\begin{aligned}\|\widetilde{\mathcal{H}_{\Phi,\beta}}\|_{LM^{\lambda,\alpha}_{p,q}(\mathbb{R}^n)\to LM^{\lambda,\gamma}_{p,q}(\mathbb{R}^n)} &\geq \frac{\|\widetilde{\mathcal{H}_{\Phi,\beta}}(f_\delta)\|_{LM^{\lambda,\gamma}_{p,q}}}{\|f_\delta\|_{LM^{\lambda,\alpha}_{p,q}}} \\ &\geq \epsilon^\epsilon \omega_n \int_0^{\epsilon^{-1}} \Phi(t) t^{-\lambda+\frac{\gamma+n}{p}+\epsilon-1} dt \\ &\geq (\epsilon^\epsilon)^2 \omega_n \int_\epsilon^{\epsilon^{-1}} \Phi(t) t^{-\lambda+\frac{\gamma+n}{p}-1} dt,\end{aligned}$$

where f_δ stands for f_δ or f_ϵ as defined above in different cases.

Letting $\epsilon \to 0^+$, we arrive at $C_{\Phi,1} < \infty$ and $\|\widetilde{\mathcal{H}_{\Phi,\beta}}\|_{LM^{\lambda,\alpha}_{p,q}(\mathbb{R}^n)\to LM^{\lambda,\gamma}_{p,q}(\mathbb{R}^n)} \geq C_{\Phi,1}$. This, together with the upper estimates, yields that the constant $C_{\Phi,1}$ is just the operator norm of $\widetilde{\mathcal{H}_{\Phi,\beta}}$ from $LM^{\lambda,\alpha}_{p,q}(\mathbb{R}^n)$ to $LM^{\lambda,\gamma}_{p,q}(\mathbb{R}^n)$. □

Theorem 2.2. *Assume that Φ is a nonnegative measurable function. Let $1 \leq p < \infty$, $1 \leq q \leq \infty$, $\alpha \in \mathbb{R}$, and let* (3) *be satisfied. Then $\mathcal{H}_\Phi$ is bounded on $LM^{\lambda,\alpha}_{p,q}(\mathbb{R}^n)$, that is,*

$$\|\mathcal{H}_\Phi(f)\|_{LM^{\lambda,\alpha}_{p,q}} \leq C\|f\|_{LM^{\lambda,\alpha}_{p,q}}, \quad \forall f \in LM^{\lambda,\alpha}_{p,q}(\mathbb{R}^n)$$

for some nonnegative constant C if and only if

$$C_{\Phi,2} = \int_{\mathbb{R}^n} \frac{\Phi(y)}{|y|^{\lambda-\frac{\alpha+n}{p}+n}} dy < \infty.$$

Moreover, if $C_{\Phi,2} < \infty$, then

$$\|\mathcal{H}_\Phi\|_{LM^{\lambda,\alpha}_{p,q}(\mathbb{R}^n)\to LM^{\lambda,\alpha}_{p,q}(\mathbb{R}^n)} = C_{\Phi,2}.$$

Proof. We first consider the "if" part. Since $1 \leq p < \infty$, it follows from Minkowski's inequality that

$$\begin{aligned}\left(\int_{B(0,r)} \left|\mathcal{H}_\Phi(f)(x)\right|^p |x|^\alpha dx\right)^{1/p} &= \left(\int_{B(0,r)} \left|\int_{\mathbb{R}^n} \frac{\Phi(y)}{|y|^n} f\left(\frac{x}{|y|}\right) dy\right|^p |x|^\alpha dx\right)^{1/p} \\ &\leq \int_{\mathbb{R}^n} \left(\int_{B(0,r)} \left|f\left(\frac{x}{|y|}\right)\right|^p |x|^\alpha dx\right)^{1/p} \frac{\Phi(y)}{|y|^n} dy \\ &= \int_{\mathbb{R}^n} \left(\int_{B\left(0,\frac{r}{|y|}\right)} \left|f(z)\right|^p |z|^\alpha dz\right)^{1/p} \frac{\Phi(y)}{|y|^{n-\frac{\alpha+n}{p}}} dy.\end{aligned}$$

If $1 \leq q < \infty$, by applying Minkowski's inequality again, we obtain

$$\begin{aligned}\|\mathcal{H}_\Phi(f)\|_{LM^{\lambda,\alpha}_{p,q}} &\leq \left(\int_0^\infty r^{-q\lambda} \left(\int_{\mathbb{R}^n} \left(\int_{B\left(0,\frac{r}{|y|}\right)} \left|f(z)\right|^p |z|^\alpha dz\right)^{1/p} \frac{\Phi(y)}{|y|^{n-\frac{\alpha+n}{p}}} dy\right)^q \frac{dr}{r}\right)^{1/q} \\ &\leq \int_{\mathbb{R}^n} \left(\int_0^\infty r^{-q\lambda} \left(\int_{B\left(0,\frac{r}{|y|}\right)} \left|f(z)\right|^p |z|^\alpha dz\right)^{q/p} \frac{dr}{r}\right)^{1/q} \frac{\Phi(y)}{|y|^{n-\frac{\alpha+n}{p}}} dy \\ &= \int_{\mathbb{R}^n} \frac{\Phi(y)}{|y|^{\lambda-\frac{\alpha+n}{p}+n}} dy \|f\|_{LM^{\lambda,\alpha}_{p,q}}.\end{aligned}$$

If $q = \infty$, then we have

$$\frac{1}{r^\lambda}\int_{\mathbb{R}^n}\left(\int_{B\left(0,\frac{r}{|y|}\right)}\left|f(z)\right|^p|z|^\alpha dz\right)^{1/p}\frac{\Phi(y)}{|y|^{n-\frac{\alpha+n}{p}}}dy = \int_{\mathbb{R}^n}\frac{1}{\left(\frac{r}{|y|}\right)^\lambda}\left(\int_{B\left(0,\frac{r}{|y|}\right)}\left|f(z)\right|^p|z|^\alpha dz\right)^{1/p}\frac{\Phi(y)}{|y|^{\lambda-\frac{\alpha+n}{p}+n}}dy$$
$$\leq \int_{\mathbb{R}^n}\frac{\Phi(y)}{|y|^{\lambda-\frac{\alpha+n}{p}+n}}dy\|f\|_{LM^{\lambda,\alpha}_{p,\infty}}.$$

By taking the supremum over $r > 0$, we get

$$\|\mathcal{H}_\Phi(f)\|_{LM^{\lambda,\alpha}_{p,\infty}} \leq \int_{\mathbb{R}^n}\frac{\Phi(y)}{|y|^{\lambda-\frac{\alpha+n}{p}+n}}dy\|f\|_{LM^{\lambda,\alpha}_{p,\infty}}.$$

Now, we proceed to the "only if" part and show that the constant $C_{\Phi,2}$ is the operator norm of $\mathcal{H}_\Phi$ on $LM^{\lambda,\alpha}_{p,q}(\mathbb{R}^n)$. As before, we consider the cases $1 \leq q < \infty$ and $q = \infty$, respectively.

Case 1: $1 \leq q < \infty$. For δ satisfying $0 < \delta + \frac{\alpha+n}{p} < \lambda$, by taking f_δ as in (4), we have

$$\|f_\delta\|^q_{LM^{\lambda,\alpha}_{p,q}} = \left(\frac{\omega_n}{n+p\delta+\alpha}\right)^{q/p}\frac{1}{n+p\delta+\alpha}B\left(\frac{q}{p}+1,\frac{q\lambda}{n+p\delta+\alpha}-\frac{q}{p}\right) < \infty.$$

On the other hand,

$$\begin{aligned}\|\mathcal{H}_\Phi(f_\delta)\|^q_{LM^{\lambda,\alpha}_{p,q}} &= \int_0^\infty r^{-q\lambda-1}\left(\int_{|x|<r}\left|\int_{|y|<|x|}\frac{\Phi(y)}{|y|^{n+\delta}}dy\right|^p|x|^{p\delta+\alpha}dx\right)^{q/p}dr\\ &\geq \int_{\epsilon^{-1}}^\infty r^{-q\lambda-1}\left(\int_{\epsilon^{-1}<|x|<r}\left|\int_{|y|<\epsilon^{-1}}\frac{\Phi(y)}{|y|^{n+\delta}}dy\right|^p|x|^{p\delta+\alpha}dx\right)^{q/p}dr\\ &= \left(\int_{|y|<\epsilon^{-1}}\frac{\Phi(y)}{|y|^{n+\delta}}dy\right)^q\int_{\epsilon^{-1}}^\infty r^{-q\lambda-1}\left(\int_{\epsilon^{-1}<|x|<r}|x|^{p\delta+\alpha}dx\right)^{q/p}dr\\ &= \left(\int_{|y|<\epsilon^{-1}}\frac{\Phi(y)}{|y|^{n+\delta}}dy\right)^q\epsilon^{\left(\lambda-\delta-\frac{\alpha+n}{p}\right)q}\|f_\delta\|^q_{LM^{\lambda,\alpha}_{p,q}}.\end{aligned}$$

Case 2: $q = \infty$. If $\lambda > 0$, then we have

$$\|f_\delta\|_{LM^{\lambda,\alpha}_{p,\infty}} = \left(\frac{\omega_n}{n+p\delta+\alpha}\right)^{1/p}\left(\frac{n+p\delta+\alpha}{\lambda p-n-p\delta-\alpha}\right)^{1/p}\left(\frac{\lambda p-n-p\delta-\alpha}{\lambda p}\right)^{\frac{\lambda}{n+p\delta+\alpha}} < \infty$$

and

$$\begin{aligned}\|\mathcal{H}_\Phi(f_\delta)\|_{LM^{\lambda,\alpha}_{p,\infty}} &= \sup_{r>0} r^{-\lambda}\left(\int_{|x|<r}\left|\int_{|y|<|x|}\frac{\Phi(y)}{|y|^{\delta+n}}dy\right|^p|x|^{p\delta+\alpha}dx\right)^{1/p}\\ &\geq \sup_{r>\epsilon^{-1}} r^{-\lambda}\left(\int_{\epsilon^{-1}<|x|<r}\left|\int_{|y|<\epsilon^{-1}}\frac{\Phi(y)}{|y|^{\delta+n}}dy\right|^p|x|^{p\delta+\alpha}dx\right)^{1/p}\\ &= \int_{|y|<\epsilon^{-1}}\frac{\Phi(y)}{|y|^{\delta+n}}dy\cdot\epsilon^{\lambda-\delta-\frac{\alpha+n}{p}}\|f_\delta\|_{LM^{\lambda,\alpha}_{p,\infty}}.\end{aligned}$$

If $\lambda = 0$, then some similar estimates can be obtained by taking f_ϵ as in (6).

Taking $\delta = \lambda - \frac{\alpha+n}{p} - \epsilon$ where $\epsilon > 0$ is suffficiently small, it follows from Cases 1 and 2 that

$$\begin{aligned}\|\mathcal{H}_\Phi\|_{LM^{\lambda,\alpha}_{p,q}(\mathbb{R}^n)\to LM^{\lambda,\alpha}_{p,q}(\mathbb{R}^n)} &\geq \frac{\|\mathcal{H}_\Phi(f_\delta)\|_{LM^{\lambda,\alpha}_{p,q}}}{\|f_\delta\|_{LM^{\lambda,\alpha}_{p,q}}}\\ &\geq \epsilon^\epsilon \int_{|y|<\epsilon^{-1}} \Phi(y)|y|^{-\lambda+\frac{\alpha+n}{p}+\epsilon-n}dy\\ &\geq (\epsilon^\epsilon)^2 \int_{\epsilon<|y|<\epsilon^{-1}} \Phi(y)|y|^{-\lambda+\frac{\alpha+n}{p}-n}dy,\end{aligned}$$

where f_δ stands for f_δ or f_ϵ as defined above in different cases.

Letting $\epsilon \to 0^+$, we finish the proof of Theorem 2.2. □

3. Applications

By choosing $\Phi(x)$ as the radial functions $\nu_n^{\frac{\beta}{n}-1}|x|^{\beta-n}\chi_{(1,\infty)}(|x|)$ and $\nu_n^{\frac{\beta}{n}-1}\chi_{(0,1)}(|x|)$ in (2), then $\widetilde{\mathcal{H}_{\Phi,\beta}}$ reduces to the n-dimensional fractional Hardy operator $\mathcal{H}_\beta$ and its adjoint operator $\mathcal{H}^*_\beta$ respectively, where

$$\mathcal{H}_\beta f(x) = \frac{1}{|B(0,|x|)|^{1-\frac{\beta}{n}}} \int_{|y|<|x|} f(y)dy, \quad x \in \mathbb{R}^n \setminus \{0\}$$

and

$$\mathcal{H}^*_\beta f(x) = \int_{|y|>|x|} \frac{f(y)}{|B(0,|y|)|^{1-\frac{\beta}{n}}}dy, \quad x \in \mathbb{R}^n.$$

It is well known that Hardy-type operators are basic average operators in harmonic analysis. For the studies on $\mathcal{H}_\beta$ and $\mathcal{H}^*_\beta$, we refer the reader to [12–14, 19, 20, 22, 26].

By using Theorem 2.1, we have the following results.

Corollary 3.1. *Let $1 \leq p < \infty$, $1 \leq q \leq \infty$ and let* (3) *be satisfied. Let $0 \leq \beta < n$, $\alpha, \gamma \in \mathbb{R}$ satisfy $\beta = \frac{\alpha-\gamma}{p}$ and let $np + \lambda p - \alpha - n > 0$. Then for any $f \in LM^{\lambda,\alpha}_{p,q}(\mathbb{R}^n)$,*

$$\|\mathcal{H}_\beta(f)\|_{LM^{\lambda,\gamma}_{p,q}} \leq \nu_n^{\frac{\beta}{n}} \frac{np}{np+\lambda p-\alpha-n}\|f\|_{LM^{\lambda,\alpha}_{p,q}}.$$

Moreover,

$$\|\mathcal{H}_\beta\|_{LM^{\lambda,\alpha}_{p,q}(\mathbb{R}^n)\to LM^{\lambda,\gamma}_{p,q}(\mathbb{R}^n)} = \nu_n^{\frac{\beta}{n}} \frac{np}{np+\lambda p-\alpha-n}.$$

Corollary 3.2. *Let $1 \leq p < \infty$, $1 \leq q \leq \infty$ and let* (3) *be satisfied. Let $0 \leq \beta < n$, $\alpha, \gamma \in \mathbb{R}$ satisfy $\beta = \frac{\alpha-\gamma}{p}$ and let $\gamma + n - \lambda p > 0$. Then for any $f \in LM^{\lambda,\alpha}_{p,q}(\mathbb{R}^n)$,*

$$\|\mathcal{H}^*_\beta(f)\|_{LM^{\lambda,\gamma}_{p,q}} \leq \nu_n^{\frac{\beta}{n}} \frac{np}{\gamma+n-\lambda p}\|f\|_{LM^{\lambda,\alpha}_{p,q}}.$$

Moreover,

$$\|\mathcal{H}^*_\beta\|_{LM^{\lambda,\alpha}_{p,q}(\mathbb{R}^n)\to LM^{\lambda,\gamma}_{p,q}(\mathbb{R}^n)} = \nu_n^{\frac{\beta}{n}} \frac{np}{\gamma+n-\lambda p}.$$

Remark 3.3. *In fact, the condition $\beta = \frac{\alpha-\gamma}{p}$ in Corollaries 3.1 and 3.2 is also necessary for the corresponding boundedness by using a dilation method. We leave the details to the interested reader.*

Let $\phi : [0,1] \to [0,\infty)$ be a measurable function. If we take

$$\Phi(y) = (\omega_n |y|)^{-1} \phi(|y|^{-1}) \chi_{\{|y|>1\}}(y)$$

in (1), then $\mathcal{H}_\Phi$ becomes the weighted Hardy–Littlewood average operator studied in [9, 29], which is given by

$$H_\phi(f)(x) = \int_0^1 \phi(t) f(tx) dt, \quad x \in \mathbb{R}^n.$$

By taking

$$\Phi(y) = {\omega_n}^{-1} |y|^{-n+1} \phi(|y|) \chi_{\{0<|y|<1\}}(y)$$

in (1), we obtain the weighted Cesàro operator G_ϕ considered in [15], which is defined by

$$G_\phi(f)(x) = \int_0^1 \phi(t) f\left(\frac{x}{t}\right) t^{-n} dt, \quad x \in \mathbb{R}^n.$$

As consequences of Theorem 2.2, we get the following results:

Corollary 3.4. *Let $1 \le p < \infty$, $1 \le q \le \infty$, let (3) be satisfied and let $\alpha \in \mathbb{R}$. Then H_ϕ is bounded on $LM^{\lambda,\alpha}_{p,q}(\mathbb{R}^n)$, that is,*

$$\|H_\phi(f)\|_{LM^{\lambda,\alpha}_{p,q}} \le C\|f\|_{LM^{\lambda,\alpha}_{p,q}}, \quad \forall f \in LM^{\lambda,\alpha}_{p,q}(\mathbb{R}^n)$$

for some nonnegative constant C if and only if

$$\int_0^1 \phi(t) t^{\lambda - \frac{\alpha+n}{p}} dt < \infty.$$

Moreover, if $\int_0^1 \phi(t) t^{\lambda - \frac{\alpha+n}{p}} dt < \infty$, then

$$\|H_\phi\|_{LM^{\lambda,\alpha}_{p,q}(\mathbb{R}^n) \to LM^{\lambda,\alpha}_{p,q}(\mathbb{R}^n)} = \int_0^1 \phi(t) t^{\lambda - \frac{\alpha+n}{p}} dt.$$

Corollary 3.5. *Let $1 \le p < \infty$, $1 \le q \le \infty$, let (3) be satisfied and let $\alpha \in \mathbb{R}$. Then G_ϕ is bounded on $LM^{\lambda,\alpha}_{p,q}(\mathbb{R}^n)$, that is,*

$$\|G_\phi(f)\|_{LM^{\lambda,\alpha}_{p,q}} \le C\|f\|_{LM^{\lambda,\alpha}_{p,q}}, \quad \forall f \in LM^{\lambda,\alpha}_{p,q}(\mathbb{R}^n)$$

for some nonnegative constant C if and only if

$$\int_0^1 \phi(t) t^{-\lambda + \frac{\alpha+n}{p} - n} dt < \infty.$$

Moreover, if $\int_0^1 \phi(t) t^{-\lambda + \frac{\alpha+n}{p} - n} dt < \infty$, then

$$\|G_\phi\|_{LM^{\lambda,\alpha}_{p,q}(\mathbb{R}^n) \to LM^{\lambda,\alpha}_{p,q}(\mathbb{R}^n)} = \int_0^1 \phi(t) t^{-\lambda + \frac{\alpha+n}{p} - n} dt.$$

Data Availability

No datasets were generated or analysed during the current study.

Conflict of interest

The authors declare that they have no conflict of interest.

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