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Mean-variance-skewness-kurtosis optimization portfolio selection model in uncertain random environments

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Abstract. The main focus of this paper is to address the computational challenges associated with portfolio optimization in a hybrid uncertainty (Uncertain-Random) environment. Considering the fact that investors consider different subjective criteria for choosing their portfolio, in this research in presenting the models, we have used different criteria such as skewness and kurtosis of the distribution of stock return variables, which can be very effective in investors decision-making.

The paper assumes that the total return can be characterized as a hybrid uncertain variable and investigates the problem of optimal portfolio selection under uncertain randomness.

The initial step involves defining the skewness and kurtosis of certain random variables, followed by the derivation of several important properties in specific distributions. These findings enable the transformation of models into deterministic forms and the establishment of uncertain random mean-variance-skewness-kurtosis optimization models for portfolio selection, thereby eliminating the need for investors to make subjective decisions.

Furthermore, the paper proposes the use of a capable artificial neural network that is globally convergent and stable to solve the obtained model. A numerical simulation result demonstrates the efficiency of the neural network in solving the portfolio optimization problem. The work done can be applied to solve real-life portfolio selection problems with better accuracy.

1. Introduction

The main goal in the realm of optimal portfolio selection theory is to maximize investors' profits by meticulously selecting investments that align with their preferences from a variety of potential options. Initially, the portfolio selection problem was founded upon Markowitz's mean-variance model [34], and numerous studies were conducted to optimize portfolios based on these two aspects of the return distribution. However, subsequent research demonstrated that relying solely on average and variance criteria is insufficient for achieving the optimal portfolio allocation. Factors such as skewness and kurtosis have proven to be highly influential and decisive in this regard [1, 2, 4, 9, 10, 12, 13, 18, 20–23, 29, 33, 42, 45]. As a result of these findings, researchers have recently shifted their focus towards higher-order moments.

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Keywords. Uncertain random variables, hybrid uncertainty, portfolio selection, skewness, kurtosis, mean variance skewness-kurtosis model.

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Traditionally, it was believed that security returns adhered to a stochastic nature, and probability theory was considered the most powerful tool for selecting an optimal portfolio. However, it is evident that the effectiveness of security efficiency is influenced by various factors, such as social, political, economic, and particularly psychological factors. Recent studies have shown that short-term security returns are not accurately reflected by historical data alone. Empirical evidence demonstrates that the distribution of underlying asset returns displays higher peaks and heavier tails compared to the normal probability distribution, highlighting the insufficiency of relying solely on the first two moments. To address these challenges, researchers have explored the use of fuzzy variables as securities returns in numerous studies, such as [11, 24, 35]. However, the utilization of fuzzy variables has revealed certain paradoxes, as highlighted by Liu and Huang [14, 25]. Consequently, the field of uncertain theory has garnered significant attention, with many researchers incorporating Liu's uncertain measurement theory into portfolio selection models [6, 7, 15–17, 37, 39, 46].

In many scenarios, investors are confronted with hybrid uncertainty, wherein emerging markets lacking sufficient historical data coexist with markets possessing adequate historical data. In this situation, in the absence of adequate historical data, we show security returns by uncertain variables and other security returns will show by random variables, in other words, random returns may appear with uncertain returns at the same time. Currently, several researchers are studying on uncertain random portfolio optimization, such as [5, 31, 32, 36, 40, 41, 44], and some studies applied skewness with respect to portfolio optimization in hybrid uncertain spaces [8, 43]. But they have not considered the fourth moment in their studies. In this article, our attempt to fill this gap is finding kurtosis in uncertain random environment to study mean-variance-skewness-kurtosis portfolio optimization model.

The initial step involves validating the uncertain stochastic model within the context of chance theory for portfolio selection, wherein both stochastic and uncertain returns are taken into account simultaneously. By leveraging the fact that certain security returns closely align with their true frequency distribution, while others do not, the portfolio can be optimized effectively. Additionally, taking into account the asymmetry and distinct kurtosis of financial assets, it is classified as a hybrid uncertain skewness and also examines kurtosis when dealing with random uncertain variables. Furthermore, we incorporate skewness-kurtosis into the mean-variance model in the presence of an uncertain random environment, thereby establishing a mean-variance-skewness-kurtosis uncertain random optimal portfolio selection model. The portfolio optimization problem can be addressed by formulating the hybrid uncertain MVSK model. This model allows for the simultaneous consideration of accounting return, risk, skewness, and kurtosis, providing a comprehensive understanding of the portfolio's performance. To solve this problem, an artificial neural network (NN) is proposed. By utilizing the NN, the efficiency of the portfolio selection model can be demonstrated through an illustrative example.

The structure of the paper is as follows. Section 2 provides a review of essential concepts related to uncertain and uncertain-random variables. Section 3 focuses on the examination and validation of skewness and kurtosis in two specific types of uncertain random returns. Moving on to section 4, various models for portfolio selection are presented, emphasizing mean-variance-skewness-kurtosis considerations. In section 5, a dynamic system model is introduced to address the portfolio selection model derived earlier, and an illustrative example is employed to showcase the effectiveness of the proposed neural network model. Finally, Section 6 concludes the paper by presenting some final remarks.

2. Preliminaries

Consider Γ be a non-empty set, and define the σ -algebra L be a collection of all the events $\Theta \in L$ over Γ. It could be defined a function that assigns to each event Θ the belief degree $\mathcal{M}\{\Theta\}$, signifying our belief in the occurrence of Θ. Liu [26] offered the following five axioms, in order to define uncertain measure in an axiomatic form, to ensure that the number $\mathcal{M}\{\Theta\}$ is not arbitrary and has special mathematical properties;

1: (Normality axiom) $\mathcal{M}(\Gamma) = 1$;

2: (Monotonicity axiom) $\mathcal{M}(\Theta_1) \leq \mathcal{M}(\Theta_2)$ every where $\Theta_1 \subseteq \Theta_2$;

3: (Duality axiom) $\mathcal{M}(\Theta) + \mathcal{M}(\Theta^c) = 1$ for every event Θ;

4: (Subadditivity axiom) For each sequence of events $\{\Theta_j\}$ that can be counted, we have

$$\mathcal{M}(\bigcup_{j=1}^{\infty}\Theta_j) \leq \sum_{j=1}^{\infty}\mathcal{M}(\Theta_j)$$

Definition 2.1. *[27]. The set function $\mathcal{M}$ which satisfies the above axioms, is called an uncertain measure.*

Definition 2.2. *[27]. Consider Γ be a non-empty set, the σ-algebra L be a collection of all the events over Γ, and $\mathcal{M}$ be an uncertain measure according to the above definition, the triple $(\Gamma, L, \mathcal{M})$ is named an uncertain space.*

5: (Product Measure Axiom) [27]. Let the triple $(\Gamma_k, L_k, \mathcal{M}_k)$ for $k = 1, 2, ..., n$, where $\Gamma = \Gamma_1 \times \Gamma_2 \times ...$ and $L = L_1 \times L_2 \times ...$ be uncertainty spaces, then it satisfyed in

$$\mathcal{M}(\prod_{k=1}^{\infty}\Theta_k) \leq \bigwedge_{k=1}^{\infty}\mathcal{M}_k(\Theta_k)$$

Where Θ_k, are arbitrary events and chosen from L_k for $k = 1, 2, ..., n$, respectively.

Definition 2.3. *[27]. The uncertainty distribution for an uncertain variable such as η is defined by function $\Phi : \mathbb{R} \to [0, 1]$ that $\Phi(x) = \mathcal{M}\{\eta \leq x\}$. Note that, if e and σ be real numbers and $\sigma > 0$ Then the distribution of normal uncertain variable is*

$$\Phi(\tau) = \Big(1 + exp(\frac{\pi(e-\tau)}{\sqrt{3}\sigma})\Big)^{-1}, \quad \tau \in \mathbb{R}, \tag{1}$$

For convenience, it is denoted by $\eta \sim \mathcal{N}(e, \sigma)$, and also the distribution of linear uncertain variable is denoted by $\eta \sim L(\alpha, \beta)$ where $\alpha < \beta$ and it's introduced as follows

$$\Phi(\tau) = \begin{cases} 1, & \text{if } \tau > \beta \\ (\tau-\alpha)/(\beta-\alpha), & \text{if } \alpha \leq \tau \leq \beta \\ 0, & \text{if } \tau < \alpha. \end{cases} \tag{2}$$

Theorem 2.4. *[28] Let $\Phi_1, \Phi_2, ..., \Phi_n$ be uncertainty distributions of independent uncertain variables $\eta_1, \eta_2, ..., \eta_n$, respectively. If $f(t_1, t_2, ..., t_n)$ be increasing strictly. Then*

$$\eta = f(\eta_1, \eta_2, ..., \eta_n), \tag{3}$$

is an uncertain variable with uncertainty distribution

$$\Psi(t) = \sup_{f(t_1,t_2,...,t_n)=t}\Big(\min_{1\leq i\leq n}\Phi_i(t_i)\Big), \quad t \in \mathbb{R}, \tag{4}$$

and following inverse function

$$\Psi^{-1}(\alpha) = f[\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), ..., \Phi_n^{-1}(\alpha)], \tag{5}$$

Where $\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), ..., \Phi_n^{-1}(\alpha)$ are unique for each $\alpha \in (0, 1)$.

Definition 2.5. *[26]. The expected value of an uncertain variable η is defined by*

$$E[\eta] = \int_0^{\infty}\mathcal{M}\{\eta \geq r\}dr - \int_{-\infty}^{0}\mathcal{M}\{\eta \leq r\}dr, \tag{6}$$

while at least one of the above integrals be finite.

Theorem 2.6. *[28]. Let a_1 and a_2 be real numbers and η_1 and η_2 be uncertain variables which them expected values are finite and they are independent, then*

$$E[a_1\eta_1 + a_2\eta_2] = a_1E[\eta_1] + a_2E[\eta_2]. \tag{7}$$

Definition 2.7. *[30] Let (Γ, P, Pr) be a probability space and $(\Gamma, L, \mathcal{M})$ an uncertainty space. The product $(\Gamma, L, \mathcal{M}) \times (\Gamma, P, Pr)$ is called a chance space. Now if the uncertain random event Θ be in $L \times P$, Then the chance measure of Θ is defined as*

$$Ch[\Theta] = \int_0^1 Pr\{\omega \in \Omega | \mathcal{M}\{\gamma \in \Gamma | (\gamma, \omega) \in \Theta\} \geq r\}dr.$$

Definition 2.8. *[30] An hybrid uncertain variable η is a function from $(\Gamma, L, \mathcal{M}) \times (\Gamma, P, Pr)$ to the set of real numbers, which called uncertain random variable. i.e. η for any Borel set B is an event in $L \times P$. For any $x \in \mathbb{R}$, it's chance distribution is specified by $\Phi(x) = Ch\{\eta \leq x\}$.*
Notice that $\eta + \zeta$ and $\eta\zeta$ are uncertain random variables, if ζ be an uncertain variable and η be a random variable.

Definition 2.9. *Assume that E indicated the operator of expected value such as defined in [30] and η be an uncertain random variable and $E[\eta]$ be finite. the Skewness and kurtosis of η is defined as*

$$S[\eta] = E[(\eta - E[\eta])^3] \tag{8}$$

and

$$K[\eta] = E[(\eta - E[\eta])^4] \tag{9}$$

Theorem 2.10. *Consider η as a hybrid uncertain variable possessing a finite expected value denoted as $E[\eta]$. Additionally, let Φ represent the chance distribution associated with η. Then*

$$S[\eta] = \int_{-\infty}^{+\infty} (\eta - E[\eta])^3 d\Phi(x), \tag{10}$$

and

$$K[\eta] = \int_{-\infty}^{+\infty} (\eta - E[\eta])^4 d\Phi(x), \tag{11}$$

Proof. You can find the proof of part 1 in [43]. Now for proofing the Kurtosis formula, assume that $E[\eta] = e$ is the finite expected value of η. From definition (2.9)

$$\begin{aligned} K[\eta] &= E[(\eta - E[\eta])^4] \\ &= \int_0^{+\infty} Ch\{(\eta - e)^4 \geq x\}dx - \int_{-\infty}^0 Ch\{(\eta - e)^4 \leq x\}dx \\ &= \int_0^{+\infty} Ch\{\eta - e \geq \sqrt[4]{x}\}dx - \int_{-\infty}^0 Ch\{\eta - e \leq \sqrt[4]{x}\}dx \\ &= \int_0^{+\infty} Ch\{\eta \geq \sqrt[4]{x} + e\}dx - \int_{-\infty}^0 Ch\{\eta \leq \sqrt[4]{x} + e\}dx \end{aligned}$$

Now let $\sqrt[4]{x} + e = z$, so $x = (z - e)^4$ and we have

$$\begin{aligned} \int_0^{+\infty} Ch\{\eta \geq \sqrt[4]{x} + e\}dx - \int_{-\infty}^0 Ch\{\eta \leq \sqrt[4]{x} + e\}dx &= \int_e^{+\infty} Ch\{\eta \geq z\}d(z - e)^4 - \int_{-\infty}^e Ch\{\eta \leq z\}d(z - e)^4 \\ &= \int_e^{+\infty} (1 - \Phi(z))d(z - e)^4 - \int_{-\infty}^e \Phi(z)d(z - e)^4 \\ &= \int_e^{+\infty} (z - e)^4 d\Phi(z) + \int_{-\infty}^e (z - e)^4 d\Phi(z) \\ &= \int_{-\infty}^{+\infty} (z - e)^4 d\Phi(z). \end{aligned}$$

In the result the kurtosis is

$$K[\eta] = \int_{-\infty}^{+\infty} (\eta - E[\eta])^4 d\Phi(x), \tag{12}$$

□

Theorem 2.11. *Let a and b be arbitrary real numbers and the expected value of an uncertain random variable η be finite, then*

$$S[a\eta + b] = a^4 S[\eta], \tag{13}$$

and

$$K[a\eta + b] = a^4 K[\eta], \tag{14}$$

Proof. We know that $E[ax + b] = aE[x] + b$. It follows from definition (2.9) that

$$S[a\eta + b] = E[(a\eta + b - (aE[\eta] + b))^3] = a^3 E[(\eta - E(\eta))^3] = a^3 S[\eta], \tag{15}$$

and

$$K[a\eta + b] = E[(a\eta + b - (aE[\eta] + b))^4] = a^4 E[(\eta - E(\eta))^4] = a^4 K[\eta]. \tag{16}$$

The theorem is proved. □

In prooving of next theorems two integrals are used, which are stated as following remarks.

Remark 2.12.

$$\int_0^{\infty} \lambda^k e^{-c\lambda^2} d\lambda = \begin{cases} \frac{\Gamma(\frac{k+1}{2})}{2(c^{\frac{k+1}{2}})}, & k > -1,\ c > 0, \\ \frac{(2m-1)!!}{2^{m+1}c^m} \sqrt{\frac{\pi}{c}}, & k = 2m,\ c > 0, \\ \frac{m!}{2(c^{m+1})}, & k = 2m+1,\ c > 0. \end{cases} \tag{17}$$

Remark 2.13. *(Sommerfeld formula)*
Following integral vanishes for odd k, but for even k

$$\begin{aligned} \int_{-\infty}^{+\infty} \lambda^k \frac{e^{\lambda}}{(e^{\lambda}+1)^2} d\lambda &= 2\int_0^{+\infty} \lambda^k \frac{e^{\lambda}}{(e^{\lambda}+1)^2} d\lambda \\ &= 2\int_0^{+\infty} d\lambda \sum_{i=0}^{\infty} e^{\lambda}\lambda^k \times \left[(i+1)(-1)^{i+1} e^{-i\lambda}\right] \\ &= 2\sum_{i=1}^{\infty} (-1)^{i+1} i \int_0^{+\infty} \lambda^k e^{-i\lambda} d\lambda \\ &= 2(k!) \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i^k} \end{aligned} \tag{18}$$

3. Explanation of the problem

In this section, the situation of uncertainty and randomness is assumed, which will be called hybrid uncertainty; and the skewness and kurtosis of uncertain random returns will be discussed.

Consider a financial market which has m number of risky securities with adequate historical evidence and n number of new securities which have inadequate historical evidence. Investors prefer to allocate their wealth among these assets.

In the remain of paper we use of following symbols: ($i = 1, 2, ..., m.$ $j = 1, 2, ..., n.$)
η_i is a random variable which shows the return of the ith risky security with sufficient historical data in future.
λ_i is the expected value of η_i.
$\lambda = (\lambda_1, \lambda_2, ..., \lambda_m)^T$ is expected vector of random variable η.
σ_{ij} is the covariance of η_i and η_j.
$\Sigma = (\sigma_{ij})_{m\times m}$ is the covariance matrix of η.
ζ_j is an uncertain variable which shows the return of the jth new risky seurity with inadequate historical data in the future.
ν_j is the expected value of ζ_j.
δ_j^2 is the variance of uncertain return of ζ_j.
x_i is the holding proportion of ith security, which have adequate historical evidences.
y_j is the holding proportion of jth new security, which have inadequate historical evidences.
ψ_i is the distribution of random return η_i in the probability space.
γ_j is the distribution of uncertain return ζ_j in uncertainty space.
$x = (x_1, x_2, ..., x_m)^T$ is the portfolio vector of existing securities.
$\eta = (\eta_1, \eta_2, ..., \eta_m)^T$ is the vector of random returns of existing securities.
$y = (y_1, y_2, ..., y_n)^T$ is the portfolio vector of newly listed securities.
$\zeta = (\zeta_1, \zeta_2, ..., \zeta_n)^T$ is the vector of securities with uncertain returns.
$(x^T, y^T) = (x_1, x_2, x_3..., x_m, y_1, y_2, ..., y_n)^T$ is the portfolio vector of all demandant risky securities.
So

$$\mu(x, y; \eta, \zeta) = x^T\eta + y^T\zeta = x_1\eta_1 + x_2\eta_2 + ... + x_m\eta_m + y_1\zeta_1 + y_2\zeta_2 + ... + +y_n\zeta_n.$$

where $x^T\eta$ and $y^T\zeta$ respectively are portfolio total returns of existing securities with adequate historical data and securities which are newly listed. $x^T\eta$ is a random variable and $y^T\zeta$ is an uncertain variable.

Similar to those in Qin [40] and Zhai et al.[43] researches, the following remarks have been used in this manuscript.

Remark 3.1. *: The probability density function of random vector η which has a multivariate normal distribution, defined as follow*

$$\psi_\eta(z) = \frac{1}{\sqrt{(2\pi)^m|\Sigma|}} e^{-\frac{1}{2}(z-\lambda)'\Sigma^{-1}(z-\lambda)}.$$

Remark 3.2. *: For any Borel set B_j of real numbers,in the sense of uncertain measure, the uncertain returns $\zeta_j, j = 1, 2, ..., k$, are independent.*

$$\mathcal{M}\Big\{\bigcap_{j=1}^{k}\{\zeta_j \in B_j\}\Big\} = \bigwedge_{j=1}^{k} \mathcal{M}\{\zeta_j \in B_j\}$$

Where $\wedge$ is the minimum operator.

According to remark (3.1), $\lambda^T x = x_1\lambda_1 + x_2\lambda_2 + ... + x_m\lambda_m$ is that expected value and $x^T\Sigma x = \Sigma_{i=1}^m \Sigma_{j=1}^n x_i x_j \sigma_{ij}$ is variance and $\sigma(x) = \sqrt{x^T\Sigma x}$ and the probability density function is as follow

$$\psi(x) = \frac{1}{\sqrt{2\pi}\sigma(x)} e^{-\frac{(w-\lambda^T x)^2}{2\sigma^2(x)}}. \tag{19}$$

Based on remark (3.2) and using linearity feature of the expected value

$$E[\mu(x, y; \eta, \zeta)] = E[x^T\eta] + E[y^T\zeta] = x^T\lambda + y^T\nu \tag{20}$$

Now from theorem (2.10) follows that the skewness and kurtosis are

$$S[\mu(x,y;\eta,\zeta)] = \int_{-\infty}^{+\infty} [u-(x^T\lambda + y^T\nu)]^3 d\Phi(u), \tag{21}$$

and

$$K[\mu(x,y;\eta,\zeta)] = \int_{-\infty}^{+\infty} [u-(x^T\lambda + y^T\nu)]^4 d\Phi(u). \tag{22}$$

where the chance distribution of $\mu(x,y;\eta,\zeta)$ denotes by $\Phi(u)$ and defined as

$$\Phi(u) = \int_{-\infty}^{+\infty} \gamma(u-w)d\psi(w). \tag{23}$$

Theorem 3.3. *Assume that $\zeta_j \in L(\alpha_j,\beta_j)$, $j = 1,2,...,n$ be a linear uncertain variable then*

$$Var[\mu(x,y;\eta,\zeta)] = \sigma^2(x) + \frac{\beta^T y - \alpha^T y}{4} \tag{24}$$

Proof. Note that from properties of uncertain variables $y^T\eta$ has linear uncertain distribution as follow and its expected value is $\frac{(\beta^T y - \alpha^T y)^2}{2}$

$$\gamma(z) = I_{\{z\geq\beta^T y\}} + \frac{z-\alpha^T y}{\beta^T y - \alpha^T y} \times I_{\{\alpha^T y \leq z \leq \beta^T y\}} \tag{25}$$

$I_{\{.\}}$ represent the indictor function of set $\{.\}$.
Therefore

$$\begin{aligned}\Phi(u) &= \int_{-\infty}^{+\infty} (I_{\{u-w\geq\beta^T y\}} + \frac{u-(w+\alpha^T y)}{\beta^T y - \alpha^T y} \times I_{\{\alpha^T y \leq u-w \leq \beta^T y\}})\Psi(w)dw \\ &= \frac{1}{\sqrt{2\pi}\sigma(x)} \int_{-\infty}^{u-\beta^T y} e^{-\frac{(w-\lambda^T(x))^2}{2\sigma^2(x)}} dw - \frac{1}{\sqrt{2\pi}\sigma(x)} \int_{u-\beta^T y}^{u-\alpha^T y} \frac{w-(u+\alpha^T y)}{\beta^T y - \alpha^T y} e^{-\frac{(w-\lambda^T(x))^2}{2\sigma^2(x)}} dw\end{aligned} \tag{26}$$

So the Variance is

$$\begin{aligned}Var[\mu(x,y;\eta,\zeta)] &= \int_{-\infty}^{+\infty} [u-(\lambda^T(x) + \frac{\alpha^T y + \beta^T y}{2})]^2 d\Phi(u) \\ &= \frac{1}{\sqrt{2\pi}(\beta^T y - \alpha^T y)\sigma(x)} \int_{-\infty}^{+\infty} \int_{u-\beta^T y}^{u-\alpha^T y} \left[u-(\lambda^T(x) + \frac{\alpha^T y + \beta^T y}{2})\right]^2 e^{-\frac{(w-\lambda^T(x))^2}{2\sigma^2(x)}} dwdu \\ &= \frac{1}{\sqrt{2\pi}(\beta^T y - \alpha^T y)\sigma(x)} \int_{-\infty}^{+\infty} e^{-\frac{(w-\lambda^T(x))^2}{2\sigma^2(x)}} (\int_{w+\alpha^T y}^{w+\beta^T y} \left[u-(\lambda^T(x) + \frac{\alpha^T y + \beta^T y}{2})\right]^2 du)dw \\ &= \frac{1}{\sqrt{2\pi}\sigma(x)} \int_{-\infty}^{+\infty} [\frac{\beta^T y - \alpha^T y}{4} + (w-\lambda^T(x))^2] e^{-\frac{(w-\lambda^T(x))^2}{2\sigma^2(x)}} dw \\ &= \frac{1}{\sqrt{2\pi}\sigma(x)} \left[\sigma^3(x)\sqrt{2\pi} + \sqrt{2\pi}\sigma(x)\frac{\beta^T y - \alpha^T y}{4}\right] \\ &= \sigma^2(x) + \frac{\beta^T y - \alpha^T y}{4}.\end{aligned} \tag{27}$$

□

Theorem 3.4. *Assume that $\zeta_j \in N(e_j, \delta_j)$, $j = 1, 2, ..., n$ be a normal uncertain variable then*

$$Var[\mu(x, y; \eta, \zeta)] = \sigma^2(x) + 3(\delta^T y)^2 \tag{28}$$

Proof. Note that from properties of uncertain variables $y^T\zeta$ has also a normal uncertain distribution as follow and it's expected value is $e^T y$.

$$\gamma(z) = (1 + e^{\frac{\pi(e^T y - z)}{\sqrt{3}\delta^T y}})^{-1} \tag{29}$$

Then

$$\Phi(u) = \frac{1}{\sqrt{2\pi}\sigma(x)} \int_{-\infty}^{+\infty} (1 + e^{\frac{\pi(e^T y - u + w)}{\sqrt{3}\delta^T y}})^{-1} e^{-\frac{(w - \lambda^T(x))^2}{2\sigma^2(x)}} dw. \tag{30}$$

So the Variance is

$$\begin{aligned} Var[\mu(x, y; \eta, \zeta)] &= \int_{-\infty}^{+\infty} [u - (\lambda^T(x) + e^T y)]^2 d\Phi(u) \\ &= \frac{1}{\delta^T y \sigma(x)} \sqrt{\frac{\pi}{6}} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left[u - (\lambda^T(x) + e^T y)\right]^2 (1 + e^{\frac{\pi(w - u + e^T y)}{\sqrt{3}\delta^T y}})^{-2} e^{\frac{\pi(w - u + e^T y)}{\sqrt{3}\delta^T y}} \\ &\times e^{-\frac{(w - \lambda^T(x))^2}{2\sigma^2(x)}} dw du \\ &= \frac{1}{\delta^T y \sigma(x)} \sqrt{\frac{\pi}{6}} \int_{-\infty}^{+\infty} e^{-\frac{(w - \lambda^T(x))^2}{2\sigma^2(x)}} \int_{-\infty}^{+\infty} \left[u - (\lambda^T(x) + e^T y)\right]^2 (1 + e^{\frac{\pi(w - u + e^T y)}{\sqrt{3}\delta^T y}})^{-2} \\ &\times e^{\frac{\pi(w - u + e^T y)}{\sqrt{3}\delta^T y}} du dw. \end{aligned} \tag{31}$$

By changing variable $\frac{\pi(w - u + e^T y)}{\sqrt{3}\delta^T y}$ to t we obtain $du = \frac{\sqrt{3}\delta^T y}{-\pi} dt$ and using remarks (2.12, 2.13), then

$$\begin{aligned} Var[\mu(x, y; \eta, \zeta)] &= \frac{1}{\sqrt{2\pi}\sigma(x)} \int_{-\infty}^{+\infty} e^{-\frac{(w - \lambda^T x)^2}{2\sigma^2(x)}} [(w - \lambda^T x)^2 + 2(w - \lambda^T x)\frac{3t\delta^T y}{\pi} + \frac{9t^2(\delta^T y)^2}{\pi^2}] \frac{e^t}{(1 + e^t)^2} dt dw \\ &= \frac{1}{\sqrt{2\pi}\sigma(x)} \int_{-\infty}^{+\infty} e^{-\frac{(w - \lambda^T x)^2}{2\sigma^2(x)}} [(w - \lambda^T x)^2 + 3(\delta^T y)^2] dw \\ &= \frac{1}{\sqrt{2\pi}\sigma(x)} [\sigma^3(x)\sqrt{2\pi} + 3(\delta^T y)^2 \sigma(x)\sqrt{2\pi}] \\ &= \sigma^2(x) + 3(\delta^T y)^2. \end{aligned} \tag{32}$$

□

Theorem 3.5. *Assume that $\zeta_j \in L(\alpha_j, \beta_j)$, $j = 1, 2, ..., n$ be a linear uncertain variable, then*

$$S[\mu(x, y; \eta, \zeta)] = 0, \tag{33}$$

and

$$K[\mu(x,y;\eta,\zeta)] = 3\sigma^4(x) + \sigma^2(x)\frac{(\beta^T y - \alpha^T y)^2}{2} + \frac{1}{80}(\beta^T y - \alpha^T y)^4. \tag{34}$$

Proof. From that $y^T\zeta$ is a linear uncertain variable using the operational law of uncertain variables the expected value is $\frac{(\beta^T y - \alpha^T y)^2}{2}$ and the uncertainty distribution is as follows:

$$\gamma(z) = \frac{z - \alpha^T y}{\beta^T y - \alpha^T y} \times I_{\{\alpha^T y \le z \le \beta^T y\}} + I_{\{z \ge \beta^T y\}} \tag{35}$$

$I_{\{.\}}$ represent the indictor function of set $\{.\}$.
So

$$\begin{aligned}\Phi(u) &= \int_{-\infty}^{+\infty} \left(\frac{u - (w + \alpha^T y)}{\beta^T y - \alpha^T y} \times I_{\{\alpha^T y \le u - w \le \beta^T y\}} + I_{\{u - w \ge \beta^T y\}}\right)\Psi(w)dw \\ &= \frac{1}{\sqrt{2\pi}\sigma(x)}\int_{-\infty}^{u-\beta^T y} e^{-\frac{(w-\lambda^T(x))^2}{2\sigma^2(x)}} dw - \frac{1}{\sqrt{2\pi}\sigma(x)}\int_{u-\beta^T y}^{u-\alpha^T y} \frac{w - (u + \alpha^T y)}{\beta^T y - \alpha^T y} e^{-\frac{(w-\lambda^T(x))^2}{2\sigma^2(x)}} dw \end{aligned} \tag{36}$$

So the Kurtosis is

$$\begin{aligned} K[\mu(x,y;\eta,\zeta)] &= \int_{-\infty}^{+\infty} [u - (\lambda^T(x) + \frac{\alpha^T y + \beta^T y}{2})]^4 d\Phi(u) \\ &= \frac{1}{\sqrt{2\pi}(\beta^T y - \alpha^T y)\sigma(x)} \int_{-\infty}^{+\infty}\int_{u-\beta^T y}^{u-\alpha^T y} \left[u - (\lambda^T(x) + \frac{\alpha^T y + \beta^T y}{2})\right]^4 e^{-\frac{(w-\lambda^T(x))^2}{2\sigma^2(x)}} dwdu \\ &= \frac{1}{\sqrt{2\pi}(\beta^T y - \alpha^T y)\sigma(x)} \int_{-\infty}^{+\infty} e^{-\frac{(w-\lambda^T(x))^2}{2\sigma^2(x)}} \left(\int_{w+\alpha^T y}^{w+\beta^T y} \left[u - (\lambda^T(x) + \frac{\alpha^T y + \beta^T y}{2})\right]^4 du\right)dw \\ &= \frac{1}{\sqrt{2\pi}\sigma(x)} \int_{-\infty}^{+\infty} e^{-\frac{(w-\lambda^T(x))^2}{2\sigma^2(x)}} \left((w - \lambda^T(x))^4 + (w - \lambda^T(x))^2\frac{(\beta^T y - \alpha^T y)^2}{2} + \frac{1}{80}(\beta^T y - \alpha^T y)^4\right)dw \end{aligned} \tag{37}$$

By changing variable $w - \lambda^T(x)$ to t and using remark (2.12), we obtain

$$\begin{aligned} K[\mu(x,y;\eta,\zeta)] &= \frac{1}{\sqrt{2\pi}\sigma(x)} \int_{-\infty}^{+\infty} e^{-\frac{t^2}{2\sigma^2(x)}} \left(t^4 + t^2\frac{(\beta^T y - \alpha^T y)^2}{2} + \frac{1}{80}(\beta^T y - \alpha^T y)^4\right)dt \\ &= 3\sigma^4(x) + \sigma^2(x)\frac{(\beta^T y - \alpha^T y)^2}{2} + \frac{1}{80}(\beta^T y - \alpha^T y)^4. \end{aligned} \tag{38}$$

proofing of skewness can be find in [43]. □

Theorem 3.6. *Assume that $\zeta_j \in N(e_j, \delta_j)$ for $j = 1, 2, ..., n$ be a normal uncertain variable, then*

$$S[\mu(x,y;\eta,\zeta)] = 0, \tag{39}$$

and

$$K[\mu(x,y;\eta,\zeta)] = 3\sigma^4(x) + 18(\delta^T y)^2\sigma^2(x) + \frac{189}{5}(\delta^T y)^4. \tag{40}$$

Proof. From that $y^T\zeta$ is a normal uncertain variable using the operational law of uncertain variables the expected value is e^Ty and the uncertainty distribution is as follows:

$$\gamma(z) = (1 + e^{\frac{\pi(e^Ty - z)}{\sqrt{3}\delta^Ty}})^{-1} \tag{41}$$

Then

$$\Phi(u) = \frac{1}{\sqrt{2\pi}\sigma(x)}\int_{-\infty}^{+\infty}(1 + e^{\frac{\pi(e^Ty - u + w)}{\sqrt{3}\delta^Ty}})^{-1}e^{-\frac{(w-\lambda^T(x))^2}{2\sigma^2(x)}}dw. \tag{42}$$

So the Kurtosis is

$$\begin{aligned}
K[\mu(x, y; \eta, \zeta)] &= \int_{-\infty}^{+\infty}[u - (\lambda^T(x) + e^Ty)]^4 d\Phi(u)\\
&= \frac{1}{\delta^Ty\sigma(x)}\sqrt{\frac{\pi}{6}}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left[u - (\lambda^T(x) + e^Ty)\right]^4(1 + e^{\frac{\pi(w-u+e^Ty)}{\sqrt{3}\delta^Ty}})^{-2}\\
&\quad\times e^{\frac{\pi(w-u+e^Ty)}{\sqrt{3}\delta^Ty}}e^{-\frac{(w-\lambda^T(x))^2}{2\sigma^2(x)}}dwdu\\
&= \frac{1}{\delta^Ty\sigma(x)}\sqrt{\frac{\pi}{6}}\int_{-\infty}^{+\infty}e^{-\frac{(w-\lambda^T(x))^2}{2\sigma^2(x)}}\int_{-\infty}^{+\infty}\left[u - (\lambda^T(x) + e^Ty)\right]^4(1 + e^{\frac{\pi(w-u+e^Ty)}{\sqrt{3}\delta^Ty}})^{-2}\\
&\quad\times e^{\frac{\pi(w-u+e^Ty)}{\sqrt{3}\delta^Ty}}dudw.
\end{aligned} \tag{43}$$

By changing variable $\frac{\pi(w-u+e^Ty)}{\sqrt{3}\delta^Ty}$ to t we have $du = \frac{\sqrt{3}\delta^Ty}{-\pi}dt$ and using remark (2.12, 2.13), then

$$\begin{aligned}
K[\mu(x, y; \eta, \zeta)] &= \frac{-1}{\sqrt{2\pi}\sigma(x)}\int_{-\infty}^{+\infty}e^{-\frac{(w-\lambda^T(x))^2}{2\sigma^2(x)}}\int_{-\infty}^{+\infty}\left[w - \frac{3t\delta^Ty}{\pi}\lambda^T(x)\right]^4\frac{e^t}{(1+e^t)^2}dtdw\\
&= \frac{-1}{\sqrt{2\pi}\sigma(x)}\int_{-\infty}^{+\infty}e^{-\frac{(w-\lambda^T(x))^2}{2\sigma^2(x)}}\left[(w - \lambda^T(x))^4 + 18(\delta^Ty)^2(w - \lambda^T(x))^2 + \frac{189}{5}(\delta^Ty)^4\right]dw
\end{aligned} \tag{44}$$

By changing variable $w - \lambda^T(x)$ to t we obtain

$$\begin{aligned}
K[\mu(x, y; \eta, \zeta)] &= \frac{-1}{\sqrt{2\pi}\sigma(x)}\int_{-\infty}^{+\infty}e^{-\frac{t^2}{2\sigma^2(x)}}\left[t^4 + 18(\delta^Ty)^2t^2 + \frac{189}{5}(\delta^Ty)^4\right]dt\\
&= 3\sigma^4(x) + 18(\delta^Ty)^2\sigma^2(x) + \frac{189}{5}(\delta^Ty)^4
\end{aligned} \tag{45}$$

proofing of skewness can be find in [43]. □

4. Portfolio Selection Problem

A mean variance skewness-kurtosis model for selecting optimal portfolio problems in uncertain random environment and its equivalent models will be represented in this section. Assume that 1_m and 1_n be a m and n-dimensional row vectors, respectively which all of the elements are 1. Generally, an investor considers a preset credibility level ϑ and skewness desired level ϱ and kurtosis level κ and then requests to minimize the investment risk for finding the suitable portfolio. Hence, the selection of an optimal hybrid uncertain portfolio can be expressed as a mean-variance-skewness-kurtosis model as follow:

$$\text{minimize } \; Var[\mu(x,y;\eta,\zeta)] = Var[x^T\eta + y^T\zeta] \tag{46}$$

$$\text{subject to} \begin{cases} E[\mu(x,y;\eta,\zeta)] = E[x^T\eta + y^T\zeta] \geq \vartheta, \\ S[\mu(x,y;\eta,\zeta)] = S[x^T\eta + y^T\zeta] \geq \varrho, \\ K[\mu(x,y;\eta,\zeta)] = K[x^T\eta + y^T\zeta] \leq \kappa, \\ 1_m^T x + 1_n^T y = 1, \\ x \geq 0, \;\; y \geq 0. \end{cases}$$

The main purpose behind the utilization of this model is to minimize the potential risks, while ϑ representing the minimum anticipated return on investment that investors are willing to embrace and ϱ is the admissible skewness level and κ is the maximum kurtosis level that can be tolerated. Alternatively, another optimal portfolio can be one which maximize expected return on the limitation that the skewness is rather than or equal to the admissible level and the risk does not surpass the predefined risk threshold ϖ and kurtosis does not exceed a preset level κ in advance.

$$\text{maximize } \; E[\mu(x,y;\eta,\zeta)] = E[x^T\eta + y^T\zeta] \tag{47}$$

$$\text{subject to} \begin{cases} Var[\mu(x,y;\eta,\zeta)] = Var[x^T\eta + y^T\zeta] \leq \varpi, \\ S[\mu(x,y;\eta,\zeta)] = S[x^T\eta + y^T\zeta] \geq \varrho, \\ K[\mu(x,y;\eta,\zeta)] = K[x^T\eta + y^T\zeta] \leq \kappa, \\ 1_m^T x + 1_n^T y = 1, \\ x \geq 0, \;\; y \geq 0. \end{cases}$$

This optimization problem can be formulated in some other different kinds, such as maximizing skewness or minimizing kurtosis or multi-objective nonlinear programming model as

$$\begin{aligned} &\text{maximize } \; E[\mu(x,y:\eta,\zeta)] = E[x^T\eta + y^T\zeta] \\ &\text{maximize } \; S[\mu(x,y:\eta,\zeta)] = S[x^T\eta + y^T\zeta] \\ &\text{minimize } \; Var[\mu(x,y:\eta,\zeta)] = Var[x^T\eta + y^T\zeta] \\ &\text{minimize } \; K[\mu(x,y:\eta,\zeta)] = K[x^T\eta + y^T\zeta] \end{aligned}$$

$$\text{subject to} \begin{cases} 1_m^T x + 1_n^T y = 1, \\ x, y \geq 0. \end{cases}$$

In order to solve this problem, consider $w_i, i = 1, 2, 3, 4$ be positive real numbers which indicate the weights of the four appropriated objectives, and $w_i \in [0, 1]$, so this multi-objective model can be transformed into a single-objective optimization model as

$$\text{minimize } \; w_1 Var[\mu(x,y;\eta,\zeta)] - w_2 E[\mu(x,y;\eta,\zeta)] - w_3 S[\mu(x,y;\eta,\zeta)] + w_4 K[\mu(x,y;\eta,\zeta)]$$

$$\text{subject to} \begin{cases} 1_m^T x + 1_n^T y = 1, \\ x \geq 0, \;\; y \geq 0. \end{cases}$$

Note that if x^* be an optimal solution of model (48), it will also be a pareto optimal solution of multi-objective nonlinear model (48).

Theorem 4.1. *Let $\eta_i, i = 1, 2, ..., m$ be a linear random variable and $\zeta_j \in L(\alpha_j, \beta_j)$ for $j = 1, 2, ..., n$ be a linear uncertain variable. Then model (46) can be changed to the crisp equivalent as following form*

$$\text{minimize } \sigma^2(x) + \frac{\beta^T y - \alpha^T y}{4} \tag{48}$$

$$\text{subject to} \begin{cases} x^T\lambda + y^T\frac{\alpha+\beta}{2} \geq \vartheta, \\ 3\sigma^4(x) + \sigma^2(x)\dfrac{(\beta^T y - \alpha^T y)^2}{2} + \dfrac{1}{80}(\beta^T y - \alpha^T y)^4 \leq \kappa, \\ \varrho \leq 0, \\ 1_m^T x + 1_n^T y = 1, \\ x \geq 0, \quad y \geq 0. \end{cases}$$

Proof. Since, all of uncertain variables are linear in this mean that $\zeta_j \in L(\alpha_j, \beta_j)$ for $j = 1, 2, ..., n$, $y^T\zeta$ is a linear uncertain variable too. Moreover, the expected value have obtained as $E[\mu(x, y; \eta, \zeta)] = x^T\lambda + y^T\frac{\alpha+\beta}{2}$ and the variance $Var[\mu(x, y; \eta, \zeta)] = \sigma^2(x) + \frac{\beta^T y-\alpha^T y}{4}$ and the skewness $S[\mu(x, y; \eta, \zeta)] = 0$ and the kurtosis $K[\mu(x, y; \eta, \zeta)] = 3\sigma^4(x) + \sigma^2(x)\dfrac{(\beta^T y - \alpha^T y)^2}{2} + \dfrac{1}{80}(\beta^T y - \alpha^T y)^4$. Substituting the above formulas into model (46), the theorem will be proved. □

Theorem 4.2. *Let η_i for $i = 1, 2, ..., m$ be a normal random variable and $\zeta_j \in N(e_j, \delta_j)$ for $j = 1, 2, ..., n$ be a normal uncertain variable. Then model (46) can be changed into the crisp equivalent form as follow*

$$\text{minimize } \sigma^2(x) + 3(\delta^T y)^2 \tag{49}$$

$$\text{subject to} \begin{cases} x^T\lambda + y^T e \geq \vartheta, \\ 3\sigma^4(x) + 18(\delta^T y)^2\sigma^2(x) + \dfrac{189}{5}(\delta^T y)^4 \leq \kappa, \\ \varrho \leq 0, \\ 1_m^T x + 1_n^T y = 1, \\ x \geq 0, \quad y \geq 0. \end{cases}$$

Proof. Since, all of uncertain variables are normal, in this mean that $\zeta_j \in N(e_j, \delta_j)$ for $j = 1, 2, ..., n$, $y^T\zeta$ is a normal uncertain variable too. Moreover, the expected value have obtained as $E[\mu(x, y; \eta, \zeta)] = x^T\lambda + y^T e$ and the variance $Var[\mu(x, y; \eta, \zeta)] = \sigma^2(x) + 3(\delta^T y)^2$ and the skewness $S[\mu(x, y; \eta, \zeta)] = 0$ and the kurtosis $K[\mu(x, y; \eta, \zeta)] = 3\sigma^4(x) + 18(\delta^T y)^2\sigma^2(x) + \dfrac{189}{5}(\delta^T y)^4$. Substituting the above formulas into model (46), the theorem will be proved. □

5. A dynamic system model

Since this portfolio selection programming model is a geometric programming problem, so we explain a brief of geometric program which is in the form

$$\min \ F_0(x) \tag{50}$$

s.t.

$$F_k(x) \leq 1, \quad (k = 1, 2, ..., m), \tag{51}$$

$$G_p(x) = 1, \quad (p = 1, 2, ..., l). \tag{52}$$

where F_ks are posynomial for $k = 0, 1, ..., m$ in the mean of a real valued function F of x which it's form is

$$F(x) = \sum_{k=1}^{K} c_k x_1^{a_{1k}} x_2^{a_{2k}} ... x_n^{a_{nk}}$$

where $a_i \in \mathbb{R}$ and each c_k is positive.
A geometric programming with using logarithmic transformation of variables $x_i, w_i = log(x_i)$ (so $x_i = e^{w_i}$) can be transformed to a nonlinear convex optimization problem. So the transformation optimization problem is as follow

$$\min \; logF_0(e^w) \tag{53}$$

s.t.

$$logF_k(e^w) \leq 0, \quad (k = 1, 2, ..., m), \tag{54}$$

$$logG_p(e^w) = 0, \quad (p = 1, 2, ..., l), \tag{55}$$

which is a convex nonlinear programming problem. thus we consider following model

$$\min \; f(w) \tag{56}$$

s.t.

$$g(w) \leq 0, \tag{57}$$

$$h(w) = 0, \tag{58}$$

where $w \in \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}$, $g(w) = (g_1(w), g_2(w), ..., g_m(w))^T$ is an m-dimensional vector-valued continuous function of n variables, and the functions $f, g_1, \ldots, g_m$ are assumed to be convex and twice differentiable, $h(w) = Aw - b$, $A \in \mathbb{R}^{l \times n}$, $\text{rank}(A) = l$ $(0 \leq l < n)$ and $b \in \mathbb{R}^l$. according to [19, 38] and using standard optimization techniques, the above model transform into a nonlinear dynamic system.

Theorem 5.1. *[3] An optimal solution of (56)-(58) is $w \in \mathbb{R}^n$ if and only if there exist $u^* \in \mathbb{R}^m$ and $v^* \in \mathbb{R}^l$ such a way that $(w^{*^T}, u^{*^T}, v^{*^T})^T$ satisfies the following KKT[1)] system*

$$\begin{cases} u^* \geq 0, \;\; g(w^*) \leq 0, \;\; h(w^*) = 0, \\ \nabla f(w^*) + \nabla g(w^*)^T u^* + \nabla h(w^*)^T v^* = 0, \\ h(w^*) = 0. \end{cases} \tag{59}$$

w^* is a KKT point of (56)-(58 and $(u^{*^T}, v^{*^T})^T$ is corresponding to w^* and called the Lagrangian multiplier vector.

Theorem 5.2. *[3] If f and g_k, $k = 1, 2, ..., m$ all be convex, w^* will be an optimal solution of (56)-(58), if and only if w^* be a KKT point of (56)-(58).*

Now, let $w(.)$, $u(.)$ and $v(.)$, to be variables which are time dependent. Purpose is to construct a dynamic system which is continuous-time and settle down to the KKT point of the problem (56)-(58) and it's dual. A recurrent neural network model is suggested for solving model (56)-(58) and its dual, which its dynamical equation is as follows:

$$\frac{dw}{dt} = -(\nabla f(w) + \nabla g(w)^T (u + g(w))^+ + \nabla h(w)^T v), \tag{60}$$

$$\frac{du}{dt} = (u + g(w))^+ - u, \tag{61}$$

$$\frac{dv}{dt} = h(w), \tag{62}$$

with an initial point $(w_0^T, u_0^T, v_0^T)^T$

$$\zeta(y) = \begin{bmatrix} -(\nabla f(w) + \nabla g(w)^T (u + g(w))^+ + \nabla h(w)^T v) \\ (u + g(w))^+ - u \\ h(w) \end{bmatrix}. \tag{63}$$

[1)]Karush-Kuhn-Tucker

Thus neural network (60)-(62) can be written as:

$$\frac{dy}{dt} = \tau\zeta(y), \tag{64}$$

$$y(t_0) = y_0. \tag{65}$$

The stability and convergence of this network have been proven in [38].

Example 5.3. *Suppose that there is 9 stocks which the monthly return of 5 existing securities is the natural logarithm of the price ratio for two consecutive months and assume that there are 4 new stocks which their monthly return rates are estimated by experienced experts and they are Linear uncertain variables. Table 1 represents the returns of 5 existing stocks and the simulated expected values of 4 newly listed stocks. The covariance matrix of the return of 5 existing stocks is calculated and indicated in Table 2. An investor would like to create an optimal portfolio, and he wishes to minimize variance, which is accepted as risk, So solving model 48 to obtain the optimal portfolio is the main concern.*

Table 1: data of securities.

stocks	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
						$L(1.2, 1.5)$	$L(1.8, 2)$	$L(2.2, 2.7)$	$L(2.5, 3.5)$
expected value	0.45	0.30	1.26	0.61	0.24	1.35	1.9	2.45	3

Table 2: The sample covariance matrix of the 5 stocks.

0.75	0.32	0.39	0.32	0.38
0.32	0.68	0.26	0.03	0.34
0.39	0.26	0.57	0.28	0.26
0.32	0.03	0.28	0.63	0.24
0.38	0.34	0.26	0.24	1.58

Consider that in investor's mind, the minimum expected return that can accept is 2.5, and the kurtosis is not allowed to exceed 2. then the model 48 will be as follows:

$$\text{minimize } \sigma^2(x) + \frac{\beta^T y - \alpha^T y}{4} \tag{66}$$

$$\text{subject to} \begin{cases} x^T\lambda + y^T\frac{\alpha+\beta}{2} \geq 2.5, \\ 3\sigma^4(x) + \sigma^2(x)\dfrac{(\beta^T y - \alpha^T y)^2}{2} + \dfrac{1}{80}(\beta^T y - \alpha^T y)^4 \leq 2, \\ \varrho \leq 0, \\ 1_m^T x + 1_n^T y = 1, \\ x, y \geq 0. \end{cases}$$

By using of introduced NN in last section and solving this model, the investor's capital should be assigned in line with the optimal solution $x^* = (0, 0, 0, 0, 0, 0, 0, 0.9090, 0.0909)$. The corresponding minimum risk is 0.1363. Figure 1 and figure 2 show that the paths of this NN, regardless of the initial point chosen, convergence will ultimately lead to the optimal solution of the problem.

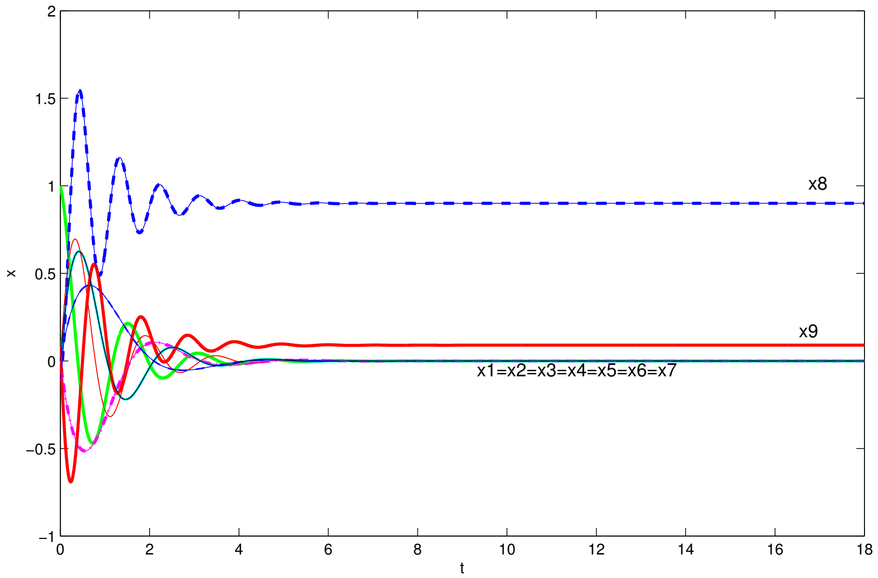

Figure 1: Transient behavior of x_is for $i = 1, 2, ..., 9$

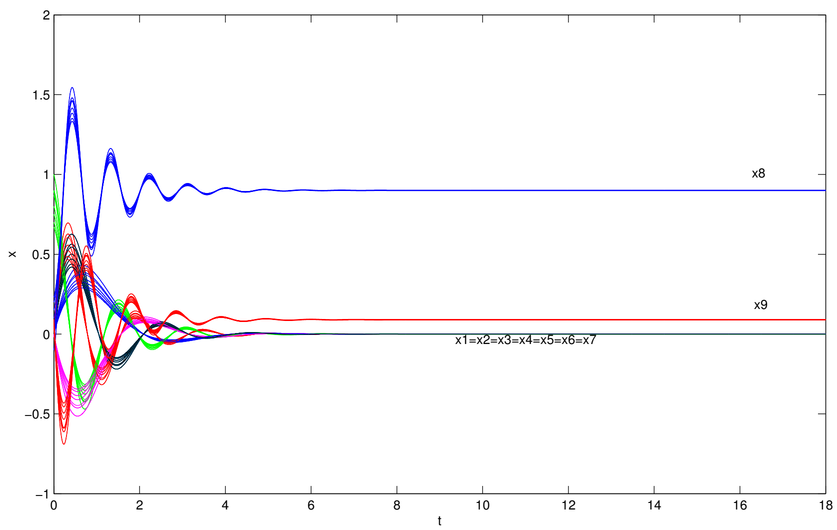

Figure 2: Transient behaviors of x(t) of the porposed neural network with various initial points

6. Managerial Insights

The managerial insights contemplated from the solution of the portfolio optimization problem are placed as follows:

To get higher returns, an investor has to concede higher risk; it is not possible to earn profit without taking risk.

For an investor whom the minimum expected return that can accept is 2.5, and the kurtosis is not allowed to exceed 2, this models can be formed and solved for find the best portfolio according to mental investors preferences. If he wants to keep the risk of investment at the minimum level, the policy of investment would be as follows: 90/9 % in stock 8 and 9/09 % in stock 9. For such an investment, the minimum risk is 0.1363.

Further, using the proposed model of portfolio optimization, it is also possible to find the investment strategy of an investor who can specify his or her target(s) about return expectancy, risk, skewness and kurtosis tolerance, in between their respective pessimistic and ideal values.

The proposed artificial neural network, is one of the best models for solving optimization problems because it's Lyapunov stable and globally covergent to unique optimal solution with any initial point. Also it do parallel calculations so is very fast and it can be used in real situations that investors have many cases for choose that make big optimization problems which must be solved.

The limitations of the proposed model are:

For an investor who wants to adopt an intermediate policy between maximizing total expected return and minimizing total risk that is not crisply defined, the determination of the optimal portfolio is not possible.

With the increase in skewness values of the return distribution from negative to positive values, the variance of the return decreases, but the total expected return may not increase. and so on about increase or decreasing of kurtosis.

Striking a balance between return, risk, Skewness and kurtosis is possible by the proposed method only when it is crisply defined.

7. Conclusion

In this manuscript, the consideration is given to the fact that the investor's ability to select securities and assets is influenced by the availability of historical data or not. While certain securities and assets have sufficient historical data, others may have invalid or inadequate data. As a result, it is deemed that the returns on these assets can be characterized as hybrid uncertain or uncertain random variables.
The concepts of skewness and kurtosis are incorporated into the framework of chance theory to analyze uncertain random variables by some theorems and proofing them and described an uncertain random model with respect to meanvarianceskewness-kurtosis for optimal portfolio selection.
So portfolios are formed according to investor preferences with applying of higher moments in uncertain random environment. The obtained model is a nonlinear geometric model that an efficient dynamic artificial neural network model is proposed for solving this model. The results obtained through the development of models for portfolio selection problems with uncertain random returns will hold substantial value in the disciplines of financial mathematics and economics, serving both theoretical advancements and practical applications.

Data availability and conflict of interest statement

This work is of theoretical nature and has not analyzed or generated any datasets.
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Author contribution statement

F.O., L.T. and K.N. contributed in designing the model and computational framework, organizing the research and performing numerical simulations and reviewing the results and writing of the manuscript.

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