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Spectral characteristics of A-right generalized weakly demicompact operators

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Abstract. In this research work, the central focus is upon elaborating certain results in Banach spaces for a new notion called A-right ascent of linear operators (where A is a given operator) which generalize an important notion in operator theory: ascent. Furthermore, if A is a bounded linear operator, we investigate the concept of A-right generalized weakly demicompact operators which in turn generalize the concept of generalized weakly demicompact operators introduced in [8]. In particular, a characterization using upper semi-Browder spectrum is set forward. Additionally, we identify some sufficient conditions on the inputs of a closable block operator matrix to ensure the generalized weak demicompactness of its closure. Moreover, some theorems in this paper generalize multiple known ones in the literature in particular [8].

1. Introduction

Let $\mathcal{X}$ and $\mathcal{Y}$ be two Banach spaces. We denote by $\mathcal{C}(\mathcal{X},\mathcal{Y})$ (resp. $\mathcal{L}(\mathcal{X},\mathcal{Y})$) the set of all closed (resp. bounded) linear operators defined from $\mathcal{X}$ into $\mathcal{Y}$. We denote by $\mathcal{K}(\mathcal{X},\mathcal{Y})$ the closed subspace of compact operators of $\mathcal{L}(\mathcal{X},\mathcal{Y})$. Given an operator $T \in \mathcal{C}(\mathcal{X},\mathcal{Y})$, we denote by $\mathcal{D}(T)$ its domain, by $\alpha(T)$ the dimension of its kernel $\mathcal{N}(T)$ and by $\beta(T)$ the codimension of the range $\mathcal{R}(T)$ in $\mathcal{Y}$. If $\mathcal{X} = \mathcal{Y}$, $\mathcal{C}(\mathcal{X},\mathcal{Y})$ and $\mathcal{L}(\mathcal{X},\mathcal{Y})$ are replaced by $\mathcal{C}(\mathcal{X})$ and $\mathcal{L}(\mathcal{X})$ respectively. For $T \in \mathcal{C}(\mathcal{X})$, we recall that the resolvent set of T is defined by:

$$\rho(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ has a bounded inverse}\},$$

and the spectrum of T is

$$\sigma(T) := \mathbb{C} \backslash \rho(T).$$

The classes of upper semi-Fredholm, lower semi-Fredholm, Fredholm and semi-Fredholm operators from $\mathcal{X}$ into $\mathcal{Y}$ are, respectively, defined by:

$$\Phi_+(\mathcal{X},\mathcal{Y}) := \{T \in \mathcal{C}(\mathcal{X},\mathcal{Y}) \colon \alpha(T) < \infty \text{ and } \mathcal{R}(T) \text{ closed in } \mathcal{Y}\},$$
$$\Phi_-(\mathcal{X},\mathcal{Y}) := \{T \in \mathcal{C}(\mathcal{X},\mathcal{Y}) \colon \beta(T) < \infty \text{ and } \mathcal{R}(T) \text{ closed in } \mathcal{Y}\},$$

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$$\Phi(\mathcal{X},\mathcal{Y}) := \Phi_-(\mathcal{X},\mathcal{Y}) \cap \Phi_+(\mathcal{X},\mathcal{Y}),$$

and

$$\Phi_\pm(\mathcal{X},\mathcal{Y}) := \Phi_-(\mathcal{X},\mathcal{Y}) \cup \Phi_+(\mathcal{X},\mathcal{Y}).$$

The set of bounded upper (resp. lower) semi-Fredholm operators from $\mathcal{X}$ into $\mathcal{Y}$ is defined by:

$$\Phi_+^b(\mathcal{X},\mathcal{Y}) = \Phi_+(\mathcal{X},\mathcal{Y}) \cap \mathcal{L}(\mathcal{X},\mathcal{Y}) \quad (\text{resp.}\Phi_-^b(\mathcal{X},\mathcal{Y}) = \Phi_-(\mathcal{X},\mathcal{Y}) \cap \mathcal{L}(\mathcal{X},\mathcal{Y})).$$

$\Phi^b(\mathcal{X},\mathcal{Y}) = \Phi(\mathcal{X},\mathcal{Y}) \cap \mathcal{L}(\mathcal{X},\mathcal{Y})$ stands for the set of bounded Fredholm operators from $\mathcal{X}$ into $\mathcal{Y}$. An operator $F \in \mathcal{L}(\mathcal{X},\mathcal{Y})$ is called an upper semi-Fredholm perturbation if $T + F \in \Phi_+(\mathcal{X},\mathcal{Y})$, whenever $T \in \Phi_+(\mathcal{X},\mathcal{Y})$. The sets of upper semi-Fredholm perturbations are denoted by $\mathcal{F}_+(\mathcal{X},\mathcal{Y})$. If $\mathcal{X} = \mathcal{Y}$, the sets $\mathcal{F}_+(\mathcal{X},\mathcal{Y})$ are replaced by $\mathcal{F}_+(\mathcal{X})$. The index of an operator $T \in \Phi_\pm(\mathcal{X},\mathcal{Y})$ is defined by $ind(T) := \alpha(T) - \beta(T)$. For $T \in \mathcal{C}(\mathcal{X},\mathcal{Y})$, T is said to have a left inverse modulo compact operator if there are $T_l \in \mathcal{L}(\mathcal{Y},\mathcal{X})$ and $K \in \mathcal{K}(\mathcal{X})$ such that $T_lT = I - K$ on $\mathcal{D}(T)$, where I is the identity operator. The operators, T_l are called left inverse modulo compact operators of T. It is well known that an operator $T \in \Phi_+(\mathcal{X},\mathcal{Y})$, if it possesses a left inverse modulo compact operators. The following two propositions provide some useful properties of $\Phi_+(\mathcal{X},\mathcal{Y})$ and $\Phi_-(\mathcal{X},\mathcal{Y})$ which will be needed in the sequel. The set of semi-Weyl operators is defined by:

$$\mathcal{W}_+^-(\mathcal{X}) := \{T \in \Phi_+(\mathcal{X}) : \text{ind}(T) \leq 0\}.$$

Proposition 1.1. *[1, 21] Let $\mathcal{X}$ be a Banach space. Therefore,*
(i) The sets $\Phi_+^b(\mathcal{X})$, $\Phi_-^b(\mathcal{X})$ and $\Phi^b(\mathcal{X})$ are open.
(ii) The index is constant on every component of each of the sets: $\Phi_+^b(\mathcal{X})$, $\Phi_-^b(\mathcal{X})$ and $\Phi^b(\mathcal{X})$.

Proposition 1.2. *[21] Let $T \in \mathcal{L}(\mathcal{Y},\mathcal{Z})$ and $B \in \mathcal{L}(\mathcal{X},\mathcal{Y})$, where $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ are three Banach spaces.*
(i) If $TB \in \Phi_+(\mathcal{X},\mathcal{Z})$, then $B \in \Phi_+(\mathcal{X},\mathcal{Y})$.
(ii) If $TB \in \Phi_-(\mathcal{X},\mathcal{Z})$, then $T \in \Phi_-(\mathcal{Y},\mathcal{Z})$.

Now, we shall recall that for any closed linear operator T and for every $x \in \mathcal{D}(T)$, the graph norm $\|.\|_T$ of x is defined by $\|x\|_T = \|x\| + \|Tx\|$. It follows from the closedness of T that $\mathcal{X}_T := (\mathcal{D}(T), \|.\|_T)$ is a Banach space. Clearly, for every $x \in \mathcal{D}(T)$, we have $\|Tx\| \leq \|x\|_T$, then $T \in \mathcal{L}(\mathcal{X}_T,\mathcal{X})$. If S_0 is a linear operator such that $\mathcal{D}(T) \subset \mathcal{D}(S_0)$, then S_0 is called T-defined. The restriction of S_0 to $\mathcal{D}(T)$ is denoted by $\widehat{S_0}$. Furthermore, we have the obvious relations

$$\begin{cases} \alpha(\widehat{T}) = \alpha(T), \beta(\widehat{T}) = \beta(T), \mathcal{R}(\widehat{T}) = \mathcal{R}(T), \\ \alpha(\widehat{T} + \widehat{S_0}) = \alpha(T + S_0),\ \beta(\widehat{T} + \widehat{S_0}) = \beta(T + S_0),\ \text{and} \\ \mathcal{R}(\widehat{T} + \widehat{S_0}) = \mathcal{R}(T + S_0). \end{cases}$$

Let asc(T)(resp. desc(T)) denote the ascent (resp. the descent) of T, i.e., the smallest non-negative integer n such that $\mathcal{N}(T^n) = \mathcal{N}(T^{n+1})$ (resp. $\mathcal{R}(T^n) = \mathcal{R}(T^{n+1})$). In case no such number exists, the ascent or descent of T is said to be infinite. The notation $\mathcal{N}_A^r(T)$ for linear operators $A, T \in \mathcal{L}(\mathcal{X})$ denote the A-right kernel of T, defined by

$$\mathcal{N}_A^r(T) = \mathcal{N}(TA).$$

We assume that

$$\mathcal{L}_M(\mathcal{X}) := \{T \in \mathcal{L}(\mathcal{X}) : T(M) \subset M\},$$

where M is a linear subspace of $\mathcal{X}$. An operator $A \in \mathcal{L}(\mathcal{X})$ denotes the A-right ascent of $T \in \mathcal{L}_{\mathcal{R}(A)}(\mathcal{X})$ (see [9]) defined by

$$\text{asc}_A^r(T) = \inf\{k \in \mathbb{N} : \mathcal{N}_A^r(T^k) = \mathcal{N}_A^r(T^{k+1})\},$$

whenever the minima exist. If no such numbers exist, the A-right ascent of T are defined to be ∞. The case when $A = I$, represent the ascent of T, that is $\text{asc}_A^r(T) = \text{asc}(T)$.

Remark 1.3. *[9] We demonstrate that if A is an operator surjective, $asc^r_A(T) = asc(T)$.*

An operator $T \in \mathcal{L}(\mathcal{X})$ is called upper semi-Browder if it is upper semi-Fredholm and has a finite ascent. For A, $T \in \mathcal{L}(X)$ and $\lambda \in \mathbb{C}$, an eigenvalue of T, the $\dim \mathcal{N}^{r,\infty}_A((\lambda - T)) = \dim \bigcup_{n\in\mathbb{N}} \mathcal{N}^r_A((\lambda - T)^n)$ is called the A-right algebraic multiplicity of T and is indicated by $\text{mult}^r_A(T, \lambda)$. We express by, $\mathcal{WK}(\mathcal{X})$ the subset of weakly compact operators of $\mathcal{L}(\mathcal{X})$. In addition, $\mathcal{WK}(\mathcal{X})$ is a closed two-sided ideal of $\mathcal{L}(\mathcal{X})$ containing $\mathcal{K}(\mathcal{X})$. Over the last years, the concept of demicompact has appeared in literature since 1966 as a result to the growing efforts to discuss fixed points. It was firstly identified by W. V. Petryshyn in [19] as follows:

Definition 1.4. *An operator $T : \mathcal{D}(T) \subset \mathcal{X} \to \mathcal{X}$, is said to be demicompact if every bounded sequence $(x_n)_n$ in $\mathcal{D}(T)$ such that $(x_n - Tx_n)_n$ converges in $\mathcal{X}$, has a convergent subsequence.*

The class of such operators is indicated by $\mathcal{DC}(\mathcal{X})$. We note that $\mathcal{K}(\mathcal{X}) \subset \mathcal{DC}(\mathcal{X})$.
It is work noting that this class invested by W. V. Petryshyn [19] and W. Y. Akashi [2] played a chief role in Fredholm and B-Fredholm theory (see [2, 5, 12–15, 19]). In 2014, B. Krichen foregrounded in [12] the concept of relative demicompactness for a closed linear operator. In 2021, B. Krichen and B. Trabelsi elaborated in [14] some B-Fredholm results including the class of demicompact linear operators. In addition, they tackled the relationship between this class and measures of weak noncompactness of linear operators with regard to an axiomatic one.
At this stage, a new class will be introduced called A-right upper semi-Browder operators, where A is a given linear operator on a Banach space $\mathcal{X}$ defined by:

$$\mathcal{B}^r_{+,A}(\mathcal{X}) := \{T \in \Phi_+(\mathcal{X}) : \text{asc}^r_A(T) < \infty\}.$$

In this research paper, the A-right upper semi-Browder spectrum of T is introduced as

$$\sigma^r_{ub,A}(T) := \{\lambda \in \mathbb{C} : \lambda - T \notin \mathcal{B}^r_{+,A}(\mathcal{X})\}.$$

We are basically interested in the following essential spectra:

$$\sigma_{e_1}(T) := \{\lambda \in \mathbb{C} : \lambda - T \notin \Phi_+(\mathcal{X})\} := \mathbb{C}\backslash\Phi_{+T},$$

$$\sigma_{e_4}(T) := \{\lambda \in \mathbb{C} : \lambda - T \notin \Phi(\mathcal{X})\} := \mathbb{C}\backslash\Phi_T,$$

$$\sigma_{e_5}(T) := \mathbb{C}\backslash\rho_5(T),$$

$$\sigma_{eap}(T) := \bigcap_{K\in\mathcal{K}(\mathcal{X})} \sigma_{ap}(T + K),$$

$$\sigma_{abp}(T) := \bigcap_{K\in\mathcal{K}(\mathcal{X}) \text{ and } KT=TK} \sigma_{ap}(T + K),$$

$$\sigma_{ap}(T) := \Big\{\lambda \in \mathbb{C} : \inf_{x\in\mathcal{D}(T) \text{ and } \|x\|=1} \|(\lambda - T)x\| = 0\Big\},$$

where $\rho_{e_5}(T) := \{\lambda \in \mathbb{C} : \lambda \in \Phi_T \text{ and } \text{ind}(\lambda - T) = 0\}$. The subsets $\sigma_{ap}(.)$, $\sigma_{e_1}(.)$ and $\sigma_{e_4}(.)$ are, respectively, the approximate point spectrum, the Gustafson and the Wolf essential spectra. $\sigma_{eap}(T)$ is a non-empty compact subset of the set of complex numbers $\mathbb{C}$ and it is called the essential approximate point spectrum of T and $\sigma_{abp}(T)$ is called Browder's essential approximate point spectrum of T.

Theorem 1.5. *[18]*Let $\mathcal{X}$ be a Banach space and $T \in \mathcal{L}(\mathcal{X})$.

$$\sigma_{ub}(T) = \sigma_{e_1}(T) \cup \text{acc}\, \sigma_{ap}(T),$$

where acc $\sigma(T)$ corresponds to the set of accumulation points of $\sigma(T)$.

The theory of block operator matrices is highly needed in numerous areas of mathematics and has a wide range of applications, namely in systems theory as Hamiltonians (see [7]), in the discretization of partial differential equations as large partitioned matrices due to sparsity patterns, in saddle point problems in non-linear analysis (see [4]), in evolution problems as linearization of second-order Cauchy problems and as linear operators addressing coupled systems of partial differential equations. Such systems are frequently invested in mathematical physics, e.g., in fluid mechanics (see [6]), magnetohydrodynamics (see [17]) and quantum mechanics (see [24]). In all these applications, the spectral traits of the corresponding block operator matrices are of crucial significance as they govern, for example, the time evolution and thus the stability of the underlying physical systems. Among the most outstanding works on spectral theory of block operator matrices, we state [10], where the author elaborated the essential spectra of 2×2 and 3×3 block operator matrices. We equally state [25], where a wide panorama of methods was set forward in order to examine the essential spectra of block operator matrices. As far as this research work is concerned, our central focus is upon the generalized weak demicompactness properties of the following matrix operator L_0 acting on the Banach space product $\mathcal{X} \times \mathcal{X}$ which is indicated by:

$$L_0 = \begin{pmatrix} T & B \\ C & D \end{pmatrix}.$$

Notably, the entries of L_0 are unbounded. The operator T acts on the Banach space $\mathcal{X}$ and has the domain $\mathcal{D}(T)$. D acts on the same Banach space $\mathcal{X}$ and is defined on $\mathcal{D}(D)$ and the intertwining operator B (resp. C) is defined on $\mathcal{D}(B)$ (resp. $\mathcal{D}(C)$) and acts from $\mathcal{X}$ into itself. We shall consider below that $\mathcal{D}(T) \subset \mathcal{D}(C)$ and $\mathcal{D}(B) \subset \mathcal{D}(D)$. It is to be noted in general that the operator L_0 is neither a closed nor a closable operator even if its entries are closed operators. In [3], it was corroborated that under some conditions, L_0 is closable and its closure is expressed by L. In the literature, multiple prominent findings were obtained concerning the spectral theory of this type of operators. Among these works, we mention [25] where the author addressed the essential spectra of the matrix operator using an abstract non-zero two-sided ideal. The main objective of this work is to use the concept of generalized weak demicompactness to examine the essential spectra of L, the closure of L_0.

This paper is laid out as follows. In the next section, we recall some definitions and results needed in the sequel of the paper. In particular, we discuss the relationship between A-right generalized weakly demicompact operators and Fredholm and upper semi-Fredholm operators. In the third section, we provide some sufficient conditions on the inputs of a closable block operator matrix so as to ensure the A-right generalized weak demicompactness of its closure. Furthermore, we establish some perturbation results for essential spectrum of matrix operators.

2. A-right Generalized Weak Demicompactness

In the remaining part of this paper, we denote $\rightarrow$ for the strong convergence and $\rightharpoonup$ for the weak convergence. We open this section with the following definitions:

Definition 2.1. *[13] Let $\mathcal{X}$ be a Banach space and let $T : \mathcal{D}(T) \subset \mathcal{X} \rightarrow \mathcal{X}$ be a linear operator. The operator T is said to be weakly demicompact, if for every bounded sequence $(x_n)_n \subset \mathcal{D}(T)$ such that $x_n - Tx_n \rightharpoonup x \in \mathcal{X}$. Then, there exists a weakly convergent subsequence of $(x_n)_n$.*

Definition 2.2. *[8] Let $\mathcal{X}$ be a Banach space. An operator A and $T \in \mathcal{L}(\mathcal{X})$ is called a generalized weakly demicompact operator if there exists a finite subset E of $\mathbb{C}$ containing* 0 *such that*
(i) For all $\lambda \in \mathbb{C}\backslash E$, $\frac{1}{\lambda}$ T is a weakly demicompact operator,
(ii) $\lambda \in \mathbb{C}\backslash E$, $T - \lambda I$ has a finite ascent, and
(iii) all $\lambda \in \sigma(T)\backslash E$, are eigenvalues of finite multiplicity and have no accumulation points except possibly points of E.

Now, we shall introduce the following definition:

Definition 2.3. *Let $\mathcal{X}$ be a Banach space. An operator A and $T \in \mathcal{L}(\mathcal{X})$ is called an A-right generalized weakly demicompact operator if there exists a finite subset E of $\mathbb{C}$ containing 0 such that*
(i) For all $\lambda \in \mathbb{C}\backslash E$, $\frac{1}{\lambda}\, T$ is a weakly demicompact operator,
(ii) $\lambda \in \mathbb{C}\backslash E$, $T - \lambda I$ has a finite A-right ascent, and
(iii) all $\lambda \in \sigma(T)\backslash E$, are eigenvalues of finite multiplicity and have no accumulation points except possibly points of E.

The set E is called an A-right generalized set of T.

Remark 2.4. *It is noteworthy that if, E is an A-right generalized set of T and D is a finite subset of $\mathbb{C}$ containing E, then D is an A-right generalized set of T as well.*

Lemma 2.5. *[9] Let $\mathcal{X}$ be a Banach space. Let $A, T \in \mathcal{L}(\mathcal{X})$. Hence,*

$$\mathcal{N}_A^r(T^n) \subset \mathcal{N}_A^r(T^{n+m})\ \forall\ n,\ m \in \mathbb{N} \tag{1}$$

Theorem 2.6. *Let $\mathcal{X}$ be a Banach space and $T \in \mathcal{L}_{\mathcal{R}(A)}(\mathcal{X})$. If T is a generalized weakly demicompact operator, then T is an A-right generalized weakly demicompact operator.*

Proof. Let $T \in \mathcal{L}_{\mathcal{R}(A)}(\mathcal{X})$ and let T be a generalized weakly demicompact operator. Now, assume that $\mathrm{asc}(T) = n < +\infty$ and $x \in \mathcal{N}_A^r(T^{n+1})$. Therefore, we obtain $Ax \in \mathcal{N}(T^{n+1}) = \mathcal{N}(T^n)$. Thus, $x \in \mathcal{N}_A^r(T^n)$ and relying on Lemma 2.5 Eq. (1), we deduce that $\mathcal{N}_A^r(T^{n+1}) = \mathcal{N}_A^r(T^n)$. Hence, T is an A-right generalized weakly demicompact operator. □

Theorem 2.7. *Let $\mathcal{X}$ be a Banach space and $T \in \mathcal{L}_{\mathcal{R}(A)}(\mathcal{X})$. Then, the following conditions are equivalent:*
(i) T is an A-right generalized weakly demicompact operator.
(ii) There exists a finite subset $E \subset \mathbb{C}$ containing 0 such that $\lambda I - T \in \Phi_+(\mathcal{X})$ for all $\lambda \in \mathbb{C}\backslash E$.

Proof. $(i) \Rightarrow (ii)$ Assume that T is an A-right generalized weakly demicompact operator. We denote by $m = \mathrm{asc}_A^r(T) < \infty$. Furthermore, for any eigenvalue $\lambda \in \mathbb{C}\backslash E$, we have $\mathrm{mult}_A^r(T, \lambda) < \infty$ which implies that

$$\mathrm{mult}_A^r(T, \lambda) = \dim \mathcal{N}_A^{r,\infty}((\lambda - T)) = \dim\ \mathcal{N}_A^r((\lambda - T)^m) < \infty.$$

Hence, grounded on Lemma 2.5 Eq. (1), we get

$$\mathcal{N}(\lambda - T) \subset \mathcal{N}_A^r((\lambda - T)^m).$$

Then, $\dim\ \mathcal{N}((\lambda - T)) = \alpha(\lambda - T) < \infty$.
Now, in the second part of this proof, we attempt to demonstrate that $\mathcal{R}((\lambda - T))$ is closed. Let $(x_n)_n$ be any sequence included in $\mathcal{X}$. Let us take $y_n = (\lambda - T)x_n$ such that $y_n \to y \in \mathcal{X}$. As $\frac{1}{\lambda}\, T$ is weakly demicompact and

$$x_n - \frac{1}{\lambda}Tx_n \rightharpoonup \frac{1}{\lambda}y,$$

then there exists a subsequence $(x_{\varphi(n)})_n$ of $(x_n)_n$ such that $x_{\varphi(n)} \rightharpoonup x \in \mathcal{X}$. Using both

$$x_{\varphi(n)} \rightharpoonup x \text{ and } (\lambda - T)x_{\varphi(n)} \rightharpoonup y,$$

together with the closedness of $(\lambda - T)$, we infer that $y = (\lambda - T)x \in \mathcal{R}(\lambda - T)$. Hence, $\mathcal{R}(\lambda - T)$ is closed.
$(ii) \Rightarrow (i)$ Assume that, there exists a finite subset $E \subset \mathbb{C}$ containing 0 such that $\lambda I - T \in \Phi_+(\mathcal{X})$ for all $\lambda \in \mathbb{C}\backslash E$. Then, T is a generalized Riesz operator. Thus, denote by $p = \mathrm{acs}(\lambda - T) < \infty$ for all $E \subset \mathbb{C}$ and all $\lambda \in \sigma(T)\backslash E$ which are eigenvalues of finite multiplicity and have no accumulation point except possibly points of E. Now, as $T \in \mathcal{L}_{\mathcal{R}(A)}(\mathcal{X})$, $p = \mathrm{acs}(\lambda - T) < \infty$ and $x \in \mathcal{N}_A^r(T^{n+1})$, then $Ax \in \mathcal{N}(T^{p+1}) = \mathcal{N}(T^p)$ and based upon Lemma 2.5 Eq. (1), we deduce that

$$\mathcal{N}_A^r(T^p) = \mathcal{N}_A^r(T^{p+1}).$$

Hence, $\mathrm{acs}^r_A(T) \leq \mathrm{acs}(T) \leq p$ and $\mathrm{acs}^r_A(T) < \infty$. On the other side, using Theorem 2.1 in [11], we infer that $\lambda - T \in \Phi_+(\mathcal{X})$ for all $\lambda \in \mathbb{C}\backslash E$ implies that $\frac{1}{\lambda}T$ is a demicompact operator for all $\lambda \in \mathbb{C}\backslash E$. As a matter of fact, from Lemma 1 in [15], we obtain $\frac{1}{\lambda}$ T which is a weakly demicompact operator for all $\lambda \in \mathbb{C}\backslash E$. Thus, we conclude that T is an A-right generalized weakly demicompact operator. □

Theorem 2.8. *Let $\mathcal{X}$ be a Banach space and $T \in \mathcal{L}(\mathcal{X})$. If μT is an A-right generalized weakly demicompact operator for each $\mu \in [0,1]$ with an A-right generalized subset E, then $\lambda - T \in \Phi(\mathcal{X})$ and $ind(\lambda - T) = 0$, for all $\lambda \in \mathbb{C}\backslash E$.*

Proof. Since μT is an A-right generalized weakly demicompact operator for each $\mu \in [0,1]$, then $\lambda - \mu T \in \Phi_+(\mathcal{X})$ for all $\lambda \in \mathbb{C}\backslash E$. Now, since the index $\mathrm{ind}(\lambda - \mu T)$ is continuous in μ and as it is an integer, including infinite values, it must be constant for every $\mu \in [0,1]$. It follows that $\mathrm{ind}(\lambda - \mu T) = \mathrm{ind}(\lambda - T) = \mathrm{ind}(\lambda) = 0$. Consequently, $\lambda - T \in \Phi(\mathcal{X})$ and $\mathrm{ind}(\lambda - T) = 0$, for all $\lambda \in \mathbb{C}\backslash E$. □

Theorem 2.9. *[20] Let $\mathcal{X}$ be a Banach space and $T \in \mathcal{L}(\mathcal{X})$. $\lambda \notin \sigma_{abp}(T)$ if, and only if, $\lambda - T \in \mathcal{W}^-_+(\mathcal{X})$ and $asc(T) < \infty$.*

The following Lemma can be drawn from [23].

Lemma 2.10. *Let $\mathcal{X}$ be a Banach space and $T \in \mathcal{L}(\mathcal{X})$. Suppose there exists a nonnegative integer N such that $dim(\mathcal{N}(T^k)) \leq N$ when $k \in \{0,1,2.....\}$. Then, $asc(T) \leq N$.*

Theorem 2.11. *Let $\mathcal{X}$ be a Banach space and $A, T \in \mathcal{L}(\mathcal{X})$. Then, the following conditions are equivalent:*
(i) $\lambda \notin \sigma^r_{ub,A}(T)$.
(ii) $\lambda - T \in \mathcal{W}^-_+(\mathcal{X})$ and $asc^r_A(T) < \infty$.

Proof. (*i*) ⇒ (*ii*) Assume that $\lambda \notin \sigma^r_{ub,A}(T)$, then $\lambda - T \in \Phi_+(\mathcal{X})$ and $\mathrm{asc}^r_A(T) < \infty$. Now, let's demonstrate that $\mathrm{ind}(\lambda - T) \leq 0$. By contradiction, suppose that $\mathrm{ind}(\lambda - T) > 0$. Then,

$$\lambda \in \sigma_{eap}(T) \subset \sigma_{abp}(T).$$

Resting on Theorem 2.9, $\lambda - T \notin \mathcal{W}^-_+(\mathcal{X})$ or $\mathrm{asc}(\lambda - T) = \infty$. First case, if $\lambda - T \notin \mathcal{W}^-_+(\mathcal{X})$, this contradicts the fact that $\lambda - T \in \Phi_+(\mathcal{X})$ and $\mathrm{ind}(\lambda - T) > 0$. Second case, if $\mathrm{asc}(\lambda - T) = \infty$, then $\mathrm{asc}(\lambda - T) > N$, for all $N \in \mathbb{N}$. Hence, departing from Lemma 2.10, we deduce that $\dim\mathcal{N}(\lambda - T)$ is not finite, which yields a contradiction. Thus, we conclude that $\mathrm{ind}(\lambda - T) \leq 0$.
(*ii*) ⇒ (*i*) The sense of this Theorem is trivial. □

Theorem 2.12. *Let $\mathcal{X}$ be a Banach space and $A, T \in \mathcal{L}(\mathcal{X})$ such that $0 \in \sigma^r_{ub,A}(T)$. If T is an A-right generalized weakly demicompact operator, then T has a finite A-right upper semi-Browder spectrum.*

Proof. Assume that T is an A-right generalized weakly demicompact. Then, referring to Theorem 2.7, there exists a finite subset $E \subset \mathbb{C}$ containing 0 such that $\lambda I - T \in \Phi_+(\mathcal{X})$ for all $\lambda \in \mathbb{C}\backslash E$, with a finite A-right ascent. Therefore, $\sigma^r_{ub,A}(T) \subset E$ and hence $\sigma^r_{ub,A}(T)$ is a finite subset. □

Theorem 2.13. *Let $\mathcal{X}$ be a Banach space and $A, T \in \mathcal{L}(\mathcal{X})$ such that $0 \in \sigma^r_{ub,A}(T)$ and A is surjective. If T has a finite A-right upper semi-Browder spectrum, then T is an A-right generalized weakly demicompact operator.*

Proof. Assume that $\sigma^r_{ub,A}(T)$ is a finite subset of $\mathbb{C}$. For all $\lambda \in \mathbb{C}\backslash\sigma^r_{ub,A}(T)$, $\lambda - T$ is an A-right upper semi-Browder. Then, $\lambda - T$ is an upper semi-Fredholm which implies that $\frac{1}{\lambda}T$ is demicompact for all $\lambda \in \mathbb{C}\backslash\sigma^r_{ub,A}(T)$. Hence, from Lemma 1 in [15], we infer that $\frac{1}{\lambda}T$ is weakly demicompact for all $\lambda \in \mathbb{C}\backslash\sigma^r_{ub,A}(T)$. Furthermore, we have $\mathrm{asc}^r_A(\lambda - T) = p < \infty$, for all $\lambda \in \mathbb{C}\backslash\sigma^r_{ub,A}(T)$. Additionally, as A is surjective, $\mathrm{asc}(\lambda - T) = p < \infty$, for all $\lambda \in \mathbb{C}\backslash\sigma_{ub}(T)$ (see [9], Remark 2.4). Let $\lambda \in \sigma(T)\backslash\sigma^r_{ub,A}(T)$. Since $\lambda \in \sigma(T)$, then λ is an isolated point of $\sigma_{ap}(T)$ (see [18], Corollary 20). Thus, $\lambda \in \sigma_{ap}(T)\backslash\sigma^r_{ub,A}(T)$. Now, we deduce that λ is an eigenvalue of T of finite multiplicity (see [20], Corollary 2.2). Again, we get all $\lambda \in \sigma(T)\backslash\sigma^r_{ub,A}(T)$ which have no accumulation points except possibly points of $\sigma^r_{ub,A}(T)$ (see [18], Corollary 20). From this perspective, T is A-right generalized weakly demicompact and an A-right generalized set $\sigma^r_{ub,A}(T)$. □

Theorem 2.14. *Let $\mathcal{X}$ be a Banach space, $T \in \mathcal{L}_{\mathcal{R}(A)}(\mathcal{X})$ and $P(x) = \sum_{k=0}^{n} a_k x^k$ be a non-zero complex polynomial, with $P(0) = 0$, $a_k = \frac{1}{deg(P)\lambda^{k-1}}$ for all $\lambda \in \mathbb{C}\backslash\{0\}$ and $x_2 \ldots x_n$ its zeros, such that $P(T)$ is A-right generalized weakly demicompact with an A-right generalized set $E = \{0\}$. Then, T is A-right generalized weakly demicompact with an A-right generalized set $E' = \{0, x_2, ..., x_n\}$.*

Proof. Assume that there exists P a non-zero complex polynomial such that $P(T)$ is A-right generalized weakly demicompact with an A-right generalized set $E = \{0\}$. Let $P(x) = \sum_{k=0}^{n} a_k x^k$. Hence, for $\nu \in \mathbb{C}\backslash\{0, x_2, ..., x_n\}$, we draw the following equality:

$$P(\nu) - P(T) = \sum_{k=1}^{n} a_k(\nu^k - T^k).$$

On the other side, for any $k \in \{1, ..., n\}$

$$\nu^k - T^k = (\nu - T)\sum_{j=0}^{k-1} \nu^j T^{k-j-1}.$$

This allows us to state that

$$P(\nu) - P(T) = (\nu - T)B(T) = B(T)(\nu - T), \tag{2}$$

where

$$B(T) = \sum_{j=0}^{k-1} \nu^j T^{k-j-1}.$$

Let $m \in \mathbb{N}$. Using Eq. (2), we obtain

$$(P(\nu) - P(T))^m = (\nu - T)^m [B(T)]^m = [B(T)]^m (\nu - T)^m. \tag{3}$$

Grounded on Eq. (3), we deduce

$$\mathcal{N}_A^r\Big((\nu - T)^m\Big) \subset \mathcal{N}_A^r\Big((P(\nu) - P(T))^m\Big), \text{ for all } m \in \mathbb{N}.$$

This implies

$$\bigcup_{m\in\mathbb{N}} \mathcal{N}_A^r\Big((\nu - T)^m\Big) \subset \bigcup_{m\in\mathbb{N}} \mathcal{N}_A^r\Big((P(\nu) - P(T))^m\Big). \tag{4}$$

Departing from Definition 2.3, we have $\mathrm{asc}_A^r(\mu - P(T)) < \infty$, for all $\mu \in \mathbb{C}\backslash\{0\}$. In particular, for $\mu = P(\nu) \neq 0$, we get $\mathrm{asc}_A^r(P(\nu) - P(T)) < \infty$. Suppose that $\mathrm{asc}_A^r(P(\nu) - P(T)) = m_0$. Then,

$$\dim \bigcup_{m\in\mathbb{N}} \mathcal{N}_A^r\Big((P(\nu) - P(T))^m\Big) = \dim \mathcal{N}_A^r\Big((P(\nu) - P(T))^{m_0}\Big) < \infty.$$

With respect to equation (4), we obtain

$$\dim \bigcup_{m\in\mathbb{N}} \mathcal{N}_A^r\Big((\nu - T)^m\Big) < \infty,$$

which allows us to get $\mathrm{asc}_A^r(\mu - T) < \infty$. Now, we need to demonstrate that $\alpha(\nu - T) < \infty$ and $\mathcal{R}(\nu - T)$ is closed. Let $x \in \mathcal{N}(\lambda - T)$ for all $\lambda \in \mathbb{C}\backslash\{0\}$. Then, $[\lambda - \sum_{k=0}^{m} \frac{1}{deg(P)} \frac{1}{\lambda^{k-1}} T^k]x = 0$ for all $\lambda \in \mathbb{C}\backslash\{0\}$. We deduce that, $x \in \mathcal{N}(\lambda - P(T))$ for all $\lambda \in \mathbb{C}\backslash\{0\}$. Again, using Definition 2.3, we infer that $\alpha(\lambda - T) < \infty$, for all $\lambda \in \mathbb{C}\backslash\{0\}$.

Therefore, in particular, for $\lambda = \nu$, we get $\alpha(\nu - T) < \infty$, which implies that there exists a closed subspace $\mathcal{X}_1$ of $\mathcal{X}$ such that

$$\mathcal{X} = \mathcal{N}(\nu - T) \oplus \mathcal{X}_1.$$

Hence, it is sufficient to confirm that there is a constant $M > 0$ such that for every $x \in \mathcal{X}_1$, $\|(\nu - T)x\| \geq M\|x\|$ (see [21], Theorem 3.14). Otherwise, there exists a sequence $(x_p)_p$ of $\mathcal{X}_0$ such that $\|(\nu - T)x_p\| \to 0$ and $\|x_p\| = 1$. Again, since $\alpha(P(\nu) - P(T)) < \infty$, then there exists a closed subspace $\mathcal{X}_2$ of $\mathcal{X}$ such that

$$\mathcal{X} = \mathcal{N}(P(\nu) - P(T)) \oplus \mathcal{X}_2.$$

Therefore, $(P(\nu) - P(T)) : \mathcal{X}_2 \longrightarrow \mathcal{R}(P(\nu) - P(T))$ is invertible with a bounded inverse on $\mathcal{R}(P(\nu) - P(T))$. Additionally, we obtain

$$(\nu - T)x_p \to 0. \tag{5}$$

Thus, relying on equations (2) and (5), we deduce that

$$(P(\nu) - P(T))x_p \to 0.$$

Departing from the boundedness of $[P(\nu) - P(T))_{|\mathcal{X}_2}]^{-1}$ on $\mathcal{R}(P(\nu) - P(T))$, we deduce that $x_p \to 0$. This contradicts the continuity of the norm. As a result, we conclude that $(\nu - T) \in \Phi_+(\mathcal{X})$, for all $\nu \in \mathbb{C}\backslash\{0, x_2 ... x_n\}$. Hence, from Theorem 2.7, T is an A-right generalized weakly demicompact with an A-right generalized set $E = \{0, x_2 ... x_n\}$. $\square$

Proposition 2.15. *[15] Let $\mathcal{X}$ be a Banach space. Let $T,\ S_0 \in \mathcal{L}(\mathcal{X})$, w be an nth root of the unit with $n \in \mathbb{N}\backslash\{0\}$ and let $B \in \mathcal{L}(\mathcal{X})$ such that $B^n = S_0^{-1}$ and $BT = TB$. Then, $T^n \in \mathcal{DC}_{S_0}(\mathcal{X})$ if, and only if, $\omega^k BT \in \mathcal{DC}(\mathcal{X})$, for all $0 \leq k \leq n - 1$.*

Example 2.16. *Consider the space C^∞, which consists of continuous ω-periodic functions $y : [0, t] \longrightarrow \mathbb{R}$, and C_0^∞, the space of continuously differentiable ω-periodic functions $y : [0, t] \longrightarrow \mathbb{R}$. The space C^∞ with the maximum norm $\|.\|_\infty$ and C_0^∞ with the norm defined as $\|f\|_1 = \|f\|_\infty + \|f'\|_\infty$ are both Banach spaces. Assume that A: $\mathbb{R} \longrightarrow \mathbb{R}$ is surjective. Let T be the operator defined by*

$$Ty(t) - \theta I y(t) = -\Big[{}_{GL}F^{\frac{1}{s}} y(t) - \frac{y(0)t^{-\frac{1}{s}}}{\Gamma(1-\frac{1}{s})}\Big]^{\frac{1}{2}},$$

for all $s \in \mathbb{N} \setminus \{0, 1\}$ and $\theta \in \mathbb{C} \setminus \{1\}$, where ${}_{GL}F^{\frac{1}{s}}$ represents the Grünwald-Letnikov fractional derivative with fractional order $\frac{1}{s}$, and $K : C_0^\infty \to C^\infty$ is given by

$$(Ky)(t) = y'(t).$$

From this perspective, we get this equality

$${}_{GL}F^{\frac{1}{s}} y(t) = \frac{y(0)t^{-\frac{1}{s}}}{\Gamma(1-\frac{1}{s})} + F^{-(1-\frac{1}{s})} y'(t),$$

where $F^{-(1-\frac{1}{s})}$ denotes the fractional integral with fractional order $(1 - \frac{1}{s})$. Observe that

$${}_{GL}F^{\frac{1}{s}} y(t) = \frac{y(0)t^{-\frac{1}{s}}}{\Gamma(1-\frac{1}{s})} +_C F^{(\frac{1}{s})} y(t),$$

where ${}_C F^{(\frac{1}{s})}$ denotes the Caputo derivative of fractional order $\frac{1}{s}$, leading to

$$\Big(\theta I - T\Big)^{2s} y(t) = \Big[{}_{GL}F^{\frac{1}{s}} y(t) - \frac{y(0)t^{-\frac{1}{s}}}{\Gamma(1-\frac{1}{s})}\Big]^s = [{}_C F^{(\frac{1}{s})}]^s y(t),$$

for all $\theta \in \mathbb{C} \setminus \{1\}$. According to [[16], as well as Theorem 3.4], it follows that

$$(\theta I - T)^{2s} y(t) = y'(t),$$

for all $\theta \in \mathbb{C} \setminus \{1\}$. Clearly, K is a bounded linear operator and $\|(\theta I - T)^{2s}\| < 1$ for all $\theta \in \mathbb{C} \setminus \{1\}$. Therefore, $(\theta I - T)^{2s}$ is a demicompact operator for all $\theta \in \mathbb{C} \setminus \{1\}$. By applying Proposition 2.15, we can conclude that $(\theta I - T)$ is a demicompact operator for all $\theta \in \mathbb{C} \setminus \{1\}$. Furthermore, using Theorem 2.1 [11], we obtain $Iy(t) - \theta I y(t) + Ty(t) = -\left[{}_{GL}F^{\frac{1}{s}} y(t) - \frac{y(0)t^{-\frac{1}{s}}}{\Gamma(1-\frac{1}{s})}\right]^{\frac{1}{2}}$ which is an upper semi-Fredholm operator for all $\theta \in \mathbb{C} \setminus \{1\}$. Hence, for $\lambda = \theta - 1$, $\lambda I y(t) - Ty(t) = \left[{}_{GL}F^{\frac{1}{s}} y(t) - \frac{y(0)t^{-\frac{1}{s}}}{\Gamma(1-\frac{1}{s})}\right]^{\frac{1}{2}} \in \Phi_+(X)$ for all $\lambda \in \mathbb{C} \setminus \{0\}$. Given that A is surjective, it follows from Theorem 2.7 that T is an A-right generalized weakly demicompact operator.

3. A-right Generalized Weak Demicompactness Results for Matrix Operator

As far as this section is concerned, we shall use some conditions reported in [22], imposed on the entries of the operator L_0 to guarantee its closedness. Having obtained the closure of the operator L_0, we shall examine its essential spectra. For this reason, we shall consider, in the product space $\mathcal{X} \times \mathcal{X}$, the following matrix operator

$$L_0 = \begin{pmatrix} T & B \\ C & D \end{pmatrix}, \tag{6}$$

where the operator T acts on $\mathcal{X}$ and has domain $\mathcal{D}(T)$, B is defined on $\mathcal{D}(B)$ and acts on the Banach space $\mathcal{X}$, and the intertwining operator C (resp. D) is defined on the domain $\mathcal{D}(C)$ (resp., $\mathcal{D}(D)$) and acts on $\mathcal{X}$. At this stage, let us take into account the following assumptions [22].
(H_1) T is a closed, densely defined linear operator on $\mathcal{X}$ with a nonempty resolvent set $\rho(T)$.
(H_2) The operator B is a densely defined linear operator on $\mathcal{X}$ and for some (hence all) $\mu \in \rho(T)$, the operator $(T-\mu)^{-1}B$ is closable. (In particular, if B is closable, then $(T-\mu)^{-1}B$ is closable).
(H_3) The operator C satisfies $\mathcal{D}(T) \subset \mathcal{D}(C)$, and for some (hence all) $\mu \in \rho(T)$, the operator $C(T-\mu)^{-1}$ is bounded. (In particular, if C is closable, then $C(T-\mu)^{-1}$ is bounded).
(H_4) $\mathcal{D}(B) \subset \mathcal{D}(D)$.
(H_5) For some (hence all) $\mu \in \rho(T)$, the operator $D - C(T-\mu)^{-1}B$ is closable. We shall indicate by $S(\mu)$ its closure.

Remark 3.1. (*i*) *With respect to assumptions (H_1) and (H_2), we infer that for each $\mu \in \rho(T)$ the operator $G(\mu) := \overline{(T-\mu)^{-1}B}$ is bounded on $\mathcal{X}$.*
(*ii*) *Departing from the assumption (H_3), we deduce that the operator: $F(\mu) := C(T-\mu)^{-1}$ is bounded on $\mathcal{X}$.*

We recall the following result which describes the closure of L_0.

Theorem 3.2. *[3] Let conditions $(H_1) - (H_3)$ be satisfied and the lineal $\mathcal{D}(B) \cap \mathcal{D}(D)$ be dense in $\mathcal{X}$. Then, the operator L_0 is closable and the closure L of L_0 is provided by:*

$$L = \mu - \begin{pmatrix} I & 0 \\ F(\mu) & I \end{pmatrix} \begin{pmatrix} \mu - T & 0 \\ 0 & \mu - S(\mu) \end{pmatrix} \begin{pmatrix} I & G(\mu) \\ 0 & I \end{pmatrix}. \tag{7}$$

In other words,

$$\begin{array}{rcl} L : \mathcal{D}(L) \subset (\mathcal{X} \times \mathcal{X}) & \longrightarrow & \mathcal{X} \times \mathcal{X} \\ \begin{pmatrix} x \\ y \end{pmatrix} & \longmapsto & L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} T(x + G(\mu)y) - \mu G(\mu)y \\ C(x + G(\mu)y) - S(\mu)y \end{pmatrix}. \end{array}$$

with $\mathcal{D}(L) = \left\{ L\begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{X} \times \mathcal{X} : x + G(\mu)y \in \mathcal{D}(T) \text{ and } y \in \mathcal{D}(S(\mu)) \right\}$.
It is to be noted that the description of the operator L is not dependent on the choice of the point $\mu \in \rho(T)$.

Remark 3.3. *Let $\lambda \in \mathbb{C}$. Relying on Eq. (4.2), it follows that*

$$L = \mu - \begin{pmatrix} I & 0 \\ F(\mu) & I \end{pmatrix} \begin{pmatrix} \mu - T & 0 \\ 0 & \mu - S(\mu) \end{pmatrix} \begin{pmatrix} I & G(\mu) \\ 0 & I \end{pmatrix}, \tag{8}$$

where $M(\mu) = \begin{pmatrix} 0 & G(\mu) \\ F(\mu) & F(\mu)G(\mu) \end{pmatrix}$.

We begin this section with a finding for a particular representation of L_0.

Lemma 3.4. *Let $\mathcal{X}$ be a Banach space and $A \in \mathcal{L}(\mathcal{X})$. Consider the 2×2 operator matrix*

$$M_D = \begin{pmatrix} B & D \\ 0 & C \end{pmatrix},$$

where $M_D \in \mathcal{L}_{(\mathcal{R}(A),\mathcal{R}(A))}(\mathcal{X} \times \mathcal{X}) = \left\{M_D \in \mathcal{L}(\mathcal{X} \times \mathcal{X}) : B, C, D \in \mathcal{L}_{\mathcal{R}(A)}(\mathcal{X})\right\}$. Hence, B and C are A-right generalized weakly demicompact operators if, and only if, M_D is an A-right generalized weakly demicompact operator.

Proof. Let $\lambda \in \mathbb{C}$. Clearly, we have

$$\lambda - M_D = \begin{pmatrix} \lambda - B & -D \\ 0 & \lambda - C \end{pmatrix}.$$

Since B and C are A-right generalized weakly demicompact, then there exist two finite subsets E_0 and E_1 of $\mathbb{C}$ including 0 such that $\lambda - B \in \Phi_+(\mathcal{X})$ for all $\lambda \in \mathbb{C} \setminus \{E_0\}$ and $\lambda - C \in \Phi_+(\mathcal{X})$ for all $\lambda \in \mathbb{C} \setminus \{E_1\}$. With regard to Remark 2.4, it follows that B and C are also A-right generalized weakly demicompact with the generalized set $E = E_0 \cup E_1$. This allows us to get $\lambda - B \in \Phi_+(\mathcal{X})$ for all $\lambda \in \mathbb{C} \setminus \{E\}$ and $\lambda - C \in \Phi_+(\mathcal{X})$ for all $\lambda \in \mathbb{C} \setminus \{E\}$. Now, when applying Lemma 6.6.1 [10], we get $\lambda - M_D \in \Phi_+(\mathcal{X} \times \mathcal{X})$ for all $\lambda \in \mathbb{C} \setminus \{E\}$. As $M_D \in \mathcal{L}_{(\mathcal{R}(A),\mathcal{R}(A))}(\mathcal{X} \times \mathcal{X})$ and with to reference Theorem 2.7, we obtain M_D which is A-right generalized weakly demicompact with the A-right generalized set E.
Conversely, suppose that M_D is an A-right generalized weakly demicompact operator. According to Theorem 2.7, there exists a finite subset E of $\mathbb{C}$ which contains 0 such that $\lambda - M_D$ is an upper semi-Fredholm operator, for all $\lambda \in \mathbb{C} \setminus (E \cap \sigma(C))$. First, we provide the following factorization

$$\lambda - M_D = \begin{pmatrix} \lambda - B & -D \\ 0 & \lambda - C \end{pmatrix} = \begin{pmatrix} I & -D(\lambda - C)^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} \lambda - B & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & \lambda - C \end{pmatrix}.$$

Since

$$\begin{pmatrix} \lambda - B & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & \lambda - C \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & \lambda - C \end{pmatrix} \begin{pmatrix} \lambda - B & 0 \\ 0 & I \end{pmatrix},$$

and using the fact that $\lambda - M_D \in \Phi_+(\mathcal{X} \times \mathcal{X})$, from Lemma 6.6.2 in [10], it follows that $\lambda - B$ and $\lambda - C$ are upper semi-Fredholm, for all $\lambda \in \mathbb{C} \setminus \{E\}$. Hence, referring to Theorem 2.7, B and C are A-right generalized weakly demicompact operators. □

Now, let's consider the following set:
$\Lambda^r_{\mathcal{X},A} = \{J \in \mathcal{L}_{\mathcal{R}(A)}(\mathcal{X}) : \omega J$ is A-right generalized weakly demicompact for every $\omega \in [0, 1]\}$,
the Frobenius Schur factorization can be expressed in terms of

$$\alpha L = \alpha\mu - \begin{pmatrix} I & 0 \\ F(\mu) & I \end{pmatrix} \begin{pmatrix} \alpha\mu - \alpha T & 0 \\ 0 & \alpha\mu - \alpha S(\mu) \end{pmatrix} \begin{pmatrix} I & G(\mu) \\ 0 & I \end{pmatrix}.$$

Let $\lambda \in \mathbb{C}$. We therefore have

$$\lambda - \alpha L := UV_\alpha(\lambda)W - (\lambda - \alpha\mu)M(\mu), \tag{9}$$

where $U = \begin{pmatrix} I & 0 \\ F(\mu) & I \end{pmatrix}$, $V_\alpha(\lambda) = \begin{pmatrix} \lambda - \alpha T & 0 \\ 0 & \lambda - \alpha S(\mu) \end{pmatrix}$ and $W = \begin{pmatrix} I & G(\mu) \\ 0 & I \end{pmatrix}$.
It is possible at this stage of analysis to express the main result of this section.

Theorem 3.5. *Let $\mathcal{X}$ be a Banach space, let $A \in \mathcal{L}(\mathcal{X})$ and let T, $G(\mu)$, $F(\mu)$ and $S(\mu) \in \mathcal{L}_{\mathcal{R}(A)}(\mathcal{X})$. Suppose that the operator L_0 defined in Eq. (6) satisfies $(H_1) - (H_5)$. Let $\mu \in \rho(T)$ and $\lambda \in \mathbb{C}$. If for $\alpha \in [0,1]$ such that $\lambda \neq \alpha\mu$, the operators αT and $\alpha S(\mu)$ are A-right generalized weakly demicompact, there exists $H^{\alpha}_{\lambda l}$ a left Fredholm inverse of $UV_{\alpha}(\lambda)W$ such that $M(\mu)H^{\alpha}_{\lambda l} \in \Lambda^{r}_{\mathcal{X},A}$, then αL is A-right generalized weakly demicompact for all $\alpha \in [0,1]$.*

Proof. We suppose that the hypothesis holds. Let $\mu \in \rho(T)$ and $\lambda \in \mathbb{C}$. Using Eq. (9), we obtain

$$\lambda - \alpha L := UV_{\alpha}(\lambda)W - (\lambda - \alpha\mu)M(\mu).$$

Now, let $H^{\alpha}_{\lambda} := UV_{\alpha}(\lambda)W$. Since T, $S(\mu) \in \mathcal{L}_{\mathcal{R}(A)}(X)$, αT and $\alpha S(\mu)$ are A-right generalized weakly demicompact, using Lemma 3.4, we deduce that there exists a finite subset E containing 0 such that $V_{\alpha}(\lambda)$ is an upper semi-Fredholm operator, for all $\lambda \in \mathbb{C}\backslash\{E\}$. Let $E_1 := E \cup E_0$, where the set $E_0 := \{\frac{1}{\alpha} : \alpha \in E\backslash\{0\}\}$. It is clear that $V_{\alpha}(\lambda)$ is an upper semi-Fredholm operator, for all $\lambda \in \mathbb{C}\backslash\{E_1\}$. Consequently, there exists a left Fredholm inverse $V^{\alpha}_{\lambda l}$ of $V_{\alpha}(\lambda)$. Resting upon Proposition 6.7.1 [10], it is defined as follows:

$$V^{\alpha}_{\lambda l} = \begin{pmatrix} T^{\alpha}_{\lambda l} & K_0 \\ K_1 & S^{\alpha}_{\lambda l}(\mu) \end{pmatrix},$$

where $T^{\alpha}_{\lambda l}$, $S^{\alpha}_{\lambda l}(\mu)$ are left Fredholm inverse, respectively, of $\lambda - \alpha T$ and $\lambda - \alpha S(\mu)$ and K_0, K_1 are compact operators. Now, based on Lemma 6.7.1 [10], we deduce that $H^{\alpha}_{\lambda l} := W^{-1}V^{\alpha}_{\lambda l}U^{-1}$ is a left Fredholm inverse of H^{α}_{λ}. Then, there exists $K_2 \in \mathcal{K}(\mathcal{X} \times \mathcal{X})$ such that

$$H^{\alpha}_{\lambda l}H^{\alpha}_{\lambda} = I - K_2.$$

It follows that

$$\lambda - \alpha L := \Big[I - (\lambda - \alpha\mu)M(\mu)H^{\alpha}_{\lambda l}\Big]H^{\alpha}_{\lambda} - (\lambda - \alpha\mu)M(\mu)K_2. \tag{10}$$

Since $M(\mu)H^{\alpha}_{\lambda l} \in \Lambda^{r}_{\mathcal{X},A}$, then from Theorem 2.7, we get $(\gamma - \omega M(\mu)H^{\alpha}_{\lambda l})$ which is an upper semi-Fredholm operator, for all $\gamma \in \mathbb{C}\backslash\{E_1\}$. As $\mu \in \rho(T)$ implies that $\alpha(\mu - T) \in \Phi(\mathcal{X})$, $\alpha\mu \in \rho(T)\backslash\{E_1\}$. Hence, we obtain $\frac{1}{\lambda - \alpha\mu} \in \mathbb{C}\backslash\{E_1\}$. Consequently, we get $\Big[I - (\lambda - \alpha\mu)M(\mu)H^{\alpha}_{\lambda l}\Big]$ which is an upper semi-Fredholm operator. Applying Theorem 5.26 [21] on Eq. (10), we therefore have $\Big[I - (\lambda - \alpha\mu)M(\mu)H^{\alpha}_{\lambda l}\Big]H^{\alpha}_{\lambda}$ which is an upper semi-Fredholm. Again applying Theorem 5.26 [21] on Eq. (10), we deduce that the first product term is an upper semiFredholm. Since $(\lambda - \alpha\mu)M(\mu)K_2 \in \mathcal{F}_+(\mathcal{X} \times \mathcal{X})$, it follows that $\lambda - \alpha L \in \Phi_+(\mathcal{X} \times \mathcal{X})$ for all $\lambda \in \mathbb{C}\backslash\{E_1\}$. Hence, from Theorem 2.7, αL is an A-right generalized weakly demicompact operator with the A-right generalized set E_1. □

Corollary 3.6. *Let $\mathcal{X}$ be a Banach space, let $A \in \mathcal{L}(\mathcal{X})$ and let T, $G(\mu)$, $F(\mu)$ and $S(\mu) \in \mathcal{L}_{\mathcal{R}(A)}(\mathcal{X})$. Suppose that the operator L_0 defined in Eq. (6) satisfies $(H_1) - (H_5)$. Let $\mu \in \rho(T)$ and let $\lambda \in \mathbb{C}$ such that $\lambda \neq \mu$. If T and $S(\mu)$ are A-right generalized weakly demicompact, there exists $H_{\lambda l}$, a left Fredholm inverse of $UV_{\alpha}(\lambda)W$ such that $M(\mu)H_{\lambda l} \in \Lambda^{r}_{\mathcal{X},A}$. Then, L is an A-right generalized weakly demicompact operator.*

Proof. The proof is a direct application of Theorem 3.5 for $\alpha = 1$. □

Proposition 3.7. *Let $\mathcal{X}$ be a Banach space, $A \in \mathcal{L}(\mathcal{X})$ and let $\lambda \in \mathbb{C}$ such that T, $G(\lambda)$, $F(\lambda)$ and $S(\lambda) \in \mathcal{L}_{\mathcal{R}(A)}(\mathcal{X})$. Suppose that the operator L_0 defined in Eq. (6) satisfies $(H1) - (H5)$. If D and T are A-right generalized weakly demicompact and $B \in \mathcal{F}_+(\mathcal{X})$, then L is an A-right generalized weakly demicompact operator.*

Proof. We suppose that the hypothesis holds. According to the Frobenius Schur factorization, we get

$$L = \lambda - PQ(\lambda)R,$$

where $P = \begin{pmatrix} I & 0 \\ F(\lambda) & I \end{pmatrix}$, $Q(\lambda) = \begin{pmatrix} \lambda - T & 0 \\ 0 & \lambda - S(\lambda) \end{pmatrix}$, $R = \begin{pmatrix} I & G(\lambda) \\ 0 & I \end{pmatrix}$, $F(\lambda) = C(T - \lambda)^{-1}$, $S(\lambda) = \overline{D - C(T - \lambda)^{-1}B}$ and $G(\lambda) = \overline{(T - \lambda)^{-1}B}$. D is A-right generalized weakly demicompact. Then, there exists a finite subset

$E_1 \in \mathbb{C}$ containing 0 such that $\lambda - D \in \Phi_+(\mathcal{X})$ for all $\lambda \in \mathbb{C}\backslash\{E_1\}$, which is equivalent to get $\widehat{\lambda - D} \in \Phi_+(\mathcal{X}_B, \mathcal{X})$ for all $\lambda \in \mathbb{C}\backslash\{E_1\}$. Let $N := C(T-\lambda)^{-1}B$, $Q := \lambda - D$ and $R := \lambda - S(\lambda)$. Since $\widehat{B} \in \mathcal{F}_+(\mathcal{X}_B, \mathcal{X})$ and $C(T-\lambda)^{-1}$ is bounded, it follows that $\widehat{N} \in \mathcal{F}_+(\mathcal{X}_B, \mathcal{X})$. Thus, we obtain for all $\lambda \in \mathbb{C}\backslash\{E_1\}$, $\widehat{Q} - \widehat{N} \in \Phi_+(\mathcal{X}_B, \mathcal{X})$. Hence, we have $\widehat{Q - N} \in \Phi_+(\mathcal{X}_{S(\lambda)}, \mathcal{X})$. As a consequence, $\widehat{R} \in \Phi_+(\mathcal{X}_{S(\lambda)}, \mathcal{X})$, for all $\lambda \in \mathbb{C}\backslash\{E_1\}$, which allows us to deduce that $\lambda - S(\lambda) \in \Phi_+(\mathcal{X})$ for all $\lambda \in \mathbb{C}\backslash\{E_1\}$. On the other side, since T is A-right generalized weakly demicompact, it follows that there exists a finite subset E_2 of $\mathbb{C}$ containing 0 such that $\lambda - T \in \Phi_+(\mathcal{X})$ for all $\lambda \in \mathbb{C}\backslash\{E_2\}$. Therefore, based on Lemma 3.4, we get $Q(\lambda) \in \Phi_+(\mathcal{X}\times\mathcal{X})$, for all $\lambda \in \mathbb{C}\backslash\{E\}$, where $E = E_1 \cup E_2$. Moreover, the boundedness of the operators P and R yields $\lambda - L$ which is an upper semi-Fredholm operator for all $\lambda \in \mathbb{C}\backslash\{E\}$. As T, $G(\lambda)$, $F(\lambda)$ and $S(\lambda) \in \mathcal{L}_{\mathcal{R}(A)}(\mathcal{X})$, it follows from Theorem 2.7 that L is A-right generalized weakly demicompact. $\square$

In the last part, we shall enact the relationship between the essential spectrum of the matrix operator L and the essential spectrum of its entries. First, let us recall the following decomposition

$$\lambda - \alpha L = UV_\alpha(\lambda)W - (\lambda - \alpha\mu)M(\mu),$$

where $U = \begin{pmatrix} I & 0 \\ F(\mu) & I \end{pmatrix}$, $V_\alpha(\lambda) = \begin{pmatrix} \lambda - \alpha T & 0 \\ 0 & \lambda - \alpha S(\mu) \end{pmatrix}$, $W = \begin{pmatrix} I & G(\mu) \\ 0 & I \end{pmatrix}$, $F(\lambda) = C(T-\lambda)^{-1}$, $S(\lambda) = \overline{D - C(T-\lambda)^{-1}B}$ and $G(\lambda) = \overline{(T-\lambda)^{-1}B}$.

Theorem 3.8. *Let $\mathcal{X}$ be a Banach space, $A \in \mathcal{L}(\mathcal{X})$ and let T, $G(\mu)$, $F(\mu)$ and $S(\mu) \in \mathcal{L}_{\mathcal{R}(A)}(\mathcal{X})$. Suppose that the operator L_0 defined in Eq. (6) satisfies $(H_1) - (H_5)$. Then,*
(i) If for every $\lambda \in [\Phi_{+T} \cap \Phi_{+S(\mu)}]\backslash\{0\}$, there exists $H_{\lambda l}$ a left Fredholm inverse of $UV(\lambda)W$ such that $M(\mu)H_{\lambda l} \in \Lambda^r_{\mathcal{X},A}$, then

$$\sigma_{e_1}(L)\backslash\{0\} \subset [\sigma_{e_1}(T) \cup \sigma_{e_1}(S(\mu))]\backslash\{0\}, \text{ where } L \text{ is the closure of } L_0.$$

(ii) If for all $\alpha \in [0,1]$ and $\lambda \in [\Phi_{\alpha T} \cap \Phi_{\alpha S(\mu)}]\backslash\{0\}$, there exists $H^\alpha_{\lambda l}$ a left Fredholm inverse of $UV_\alpha(\lambda)W$ such that $M(\mu)H^\alpha_{\lambda l} \in \Lambda^r_{\mathcal{X},A}$, then

$$\sigma_{e_i}(L)\backslash\{0\} \subset [\sigma_{e_i}(\alpha T) \cup \sigma_{e_i}(\alpha S(\mu))]\backslash\{0\}, \text{ where } i \in \{4,5\}.$$

Proof. (*i*) Let $\lambda \notin [\Phi_{+T} \cap \Phi_{+S(\mu)}]\backslash\{0\}$. Then, $\lambda - T \in \Phi_+(\mathcal{X})$ and $\lambda - S(\mu) \in \Phi_+(\mathcal{X})$. As T and $S(\mu) \in \mathcal{L}_{\mathcal{R}(A)}(\mathcal{X})$, it follows from Theorem 2.7, that T and $S(\mu)$ are A-right generalized weakly demicompact. Using the fact that there exists $H_{\lambda l}$, a left Fredholm inverse of $UV(\lambda)W$ such that $M(\mu)H_{\lambda l} \in \Lambda^r_{\mathcal{X},A}$, and applying Corollary 3.6, we deduce that the matrix operator L is A-right generalized weakly demicompact. Thus, according to Theorem 2.7, $\lambda - L$ is an upper semi-Fredholm operator. Hence, $\lambda \notin \sigma_{e_1}(L)\backslash\{0\}$. Thus,

$$\sigma_{e_1}(L)\backslash\{0\} \subset [\sigma_{e_1}(T) \cup \sigma_{e_1}(S(\mu))]\backslash\{0\}.$$

(*ii*) Let $\lambda \notin \sigma_{e_4}(\alpha T) \cup (\alpha S(\mu)) \cup \{0\}$. Then, $\lambda - \alpha T \in \Phi(\mathcal{X})$ and $\lambda - \alpha S(\mu) \in \Phi(\mathcal{X})$, which implies that $\lambda - \alpha T \in \Phi_+(\mathcal{X})$ and $\lambda - \alpha S(\mu) \in \Phi_+(\mathcal{X})$. Therefore, grounded on Theorem 2.7, we have αT and $\alpha S(\mu)$ which are A-right generalized weakly demicompact operators, for all $\alpha \in [0,1]$. Now, using the fact that there exists $H^\alpha_{\lambda l}$ a left Fredholm inverse of $UV_\alpha(\lambda)W$ such that $M(\mu)^\alpha_{\lambda l} \in \Lambda^r_{\mathcal{X},A}$ and applying Theorem 3.5, we obtain the matrix operator αL which is A-right generalized weakly demicompact for all $\alpha \in [0,1]$. Hence, by applying Theorem 2.8, $\lambda - L \in \Phi(\mathcal{X}\times\mathcal{X})$. Hence, $\lambda \notin \sigma_{e_4}(L)\backslash\{0\}$. Thus,

$$\sigma_{e_4}(L)\backslash\{0\} \subset [\sigma_{e_4}(\alpha T) \cup \sigma_{e_4}(\alpha S(\mu))]\backslash\{0\}.$$

For the Schechter's spectrum, let $\lambda \notin \sigma_{e_5}(\alpha T) \cup \sigma_{e_5}(\alpha S(\mu)) \cup \{0\}$. Then, $\lambda - \alpha T$ and $\lambda - \alpha S(\mu)$ are Fredholm with index zero. Consequently, we get $\lambda - \alpha T \in \Phi_+(\mathcal{X})$ and $\lambda - \alpha S(\mu) \in \Phi_+(\mathcal{X})$. As T and $S(\mu) \in \mathcal{L}_{\mathcal{R}(A)}(\mathcal{X})$, it follows from Theorem 2.7 that αT and $\alpha S(\mu)$ are A-right generalized weakly demicompact for all $\alpha \in [0,1]$. Now, using the fact that there exists $H^\alpha_{\lambda l}$, a left Fredholm inverse of $UV_\alpha(\lambda)W$ such that $M(\mu)H^\alpha_{\lambda l} \in \Lambda^r_{\mathcal{X},A}$ and applying Theorem 3.5, we deduce that the matrix operator αL is A-right generalized weakly demicompact for all $\alpha \in [0,1]$. Hence, departing from Theorem 2.8, $\lambda - L \in \Phi(\mathcal{X}\times\mathcal{X})$ with index zero. Thus, $\lambda \notin \sigma_{e_5}(L)\backslash\{0\}$. $\square$

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