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Hypergraph-theoretic perspectives on prime ideal sum graphs

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Abstract. Graph models are effective for capturing pairwise relationships in relational data, but many real-world relationships involve multiple entities and may be more accurately represented using hypergraphs. Consequently, the construction and analysis of hypergraph models for algebraic structures has recently attracted growing interest among algebraists. Motivated by this fact, in this work we study the hypergraph version of prime ideal sum graphs. As part of our investigation, we delved into the hypergraph-theoretic properties of completeness, connectedness, diameter, and girth concepts. Furthermore, we analyzed the homomorphic and isomorphic characteristics of these hypergraphs. At the end, we give an algorithmic design to obtain the decomposition sets of the hyperedges of a prime ideal sum hypergraph over $\mathbb{Z}_n$ which can be developed to compute the topological parameters of the graph directly.

1. Introduction

Recognizing the critical role of relational data in decision-making, researchers have developed graph algorithms that rely on simplified pairwise relationships to address various problems. However, studies have shown that hypergraphs-an extension of graphs where edges (hyperedges) can connect multiple vertices simultaneously-offer a more effective way to model complex, non-pairwise relationships, ultimately enabling more informed decisions.

In the field of chemistry, hypergraphs are utilized to represent molecules or chemical reactions that involve the simultaneous bonding of several atoms. Hypergraphs differ from graphs in that they can model interactions involving more than two atoms, which is particularly significant for reactions characterized by complex bonding or for representing molecular features that stem from the aggregation of multiple atoms [18, 20, 25, 36].

Hypergraph theory was introduced by the French mathematician Berge as a generalization of graph theory. A *graph* $G = (V(G), E(G))$ consists of a pair of sets, where $V(G)$ represents the set of a finite number of vertices, and $E(G)$ denotes the set of edges linking the associated vertices [38]. And a *hypergraph* $H = (V(H), E(H))$ is a pair of sets where $V(H)$ indicates the set of finitely many vertices and $E(H)$ indicates the finitely many sets of edges E_i $(1 \leq i \leq n)$, provided $\cup_{i=1}^{n} E_i = V(H)$, whose elements consist of non-empty subsets of $V(H)$. An edge can connect two related nodes in a graph, while a hyperedge can connect more than two vertices in a

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hypergraph. Therefore, hypergraphs are more powerful than graphs when it comes to representing complex systems [12]. A vertex in a hypergraph is considered *isolated* if there are no hyperedges that include it. A hypergraph that contains no hyperedge that covers any of its vertices is termed *null*. In a hypergraph, two distinct vertices that are part of the same hyperedge are referred to as *adjacent*. If no hyperedge is contained within any others, the hypergraph is described as *simple* [9]. A hypergraph is termed *linear* if any two distinct edges in H share at most one vertex.

The concept of zero divisor graphs in commutative rings marks the introduction of graph theory into the realm of pure algebra [8, 10]. Subsequently, the exploration of various graph-theoretic properties of algebraic structures has become a compelling area of interest for numerous researchers. It is feasible to find a selection of graphical structures pertaining to rings in the existing literature[1–3, 6–8, 11, 13, 23, 26, 28, 30–33]. In addition to these, as ideal theory is the heart of ring theory, several ideal-based graphs have been defined on commutative rings. For example, the inclusion ideal graph, the intersection graph of ideals, the ideal-relation graph, the co-maximal ideal graph, and the prime ideal sum graph are among them [4, 16, 27, 34, 39].

In this article, inspired by [5, 34], we investigate hypergraph-theoretic properties of prime ideal sum hypergraphs over commutative rings with unity, such as completeness, connectedness, diameter, and girth.

In Section 2, we present the necessary background and preliminary definitions from hypergraph theory and ideal theory that will be used throughout the paper, including notions of hyperpaths, cycles, diameter, girth, and connectedness.

In Section 3, we introduce the prime ideal sum hypergraph $PISH(R)$ and conduct a detailed investigation of its structural properties. Specifically, we characterize completeness, connectedness, diameter, and girth of $PISH(R)$, supported by illustrative examples and comparisons with the classical prime ideal sum graph.

In Section 4, we study homomorphisms and isomorphisms between prime ideal sum hypergraphs. We show how ring epimorphisms induce hypergraph homomorphisms and analyze the extent to which hypergraph structures are preserved under quotient constructions.

In Section 5, we focus on prime ideal sum hypergraphs over general *ZPI*-rings. We classify the possible hypergraph structures arising in this setting and derive results concerning diameter and girth in terms of associated prime ideals.

Finally, in Section 6, we propose a SageMath-based algorithmic framework for constructing the vertex and hyperedge sets of prime ideal sum hypergraphs over $\mathbb{Z}_n$. This computational approach provides an effective tool for visualizing and computing structural parameters of these hypergraphs.

While graph-theoretic approaches to algebraic structures have been extensively studied and continue to attract significant attention, research based on hypergraph theory remains comparatively limited [15, 22, 35, 37]. In this context, the present study aims to bridge this gap by extending the prime ideal sum framework from graphs to hypergraphs. In addition to the theoretical analysis of structural properties, the incorporation of a SageMath-based computational framework enables explicit construction and exploration of prime ideal sum hypergraphs. This combined theoretical and algorithmic perspective not only enhances the understanding of algebraic hypergraph models but also provides a practical foundation for future computational and experimental studies in the area.

2. Preliminaries

In this section we give a list of definitions which we need in the next section.

Definition 2.1. *The prime ideal sum graph (abbreviated by PIS) of a ring is described as a graph whose vertices are all non-trivial ideals of the ring, and edges connect two vertices whenever the sum of these vertices is a prime ideal [34].*

Definition 2.2. *Let $H = (V(H), E(H))$. A sequence of vertices $a = v_0, v_1, v_2, ..., v_k = b \in V(H)$ which are adjacent succesively and hyperedges $E_1, E_2, ..., E_k \in E(H)$ (both possible repeated) such that $v_0E_1v_1E_2v_2...v_{k-1}E_{k-1}v_k$ is a (a, b)-walk in H [9].*

Definition 2.3. *Let $H = (V(H), E(H))$. A sequence of distinct vertices $v_1, v_2, ..., v_k \in V(H)$ which are adjacent succesively and hyperedges $E_1, E_2, ..., E_k \in E(H)$ such that $v_1E_1v_2E_2v_3...v_{k-1}E_{k-1}v_k$ is a hyperpath in H [19].*

Definition 2.4. *Let $H = (V(H), E(H))$. A sequence $v_1E_1v_2E_2v_3...v_{k-1}E_{k-1}v_kE_kv_1$ is called a cycle in a hypergraph H where $v_1, v_2, v_3, ..., v_{k-1}, v_k \in V(H)$ are distinct vertices, $E_1, E_2, ..., E_k \in E(H)$ are distinct hyperedges, $v_i, v_{i+1} \in E_i$ for each $i \in \{1, .., k-1\}$ and $v_k, v_1 \in E_k$ [12].*

Definition 2.5. *The diameter, $diam(H)$, of a hypergraph $H = (V(H), E(H))$ is the maximum distance between any two vertices in H [29].*

Definition 2.6. *The girth of a hypergraph H, containing a cycle, is the minimum length of a cycle in H [9].*

Definition 2.7. *If there exists a (u, v)-walk between any two vertices $u, v \in V(H)$, then the hypergraph $H = (V(H), E(H))$ is called connected [9].*

A hypertree is a connected hypergraph with no hypercycle. A hypergraph with a central vertex contained in every hyperedge is called a hyperstar.

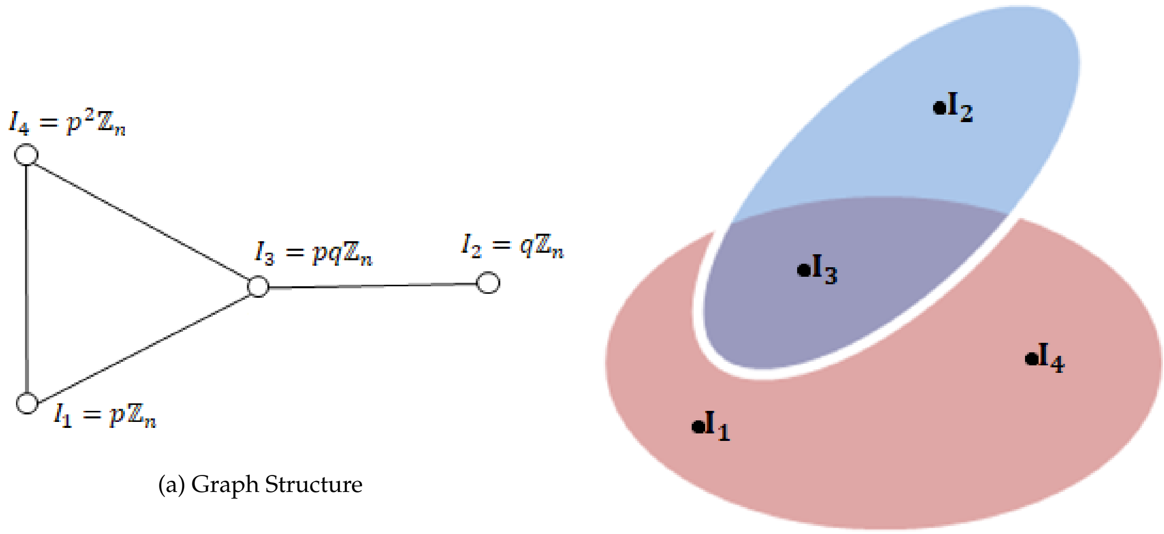
3. Main results

3.1. Completeness, Connectedness, Diameter and Girth of $PISH(R)$

Definition 3.1. *For a ring R the prime ideal sum hypergraph of R, abbreviated by $PISH(R) = (V_H(R), E_H(R))$, is a graph whose vertices are all non-trivial ideals of R and a subset E_i with at least two elements is a hyperedge of $PISH(R)$ provided $I + J$ is a prime ideal of R for each non-trivial ideal I, J in E_i and E_i is maximal with respect to this property [5].*

It is possible to re-express the hypergraph structure we defined, on a 'graph' basis as follows: Let R be a ring and $V = \{I \subsetneqq R \mid I \neq \{0_R\}\}$. Let us consider the graph $G(R)$ where any maximal subset E_i of V whose elements constructs a complete subgraph of $G(R)$ is a hyperedge in $PISH(R)$. Thus, the number of maximal subsets of E_i formed according to this relation is equal to the number of hyperedges of the hypergraph. Lastly, recall that the degree of a vertex (abbreviated by 'deg') in a hypergraph is the number of hyperedges incident with it.

Example 3.2. *Let us consider the ring $R = \mathbb{Z}_n$ where $n = p^2q$ for two distinct primes p and q. Clearly $I_1 = p\mathbb{Z}_n, I_2 = q\mathbb{Z}_n, I_3 = pq\mathbb{Z}_n, I_4 = p^2\mathbb{Z}_n$ are all of the non-trivial prime ideals of R. Therefore taking $V_H(R) = \{I_1, I_2, I_3, I_4\}$ and $E_H(R) = \{E_1, E_2\}$, where $E_1 = \{I_1, I_3, I_4\}$, $E_2 = \{I_2, I_3\}$, we get the following Figure 1 comparison for prime ideal sum graph and hypergraph representations of R.*



(a) Graph Structure

(b) Hypergraph Structure

Figure 1: Graph and hypergraph modeling of $PIS(\mathbb{Z}_{p^2q})$

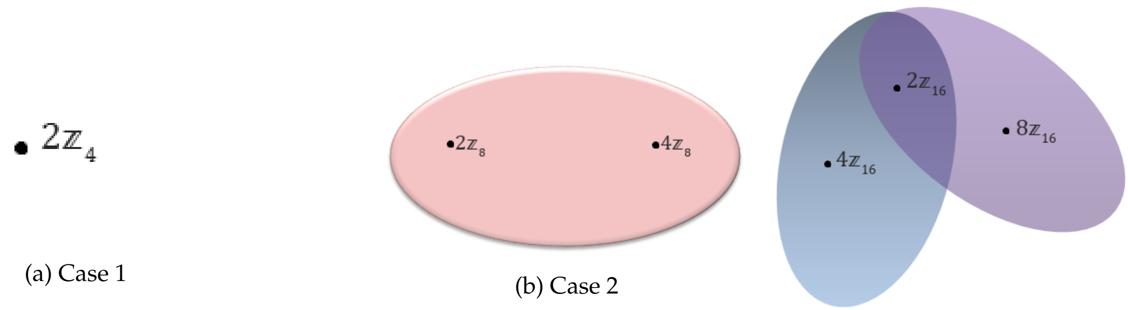
Inspired from [34, Theorem 2.1] , it can be shown that easily a universal vertex of prime ideal sum graph and hypergraph coincides.

Example 3.3. *Let us consider the non-reduced local ring* $R_1 = \mathbb{Z}_{p^k}$ $(k \geq 3)$ *and also consider the ring* $R_2 = \mathbb{Z}_{p^2q}$. *In* R_1, $Rad(R) = \langle p \rangle = p\mathbb{Z}_{p^k}$ *is the unique maximal ideal of R. Then the list of the vertex set can be determined as* $V(R_1) = V_H(R_1) = \left\{v_1 = pR_1, v_2 = p^2R_1, ..., v_{k-1} = p^{k-1}R_1\right\}$ *and the vertex* v_1 *is adjacent with all of the elements of* $V(R_1) - \{v_1\}$. *Also* $E_H(R_1) = \{E_1 = v_1, v_2, E_2 = v_1, v_3, ..., E_{k-1} = v_1, v_{k-1}\}$. *On the other hand in* R_2, pR_2 *and* qR_2 *are the only two maximal ideals provided that* $pR_2 \bigcap qR_2 = pqR_2$ *is a non-trivial minimal ideal of R and there exists no non-prime ideal properly containing* $pR_2 \bigcap qR_2 = pqR_2 = Rad(R)$. *And* $Rad(R)$ *is the universal vertex for* $PIS(R_2)$ *and* $PISH(R_2)$.

Now we give place some examples helping a construction theorem for $PISH(\mathbb{Z}_{p^\alpha})$ where $p \geqslant 2$ is a prime integer and the integer $\alpha > 1$.

Example 3.4. *Let* $R = \mathbb{Z}_n$ *where* $n = 4$, $n = 8$ *and* $n = 16$, *respectively. Case 1: For* $n = 4$ *since* $PISH(R)$ *does not have any hyperedge, then it is a null hypergraph.*

Case 2: For $n = 8$ *since* $2\mathbb{Z}_8$ *and* $4\mathbb{Z}_8$ *are both the only vertices of* $V_H(R)$ *and also these are adjacent of each other, then* $E_H(R)$ *consists of exactly one hyperedge in* $PISH(R)$. *Case 3: For* $n = 16$ *since* $\mathbb{Z}_{16}$ *is a local ring whose ideals are linearly ordered, then the only adjacent vertex of* $4\mathbb{Z}_{16}$ *and* $8\mathbb{Z}_{16}$ *is* $2\mathbb{Z}_{16}$. *Therefore taking into account a hypertree structure is obtained for* $PISH(R) = (V_H(R) = \{2\mathbb{Z}_{16}, 4\mathbb{Z}_{16}, 8\mathbb{Z}_{16}\}, E_H(R) = \{E_1 = \{2\mathbb{Z}_{16}, 4\mathbb{Z}_{16}\}, E_2 = \{2\mathbb{Z}_{16}, 8\mathbb{Z}_{16}\})$.



(a) Case 1 (b) Case 2 (c) Case 3

Figure 2: *PISH* modeling of each case

In the light of examples given in Figure 2, we develop a theoretical result for $PISH(\mathbb{Z}_{p^\alpha})$ *where p is a prime integer* $(p \geq 2, \alpha > 1)$.

Theorem 3.5. *Let* $R = \mathbb{Z}_{p^\alpha}$ *where* $p \geq 2$ *and* $\alpha > 1$. *Then* $PISH(R)$ *consists of* $(\alpha - 2)$ *hyperedges. In particular it is a hyperstar graph.*

Proof. As $\mathbb{Z}_{p^\alpha}$ is local, then $p\mathbb{Z}_{p^\alpha}$ is the only prime ideal of $\mathbb{Z}_{p^\alpha}$. When we remove $p\mathbb{Z}_{p^\alpha}$ from the family of non-tivial ideals of $p\mathbb{Z}_{p^\alpha}$, we get a vertex set $V = \{p^2\mathbb{Z}_{p^\alpha}, p^3\mathbb{Z}_{p^\alpha}, ..., p^{\alpha-1}\mathbb{Z}_{p^\alpha}\}$ and then $|V| = \alpha - 2$. Here each of vertices are adjacent with only $p\mathbb{Z}_{p^\alpha}$. Hence, $PISH(R)$ has got $(\alpha - 2)$ hyperedges. □

Now, after recalling a hypergraph-theoretic concept of completeness, we will give a theory for the completeness of $PISH(R)$.

Definition 3.6. *A hypergraph* $H = (V(H), E(H))$ *is called complete, if a hyperedge E exists in* $E(H)$ *providing* $\{v_1, v_2\} \subseteq E$ *for any* $v_1, v_2 \in V(H)$ *[17, 21].*

Theorem 3.7. *For a ring R,* $PISH(R)$ *is a complete hypergraph iff R is local and any non-prime ideal of R is minimal.*

Proof. Let $PISH(R)$ is a complete hypergraph. Thus, there exists a hyperedge consisting of each pair of non-trivial ideals. If P_1, P_2 are distinct maximal ideals of R, then $\{P_1, P_2\} \subseteq E_0$ where E_0 is a hyperedge in $PISH(R)$. For the inclusion $P_1, P_2 \subseteq P_1 + P_2 \subseteq R$, it must be true that $P_1 + P_2 \neq P_1, P_2$ from the maximalities of P_1 and P_2. Hence, R contains exactly one maximal ideal $Rad(R)$ and so R is local. Now let us take a non-prime proper ideal $I \subseteq R$ and assume that $\{0\} \subsetneq J \subsetneq I$ for an ideal $J \subseteq R$. Thus, $\{I, J\}$ can not be included by any of the hyperedges of $PISH(R)$. This contradicts with the completeness of the hypergraph. Hence I is prime.

For the sufficiency part, let $\{I_1, I_2\}$ be a pair of nonzero proper ideals which is not included by any of the hyperedges in $PISH(R)$. Then, $I_1 + I_2$ is not prime in R. Therefore, it must be minimal in R by hypothesis. But this causes the contradiction that $I_1 = I_2$. Hence, each pair of vertices in $V_H(R)$ must be included by a hyperedge in $E_H(R)$, that is, $PISH(R)$ is a complete. □

The following theorem is useful to deduce on the connectedness of $PISH(R)$ of a ring R.

Theorem 3.8. *Let R be a ring and I is a non-trivial ideal of R. Then $\deg_{PISH(R)} I = 0$ if and only if I is both a maximal and a minimal ideal of R.*

Proof. Let $\deg_{PISH(R)} I = 0$. Then I is not covered by a hyperedge of $PISH(R)$. And, I is not connected with any of the vertices of $V(H)$. However, I is a maximal ideal of R. Otherwise, there would be a maximal ideal P in R such that $I \subseteq P$. In this case, I and P must be covered by a hyperedge as $I + P = P$ is prime which is a contradiction. On the other hand, I is a minimal ideal of R as it is maximal. Because, if J were a proper ideal of R satisfying that $0 \subsetneq J \subsetneq P$, then J and P would be covered by a hyperedge in $PISH(R)$ as $J + P = P$ is prime. Conversely, let I is both a maximal and a minimal ideal of R. Then it is an isolated vertex in $PIS(R)$ by [34, Theorem 2.4]. Thus, there exists no ideal of R whose sum with I is prime. Hence, I is not contained by a hyperedge in $PISH(R)$ and so $\deg_{PISH(R)} I = 0$. □

Corollary 3.9. *For a ring R, $PISH(R)$ is connected if and only if R has no decomposition of two fields.*

Corollary 3.10. *The $PISH(R)$ of a ring R is null if and only if R is a direct sum of two fields.*

Proof. For necessity, by hypothesis, there is no hyperedge in $PISH(R)$. Hence, it is a null hypergraph. For the converse implication, R has only two non-trivial prime ideal by assumption. Since there is no hyperpath between these ideals, then there is no hyperedge linking them. Hence $PISH(R)$ is a null hypergraph. □

Example 3.11. *Let $R = \mathbb{Z}_6$. It is clear that $\mathbb{Z}_6 = 2\mathbb{Z}_6 \oplus 3\mathbb{Z}_6 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_3$. The PIS hypergraph structure of it is null, since there is no hyperedge covering the only non-trivials $2\mathbb{Z}_6, 3\mathbb{Z}_6$ of $\mathbb{Z}_6$ together.*

Theorem 3.12. *Let R be a ring. If $PISH(R)$ is a connected hypergraph, then $diam(PISH(R)) \leq 4$.*

Proof. Let $PISH(R)$ be a connected hypergraph. Then for the distint prime ideals I, J of R let us examine the cases given below.

Case 1: Let $I + J$ be prime. In this case, they can be included by the same hyperedge E_1 in $E_H(R)$. Then we get the hyperwalk IE_1J and so $diam(PISH(R)) = 1$.

Case 2: Let $I + J$ be not prime. Since R is finitely generated, there exists some maximal ideals P_1, P_2 of R such that $I \subseteq P_1$ and $J \subseteq P_2$.

If $P_1 = P_2$, then $I + P_1$, $J + P_1$ is prime and so there exists some hyperedges E_1, E_2 such that $\{I, P_1\} \subseteq E_1$ and $\{J, P_1\} \subseteq E_2$. Hence we get $IE_1P_1E_2J$ and so $diam(PISH(R)) = 2$.

If $P_1 \neq P_2$, taking into account the ideal $P_1 \cap P_2$ of R, the existence of some hyperedges E_1, E_2, E_3, E_4 can be seen such that $\{I, P_1\} \subseteq E_1$, $\{P_1, P_1 \cap P_2\} \subseteq E_2$, $\{P_1 \cap P_2, P_2\} \subseteq E_3$, $\{P_2, J\} \subseteq E_4$. So we get the hyperpath $IE_1P_1E_2(P_1 \cap P_2)E_3P_2E_4J$. Then, $diam(PISH(R)) = 4$ as $P_1 \cap P_2 \neq 0$; because of the connectivity of the hypergraph. Otherwise, $R/P_1, R/P_2$ would be isomorphic to a direct summand of the ring R which contradicts with the connectivity of the hypergraph. □

Corollary 3.13. *The diameter of $PISH(R)$ of a principal ideal ring R is less than or equal to 2.*

Proof. If I and J are the distinct primes contained by the same maximal ideal of R, then, $diam(PISH(R)) = 1$, clearly. Let P_1, P_2 be the distinct maximals containing I and J, respectively. By hypothesis, for some $x, y, z, t \in R$ we have $I = Rx \subseteq Rz = P_1$, $J = Ry \subseteq Rt = P_2$. Note that as $P_1 \cap P_2 = P_1P_2 \neq \{0\}$ because of the connectivity of the hypergraph, $K = Rzt$ is a non-trivial ideal of R. However, there exists a hyperedge E_1 containing $\{I, K, P_1\}$ and a hyperedge E_2 connecting $\{J, K, P_2\}$ by Theorem 2.6 [34]. Hence, we get IE_1KE_2J, that is $diam(PISH(R)) = 2$. □

Corollary 3.14. *$diam(PISH(R)) \leq 4$ whenever R is an integral domain.*

Remark 3.15. *For a ring R, although $PIS(R)$ has a cycle, $PISH(R)$ may not have a hypercycle.*

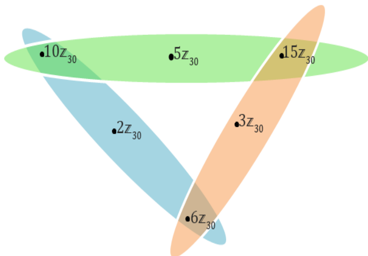
Example 3.16. *In Example 3.2, $girth(PIS(\mathbb{Z}_{12})) = 3$ as I_1, I_3 and I_4 construct a cycle. But $PISH(\mathbb{Z}_{12})$ has no cylce as it is a linear hypergraph as a hypertree.*

Theorem 3.17. *Let R be a ring containing three distinct maximal ideals. Then $girth(PISH(R)) = 3$.*

Proof. Let P_1, P_2, P_3 be three distinct maximal ideals ideals satisfying the mentioned hypothesis. Then there exists hyperedges E_1, E_2, E_3 satisfying $\{P_1 \cap P_2, P_1, P_1 \cap P_3\} \subseteq E_1$, $\{P_1 \cap P_2, P_2, P_2 \cap P_3\} \subseteq E_2$, $\{P_2 \cap P_3, P_3, P_1 \cap P_3\} \subseteq E_3$. Hence, $(P_1 \cap P_2)E_2(P_2 \cap P_3)E_3(P_1 \cap P_3)E_1(P_1 \cap P_2)$ is a cycle in $PISH(R)$. Hence, $girth(PISH(R)) = 3$. □

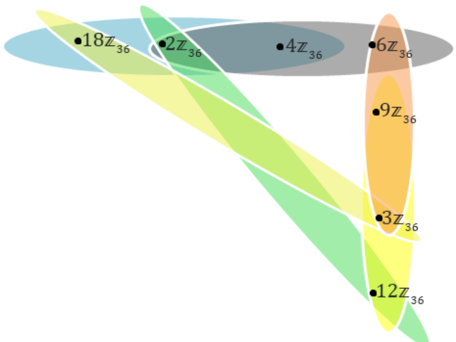
Example 3.18. *Let $R = \mathbb{Z}_{30}$. Since, $u_1 = 2\mathbb{Z}_{30}, u_2 = 3\mathbb{Z}_{30}, u_3 = 5\mathbb{Z}_{30}, u_4 = 6\mathbb{Z}_{30}, u_5 = 10\mathbb{Z}_{30}, u_6 = 15\mathbb{Z}_{30}$ are all the vertices of the PIS-hypergraph of $\mathbb{Z}_{30}$ where u_1, u_2, u_3 are the prime ones, then the hyperedges of $PISH(R)$ as follows. $E_1 = \{u_1, u_4, u_5\}, E_2 = \{u_2, u_4, u_6\}, E_3 = \{u_3, u_5, u_6\}$. It can be seen from Figure 3 $girth(PISH(R)) = 3$.*

Figure 3: *PISH* structure of $\mathbb{Z}_{30}$



Example 3.19. *Let $R = \mathbb{Z}_{p^2q^2}$ where p and q are the distinct prime integers. Here, $V(H) = \{u_1 = pR, u_2 = qR, u_3 = pqR, u_4 = p^2qR, u_5 = pq^2R, u_6 = p^2R, u_7 = q^2R\}$ where $u_1 = pR, u_2 = qR$ are the prime ideals of R. Then $E(H) = \{E_1 = \{u_1, u_3, u_6\}, E_2 = \{u_1, u_5, u_6\}, E_3 = \{u_2, u_3, u_7\}, E_4 = \{u_2, u_4, u_7\}, E_5 = \{u_1, u_4\}, E_6 = \{u_2, u_5\}\}$. By Figure 4, $girth(PISH(R)) = 4$.*

Figure 4: *PISH* structure of $\mathbb{Z}_{36}$



4. Homomorphism on $PISH(R)$

In this section we give some anecdotes on PIS-hypergraph modeling for a ring R. But firstly, some terminological concepts to be needed later will be given which are deeply examined from [14] by the interested readers.

Definition 4.1. *Let V_1, V_2 be two finite sets and $\phi : V_1 \longrightarrow V_2$ be a mapping. The mapping $\widehat{\phi} : 2^{V_1} \longrightarrow 2^{V_2}$ defined by $\widehat{\phi}(V^{/}) = \cup_{v\in V^{/}}\{\phi(v)\}$ for every $V^{/} \subseteq V_1$ is called an extension of ϕ.*

Definition 4.2. *Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two hypergraphs. a mapping $\phi : V(G) \longrightarrow V(H)$ is called a homomorphism from G to H if, for any edge $e \in E(G)$, we have that $\widehat{\phi}(e) \in E(H)$.*

Theorem 4.3. *Let $f : R \longrightarrow S$ be an epic homomorphism and let $PISH(R) = (V(R), E(R))$ and $PISH(S) = (V(S), E(S))$ be two hypergraphs. Then, $g : V(S) \longrightarrow V(R)$ is a hypergraph homomorphism provided $g(J) = f^{-1}(J)$ for any ideal J of S.*

Proof. Let E_1 be an arbitrary edge of $E(S)$. Then E_1 is of the form such that $E_1 = \{J_1, J_2, ..., J_n\}$ where the sum of each distinct pair of ideals is prime in S and E_1 is maximal. Let us consider the mapping $g : V(S) \longrightarrow V(R)$. Then the map $\widehat{g} : 2^{V(S)} \longrightarrow 2^{V(R)}$ can be defined as $\widehat{g}(E_s) = \cup_{J\in E_s}\{g(J)\}$ for every $E_s \subseteq V(S)$. In particular,

$$\widehat{g}(E_1) = \{g(J_1), g(J_2), ..., g(J_n)\} = \{f^{-1}(J_1), f^{-1}(J_2), ..., f^{-1}(J_n)\}.$$

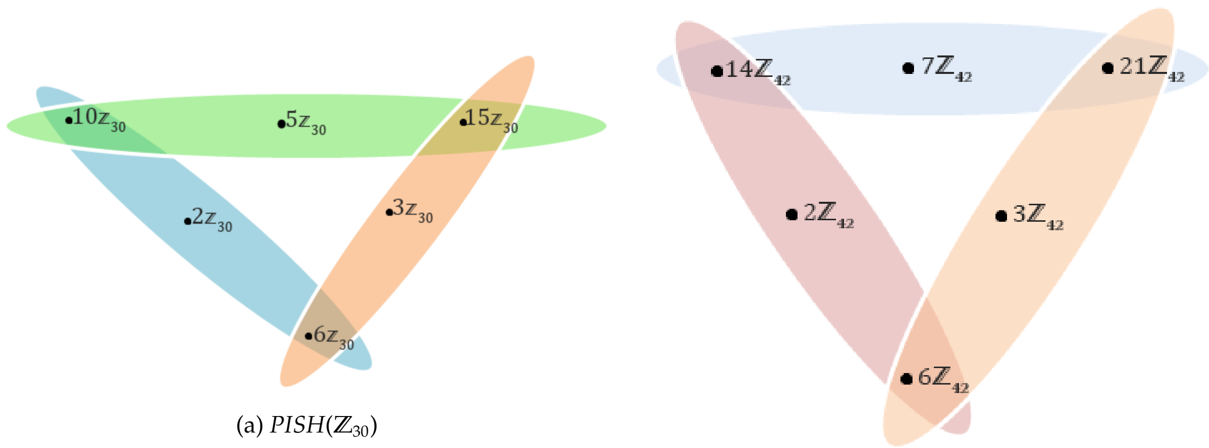
Let for the distint indices $a, b \in \{1, 2, ..., n\}$,

$$g(J_a) + g(J_b) = f^{-1}(J_a) + f^{-1}(J_b) = f^{-1}(J_a + J_b)$$

is prime as J_a and J_b are included by the same hyperedge. E_i. Then each distinct pair of ideals in $\widehat{g}(E_1)$ is adjacent. Now, the maximality of $\widehat{g}(E_1)$ must be verified. Let K be a nontrivial ideal of S satisfying $g(J_i) + K = f^{-1}(J_i) + K$ is prime for each element $g(J_i)$ in $\widehat{g}(E_1)$ for $i = 1, 2, ..., n$. Then, $J_i + f(K) = f(f^{-1}(J_i + K))$ is also prime as f is an epimorphism. Thus $f(K)$ must be included by E_1 which contradicts with the maximality of hyperedge set. Therefore, $\widehat{g}(E_1)$ is maximal in $E(R)$ and so it is a hyperedge in $PISH(R)$. Hence, g is a hypergraph homomorphism. □

Corollary 4.4. *The prime ideal sum hypergraphs of two isomorphic rings are also isomorphic.*

Remark 4.5. *Note that there are some rings which are non-isomorphic although their prime ideal sum hypergraphs are isomorphic. For example, $\mathbb{Z}_{30}$ and $\mathbb{Z}_{42}$ are such rings. The hypergraph structure of them can be seen in Figure 5.*



(a) $PISH(\mathbb{Z}_{30})$

(b) $PISH(\mathbb{Z}_{42})$

Figure 5: Hypergraph modeling of $PISH(\mathbb{Z}_{30})$ and $PISH(\mathbb{Z}_{42})$

Definition 4.6. *A complete subhypergraph with $t-$vertices is called a clique of order t. A clique is said to be maximum if it has maximum cardinality. The clique number of a hypergraph H is defined as the cardinality of the maximum clique of H.*

Corollary 4.7. *Let $f : R \longrightarrow S$ be an epimorphism of rings. The PIS hypergraph homomorphism between the rings preserves cliques of hypergraphs.*

Proof. Clear by Thorem 4.3 and applying the similar technique used in the proof of [34, Theorem 2.17]. □

Definition 4.8. *Given two hypergraphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$, an isomorphism between them is defined by any bijection $\phi : V(G) \longrightarrow V(H)$ for which $\{i_1, i_2, ..., i_k\} \in E(G) \Leftrightarrow \{\phi(i_1), \phi(i_2), ..., \phi(i_k)\} \in E(H), \forall\{i_1, i_2, ..., i_k\} \in E(G)$. Two hypergraphs are said to be isomorphic if there exists an isomorphism between them.*

Theorem 4.9. *Let R be a ring and I be an ideal of R. Then two hypergraphs $PISH(R) = (V(R), E(R))$ where $V(R)$ includes all nontrivial ideals of R containing I as nodes and $PISH(R/I) = (V(R/I), E(R/I))$ are isomorphic.*

Proof. Consider the map $\phi : V(R) \longrightarrow V(R/I)$, defined by $\phi(J) = J/I$ for every vertex $J \in V(R)$. For any $E_1 = \{J_1, J_2, ..., J_k\} \in E(R)$, since the sum of any two distinct vertices is prime. Then $\{\phi(J_1), \phi(J_2), ..., \phi(J_k)\} \in E(R/I)$ is provided. Similarly, vice versa. This completes the proof. □

5. *PISH* of a General *ZPI-Ring*

An ring R is called general ZPI-ring if every proper ideal of R is written as a finite product of its prime ideals. A prime ideal P in a Noetherian commutative ring is called an associated prime if P is an annihilator of a non-zero element in the ring.

Through this part of the study, we examine some hypergraph-theoretic properties of $PISH(R)$ where R is a general ZPI-ring unless we specify otherwise.

Theorem 5.1. *Let R be a ring which is not a field with only one associated prime. Then $PISH(R)$ is null or complete or a hyperstar graph.*

Proof. Let $P \subseteq R$ be the only associated prime in R. Then, as R is not a field there exists a positive integer $i > 1$ for the trivial submodule $\{0\}$ such that $P^i = 0$.

If $i = 2$, that is $P^2 = 0$, then any nontrivial ideal I of R satisfies the inclusion $P^2 \subsetneq I \subseteq P$ which forces I to be equal with P as R is a general ZPI-ring by [24, Theorem 39.2]. Therefore, $PISH(R)$ includes only one vertex P which does not contained by any hyperdege. Hence $PISH(R)$ is null.

If $i = 3$, then $PISH(R)$ has only a single hyperedge with two vertices P and P^2. Then it is complete.

Now, let $i > 3$. Then for any non trivial two ideals have the forms $I = P^{i_1}, J = P^{i_2}$ where $i_1 \neq i_2 \neq 1$. Here non-prime sum of these ideals is P^{i_1} or P^{i_2} because of the inclusion relation between them. And so, the set of the hyperedges has the form $E_H(R) = \{E_1 = \{P, P^{i_1}\}, E_2 = \{P, P^{i_2}\}, ..., E_n = \{P, P^{i_n}\}\}$.Hence, $PISH(R)$ is a hyperstar graph. □

Corollary 5.2. *Let R be a non-field principal ideal domain with only one associated prime. Then $diam(PISH(R)) = 0 \vee 1 \vee 2$.*

Proof. Let $Ass(R) = \{P\}$.

If $P^2 = 0$, then $diam(PISH(R)) = 0$ by nullness of $PISH(R)$ given in Theorem 5.1.

If $P^3 = 0$, then $diam(PISH(R)) = 1$ as $PISH(R)$ consists of only one hyperdege covering all vertices of $V_H(R)$.

If $P^i = 0$ $(i > 3)$, then $diam(PISH(R)) = 2$ as $PISH(R)$ is a hyperstar graph by Theorem 5.1. □

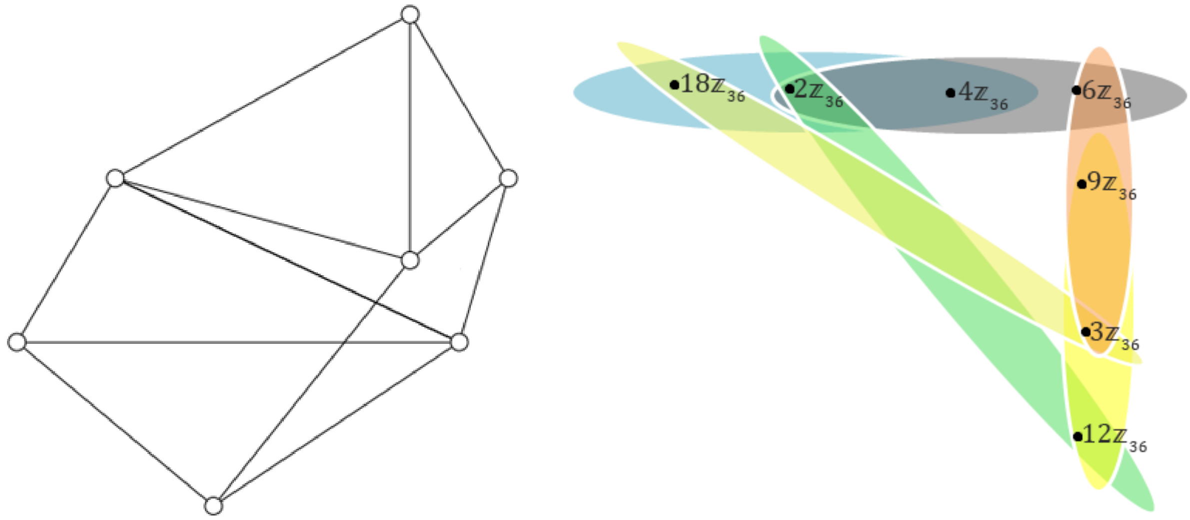
Corollary 5.3. *Let R be a non-field general ZPI-ring with only one associated prime. Then $diam(PISH(R)) = \infty$.*

Proof. Clear as $PISH(R)$ is acyclic by Theorem 5.1. $\square$

Theorem 5.4. *Let R be a non-field generalized ZPI-ring.Then $girth(PISH(R)) = 3 \vee 4 \vee \infty$.*

Proof. Whenever $Ass(R)$ consists of one associated prime P_1 or two associated primes P_1, P_2 with $P_1P_2 = 0$, $girth(PISH(R)) = \infty$ as it has no cycle. And also $girth(PISH(R)) = 3$ whenever R contains at least three maximal ideals by Theorem 3.17. Now, assume that P_1, P_2 be two associated primes of R providing $P_1P_2 \neq 0$. Then, P_1^2 or P_2^2 is a product of 0. Suppose P_2^2 satisfies this option. So $P_1E_1(P_1P_2)E_2P_2E_3(P_2^2P_1)E_4P_1$ is a rectangle in $PISH(R)$. Hence $girth(PISH(R)) = 4$. $\square$

Example 5.5. *Let us consider the ring $\mathbb{Z}_{36}$ as a general ZPI-ring. As it is proven in [34, Theorem 3.4] $girth(PIS(\mathbb{Z}_{36})) = 3$. However as $Ass(\mathbb{Z}_{36}) = \{P_1 = 2\mathbb{Z}_{36}, P_2 = 3\mathbb{Z}_{36}\}$ and $P_1P_2 \neq 0$, $2\mathbb{Z}_{36}E_16\mathbb{Z}_{36}E_33\mathbb{Z}_{36}E_618\mathbb{Z}_{36}E_22\mathbb{Z}_{36}$ forms a rectangle and so, $girth(PISH(\mathbb{Z}_{36})) = 4$ (See Figure 6).*



(a) Graph Structure

(b) Hypergraph Structure

Figure 6: Graph and hypergraph comparison of $PIS(\mathbb{Z}_{p^2q^2})$

6. Decomposition Code for Prime Ideal Sum Hypergraphs with a SageMath-Based Approach

In this section, we present a SageMath code that constructs the sets $V(H)$ and $E(H)$ of prime ideal sum hypergraphs for different values of n in the commutative rings $\mathbb{Z}_n$. Line 1 defines the function `find_proper_ideals(n)`, which is responsible for identifying all proper ideals in $\mathbb{Z}_n$. Line 2 initializes an empty list to store the proper ideals. Lines 3 to 5 iterate over all integers from 2 to $\lfloor n/2 \rfloor$, checking whether each integer divides n. If so, the divisor is added to the list. Line 6 returns the final list of proper ideals. Line 8 begins the definition of the helper function `gcd_all(subset)`, which computes the greatest common divisor (GCD) of all elements in a given subset. Line 9 converts the input subset to a list, and Line 10 initializes the GCD computation with the first element. Lines 11 and 12 iteratively apply the built-in `gcd` function to compute the total GCD. Line 13 returns the result. Lines 26 to 29 filter these candidates to retain only the maximal ones, that is, those that are not strictly contained in any other candidate. Line 31 returns the final list of maximal hyperedges. Line 33 defines the main function `prime_ideal_sum_hypergraph(n)`, which manages the overall process of constructing and printing the hypergraph. Line 34 prints a message indicating the start of the computation. Line 36 computes the proper ideals. If none are found (that is, if n is prime), Lines 37 and 38 print an appropriate message and terminate the function. Line 41 prints the title of the vertex set, while Line 42 outputs the list of proper ideals as the vertex set $V_H(R)$. Line 44 calculates the hyperedges using the earlier function. If no hyperedges are found, Lines 45 and 46 print a corresponding message. Otherwise, Lines 47 to 50 format and display the edge set $E_H(R)$. Line 52 reads the input n from the user. Finally, Line 53 calls the main function to initiate the hypergraph construction.

6.1. SageMath Code

```
1  def find_proper_ideals(n):
2      ideals = []
3      for d in range(2, n // 2 + 1):
4          if n % d == 0:
5              ideals.append(d)
6      return ideals
7
8  def gcd_all(subset):
9      subset = list(subset)
10     g = subset[0]
11     for x in subset[1:]:
12         g = gcd(g, x)
13     return g
14
15 def find_hyperedges(proper_ideals):
16     candidate_hyperedges = []
17     for r in range(2, len(proper_ideals) + 1):
18         for subset in Combinations(proper_ideals, r):
19             subset = set(subset)
20             gcd_value = gcd_all(subset)
21
22             all_gcds = {gcd(a, b) for a in subset for b in subset if a != b}
23             if len(all_gcds) == 1 and is_prime(list(all_gcds)[0]):
24                 candidate_hyperedges.append(subset)
25
26     maximal_hyperedges = []
27     for edge in candidate_hyperedges:
28         if not any(edge < other for other in candidate_hyperedges):
29             maximal_hyperedges.append(edge)
30
31     return maximal_hyperedges
32
33 def prime_ideal_sum_hypergraph(n):
34     print(f Computing PISH(Z_{n})...\n )
35
36     proper_ideals = find_proper_ideals(n)
37     if not proper_ideals:
38         print ( No proper ideals found. The hypergraph cannot be formed . )
39         return
40
41     print ( Vertices (Proper Ideals): )
42     print(f V_H(R) = {{ {', '.join(f'<{d}>' for d in proper_ideals)} }} )
43
44     hyperedges = find_hyperedges(proper_ideals)
```

```
    if not hyperedges:
        print(\nNo hyperedges found.)
    else:
        print(\nHyperedges (Maximal Prime Ideal Sums):)
        edge_strings = [f{{{', '.join(f'<{x}>' for x in e)}}} for e in
            hyperedges]
        print(f E_H(R) = {{ {', '.join(edge_strings)} }})
n = input ()
prime_ideal_sum_hypergraph(n)
```

6.2. The Output for PISH for $\mathbb{Z}_{30}$ Based on the SageMath Code

Computing $PISH(\mathbb{Z}_{30})$...

Vertices (Proper Ideals):
$V_H(R) = \{\langle 2\rangle, \langle 3\rangle, \langle 5\rangle, \langle 6\rangle, \langle 10\rangle, \langle 15\rangle\}$

Hyperedges (Maximal Prime Ideal Sums):
$E_H(R) = \{\{\langle 2\rangle, \langle 10\rangle, \langle 6\rangle\}, \{\langle 3\rangle, \langle 6\rangle, \langle 15\rangle\}, \{\langle 10\rangle, \langle 5\rangle, \langle 15\rangle\}\}$

7. Main Contribution

The main contributions of this paper can be summarized as follows:

- We introduce and formalize the concept of the prime ideal sum hypergraph associated with a commutative ring and establish its basic hypergraph-theoretic framework.
- We provide a comprehensive analysis of completeness, connectedness, diameter, and girth of $PISH(R)$, highlighting key differences from the corresponding graph-theoretic model.
- We investigate homomorphisms and isomorphisms between prime ideal sum hypergraphs induced by ring epimorphisms and quotient rings.
- We classify the structure of $PISH(R)$ for general ZPI-rings, identifying conditions under which the hypergraph is null, complete, or a hyperstar.
- We develop and implement a SageMath-based algorithmic framework to explicitly compute the vertex and hyperedge sets of $PISH(\mathbb{Z}_n)$, enabling the direct computation and visualization of structural and topological parameters such as connectedness, diameter, and girth.

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