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Characterizations of non-additive mixed skew and η-Jordan triple higher derivations on prime $*$-algebras

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Abstract. This article studied the structure of non-additive mixed skew and η-Jordan triple higher derivations $\Psi = \{\psi_n\}_{n\in\mathbb{N}}$ of $*$-algebras $\mathcal{A}$ and proved that under some conditions each non-additive mixed skew and η-Jordan triple higher derivation $\Psi = \{\psi_n\}_{n\in\mathbb{N}}$ is an additive $*$-higher derivation and $\psi_n(\eta x) = \eta\psi_n(x)$ for $-1 \neq \eta \in \mathbb{R}$. As applications, non-additive mixed skew and η-Jordan triple higher derivations on some classical operator algebras are characterized, such that factor von Neumann algebras, standard operator algebras and so on.

1. Brief historical development

This article seeks to construct a self-contained knowledge architecture, so we begin by introducing some concepts related to the topic. Let $\mathcal{A}$ be an $*$-algebra defined over a complex field $\mathbb{C}$ with an algebraic center $Z(\mathcal{A})$, where $*$ satisfies the relation $(xy)^* = y^*x^*$ and $((x)^*)^* = x$ for all $x, y \in \mathcal{A}$. We introduce some multiplication notation, the symbols $A \circ B = AB + BA$, $A \bullet B = AB + BA^*$ and $A \circ_\eta B = AB + \eta BA$ are called Jordan product, skew Jordan product and η-Jordan product, respectively, where η is an element in the set $\mathbb{R}$ of rational numbers. Jordan and skew Jordan products are gaining importance across a number of research fields, and many authors have been interested in investigating them (see [10, 17, 18, 21]). An additive mapping $\varphi : \mathcal{A} \to \mathcal{A}$ is called additive $*$-derivation if $\varphi(AB) = \varphi(A)B + A\varphi(B)$ and $\varphi(A^*) = \varphi(A)^*$ for all $A, B \in \mathcal{A}$. In addition, an mapping $\varphi : \mathcal{A} \to \mathcal{A}$ (without the additivity assumption) is called non-additive skew derivation if $\varphi(A \bullet B) = \varphi(A) \bullet B + A \bullet \varphi(B)$ for all $A, B \in \mathcal{A}$. Khan and coauthors studied the structure of *non-additive mixed skew and η-Jordan triple derivations* (without the additivity assumption) $\varphi : \mathcal{A} \to \mathcal{A}$, which satisfies the equation

$$\varphi(A \circ_\eta B \bullet C) = \varphi(A) \circ_\eta B \bullet C + A \circ_\eta \varphi(B) \bullet C + A \circ_\eta B \bullet \varphi(C) \tag{1.1}$$

for all $A, B, C \in \mathcal{A}$, and proved that under certain conditions every non-additive mixed skew and η-Jordan triple derivation is additive $*$-derivation and satisfies the relation $\varphi(\eta A) = \eta\varphi(A)$. The idea behind their

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study came from the study of a structure called a η-Jordan triple derivation $\varphi : \mathcal{A} \to \mathcal{A}$, which satisfies the following equation

$$\varphi(A \bullet_\eta B \bullet_\eta C) = \varphi(A) \bullet_\eta B \bullet_\eta C + A \bullet_\eta \varphi(B) \bullet_\eta C + A \bullet_\eta B \bullet_\eta \varphi(C)$$

for all $A, B, C \in \mathcal{A}$. In recent years, the research on the structure of various non-additive mixed derivations has attracted much attention from scholars, and it is recommended to read references[5–7, 12, 18, 20, 23, 24]. In 1991, Daif[8] initially proved that each non-additive derivation is additive on a 2-torsion free prime ring containing a nontrivial idempotent. In 2014, Li et al.[13] showed that if $B(H)$ be the algebra of all bounded linear operators on a complex Hilbert space H and $\mathfrak{A} \subseteq B(H)$ be a von Neumann algebra without central abelian projections, then $\psi : \mathfrak{A} \to B(H)$ is a nonlinear η-$*$-Jordan derivation if and only if ψ is an additive $*$-derivation and $\psi(\eta A) = \eta\psi(A)$ for all $A \in \mathfrak{A}$. This result has been generalized to the case of nonlinear 3-$*$-Jordan triple derivations by Zhao and Li[23]. For more interesting results related to Jordan and Lie $*$-derivations, we refer the readers to [3, 13, 16, 17, 19, 22].

There is a very interesting generalization of non-additive mixed skew and η-Jordan triple derivations and $*$-derivations, which are called mixed skew and η-Jordan triple higher derivations and $*$-higher derivations, the problem of higher version structure of derivation has attracted a lot of scholars' attention, see [1, 4, 15, 21]. In this paper, we will study non-additive mixed skew and η-Jordan triple higher derivations on unital $*$-algebras with nontrivial idempotents. This does not mean that the mappings are not additive, but they are not necessarily additive. Let us first recall some basic facts related to mixed skew and η-Jordan triple higher derivations. Let $\Psi = \{\psi_n\}_{n\in\mathbb{N}}$ be a sequence of mappings on $\mathcal{A}$ such that $\psi_0 = id_{\mathcal{A}}$. Then Ψ is called

(1) an additive $*$-higher derivation if

$$\psi_n(xy) = \sum_{i+j=n} \psi_i(x)\psi_j(y), \quad \psi_n(x+y) = \psi_n(x) + \psi_n(y) \text{ and } \psi_n(x^*) = \psi_n(x)^*;$$

(2) *non-additive mixed skew and η-Jordan triple higher derivations*

$$\psi_n(x \circ_\eta y \bullet z) = \sum_{i+j+k=n} \psi_i(x) \circ_\eta \psi_j(y) \bullet \psi_k(z) \tag{1.2}$$

for any $x, y, z \in \mathcal{A}$. The idea of mixed η-Jordan triple derivation originated from the definitions of multiplicative $*$-Jordan type higher derivations[21] and nonlinear bi-skew Jordan-type higher derivations[15]. In 2019, Wani and his collaborators[21] have studied the structure of multiplicative $*$-Jordan-type higher derivations on von Neumann algebras without nonzero central abelian projections, and proved that every multiplicative $*$-Jordan-type higher derivations on von Neumann algebras is an additive $*$-higher derivation. Inspired by this paper[21], the first author and coauthors[15] characterize the structure of nonlinear bi-skew Jordan-type higher derivations on unital $*$-algebras with idempotents and show that every nonlinear bi-skew Jordan-type higher derivation is an additive $*$-higher derivation. This result[15] generalized the conclusions of the papers[2, 5] to a higher version, that is, nonlinear bi-skew Jordan-type higher derivation. In addition, we focus on the fact that Khan and coauthors[10] have studied the structure of non-additive mixed skew and η-Jordan triple derivations on an unital prime $*$-algebra under certain conditions and proved that every non-additive mixed skew and η-Jordan triple derivation on an unital prime $*$-algebra is an additive $*$-derivation and satisfies a natural equation. Inspired by articles[10, 15, 21] and the relationship between equations (1.1) and (1.2), there is a natural question:

Question 1.1. *Is a non-additive mixed skew and η-Jordan triple higher derivation on an unital prime $*$-algebra an additive $*$-higher derivation?*

This is an interesting problem. The solution of this problem not only extends the structure problem of non-additive mixed skew and η-Jordan triple derivations studied by Khan and coauthors[10] to higher version, but also describes the structure of many non-additive mixed skew and η-Jordan triple higher derivations on algebras with analytical background, such as standard operator algebras, factor von Neumann algebras and so on.

In recent years, the study of mixed Lie and Jordan triple products and derivations has received a fair amount of attention(see [14, 24–26]). Recently, Taghavi et al.[20] demonstrated that if $\psi(I)$ is skew self-adjoint(i.e., $\psi_1(I)^* = -\psi_1(I)$),then any non-linear map preserving η-Jordan triple derivation on prime $*$-algebra $\mathcal{A}$ is an additive $*$-derivation. Inspired by this conclusion, in 2024, Khan and coauthors[10] investigated the above result on more general context for a mixed η-triple Jordan derivation on a prime $*$-algebra $\mathcal{A}$. Inspired by articles [10, 15, 20], we provide a positive answer to Question 1.1 and extend Khan et al. 's conclusion[10] to a higher version. In particular, we establish the following:

Theorem 1.2. *Let $\mathcal{A}$ be an unital prime $*$-algebra with a nontrivial projection and $\eta \neq -1, \eta \in \mathbb{R}$. Then the non-additive mixed skew and η-Jordan triple higher derivation $\Psi = \{\psi_n\}_{n\in\mathbb{N}}$ formed by the non-additive mapping $\psi_n : \mathcal{A} \to \mathcal{A}$ satisfying the relation*

$$\psi_n(A \circ_\eta B \bullet C) = \sum_{i+j+k=n} \psi_i(A) \circ_\eta \psi_j(B) \bullet \psi_k(C)$$

is additive. Moreover, if $\psi_1(I)$ is skew self-adjoint, i.e., $\psi_1(I)^* = -\psi_1(I)$, then $\Psi = \{\psi_n\}_{n\in\mathbb{N}}$ is an additive $*$-higher derivation and $\psi_n(\eta A) = \eta\psi_n(A)$ for all $A \in \mathcal{A}$.

The structure of this paper is described in this part. The first part describes the background and research motivation of the article. The second part describes the concept and connotation of prime $*$-algebra, which provides the theoretical basis for the main conclusion of the paper. In the third part, the additivity of non-additive mixed skew and η-Jordan triple higher derivations is studied, and it is proved that every non-additive mixed skew and η-Jordan triple higher derivation is additive, which lays the foundation for studying the structure of non-additive mixed skew and η-Jordan triple higher derivations in the fifth part. Therefore, in the fifth part of this paper, the relationship between non-additive mixed skew and η-Jordan triple higher derivations and additive $*$-higher derivations is studied, and it is proved that all non-additive mixed derivations are additive $*$-higher derivations under certain conditions, , and naturally obtain a corollary of the main conclusion, which was proved by Khan and Coauthors. In the last section, we apply the main conclusion to some typical operator algebras, such as standard operator algebras and factor von Neumann algebras, and so on, and obtain a series of corollary.

2. Basic definitions and preliminaries

Throughout this paper, we'll use $\mathcal{A}$ to stand for unital prime $*$-algebra containing a nontrivial symmetric idempotent defined on a complex field $\mathbb{C}$. Denote by $\mathcal{Z}(\mathcal{A})$ the center of $\mathcal{A}$. Let I and $P_1 \in \mathcal{A}$ represent the unite element and nontrivial symmetric idempotent element of algebra $\mathcal{A}$, respectively, and then we know that $P_2 = I - P_1$ is also a symmetric idempotent. By means of the relation of idempotent elements P_1 and P_2, we know that algebra $\mathcal{A}$ has the following Peirce decomposition

$$\mathcal{A} = P_1\mathcal{A}P_1 + P_1\mathcal{A}P_2 + P_2\mathcal{A}P_1 + P_2\mathcal{A}P_2.$$

For the sake of further elaboration, we introduce the following symbols: $\mathcal{A}_{ij} = P_i\mathcal{A}P_j$ for $i, j \in \{1, 2\}$. So the Pierce decomposition of unital prime $*$-algebra with nontrivial symmetric idempotents $\mathcal{A}$ can be rewritten as

$$\mathcal{A} = \mathcal{A}_{11} + \mathcal{A}_{12} + \mathcal{A}_{21} + \mathcal{A}_{22}.$$

The primality of unital prime $*$-algebra $\mathcal{A}$ is expressed as follows: if $a\mathcal{A}b = 0$ for $a, b \in \mathcal{P}$, then $a = 0$ or $b = 0$. The primality of unital prime $*$-algebra $\mathcal{A}$ provides the basis for studying the structure of non-additive mixed skew and η-Jordan triple higher derivations on unital prime $*$-algebra with nontrivial symmetric idempotents in this paper. The research topic of this paper is meaningful. because under this condition, many important algebras with analytical background are regarded as its typical examples, such as standard operator algebras, factor von Neumann algebras, and so on, and then the non-additive mixed skew and η-Jordan triple higher derivations on them is also characterized.

3. $\Psi = \{\psi_n\}_{n\in\mathbb{N}}$ is additive

This section focuses on the study of the structure of non-additive mixed skew and η-Jordan triple higher derivations $\Psi = \{\psi_n\}_{n\in\mathbb{N}}$ on prime $*$-algebras, that is, non-additive mixed skew and η-Jordan triple higher derivations $\Psi = \{\psi_n\}_{n\in\mathbb{N}}$ are additive. The main conclusion of this section is presented here.

Theorem 3.1. *Let $\mathcal{A}$ be an unital prime $*$-algebra with a nontrivial symmetric idempotent and $\eta \neq -1, \eta \in \mathbb{R}$. Then non-additive mixed skew and η-Jordan triple higher derivations $\Psi = \{\psi_n\}_{n\in\mathbb{N}}$ satisfying*

$$\psi_n(x \circ_\eta y \bullet z) = \sum_{i+j+k=n} \psi_i(x) \circ_\eta \psi_j(y) \bullet \psi_k(z)$$

is additive, i.e., $\psi_n(x + y) = \psi_n(x) + \psi_n(y)$ for all $x, y, z \in \mathcal{A}$

To better prove that Theorem 3.1 holds, we introduce Proposition 3.2 obtained from Khan and coauthors[10, Theorem1.2], which proves that every non-additive mixed skew and η-Jordan triple higher derivation is additive, it is the assumptions used in conformity with the assumptions of Theorem 3.1

Proposition 3.2. *[10, Theorem 1.2] Let $\mathcal{A}$ be an unital prime $*$-algebra with a nontrivial symmetric idempotent and $\eta \neq -1, \eta \in \mathbb{R}$. Then non-additive mixed skew and η-Jordan triple higher derivations $\psi_1 : \mathcal{A} \to \mathcal{A}$ satisfying*

$$\psi_1(x \circ_\eta y \bullet z) = \psi_1(x) \circ_\eta y \bullet z + x \circ_\eta \psi_1(y) \bullet z + x \circ_\eta y \bullet \psi_1(z)$$

is additive,i.e.,$\psi_1(x + y) = \psi_1(x) + \psi_1(y)$ for all $x, y \in \mathcal{A}$

The above proposition provides the inductive basis for us to prove Theorem 3.1 by mathematical induction for the index n, that is, $n = 1$.

To obtain the theorem, we need to introduce the following structural properties of the non-additive mixed skew and η-Jordan triple derivations $\psi_1 : \mathcal{A} \to \mathcal{A}$ in the sense of article[10, Theorem 1.2]:

$$\mathbf{L}_1 = \begin{cases} \psi_1(0) = 0; \\ \psi_1(C_{11} + C_{12} + C_{22}) = \psi_1(C_{11}) + \psi_1(C_{12}) + \psi_1(C_{22}) \text{ for all } C_{11} \in \mathcal{A}_{11}, C_{12} \in \mathcal{A}_{12}, C_{22} \in \mathcal{A}_{22}; \\ \psi_1(C_{12} + D_{12}) = \psi_1(C_{12}) + \psi_1(D_{12}) \text{ for all } C_{12}, D_{12} \in \mathcal{A}_{12}; \\ \psi_1(C_{ii} + D_{ii}) = \psi_1(C_{ii}) + \psi_1(D_{ii}) \text{ for all } C_{ii}, D_{ii} \in \mathcal{A}_{ii}, i \in \{1, 2\}. \end{cases}$$

Now let $n \in \mathbb{N}$ with $n \geq 1$ and we assume that Theorem 3.1 holds for all $k < n$. This implies that $\{\psi_i\}_{i=0}^k$ is an additive mixed skew and η-Jordan triple higher derivation of order k and the mappings ψ_k satisfy the following properties:

$$\mathbf{L}_k = \begin{cases} \psi_k(0) = 0; \\ \psi_k(C_{11} + C_{12} + C_{22}) = \psi_k(C_{11}) + \psi_k(C_{12}) + \psi_k(C_{22}) \text{ for all } C_{11} \in \mathcal{A}_{11}, C_{12} \in \mathcal{A}_{12}, C_{22} \in \mathcal{A}_{22}; \\ \psi_k(C_{12} + D_{12}) = \psi_k(C_{12}) + \psi_k(D_{12}) \text{ for all } C_{12}, D_{12} \in \mathcal{A}_{12}; \\ \psi_k(C_{ii} + D_{ii}) = \psi_k(C_{ii}) + \psi_k(D_{ii}) \text{ for all } C_{ii}, D_{ii} \in \mathcal{A}_{ii}, i \in \{1, 2\}. \end{cases}$$

Below we give the main conclusion of this part, and give his proof process according to the inductive hypothesis method. In order to facilitate the proof process of the following conclusions, we split the whole proof process into the following steps.

Step 1. With notations as above, we have $\psi_n(0) = 0$.

Proof. By the induction hypothesis, it follows that For $A = B = C = 0$ in equation (1.2), we get:

$$\begin{aligned} \psi_n(0) &= \psi_n(0 \circ_\eta 0 \bullet 0) \\ &= \sum_{i+j+k=n} \psi_i(0) \circ_\eta \psi_j(0) \bullet \psi_k(0) \\ &= 0. \end{aligned}$$

So $\psi_n(0) = 0$. Thus we prove this step. □

Step 2. With notations as above, we have

$$\psi_n(A_{12}+A_{22})=\psi_n(A_{12})+\psi_n(A_{22})$$

for all $A_{12}\in\mathcal{A}_{12}$ and $A_{22}\in\mathcal{A}_{22}$.

Proof. We show that $\mathrm{T}=\psi_n(A_{12}+A_{22})-\psi_n(A_{12})-\psi_n(A_{22})$ for all $A_{12}\in\mathcal{A}_{12}$ and $A_{22}\in\mathcal{A}_{22}$.

It follows from Condition $\mathbf{L}_1$ of the induction hypothesis that $\psi_1(A_{12}+A_{22})=\psi_1(A_{12})+\psi_1(A_{22})$, we assume that Conclusion

$$\psi_k(A_{12}+A_{22})=\psi_k(A_{12})+\psi_k(A_{22})$$

holds for all $0\le k\le n-1$, and in the following we consider case n. By virtue of the definition of non-additive mixed skew and η-Jordan triple higher derivations, we have

$$\begin{aligned}
&\psi_n(P_1)\circ_\eta P_1\bullet(A_{12}+A_{22})+P_1\circ_\eta\psi_n(P_1)\bullet(A_{12}+A_{22})+P_1\circ_\eta P_1\bullet\psi_n(A_{12}+A_{22})\\
&+\sum_{i+j+k=n,0<i,j,k<n}\psi_i(P_1)\circ_\eta\psi_j(P_1)\bullet\psi_k(A_{12}+A_{22})\\
=&\psi_n(P_1\circ_\eta P_1\bullet(A_{12}+A_{22}))\\
=&\psi_n(P_1\circ_\eta P_1\bullet A_{12})+\psi_n(P_1\circ_\eta P_1\bullet A_{22})\\
=&\psi_n(P_1)\circ_\eta P_1\bullet A_{12}+P_1\circ_\eta\psi_n(P_1)\bullet A_{12}+P_1\circ_\eta P_1\bullet\psi_n(A_{12})\\
&+\sum_{i+j+k=n,0<i,j,k<n}\psi_i(P_1)\circ_\eta\psi_j(P_1)\bullet\psi_k(A_{12})\\
&+\psi_n(P_1)\circ_\eta P_1\bullet A_{22}+P_1\circ_\eta\psi_n(P_1)\bullet A_{22}+P_1\circ_\eta P_1\bullet\psi_n(A_{22})\\
&+\sum_{i+j+k=n,0<i,j,k<n}\psi_i(P_1)\circ_\eta\psi_j(P_1)\bullet\psi_k(A_{22}).
\end{aligned}$$

for all $A_{12}\in\mathcal{A}_{12}$ and $A_{22}\in\mathcal{A}_{22}$.

With the first and last equations above, it follows that equality $P_1\circ_\eta P_1\bullet T=0$ holds and then to the conclusion $T_{11}=T_{12}=T_{21}=0$.

On the other hand, we have

$$\begin{aligned}
&\psi_n(I)\circ_\eta(P_1-P_2)\bullet(A_{12}+A_{22})+I\circ_\eta\psi_n(P_1-P_2)\bullet(A_{12}+A_{22})+I\circ_\eta(P_1-P_2)\bullet\psi_n(A_{12}+A_{22})\\
&+\sum_{i+j+k=n,0<i,j,k<n}\psi_i(I)\circ_\eta\psi_j(P_1-P_2)\bullet\psi_k(A_{12}+A_{22})\\
=&\psi_n(I\circ_\eta(P_1-P_2)\bullet(A_{12}+A_{22}))\\
=&\psi_n(I\circ_\eta(P_1-P_2)\bullet A_{12})+\psi_n(I\circ_\eta(P_1-P_2)\bullet A_{22})\\
=&\psi_n(I)\circ_\eta(P_1-P_2)\bullet A_{12}+I\circ_\eta\psi_n(P_1-P_2)\bullet A_{12}+I\circ_\eta(P_1-P_2)\bullet\psi_n(A_{12})+\\
&\sum_{i+j+k=n,0<i,j,k<n}\psi_i(I)\circ_\eta\psi_j(P_1-P_2)\bullet\psi_k(A_{12})\\
&+\psi_n(I)\circ_\eta(P_1-P_2)\bullet A_{22}+I\circ_\eta\psi_n(P_1-P_2)\bullet A_{22}+I\circ_\eta(P_1-P_2)\bullet\psi_n(A_{22})\\
&+\sum_{i+j+k=n,0<i,j,k<n}\psi_i(I)\circ_\eta\psi_j(P_1-P_2)\bullet\psi_k(A_{22})
\end{aligned}$$

for all $A_{12}\in\mathcal{A}_{12}$ and $A_{22}\in\mathcal{A}_{22}$. With the first and last expressions of the above equations, it can be inferred that equation $I\circ_\eta(P_1-P_2)\bullet T=0$ is true, and then it can be obtained $T_{22}=0$, and this completes the proof of the step. □

Step 3. With notations as above, we have

$$\psi_n(A_{12}+A_{21})=\psi_n(A_{12})+\psi_n(A_{21})$$

for all $A_{12}\in\mathcal{A}_{12}$ and $A_{21}\in\mathcal{A}_{21}$.

Proof. We need to show that

$$T = \psi_n(A_{12} + A_{21}) - \psi_n(A_{12}) - \psi_n(A_{21}) = 0$$

for all $A_{12} \in \mathcal{A}_{12}$ and $A_{21} \in \mathcal{A}_{21}$.

According to $\mathbf{L}_1$, conclusion

$$\psi_1(A_{12} + A_{21}) = \psi_1(A_{12}) + \psi_1(A_{21})$$

is valid when $n = 1$ for all $A_{12} \in \mathcal{A}_{12}$ and $A_{21} \in \mathcal{A}_{21}$. Let's assume that the induction hypothesis $\mathbf{L}_k$ is also true for $1 \leq k \leq n-1$. We consider the case $k = n$ below.

In view of the definition of non-additive mixed skew and η-Jordan triple higher derivations, we obtain

$$\begin{aligned}
&\psi_n(P_1) \circ_\eta (A_{12} + A_{21}) \bullet P_1 + P_1 \circ_\eta \psi_n(A_{12} + A_{21}) \bullet P_1 + P_1 \circ_\eta (A_{12} + A_{21}) \bullet \psi_n(P_1) \\
&\quad + \sum_{i+j+k=n, 0<i,j,k<n} \psi_i(P_1) \circ_\eta \psi_j(A_{12} + A_{21}) \bullet \psi_k(P_1) \\
=&\psi_n(P_1 \circ_\eta (A_{12} + A_{21}) \bullet P_1) \\
=&\psi_n(P_1 \circ_\eta A_{12} \bullet P_1) + \psi_n(P_1 \circ_\eta A_{21} \bullet P_1) \\
=&\psi_n(P_1) \circ_\eta A_{12} \bullet P_1 + P_1 \circ_\eta \psi_n(A_{12}) \bullet P_1 + P_1 \circ_\eta A_{12} \bullet \psi_n(P_1) \\
&\quad + \sum_{i+j+k=n, 0<i,j,k<n} \psi_i(P_1) \circ_\eta \psi_j(A_{12}) \bullet \psi_k(P_1) \\
&\quad + \psi_n(P_1) \circ_\eta A_{21} \bullet P_1 + P_1 \circ_\eta \psi_n(A_{21}) \bullet P_1 + P_1 \circ_\eta A_{21} \bullet \psi_n(P_1) \\
&\quad + \sum_{i+j+k=n, 0<i,j,k<n} \psi_i(P_1) \circ_\eta \psi_j(A_{21}) \bullet \psi_k(P_1) \\
=&\psi_n(P_1) \circ_\eta (A_{12} + A_{21}) \bullet P_1 + P_1 \circ_\eta (\psi_n(A_{12}) + \psi_n(A_{21})) \bullet P_1 + P_1 \circ_\eta (A_{12} + A_{21}) \bullet \psi_n(P_1) \\
&\quad + \sum_{i+j+k=n, 0<i,j,k<n} \psi_i(P_1) \circ_\eta \psi_j(A_{12} + A_{21}) \bullet \psi_k(P_1)
\end{aligned} \tag{2.1}$$

for all $A_{12} \in \mathcal{A}_{12}$ and $A_{21} \in \mathcal{A}_{21}$.

Therefore, $T_{21} = 0$ and

$$T_{11} + T_{11}^* = 0. \tag{2.2}$$

While replacing P_1 with iP_1 in equation(2.1), using the relation $\mathrm{i}^* = -\mathrm{i}$, it follows that

$$T_{11} - T_{11}^* = 0 \tag{2.3}$$

holds. It follows from the last two equalities (2.2) and (2.3) that equality $T_{11} = 0$ holds.

Replacing P_1 with P_2 in equation (2.1) and using similar calculation methods and techniques can be obtained equality $T_{12} = 0$ and

$$T_{22} + (T_{22})^* = 0 \tag{2.4}$$

Moreover, by applying iP_2 instead of P_1 in above (2.1), we obtain

$$T_{22} - (T_{22})^* = 0 \tag{2.5}$$

It can be obtained by means of (2.4) and (2.5) that $T_{22} = 0$ holds. □

Step 4. With notations as above, we have

(1) $\psi_n(A_{12} + A_{21} + A_{22}) = \psi_n(A_{12}) + \psi_n(A_{21}) + \psi_n(A_{22})$;

(2) $\psi_n(A_{11} + A_{12} + A_{21}) = \psi_n(A_{11}) + \psi_n(A_{12}) + \psi_n(A_{21})$

for all $A_{11} \in \mathcal{A}_{11}, A_{12} \in \mathcal{A}_{12}, A_{21} \in \mathcal{A}_{21}, A_{22} \in \mathcal{A}_{22}$.

Proof. We only prove the first part. The second one can be proved similarly. We prove that

$$T = \psi_n(A_{12} + A_{21} + A_{22}) - \psi_n(A_{12}) - \psi_n(A_{21}) - \psi_n(A_{22})$$

for all $A_{12} \in \mathcal{A}_{12}, A_{21} \in \mathcal{A}_{21}, A_{22} \in \mathcal{A}_{22}$. By the inductive basis $\mathbb{L}_1$, it follows that

$$\psi_1(A_{12} + A_{21} + A_{22}) = \psi_1(A_{12}) + \psi_1(A_{21}) + \psi_1(A_{22})$$

for all $A_{12} \in \mathcal{A}_{12}, A_{21} \in \mathcal{A}_{21}, A_{22} \in \mathcal{A}_{22}$,. For $1 \leq k \leq n-1$, with the help of $\mathbb{L}_k$, we assume that the equation

$$\psi_k(A_{12} + A_{21} + A_{22}) = \psi_k(A_{12}) + \psi_k(A_{21}) + \psi_k(A_{22})$$

was established for all $A_{12} \in \mathcal{A}_{12}, A_{21} \in \mathcal{A}_{21}, A_{22} \in \mathcal{A}_{22}$. In the following we study the case $k = n$. According to the stated earlier, we can get

$$\begin{aligned}
&\psi_n(P_1) \circ_\eta P_1 \bullet (A_{12} + A_{21} + A_{22}) + P_1 \circ_\eta \psi_n(P_1) \bullet (A_{12} + A_{21} + A_{22}) + P_1 \circ_\eta P_1 \bullet \psi_n(A_{12} + A_{21} + A_{22}) \\
&+ \sum_{i+j+k=n, 0<i,j,k<n} \psi_i(P_1) \circ_\eta \psi_j(P_1) \bullet \psi_k(A_{12} + A_{21} + A_{22}) \\
=&\psi_n(P_1 \circ_\eta P_1 \bullet (A_{12} + A_{21} + A_{22})) \\
=&\psi_n(P_1 \circ_\eta P_1 \bullet (A_{12} + A_{21})) + \psi_n(P_1 \circ_\eta P_1 \bullet A_{22}) \\
=&\psi_n(P_1) \circ_\eta P_1 \bullet (A_{12} + A_{21}) + P_1 \circ_\eta \psi_n(P_1) \bullet (A_{12} + A_{21}) + P_1 \circ_\eta P_1 \bullet \psi_n(A_{12} + A_{21}) \\
&+ \sum_{i+j+k=n, 0<i,j,k<n} \psi_i(P_1) \circ_\eta \psi_j(P_1) \bullet \psi_k(A_{12} + A_{21}) \\
&+ \psi_n(P_1) \circ_\eta P_1 \bullet A_{22} + P_1 \circ_\eta \psi_n(P_1) \bullet A_{22} + P_1 \circ_\eta P_1 \bullet \psi_n(A_{22}) \\
&+ \sum_{i+j+k=n, 0<i,j,k<n} \psi_i(P_1) \circ_\eta \psi_j(P_1) \bullet \psi_k(A_{22}) \\
=&\psi_n(P_1) \circ_\eta P_1 \bullet A_{12} + P_1 \circ_\eta \psi_n(P_1) \bullet A_{12} + P_1 \circ_\eta P_1 \bullet \psi_n(A_{12}) \\
&+ \sum_{i+j+k=n, 0<i,j,k<n} \psi_i(P_1) \circ_\eta \psi_j(P_1) \bullet \psi_k(A_{12}) \\
&+ \psi_n(P_1) \circ_\eta P_1 \bullet A_{21} + P_1 \circ_\eta \psi_n(P_1) \bullet A_{21} + P_1 \circ_\eta P_1 \bullet \psi_n(A_{21}) \\
&+ \sum_{i+j+k=n, 0<i,j,k<n} \psi_i(P_1) \circ_\eta \psi_j(P_1) \bullet \psi_k(A_{21}) \\
&+ \psi_n(P_1) \circ_\eta P_1 \bullet A_{22} + P_1 \circ_\eta \psi_n(P_1) \bullet A_{22} + P_1 \circ_\eta P_1 \bullet \psi_n(A_{22}) \\
&+ \sum_{i+j+k=n, 0<i,j,k<n} \psi_i(P_1) \circ_\eta \psi_j(P_1) \bullet \psi_k(A_{22}) \\
=& + \psi_n(P_1) \circ_\eta P_1 \bullet (A_{12} + A_{21} + A_{22}) + P_1 \circ_\eta \psi_n(P_1) \bullet (A_{12} + A_{21} + A_{22}) \\
&+ P_1 \circ_\eta P_1 \bullet (\psi_n(A_{12}) + \psi_n(A_{21}) + \psi_n(A_{22})) \\
&+ \sum_{i+j+k=n, 0<i,j,k<n} \psi_i(P_1) \circ_\eta \psi_j(P_1) \bullet \psi_k(A_{12} + A_{21} + A_{22})
\end{aligned}$$

for all $A_{12} \in \mathcal{A}_{12}, A_{21} \in \mathcal{A}_{21}, A_{22} \in \mathcal{A}_{22}$. With the help of the first expression and the last expression in the last equation, we can obtain $P_1 \circ_\eta P_1 \bullet T = 0$. And then $T_{11} = T_{21} = T_{12} = 0$.

On the other hand, we can verify that

$$
\begin{aligned}
&\psi_n(P_2)\circ_\eta(A_{12}+A_{21}+A_{22})\bullet P_2+P_2\circ_\eta\psi_n(A_{12}+A_{21}+A_{22})\bullet P_2+P_2\circ_\eta(A_{12}+A_{21}+A_{22})\bullet\psi_n(P_2)\\
&\quad+\sum_{i+j+k=n,0<i,j,k<n}\psi_i(P_2)\circ_\eta\psi_j(A_{12}+A_{21}+A_{22})\bullet\psi_k(P_2)\\
=&\psi_n(P_2\circ_\eta(A_{12}+A_{21}+A_{22})\bullet P_2)\\
=&\psi_n(P_2)\circ_\eta A_{12}\bullet P_2+P_2\circ_\eta\psi_n(A_{12})\bullet P_2+P_2\circ_\eta A_{12}\bullet\psi_n(P_2)+\\
&\sum_{i+j+k=n,0<i,j,k<n}\psi_i(P_2)\circ_\eta\psi_j(A_{12})\bullet\psi_k(P_2)\\
&\quad+\psi_n(P_2)\circ_\eta A_{21}\bullet P_2+P_2\circ_\eta\psi_n(A_{21})\bullet P_2+P_2\circ_\eta A_{21}\bullet\psi_n(P_2)+\\
&\sum_{i+j+k=n,0<i,j,k<n}\psi_i(P_2)\circ_\eta\psi_j(A_{21})\bullet\psi_k(P_2)\\
&\quad+\psi_n(P_2)\circ_\eta A_{22}\bullet P_2+P_2\circ_\eta\psi_n(A_{22})\bullet P_2+P_2\circ_\eta A_{22}\bullet\psi_n(P_2)+\\
&\sum_{i+j+k=n,0<i,j,k<n}\psi_i(P_2)\circ_\eta\psi_j(A_{22})\bullet\psi_k(P_2)\\
=&\psi_n(P_2)\circ_\eta(A_{12}+A_{21}+A_{22})\bullet P_2+P_2\circ_\eta(\psi_n(A_{12})+\psi_n(A_{21})+\psi_n(A_{22}))\bullet P_2\\
&\quad+P_2\circ_\eta(A_{12}+A_{21}+A_{22})\bullet\psi_n(P_2)\\
&\quad+\sum_{i+j+k=n,0<i,j,k<n}\psi_i(P_2)\circ_\eta\psi_j(A_{12}+A_{21}+A_{22})\bullet\psi_k(P_2)
\end{aligned}
$$

Hence, we obtain $P_2\circ_\eta T\bullet P_2=0$. Then, $T_{22}+T_{22}^*=0$. Using similar computational methods and solving techniques, by applying iP_2 instead of P_2 in above equation, we have $T_{22}-T_{22}^*=0$. So, $T_{22}=0$. This completes the proof of the step. □

Step 5 For any $A_{ij}\in\mathcal{A}_{ij}$, we show that

$$\psi_n\left(\sum_{i,j=1}^{2}A_{ij}\right)=\sum_{i,j=1}^{2}\psi_n(A_{ij}).$$

Proof. $T=\psi_n\left(\sum_{i,j=1}^{2}A_{ij}\right)-\sum_{i,j=1}^{2}\psi_n\left(A_{ij}\right)$ It is easy to verify from **Step 4** that

$$
\begin{aligned}
&\psi_n(P_2)\circ_\eta P_2\bullet(\sum_{i,j=1}^{2}A_{ij})+P_2\circ_\eta\psi_n(P_2)\bullet(\sum_{i,j=1}^{2}A_{ij})+P_2\circ_\eta P_2\bullet\psi_n\left(\sum_{i,j=1}^{2}A_{ij}\right)\\
&\quad+\sum_{i+j+k=n,0<i,j,k<n}\psi_i(P_2)\circ_\eta\psi_j(P_2)\bullet\psi_k\left(\sum_{i,j=1}^{2}A_{ij}\right)\\
=&\psi_n(P_2\circ_\eta P_2\bullet(\sum_{i,j=1}^{2}A_{ij}))\\
=&\psi_n(P_2\circ_\eta P_2\bullet(A_{12}+A_{21}+A_{22}))+\psi_n(P_2\circ_\eta P_2\bullet A_{11})
\end{aligned}
$$

$$
\begin{aligned}
&=\psi_n(P_2) \circ_\eta P_2 \bullet (A_{12} + A_{21} + A_{22}) + P_2 \circ_\eta \psi_n(P_2) \bullet (A_{12} + A_{21} + A_{22}) + P_2 \circ_\eta P_2 \bullet \psi_n(A_{12} + A_{21} + A_{22})\\
&\quad + \sum_{i+j+k=n, 0<i,j,k<n} \psi_i(P_2) \circ_\eta \psi_j(P_2) \bullet \psi_k(A_{12} + A_{21} + A_{22})\\
&\quad + \psi_n(P_2) \circ_\eta P_2 \bullet A_{11} + P_2 \circ_\eta \psi_n(P_2) \bullet A_{11} + P_2 \circ_\eta P_2 \bullet \psi_n(A_{11})\\
&\quad + \sum_{i+j+k=n, 0<i,j,k<n} \psi_i(P_2) \circ_\eta \psi_j(P_2) \bullet \psi_k(A_{11})\\
&=\psi_n(P_2) \circ_\eta P_2 \bullet A_{11} + P_2 \circ_\eta \psi_n(P_2) \bullet A_{11} + P_2 \circ_\eta P_2 \bullet \psi_n(A_{11})\\
&\quad + \sum_{i+j+k=n, 0<i,j,k<n} \psi_i(P_2) \circ_\eta \psi_j(P_2) \bullet \psi_k(A_{11})\\
&\quad + \psi_n(P_2) \circ_\eta P_2 \bullet A_{12} + P_2 \circ_\eta \psi_n(P_2) \bullet A_{12} + P_2 \circ_\eta P_2 \bullet \psi_n(A_{12})\\
&\quad + \sum_{i+j+k=n, 0<i,j,k<n} \psi_i(P_2) \circ_\eta \psi_j(P_2) \bullet \psi_k(A_{12})\\
&\quad + \psi_n(P_2) \circ_\eta P_2 \bullet A_{21} + P_2 \circ_\eta \psi_n(P_2) \bullet A_{21} + P_2 \circ_\eta P_2 \bullet \psi_n(A_{21})\\
&\quad + \sum_{i+j+k=n, 0<i,j,k<n} \psi_i(P_2) \circ_\eta \psi_j(P_2) \bullet \psi_k(A_{21})\\
&\quad + \psi_n(P_2) \circ_\eta P_2 \bullet A_{22} + P_2 \circ_\eta \psi_n(P_2) \bullet A_{22} + P_2 \circ_\eta P_2 \bullet \psi_n(A_{22})\\
&\quad + \sum_{i+j+k=n, 0<i,j,k<n} \psi_i(P_2) \circ_\eta \psi_j(P_2) \bullet \psi_k(A_{22}).
\end{aligned}
$$

Equation $P_2 \circ_\eta P_2 \bullet T = 0$ holds by combining the first and last expressions of the above equation, and then the equation $T_{22} = T_{12} = 0$ is established.

By using similar calculation techniques and methods, while replacing P_2 with P_1 in above equation, it can be obtained $T_{11} = T_{21} = 0$. □

Step 6. For all $A_{ij}, B_{ij} \in \mathcal{A}_{ij}$ with $i \neq j \in \{1, 2\}$,

$$\psi_n(A_{ij} + B_{ij}) = \psi_n(A_{ij}) + \psi_n(B_{ij}).$$

Proof. It follows from Step 2 and induction hypothesis $\mathbb{L}_k$ for $1 \leq k \leq n-1$ that

$$
\begin{aligned}
\psi_n(A_{ij} + B_{ij}) &= \psi_n\left(\frac{1}{1+\eta} I \circ_\eta P_i \bullet (2P_j + A_{ij} + B_{ij})\right)\\
&= \psi_n\left(\frac{1}{1+\eta} I\right) \circ_\eta P_i \bullet (2P_j + A_{ij} + B_{ij}) +\\
&\quad \frac{1}{1+\eta} I \circ_\eta \psi_n(P_i) \bullet (2P_j + A_{ij} + B_{ij}) + \frac{1}{1+\eta} I \circ_\eta P_i \bullet \psi_n(2P_j + A_{ij} + B_{ij})\\
&\quad + \sum_{i+j+k=n, 0<i,j,k<n} \psi_i(\frac{1}{1+\eta} I) \circ_\eta \psi_j(P_i) \bullet \psi_k(2P_j + A_{ij} + B_{ij})\\
&= \psi_n\left(\frac{1}{1+\eta} I\right) \circ_\eta P_i \bullet (2P_j + A_{ij} + B_{ij}) + \frac{1}{1+\eta} I \circ_\eta \psi_n(P_i) \bullet (2P_j + A_{ij} + B_{ij}) +\\
&\quad \frac{1}{1+\eta} I \circ_\eta P_i \bullet (\psi_n(2P_j) + \psi_n(A_{ij}) + \psi_n(B_{ij}))\\
&\quad + \sum_{i+j+k=n, 0<i,j,k<n} \psi_i(\frac{1}{1+\eta} I) \circ_\eta \psi_j(P_i) \bullet (\psi_k(2P_j) + \psi_k(A_{ij}) + \psi_k(B_{ij}))
\end{aligned}
$$

$$
\begin{aligned}
=&2\psi_n\left(\frac{1}{1+\eta}I \circ_\eta P_i \bullet 2P_j\right) + \psi_n\left(\frac{1}{1+\eta}I \circ_\eta P_i \bullet A_{ij}\right) + \psi_n\left(\frac{1}{1+\eta}I \circ_\eta P_i \bullet B_{ij}\right)\\
=&\psi_n(A_{ij}) + \psi_n(B_{ij})
\end{aligned}
$$

for all $A_{ij}, B_{ij} \in \mathcal{A}_{ij}$ with $i \neq j \in \{1,2\}$. $\square$

Step 7. For any $A_{ii}, B_{ii} \in \mathcal{A}_{ii}$ such that $i \in \{1,2\}$,

$$\psi_n(A_{ii} + B_{ii}) = \psi_n(A_{ii}) + \psi_n(B_{ii}).$$

Proof. Suppose that

$$T = \psi_n(A_{ii} + B_{ii}) - (\psi_n(A_{ii}) + \psi_n(B_{ii})).$$

It is easy to find

$$
\begin{aligned}
&\psi_n(P_i) \circ_\eta (A_{ii} + B_{ii}) \bullet P_j + P_i \circ_\eta \psi_n(A_{ii} + B_{ii}) \bullet P_j + P_i \circ_\eta (A_{ii} + B_{ii}) \bullet \psi_n(P_j)\\
&\quad + \sum_{i+j+k=n,0<i,j,k<n} \psi_i(P_i) \circ_\eta \psi_j(A_{ii} + B_{ii}) \bullet \psi_k(P_j)\\
=&\psi_n(P_i \circ_\eta (A_{ii} + B_{ii}) \bullet P_j)\\
=&\psi_n(P_i \circ_\eta A_{ii} \bullet P_j) + \psi_n(P_i \circ_\eta B_{ii} \bullet P_j)\\
=&\psi_n(P_i) \circ_\eta A_{ii} \bullet P_j + P_i \circ_\eta \psi_n(A_{ii}) \bullet P_j + P_i \circ_\eta A_{ii} \bullet \psi_n(P_j)\\
&+ \psi_n(P_i) \circ_\eta B_{ii} \bullet P_j + P_i \circ_\eta \psi_n(B_{ii}) \bullet P_j + P_i \circ_\eta B_{ii} \bullet \psi_n(P_j)\\
&+ \sum_{i+j+k=n,0<i,j,k<n} \psi_i(P_i) \circ_\eta \psi_j(A_{ii}) \bullet \psi_k(P_j)\\
&+ \sum_{i+j+k=n,0<i,j,k<n} \psi_i(P_i) \circ_\eta \psi_j(B_{ii}) \bullet \psi_k(P_j).
\end{aligned}
$$

and

$$
\begin{aligned}
&\psi_n(P_j) \circ_\eta (A_{ii} + B_{ii}) \bullet P_i + P_j \circ_\eta \psi_n(A_{ii} + B_{ii}) \bullet P_i + P_j \circ_\eta (A_{ii} + B_{ii}) \bullet \psi_n(P_i)\\
&+ \sum_{i+j+k=n,0<i,j,k<n} \psi_i(P_j) \circ_\eta \psi_j(A_{ii} + B_{ii}) \bullet \psi_k(P_i)\\
=&\psi_n(P_j \circ_\eta (A_{ii} + B_{ii}) \bullet P_i)\\
=&\psi_n(P_j \circ_\eta A_{ii} \bullet P_i) + \psi_n(P_j \circ_\eta B_{ii} \bullet P_i)\\
=&\psi_n(P_j) \circ_\eta A_{ii} \bullet P_i + P_j \circ_\eta \psi_n(A_{ii}) \bullet P_i + P_j \circ_\eta A_{ii} \bullet \psi_n(P_i)\\
&+ \psi_n(P_j) \circ_\eta B_{ii} \bullet P_i + P_j \circ_\eta \psi_n(B_{ii}) \bullet P_i + P_j \circ_\eta B_{ii} \bullet \psi_n(P_i)\\
&+ \sum_{i+j+k=n,0<i,j,k<n} \psi_i(P_j) \circ_\eta \psi_j(A_{ii}) \bullet \psi_k(P_i)\\
&+ \sum_{i+j+k=n,0<i,j,k<n} \psi_i(P_j) \circ_\eta \psi_j(B_{ii}) \bullet \psi_k(P_i)
\end{aligned}
$$

for any $A_{ii}, B_{ii} \in \mathcal{A}_{ii}$ such that $i \in \{1,2\}$. The last two equations above lead to equations $P_i \circ_\eta T \bullet P_j = 0$ and $P_j \circ_\eta T \bullet P_i = 0$. And then $T_{12} = T_{21} = 0$ is valid. Next, with the aid of Step 2 and Step 6, for any $C_{ji} \in \mathcal{A}_{ji}$

for $i \neq j \in \{1,2\}$, we have

$$\begin{aligned}
&\psi_n(C_{ji}) \circ_\eta (A_{ii}+B_{ii}) \bullet P_i + C_{ji} \circ_\eta \psi_n(A_{ii}+B_{ii}) \bullet P_i + C_{ji} \circ_\eta (A_{ii}+B_{ii}) \bullet \psi_n(P_i)\\
&+ \sum_{i+j+k=n,0<i,j,k<n} \psi_i(C_{ji}) \circ_\eta \psi_j(A_{ii}+B_{ii}) \bullet \psi_k(P_i)\\
=&\psi_n(C_{ji} \circ_\eta (A_{ii}+B_{ii}) \bullet P_i)\\
=&\psi_n(C_{ji}A_{ii} + C_{ji}B_{ii} + A_{ii}^*C_{ji}^* + B_{ii}^*C_{ji}^*)\\
=&\psi_n(C_{ji}A_{ii} + A_{ii}^*C_{ji}^*) + \psi_n(C_{ji}B_{ii} + B_{ii}^*C_{ji}^*)\\
=&\psi_n(C_{ji} \circ_\eta A_{ii} \bullet P_i) + \psi_n(C_{ji} \circ_\eta B_{ii} \bullet P_i)\\
=&\psi_n(C_{ji}) \circ_\eta (A_{ii}+B_{ii}) \bullet P_i + C_{ji} \circ_\eta (\psi_n(A_{ii}) + \psi_n(B_{ii})) \bullet P_i\\
&+ C_{ji} \circ_\eta (A_{ii}+B_{ii}) \bullet \psi_n(P_i) + \sum_{i+j+k=n,0<i,j,k<n} \psi_i(C_{ji}) \circ_\eta \psi_j(A_{ii}+B_{ii}) \bullet \psi_k(P_i)
\end{aligned}$$

for any $A_{ii}, B_{ii} \in \mathcal{A}_{ii}$ such that $i \in \{1,2\}$.

For any $C_{ij} \in \mathcal{A}_{ij}$ with $i \neq j$, after similar calculation methods and techniques, we can obtain

$$\begin{aligned}
&\psi_n(C_{ij}) \circ_\eta (A_{ii}+B_{ii}) \bullet P_j + C_{ij} \circ_\eta \psi_n(A_{ii}+B_{ii}) \bullet P_j + C_{ij} \circ_\eta (A_{ii}+B_{ii}) \bullet \psi_n(P_j)\\
&+ \sum_{i+j+k=n,0<i,j,k<n} \psi_i(C_{ij}) \circ_\eta \psi_j(A_{ii}+B_{ii}) \bullet \psi_k(P_j)\\
=&\psi_n(C_{ij}) \circ_\eta (A_{ii}+B_{ii}) \bullet P_j + C_{ij} \circ_\eta (\psi_n(A_{ii}) + \psi_n(B_{ii})) \bullet P_j + C_{ij} \circ_\eta (A_{ii}+B_{ii}) \bullet \psi_n(P_j)\\
&+ \sum_{i+j+k=n,0<i,j,k<n} \psi_i(C_{ij}) \circ_\eta \psi_j(A_{ii}+B_{ii}) \bullet \psi_k(P_j).
\end{aligned}$$

Hence $C_{ji} \circ_\eta T \bullet P_i = C_{ij} \circ_\eta T \bullet P_j = 0$. This gives $T_{11} = T_{22} = 0$. Thus $T = 0$, i.e.,

$$\psi_n(A_{ii}+B_{ii}) = \psi_n(A_{ii}) + \psi_n(B_{ii})$$

for any $A_{ii}, B_{ii} \in \mathcal{A}_{ii}$ such that $i \in \{1,2\}$. This step is proved. □

By means of Steps 1-7, the proof of Theorem 1.2 is presented as follows.

Proof of Theorem 3.1. For any $A = A_{11} + A_{12} + A_{21} + A_{22} \in \mathcal{A}$ and $B = B_{11} + B_{12} + B_{21} + B_{22} \in \mathcal{A}$, in keeping with Steps 1-8, we have

$$\begin{aligned}
\psi_n(A+B) =&\psi_n(A_{11} + A_{12} + A_{21} + A_{22} + B_{11} + B_{12} + B_{21} + B_{22})\\
=&\psi_n(A_{11}+B_{11}) + \psi_n(A_{12}+B_{12})\\
&+ \psi_n(A_{21}+B_{21}) + \psi_n(A_{22}+B_{22})\\
=&\psi_n(A_{11}) + \psi_n(B_{11}) + \psi_n(A_{12}) + \psi_n(B_{12})\\
&+ \psi_n(A_{21}) + \psi_n(B_{21}) + \psi_n(A_{22}) + \psi_n(B_{22})\\
=&\psi_n(A_{11} + A_{12} + A_{21} + A_{22}) + \psi_n(B_{11} + B_{12} + B_{21} + B_{22})\\
=&\psi_n(A) + \psi_n(B).
\end{aligned}$$

That is to say, $\psi_n(A+B) = \psi_n(A) + \psi_n(B)$ for all $A, B \in \mathcal{A}$.

4. $\Psi = \{\psi_n\}_{n\in\mathbb{N}}$ is an additive ∗-higher derivation

In this part, we will study the structure of the non-additive mixed skew and η-Jordan triple higher derivations $\Psi = \{\psi_n\}_{n\in\mathbb{N}}$ under the condition "$\psi_1(I)^* = -\psi_1(I)$", and prove that the non-additive mixed skew and η-Jordan triple higher derivations $\{\psi_n\}_{n\in\mathbb{N}}$ are additive ∗-higher derivations.

Theorem 4.1. *Let $\mathcal{A}$ be an unital prime $*$-algebra with a nontrivial symmetric idempotent and $\eta \neq -1, \eta \in \mathbb{R}$. If the non-additive mixed skew and η-Jordan triple higher derivations $\Psi = \{\psi_n\}_{n\in\mathbb{N}}$ satisfies equation (1.2) and the condition that the element $\psi_1(I)$ is skew self-adjoint, i.e., $\psi_1(I)^* = -\psi_1(I)$, then the non-additive mixed skew and η-Jordan triple higher derivations $\Psi = \{\psi_n\}_{n\in\mathbb{N}}$ are additive $*$-higher derivations and satisfy $\psi_n(\eta A) = \eta\psi_n(A)$ for all $A \in \mathcal{A}$.*

To facilitate the proof that the above Theorem 4.1 holds, we need to resort to Proposition 4.2, which provides the basis for us to use mathematical induction on the index n.

Proposition 4.2. *Let $\mathcal{A}$ be an unital prime $*$-algebra with a nontrivial symmetric idempotent and $\eta \neq -1, \eta \in \mathbb{R}$. Then the map $\psi_1 : \mathcal{A} \to \mathcal{A}$ satisfying*

$$\psi_1(A \circ_\eta B \bullet C) = \psi_1(A) \circ_\eta B \bullet C + A \circ_\eta \psi_1(B) \bullet C + A \circ_\eta B \bullet \psi_1(C)$$

for any $A, B, C \in \mathcal{A}$ and skew self-adjoint $\psi_1(I)$ is an additive $*$-derivation and $\psi_1(\eta A) = \eta\psi_1(A)$ for all $A \in \mathcal{A}$

In order to obtain this theorem, we will use an induction method for the component index n. When $n = 1$, ψ_1 is a non-additive mixed skew and η-Jordan triple derivations on $\mathcal{A}$. With the help of [10, Theorem 3.1], Khan and coauthors[10] proved that every non-additive mixed skew and η-Jordan triple derivation on $\mathcal{A}$ is an additive $*$-derivation, and satisfies the following properties:

$$\mathbf{M}_1 = \begin{cases} \psi_1(\mathrm{i}I) = \psi_1(I) = 0 \text{ for any } \eta \neq -1, \eta \in \mathbb{R}; \;\; \psi_1(\mathrm{i}A) = \mathrm{i}\psi_1(A); \\ \psi_1(\eta A) = \eta\psi_1(A); \;\; \psi_1(A^*) = \psi_1(A)^*. \end{cases}$$

Now let $n \in \mathbb{N}$ with $n \geq 1$ and we assume that Theorem 3.1 holds for all $k < n$. This implies that the sequence $\{\psi_i\}_{i=0}^k$ is an additive $*$-higher derivation of order k and the mappings ψ_k satisfies the following properties:

$$\mathbf{M}_k = \begin{cases} \psi_k(\mathrm{i}I) = \psi_k(I) = 0 \text{ for any } \eta \neq -1, \eta \in \mathbb{R}; \;\; \psi_k(\mathrm{i}A) = \mathrm{i}\psi_k(A); \\ \psi_k(\eta A) = \eta\psi_k(A); \;\; \psi_k(A^*) = \psi_k(A)^*. \end{cases}$$

The proof process can be achieved through a series of claims.

Claim 1. With notations as above, we have

$$\psi_n(\mathrm{i}A) = \mathrm{i}\psi_n(A)$$

for all $A \in \mathcal{A}$.

Proof. For any $A, B, C \in \mathcal{A}$, Through simple verification, we can find that equation

$$(\mathrm{i}A) \circ_\eta B \bullet I = A \circ_\eta (\mathrm{i}B) \bullet I.$$

holds. Applying the mapping ψ_n to the above equation, we obtain

$$\begin{aligned} \psi_n(\mathrm{i}A \circ_\eta B \bullet I) =& \psi_n(\mathrm{i}A) \circ_\eta B \bullet I + \mathrm{i}A \circ_\eta \psi_n(B) \bullet I + \mathrm{i}A \circ_\eta B \bullet \psi_n(I) \\ &+ \sum_{i+j+k=n, 0<i,j,k<n} \psi_i(\mathrm{i}A) \circ_\eta \psi_j(B) \bullet \psi_k(I) \\ =& \psi_n(\mathrm{i}A) \circ_\eta B \bullet I + \mathrm{i}A \circ_\eta \psi_n(B) \bullet I + \mathrm{i}A \circ_\eta B \bullet \psi_n(I) \\ &+ \mathrm{i} \sum_{i+j+k=n, 0<i,j,k<n} \psi_i(A) \circ_\eta \psi_j(B) \bullet \psi_k(I) \end{aligned} \tag{4.1}$$

for all $A, B \in \mathcal{A}$. On the other hand,

$$\begin{aligned} \psi_n(A \circ_\eta \mathrm{i}B \bullet I) =& \psi_n(A) \circ_\eta \mathrm{i}B \bullet I + A \circ_\eta \psi_n(\mathrm{i}B) \bullet I + A \circ_\eta \mathrm{i}B \bullet \psi_n(I) \\ &+ \sum_{i+j+k=n, 0<i,j,k<n} \psi_i(A) \circ_\eta \psi_j(\mathrm{i}B) \bullet \psi_k(I) \\ =& \psi_n(A) \circ_\eta \mathrm{i}B \bullet I + A \circ_\eta \psi_n(\mathrm{i}B) \bullet I + A \circ_\eta \mathrm{i}B \bullet \psi_n(I) \\ &+ \mathrm{i} \sum_{i+j+k=n, 0<i,j,k<n} \psi_i(A) \circ_\eta \psi_j(B) \bullet \psi_k(I) \end{aligned} \tag{4.2}$$

for all $A, B \in \mathcal{A}$. Subtracting the last two equalities (4.1) and (4.2) shows that equation

$$\psi_n(\mathrm{i}A) \circ_\eta B \bullet I + \mathrm{i}A \circ_\eta \psi_n(B) \bullet I = \psi_n(A) \circ_\eta \mathrm{i}B \bullet I + A \circ_\eta \psi_n(\mathrm{i}B) \bullet I$$

holds. for all $A, B \in \mathcal{A}$. This implies

$$(\psi_n(\mathrm{i}A) - \mathrm{i}\psi_n(A)) \circ_\eta B \bullet I = A \circ_\eta (\psi_n(\mathrm{i}B) - \mathrm{i}\psi_n(B)) \bullet I \tag{4.3}$$

for all $A, B \in \mathcal{A}$. By substituting iA for A in the above equation(4.3), and using the additivity of the mapping ψ_n, we obtain

$$(-\psi_n(A) - \mathrm{i}\psi_n(iA)) \circ_\eta B \bullet I = \mathrm{i}A \circ_\eta (\psi_n(\mathrm{i}B) - \mathrm{i}\psi_n(B)) \bullet I \tag{4.4}$$

for all $A, B \in \mathcal{A}$. Multiply equation (4.3) by i and combines the so obtained relation with equation (4.4), we obtain

$$(\psi_n(A) + \mathrm{i}\psi_n(\mathrm{i}A)) \circ_\eta B \bullet I = 0$$

for all $A, B \in \mathcal{A}$. This can be written as

$$(\psi_n(\mathrm{i}A) - \mathrm{i}\psi_n(A)) \circ_\eta B \bullet I = 0$$

for all $A, B \in \mathcal{A}$. In particular, for $B = I$, we have

$$\psi_n(\mathrm{i}A) - \mathrm{i}\psi_n(A) + \psi_n(\mathrm{i}A)^* + \mathrm{i}\psi_n(A)^* = 0 \tag{4.5}$$

for all $A \in \mathcal{A}$. Setting A as iA in the last expression(4.5), we obtain

$$\psi_n(A) + \mathrm{i}\psi_n(\mathrm{i}A) + \psi_n(A)^* - \mathrm{i}\psi_n(iA)^* = 0 \tag{4.6}$$

for all $A \in \mathcal{A}$. Also, multiply equation (4.5) by i, we get

$$\mathrm{i}\psi_n(\mathrm{i}A) + \psi_n(A) + \mathrm{i}\psi_n(\mathrm{i}A)^* - \psi_n(A)^* = 0 \tag{4.7}$$

for all $A \in \mathcal{A}$. Combining equations (4.6) and (4.7) yields

$$\mathrm{i}\psi_n(\mathrm{i}A) + \psi_n(A) = 0.$$

for all $A \in \mathcal{A}$, and hence, $\psi_n(\mathrm{i}A) = \mathrm{i}\psi_n(A)$ for all $A \in \mathcal{A}$. $\square$

Claim 2 With notations as above, we have

$$\psi_n(I)^* = -\psi_n(I).$$

Proof. The main point of this conclusion is to prove that $(\psi_n(I))^* = -\psi_n(I)$ is true. Notice the hypothesis condition in the theorem: $(\psi_1(I))^* = -\psi_1(I)$, which provides us with the conditions for the inductive assumption on index n. Suppose conclusion $(\psi_k(I))^* = -\psi_k(I)$ holds for $k < n$. In the following we consider the case $k = n$.

It is easy to find

$$\begin{aligned}(1+\eta)\psi_n(I) =& \psi_n(I \circ_\eta I \bullet I)\\ =& \psi_n(I) \circ_\eta I \bullet I + I \circ_\eta \psi_n(I) \bullet I + I \circ_\eta I \bullet \psi_n(I)\\ &+ \sum_{i+j+k=n, 0<i,j,k<n} \psi_i(I) \circ_\eta \psi_j(I) \bullet \psi_k(I)\\ =& (1+\eta)(4\psi_n(I) + 2(\psi_n(I))^*) + \sum_{i+j+k=n, 0<i,j,k<n} (\psi_i(I)\psi_j(I)\psi_k(I)\\ &+ \eta\psi_j(I)\psi_i(I)\psi_k(I) + \psi_k(I)(\psi_j(I))^*(\psi_i(I))^* + \eta\psi_k(I)(\psi_i(I))^*(\psi_j(I))^*)\\ =& (1+\eta)(4\psi_n(I) + 2(\psi_n(I))^*) + \sum_{i+j+k=n, 0<i,j,k<n} (\psi_i(I)\psi_j(I)\psi_k(I)\\ &+ \eta\psi_j(I)\psi_i(I)\psi_k(I) + \psi_k(I)\psi_j(I)\psi_i(I) + \eta\psi_k(I)\psi_i(I)\psi_j(I)).\end{aligned}$$

On the other hand,

$$\begin{aligned}(1+\eta)(\psi_n(I))^* =&(1+\eta)(2\psi_n(I)+4(\psi_n(I))^*)\\ &+\sum_{i+j+k=n,0<i,j,k<n}(-\psi_k(I)\psi_j(I)\psi_i(I)\\ &-\eta\psi_k(I)\psi_i(I)\psi_j(I)-(\psi_i(I))^*(\psi_j(I))^*\psi_k(I)-\eta(\psi_j(I))^*(\psi_i(I))^*\psi_k(I))\end{aligned}$$

Thus,

$$(1+\eta)(\psi_n(I)^*+\psi_n(I))=(1+\eta)(6\psi_n(I)^*+6\psi_n(I)).$$

Hence, $5(1+\eta)(\psi_n(I)^*+\psi_n(I))=0$. Since η is not equal to -1, it follows that $\psi_n(I)^*=-\psi_n(I)$. □

Claim 3 With notations as above, we have

$$\psi_n(I)=\psi_n(\mathrm{i}I)=0$$

for any $\eta\neq(-1),\eta\in\mathbb{R}$.

Proof. By Claim 1 and the fact that $\psi_n(I)^*=-\psi_n(I)$, we have

$$\psi_n(\mathrm{i}I)^*=-\mathrm{i}\psi_n(I)^*=\mathrm{i}\psi_n(I)=\psi_n(\mathrm{i}I).$$

We also have

$$\begin{aligned}\psi_n(\mathrm{i}I\circ_\eta I\bullet I)=&\psi_n(\mathrm{i}I)\circ_\eta I\bullet I+\mathrm{i}I\circ_\eta\psi_n(I)\bullet I\\ &+\mathrm{i}I\circ_\eta I\bullet\psi_n(I)+\sum_{i+j+k=n,0<i,j,k<n}\psi_i(\mathrm{i}I)\circ_\eta\psi_j(I)\bullet\psi_k(I)\\ =&(1+\eta)\psi_n(\mathrm{i}I)\bullet I+\mathrm{i}(1+\eta)\psi_n(\mathrm{i}I)\bullet I+(1+\eta)\mathrm{i}I\bullet\psi_n(I)\\ =&4\mathrm{i}(1+\eta)\psi_n(I).\end{aligned}$$

On the other hand,

$$\mathrm{i}I\circ_\eta I\bullet I=0.$$

Since $\eta\neq-1$, hence, $\psi_n(I)=0$ and consequently $\psi_n(\mathrm{i}I)=0$. □

Claim 4 With notations as above, we have

$$\psi_n(\eta A)=\eta\psi_n(A)$$

for any $A\in\mathcal{A}$.

Proof. It follows from Claim 3 that

$$\psi_n(I\circ_\eta I\bullet A)=I\circ_\eta I\bullet\psi_n(A).$$

This yields $\psi_n(\eta A)=\eta\psi_n(A)$. □

Claim 5 With notations as above, we have

$$\psi_n(A^*)=\psi_n(A)^*$$

for all $A\in\mathcal{A}$.

Proof. Observe that

$$\psi_n(I\circ_\eta A\bullet I)=I\circ_\eta\psi_n(A)\bullet I.$$

This gives

$$(1+\eta)\psi_n(A+A^*)=(1+\eta)\psi_n(A)+(1+\eta)\psi_n(A)^*.$$

Therefore, we have $\psi_n(A^*)=\psi_n(A)^*$ for all $A\in\mathcal{A}$. □

Claim 6 With notations as above, we have the mapping ψ_n satisfies the relation

$$\psi_n(AB) = \sum_{i+j=n,0<i,j<n} \psi_i(A)\psi_j(B)$$

for all $A, B \in \mathcal{A}$.

Proof. For any $A, B \in \mathcal{A}$, we see that

$$\begin{aligned}(1+\eta)\psi_n(AB + BA^*) =& \psi_n(I \circ_\eta A \bullet B)\\ =& I \circ_\eta \psi_n(A) \bullet B + I \circ_\eta A \bullet \psi_n(B) + \sum_{i+j+k=n,0<i,j,k<n} \psi_i(I) \circ_\eta \psi_j(A) \bullet \psi_k(B)\\ =& (1+\eta)(\psi_n(A)B + B\psi_n(A)^*) + (1+\eta)(A\psi_n(B) + \psi_n(B)A^*)\\ &+ \sum_{i+j=n,0<i,j<n} I \circ_\eta \psi_i(A) \bullet \psi_j(B).\end{aligned} \tag{4.8}$$

Setting A (resp. B) as $-\mathrm{i}A$ (resp. $-\mathrm{i}B$) in the above relation gives

$$\begin{aligned}(1+\eta)\psi_n(AB - BA^*) =& (1+\eta)(\psi_n(A)B - B\psi_n(A)^*) + (1+\eta)(A\psi_n(B) - \psi_n(B)A^*)\\ &+ \sum_{i+j=n,0<i,j<n} I \circ_\eta \psi_i(A) \bullet \psi_j(-\mathrm{i}B)\end{aligned} \tag{4.9}$$

for all $A, B \in \mathcal{A}$. Addition of equations (4.8) and (4.9) yields

$$\psi_n(AB) = \psi_n(A)B + A\psi_n(B) + \sum_{i+j=n,0<i,j<n} \psi_i(A)\psi_j(B)$$

for all $A, B \in \mathcal{A}$. Therefore, it follows from the above proof that Theorem 4.1 holds. □

With the aid of Theorems 4.1, we immediately have the following corollary, which follows from [10, Theorem 3.1].

Corollary 4.3. *[10, Theorem 3.1] Let $\mathcal{A}$ be an unital prime $*$-algebra with nontrivial idempotents and $\eta \neq -1, \eta \in \mathbb{R}$. If the non-additive mixed skew and η-Jordan triple derivations $\psi_1 : \mathcal{A} \to \mathcal{A}$ satisfies condition $\psi_1(I)^* = -\psi_1(I)$, then the non-additive mixed skew and η-Jordan triple derivations $\psi_1 : \mathcal{A} \to \mathcal{A}$ are additive $*$-derivations and satisfies $\psi_1(\eta A) = \eta\psi_1(A)$ for all $A \in \mathcal{A}$.*

5. Applications

With the help of [15], we know that operator algebras with analytical background such as factor von Neumann algebras and standard operator algebra all satisfy the primality condition of prime $*$-algebra , so we can obtain the following series of conclusions.

An algebra $\mathfrak{B}(\mathcal{S})$ composed of all bounded operators is defined on a complex Hilbert space $\mathcal{S}$. If the center $Z(\mathbb{A})$ of a von Neumann algebra $\mathbb{A} \subseteq \mathfrak{B}(\mathcal{S})$ satisfies condition $Z(\mathbb{A}) = \mathbb{C}I$, then $\mathbb{A}$ is called a factor von Neumann algebra. It is clear that the algebra $\mathbb{A}$ is a prime algebra, and the following corollary holds with the help of Corollary 4.3.

Corollary 5.1. *Let $\mathbb{A}$ be a factor von Neumann algebra acting on complex Hilbert space with dim$(\mathbb{A}) \geq 2$. Then every the non-additive mixed skew and η-Jordan triple derivations $\psi_1 : \mathcal{A} \to \mathcal{A}$ satisfies condition $\psi_1(I)^* = -\psi_1(I)$, is an additive $*$-derivation.*

According to Theorem 4.1, the following corollary can be obtained by combining the above Corollary 5.1, 0¡i,j¡n.

Corollary 5.2. *Let $\mathbb{A}$ be an factor von Neumann algebra acting on complex Hilbert space with dim*$(\mathbb{A}) \geq 2$ *and* $\eta \neq -1, \eta \in \mathbb{R}$. *If the non-additive mixed skew and η-Jordan triple higher derivations* $\Psi = \{\psi_n\}_{n\in\mathbb{N}}$ *satisfies condition* $\psi_1(I)^* = -\psi_1(I)$, *then the non-additive mixed skew and η-Jordan triple higher derivations* $\Psi = \{\psi_n\}_{n\in\mathbb{N}}$ *are additive* $*$*-higher derivations and satisfies* $\psi_n(\eta A) = \eta\psi_n(A)$ *for all* $A \in \mathcal{A}$.

We use the symbol $\mathfrak{G}(\mathcal{S}) \subseteq \mathfrak{B}(\mathcal{S})$ to denote the subalgebras of the algebra $\mathfrak{B}(\mathcal{S})$ formed by all finite rank operators. If a subalgebra $\mathfrak{R}$ contains $\mathfrak{G}(\mathcal{S})$, it is called a standard operator algebra.

Corollary 5.3. *Let* $\mathfrak{S}$ *be an infinite dimensional complex Hilbert space and* $\mathfrak{R}$ *be a standard operator algebra on* $\mathfrak{S}$ *containing the identity operator I. Then every the non-additive mixed skew and η-Jordan triple derivations* $\psi_1 : \mathcal{A} \to \mathcal{A}$ *satisfies condition* $\psi_1(I)^* = -\psi_1(I)$, *is an additive* $*$*-derivation.*

With the help of Theorem 4.1, the following inference can be obtained by combining the above Corollary 5.3.

Corollary 5.4. *Let* $\mathfrak{S}$ *be an infinite dimensional complex Hilbert space and* $\mathfrak{R}$ *be a standard operator algebra on* $\mathfrak{S}$ *containing the identity operator I and* $\eta \neq -1, \eta \in \mathbb{R}$. *If the non-additive mixed skew and η-Jordan triple higher derivations* $\Psi = \{\psi_n\}_{n\in\mathbb{N}}$ *satisfies condition* $\psi_1(I)^* = -\psi_1(I)$, *then the non-additive mixed skew and η-Jordan triple higher derivations* $\Psi = \{\psi_n\}_{n\in\mathbb{N}}$ *are additive* $*$*-higher derivations and satisfies* $\psi_n(\eta A) = \eta\psi_n(A)$ *for all* $A \in \mathcal{A}$.

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