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Multiset rough approximations and their generated topologies

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Abstract. In this paper, we introduce a generalized framework for multiset neighborhoods extending the previously defined structures into eight distinct types. Based on these generalized neighborhoods, we define corresponding multiset approximations and construct eight new topologies derived from them. Additionally, we extend six classical types of rough set approximations to the multiset context and investigate their properties in detail. These developments offer a unified and expanded perspective on approximation theory within multiset environments.

Table 1: List of Symbols

Symbol	Meaning
$\eta_j(w/x)$	Mset j-neighbourhood of elemen $x \in^w \delta$
$\underline{\Gamma}^{\delta}_{\eta_j}(\chi)$	Mset lower approximation of χ
$\overline{\Gamma}^{\delta}_{\eta_j}(\chi)$	Mset upper approximation of χ
$\mu^{\delta}_{\eta_j}(\chi)$	Mset accuracy measure of χ
$\Upsilon^{\delta}_{\eta_j}$	Mset topology on δ
$\underline{\Upsilon}^{\delta}_{\eta_j}(\chi)$	Mset Lower approximation of χ generated by mset topology $\Upsilon^{\delta}_{\eta_j}$.
$\overline{\Upsilon}^{\delta}_{\eta_j}(\chi)$	Mset upper approximation of χ generated by mset topology $\Upsilon^{\delta}_{\eta_j}$.
$\mu\Upsilon^{\delta}_{\eta_j}(\chi)$	Mset accuracy measure generated by mset topology $\Upsilon^{\delta}_{\eta_j}$.
${}^{-i}\mathfrak{R}^{\delta}_{\eta_j}(\chi)$	Mset reflexive lower approximation type i of χ (i = 1, 2 or 3).
${}^{+i}\mathfrak{R}^{\delta}_{\eta_j}(\chi)$	Mset reflexive upper approximation type i 0f χ (i = 1, 2 or 3).
${}^{-\alpha}\mathfrak{R}^{\delta}_{\eta_j}(\chi)$	Mset similarity lower approximation type α of χ (α = 4, 5 or 6).
${}^{+\alpha}\mathfrak{R}^{\delta}_{\eta_j}(\chi)$	Mset similarity upper approximation type α of χ (α = 4, 5 or 6).

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1. Introduction and preliminaries

The concept of rough set theory -introduced by Pawlak in 1982 [18]- provides a mathematical framework for dealing with uncertainty and vagueness in data through lower and upper approximations based on equivalence relations. Over time, this framework has been significantly expanded by considering reflexive, similarity and arbitrary binary relations and various neighborhood-based approximations, thereby broadening its scope and applicability. In 2014, Abd El-Monsef et al. [1] proposed a comprehensive strategy for generating eight topologies using neighborhood systems, independent of base or subbase definitions. This framework has been extended by other researchers for example: in 2021, Al-Shami [6] defined eight types of containment neighborhoods. By them, he had constructed new rough set models. After that, in 2023, H. I. Mustafa et al [16] used the system of containment neighborhoods to present new rough set models generated by topology and ideals.

More recently, innovative methods have been developed for generating rough neighborhoods for accurate decision-making in medical applications, such as COVID-19 variant diagnosis [10], and in the study of primal approximation spaces [9]. In 2025, H. I. Mustafa and R. Wassef. [17] integrated soft set theory with neighborhood systems and introduced eight soft neighborhood types built from parameterized relations.

In 2011 Abo-Tabl [2] introduced three types of approximations based on reflexive relation. In 2017, Jianhua Dai et al [11] defined another three types of approximations based on similarity relation.

Parallel to these developments, S.P. Jena et al [15] introduced the concept of multiset as a powerful tool for modeling collections where element repetition is significant. Integrating multiset structures with rough approximation concepts has led to new perspectives in handling multiplicity within information systems. In 2011, Girish and John defined the multiset right neighborhood. In 2013, Abo-Tabl defined the minimal right multiset neighborhood and construct rough set approximations in multisets based on this neighborhood.

The aim of this paper is to define eight generalized multiset neighborhoods, formulate multiset approximations based on these neighborhoods, construct eight multiset topologies derived from the approximations and extend six classical rough set approximation types -That where studied in [8]- into the multiset domain using reflexive and similarity relations.

This paper is organized as follows: In Section 2 we define the multiset eight neighborhoods which are a generalization of the neighborhoods defined in [3], in Section 3 we define multiset approximations based on these neighborhoods, in Section 4 we generate new eight multiset topologies based on the multiset approximations as well as construct new multiset approximations from the interior and closure operators of these topologies, in Section 5 and 6 we define multiset reflexive and multiset similarity approximations respectively. We study the properties of the concepts we define and give an illustrative examples for them.

Now we recall the basic concepts needed for our study.

Definition 1.1. ([7]) A subset $\mathfrak{R} \subseteq \mho \times \mho$ is called a binary relation, it is said to be reflexive if $(v, v) \in \mathfrak{R} \quad \forall\, v \in \mho$, symmetric if $(u, v) \in \mathfrak{R}$ whenever $(v, u) \in \mathfrak{R}$, transitive if $(v, w) \in \mathfrak{R}$ whenever $(v, u) \in \mathfrak{R}$ and $(u, w) \in \mathfrak{R}$, equivalence if $\mathfrak{R}$ is reflexive, symmetric and transitive and serial if $(v, w) \in \mathfrak{R}$ for each $v, w \in \mho$.

At first, Pawlak associated each set with two crisp sets called lower and upper approximations. These approximations defined with respect to the equivalent classes as follows.

Definition 1.2. ([18]) If $\mathfrak{R}$ is an equivalence relation on a universe $\mho$ and $[x]_R$ is the equivalence class containing $x \in \mho$. The lower approximation $\underline{\Gamma}(\chi)$ and upper approximation $\overline{\Gamma}(\chi)$ of a set χ of $\mho$ are given by

i. $\underline{\Gamma}(\chi) = \{x \in \mho : [x]_{\mathfrak{R}} \subseteq \chi\}$.

ii. $\overline{\Gamma}(\chi) = \{x \in \mho : [x]_{\mathfrak{R}} \cap \chi \neq \phi\}$.

Proposition 1.3. *([18]) Let* $A, B \subseteq \mho$*, then:*

(L1) $\underline{\Gamma}(A) = [\overline{F}(A^c)]^c$.

(L2) $\underline{\Gamma}(\mho) = \mho$.

(L3) $\underline{\Gamma}(A \cap B) = \underline{\Gamma}(A) \cap \underline{\Gamma}(B)$.

(L4) $\underline{\Gamma}(A) \cup \underline{\Gamma}(B) \subseteq \underline{\Gamma}(A \cup B)$.

(L5) *If* $A \subseteq B$, *then* $\underline{\Gamma}(A) \subseteq \underline{\Gamma}(B)$.

(L6) $\underline{\Gamma}(\phi) = \phi$.

(L7) $\underline{\Gamma}(A) \subseteq A$.

(L8) $A \subseteq \underline{\Gamma}(\overline{\Gamma}(A))$.

(L9) $\underline{\Gamma}(A) = \underline{\Gamma}(\underline{\Gamma}(A))$.

(L10) $\overline{\Gamma}(A) = \underline{\Gamma}(\overline{\Gamma}(A))$.

(U1) $\overline{\Gamma}(A) = [\underline{\Gamma}(A^c)]^c$.

(U2) $\overline{\Gamma}(\phi) = \phi$.

(U3) $\overline{\Gamma}(A \cap B) \subseteq \overline{\Gamma}(A) \cap \overline{\Gamma}(B)$.

(U4) $\overline{\Gamma}(A \cup B) = \overline{\Gamma}(A) \cup \overline{\Gamma}(B)$.

(U5) *If* $A \subseteq B$, *then* $\overline{\Gamma}(A) \subseteq \overline{\Gamma}(B)$.

(U6) $\overline{\Gamma}(\mho) = \mho$.

(U7) $A \subseteq \overline{\Gamma}(A)$.

(U8) $A \supseteq \overline{\Gamma}(\underline{\Gamma}(A))$.

(U9) $\overline{\Gamma}(A) = \overline{\Gamma}(\overline{\Gamma}(A))$.

(U10) $\underline{\Gamma}(A) = \overline{\Gamma}(\underline{\Gamma}(A))$.

(K) $\underline{\Gamma}[A^c \cup B] \subseteq [\underline{\Gamma}(A)]^c \cup \underline{\Gamma}(B)$.

(LU) $\underline{\Gamma}(A) \subseteq \overline{\Gamma}(B)$.

Then, the equivalence relation has been replaced by specific (or arbitrary) binary relation to expand the scope of applications of rough set theory. This leads to deal with the so-called neighborhoods of an element instead of its equivalence classes. As a result, various sorts of rough-set paradigms have been introduced in the published literature. In what follows, we recall a set of these neighborhoods and paradigms that we need to show the importance and robustness of this work.

Definition 1.4. ([5, 19**?**]) Let $\Re$ be an arbitrary binary relation on a universe set $\mho$, then the j-neighborhoods of an element $x \in \mho$ are defined as follows:
$j \in \{r, l, i, u, \langle r\rangle, \langle l\rangle, \langle i\rangle, \langle u\rangle\}$.

i. $\eta_r(x) = \{y \in U : {}_xR_y\}$

ii. $\eta_l(x) = \{y \in U : {}_yR_x\}$

iii. $\eta_i(x) = \eta_r(x) \cap \eta_l(x)$

iv. $\eta_u(x) = \eta_r(x) \cup \eta_l(x)$

v. $\eta_{\langle r\rangle}(x) = \begin{cases} \bigcap\limits_{x\in\eta_r(y)} \eta_r(y) & \text{if there exists } x \in \eta_r(y) \\ \phi & \text{Otherwise} \end{cases}$

vi. $\eta_{\langle l\rangle}(x) = \begin{cases} \bigcap\limits_{x\in\eta_l(y)} \eta_l(y) & \text{if there exists } x \in \eta_l(y) \\ \phi & \text{Otherwise} \end{cases}$

vii. $\eta_{\langle i\rangle}(x) = \eta_{\langle r\rangle}(x) \cap \eta_{\langle l\rangle}(x)$

viii. $\eta_{\langle u\rangle}(x) = \eta_{\langle r\rangle}(x) \cup \eta_{\langle l\rangle}(x)$

Definition 1.5. ([1]) Let $\mathfrak{R}$ be an arbitrary binary relation on $\mho$ and $\psi_j : \mho \longrightarrow P(\mho)$ be a mapping which assigns for each z in $\mho$ its j-neighborhood in $P(\mho)$. Then $(\mho, \mathfrak{R}, \psi_j)$ is called a j-neighborhood space (j-NS).

Definition 1.6. ([5, 19**?**]) Let $(\mho, \mathfrak{R}, \psi_j)$ be a η_jS, then the η_j-lower approximation $\Gamma_{\eta_j}(\chi)$, η_j-upper approximation $\Gamma^{\eta_j}(\chi)$ and η_j-accuracy measure $\mu_{\eta_j}(\chi)$ of a set $\chi \subseteq \mho$ are defined as follows:

i. $\Gamma_{\eta_j}(\chi) = \{z \in \mho : \eta_j(z) \subseteq \chi\}$

ii. $\Gamma^{\eta_j}(\chi) = \{z \in \mho : \eta_j(z) \cap \chi \neq \phi\}$

iii. $\mu_{\eta_j}(\chi) = \frac{|\Gamma_{\eta_j}(z)\cap\chi|}{|\Gamma^{\eta_j}(X)\cup\chi|}$

Where $|\Gamma^{\eta_j}(z) \cup \chi| \neq 0$.

Definition 1.7. ([8]) Let $\mathfrak{R}$ be a reflexive relation on $\mho$ and $\Psi_j : \mho \longrightarrow P(\mho)$ be a mapping which assigns for each z in $\mho$ its j-neighborhood in $P(\mho)$. Then $(\mho, \mathfrak{R}, \Psi_j)$ is called a reflexive j-neighborhood space (j-RNS).

Definition 1.8. ([8]) Let $(\mho, \mathfrak{R}, \Psi_j)$ be a j-RNS, then:

$$\eta_j^{\cap}(u) = \begin{cases} \bigcap\limits_{u\in\eta_j(v)} \eta_j(v), & \text{If there exists } v \text{ such that } u \in \eta_j(v) \\ \phi & \text{Otherwise} \end{cases}.$$

Definition 1.9. ([8]) Let $(\mho, \mathfrak{R}, \Psi_j)$ be a j-RNS, a reflexive 1 j-neighborhood approximations of $\chi \subseteq \mho$ are defined by:
$\mathfrak{R}^{-1}_{\eta_j}(\chi) = \{u \in \mho : \eta_j^{\cap}(u) \subseteq \chi\}$.
(Called the reflexive 1-lower approximation of χ).
$\mathfrak{R}^{+1}_{\eta_j}(\chi) = \{u \in \mho : \eta_j^{\cap}(u) \cap \chi \neq \phi\}$.
(Called the reflexive 1-upper approximation of χ).

Definition 1.10. ([8]) Let $(\mho, \mathfrak{R}, \Psi_j)$ be a j-RNS, a reflexive 2 j-neighborhood approximations of $\chi \subseteq \mho$ are defined by:
$\mathfrak{R}^{-2}_{\eta_j}(\chi) = \cup\{\eta_j^{\cap}(u) : \eta_j^{\cap}(u) \subseteq \chi\}$.
(Called the reflexive 2-lower approximation of χ).
$\mathfrak{R}^{+2}_{\eta_j}(\chi) = [\mathfrak{R}^{-2}_{\eta_j}(\chi^c)]^c$.
(Called the reflexive 2-upper approximation of χ).

Definition 1.11. ([8]) Let $(\mho, \mathfrak{R}, \Psi_j)$ be a j-RNS, a reflexive 3 j-neighborhood approximations of $\chi \subseteq \mho$ are defined by:
$\mathfrak{R}^{-3}_{\eta_j}(\chi) = [\mathfrak{R}^{+3}_{\eta_j}(\chi^c)]^c$.
(Called the reflexive 3-lower approximation of χ).
$\mathfrak{R}^{+3}_{\eta_j}(\chi) = \cup\{\eta_j^{\cap}(u) : \eta_j^{\cap}(u) \cap \chi \neq \phi\}$.
(Called the reflexive 3-upper approximation of χ).

Definition 1.12. [8] Let $\mathfrak{R}$ be a similarity relation on $\mho$ and $\Psi_j : \mho \longrightarrow P(\mho)$ be a mapping which assigns for each z in $\mho$ its j-neighborhood in $P(\mho)$. Then $(\mho, \mathfrak{R}, \Psi_j)$ is called a similarity j-neighborhood space (j-SNS).

Definition 1.13. ([8]) Let $(\mho, \mathfrak{R}, \Psi_j)$ be a j-SNS, then:

$$\eta_j^{\cup}(u) = \begin{cases} \bigcup\limits_{u\in\eta_j(v)} \eta_j(v), & \text{If there exists } v \text{ such that } u \in \eta_j(v) \\ \phi & \text{Otherwise} \end{cases}.$$

Definition 1.14. ([8]) Let $(\mho, \mathfrak{R}, \Psi_j)$ be a j-SNS, then the similarity 4 j-neighborhood approximations of $\chi \subseteq \mho$ are defined by:
$\mathfrak{R}^{-4}_{\eta_j}(\chi) = \{u \in \mho : \eta_j^{\cup}(u) \subseteq \chi\}$.
(Called the similarity 4-lower approximation of χ).
$\mathfrak{R}^{+4}_{\eta_j}(\chi) = \{u \in \mho : \eta_j^{\cup}(u) \cap \chi \neq \phi\}$.
(Called the similarity 4-upper approximation of χ).

Definition 1.15. ([8]) Let $(\mho, \mathfrak{R}, \Psi_j)$ be a j-SNS, then the similarity 5 j-neighborhood approximations of $\chi \subseteq \mho$ are defined by:
$\mathfrak{R}^{-5}_{\eta_j}(\chi) = \cup\{\eta_j^{\cup}(u) : \eta_j^{\cup}(u) \subseteq \chi\}$.
(Called the similarity 5-lower approximation of χ).
$\mathfrak{R}^{+5}_{\eta_j}(\chi) = [\mathfrak{R}^{-5}_{\eta_j}(\chi^c)]^c$.
(Called the similarity 5-upper approximation of χ).

Definition 1.16. ([8]) Let $(\mho, \mathfrak{R}, \Psi_j)$ be a j-SNS, then the similarity 6 j-neighborhood approximations of $\chi \subseteq \mho$ are defined by:
$\mathfrak{R}^{-6}_{\eta_j}(\chi) = [\mathfrak{R}^{+6}_{\eta_j}(\chi^c)]^c$.
(Called the similarity 6-lower approximation of χ).
$\mathfrak{R}^{+6}_{\eta_j}(\chi) = \cup\{\eta_j^{\cup}(u) : \eta_j^{\cup}(u) \cap \chi \neq \phi\}$.
(Called the similarity 6-upper approximation of χ).

Definition 1.17. ([13, 14]) [Multiset] An mset δ drawn from the set Δ is represented by a function count C_δ defined as $C_\delta : \Delta \longrightarrow N$ where N is the set of non-negative integers. Let δ be an mset drawn from the set $\Delta = \{x_1, x_2, ..., x_\omega\}$ with x appearing n times in δ. It is denoted by $x \in^n \delta$. The mset δ drawn from the set Δ is denoted by $\delta = \{k_1/x_1, k_2/x_2, ..., k_n/x_n\}$ where δ is an mset with x_1 appearing k_1 times, x_2 appearing k_2 times and so on. $C_\delta(x)$ is the number of occurrences of the element x in the mset δ and the elements which are not included in the mset δ have zero count.

Let δ be an mset drawn from a set Δ, the support set(also called root set) of δ denoted by δ^* is a subset of Δ and $\delta^* = \{x \in \Delta : C_\delta(x) > 0\}$. i.e. δ^* is an ordinary set. An mset δ is said to be an empty mset if for all $x \in \Delta$, $C_\delta(x) = 0$. The cardinality of an mset δ drawn from a set Δ is denoted by Card(δ) and is given by $\mathrm{Card}(\delta) = \sum\limits_{x\in\Delta} C_\delta(x)$.

Definition 1.18. ([13, 14]) Let δ and ξ be two msets drawn from a set Δ, then:

i. $\delta = \xi$ if $C_\delta(x) = C_\xi(x)$ for all $x \in \Delta$.

ii. $\xi \subseteq \delta$ if $C_\xi(x) \leq C_\delta(x)$ for all $x \in \Delta$.(submset)

iii. $\beta = \delta \cup \xi$ if $C_\beta(x) = \mathrm{Max}\{C_\delta(x), C_\xi(x)\}$ for all $x \in \Delta$.

iv. $\beta = \delta \cap \xi$ if $C_\beta(x) = \mathrm{Min}\{C_\delta(x), C_\xi(x)\}$ for all $x \in \Delta$.

v. $\beta = \delta \oplus \xi$ if $C_\beta(x) = C_\delta(x) + C_\xi(x)$ for all $x \in \Delta$.

vi. $\beta = \delta \ominus \xi$ if $C_\beta(x) = Max\{C_\delta(x) - C_\xi(x), 0\}$ for all $x \in \Delta$.

Where $\oplus$ and $\ominus$ denotes the mset addition and mset subtraction respectively.

Definition 1.19. ([13, 14]) A domain Δ is defined as a set of elements from which msets are constructed. The mset space $[\Delta]^\omega$ is the set of all msets whose elements are in Δ such that no element in the mset occurs more than ω times. The set $[\Delta]^\infty$ is the set of all msets whose elements are in Δ such that there is no limit on the number of occurrences of an element in the mset.

Remark 1.20. ([13, 14]) Using Definition 1.19, the mset sum can be modified as follows:
$C_{\delta_1+\delta_2} = Min\{\omega, C_{\delta_1}(x) + C_{\delta_2}(x)\}$ for all $x \in \Delta$.

Remark 1.21. ([13, 14]) Let δ be an mset drawn from Δ with x appearing n times in δ. $[\delta]_x$ denotes that the element x belongs to the mset δ and $|[\delta]_x|$ denotes the cardinality of the element x in δ.

Definition 1.22. [Power mset] ([13, 14]) Let $\delta \in [\Delta]^\omega$ be an mset, the power mset of δ, denoted by $P(\delta)$ is the set of all submsets of δ. We have $\xi \in P(\delta)$ if and only if $\xi \subseteq \delta$. $\xi \in^k P(\delta)$ where $k = \prod_x \begin{pmatrix} |[\delta]_x| \\ |[\xi]_x| \end{pmatrix}$, the product $\prod_z$ is taken over by distinct elements of z of the mset N, $|[M]_z| = m$ iff $x \in^\omega \delta$ and $|[\xi]_x| = n$ iff $x \in^n \xi$, then

$$\begin{pmatrix} |[\delta]_x| \\ |[\xi]_x| \end{pmatrix} = \begin{pmatrix} \omega \\ n \end{pmatrix} = \frac{\omega!}{n!(\omega - n)!}$$

The power set of an mset is the support set of the power mset and is denoted by $P^*(\delta)$, where $Card(P^*(\delta)) = \prod_{i=1}^{n}(1 + \omega_i)$.

Remark 1.23. The power mset is an mset but its support set is an ordinary set whose elements are msets.

Definition 1.24. ([13, 14]) The maximum mset is denoted by H where $C_H(x) = Max\{C_\delta(x) : x \in^n \delta, \delta \in [\Delta]^\omega, n \leq \omega\}$.

Definition 1.25. ([13, 14]) Let $[\Delta]^\omega$ be an mset space and $\{\delta_1, \delta_2, ...\}$ be a collection of msets drawn from $[\Delta]^\omega$, then the following operations are possible under an arbitrary colection of msets.

i. The union
$$\bigcup_{i\in I} \delta_i = \{C_{\cup\delta_i}(x)/x : C_{\cup\delta_i}(x) = Max\{C_{\delta_i}(x) : x \in \Delta\}\}$$

ii. The intersection
$$\bigcap_{i\in I} \delta_i = \{C_{\cap\delta_i}(x)/x : C_{\cap\delta_i}(x) = Min\{C_{\delta_i}(x) : x \in \Delta\}\}$$

iii. The mset addition
$$\bigoplus_{i\in I} \delta_i = \{C_{\oplus\delta_i}(x)/x : C_{\oplus\delta_i}(x) = \sum_{i\in I} C_{\delta_i}(x) : x \in \Delta\}$$

iv. The mset complement
$$\delta^c = H \ominus \delta = \{C_{\delta^c}(x)/x : C_{\delta^c}(x) = C_\kappa(x) - C_\delta(x), x \in \Delta\}$$

Definition 1.26. ([13, 14]) Let δ_1 and δ_2 be two msets drawn from a set Δ, then the cartesian product of δ_1 and δ_2 is defined as $\delta_1 \times \delta_2 = \{(m/x, n/y)/mn : x \in^m \delta_1, y \in^n \delta_2\}$. Where x is repeated m times in δ_1, y is repeated n times in δ_2 and the pair $(m/x, n/y)$ is repeated mn times in $\delta_1 \times \delta_2$.

Definition 1.27. [Mset Relation] ([13, 14]) A submset $\mathfrak{R}$ of $\delta \times \delta$ is said to be an mset relation on δ if every member $(m/x, n/y)$ of $\mathfrak{R}$ has a count equals the product of m and n. (m/x) related to (n/y) is denoted by $(m/x)\mathfrak{R}(n/y)$.

Definition 1.28. ([13, 14]) Let $\delta \in [\Delta]^{\omega}$ and $\mathfrak{R}$ is a relation on δ then

i. $\mathfrak{R}$ is reflexive if $(m/x)\mathfrak{R}(m/x)$ for all m/x in δ, irreflexive if $(m/x)\mathfrak{R}(m/x)$ never holds for any m/x in δ.

ii. $\mathfrak{R}$ is symmetric if $(m/x)\mathfrak{R}(n/y)$ implies $(n/y)\mathfrak{R}(m/x)$, antisymmetric if $(m/x)\mathfrak{R}(n/y)$ and $(n/y)\mathfrak{R}(m/x)$ imply m/x and n/y are equal.

iii. $\mathfrak{R}$ is transitive if $(m/x)\mathfrak{R}(n/y)$ and $(n/y)\mathfrak{R}(k/z)$ imply $(m/x)\mathfrak{R}(k/z)$.

iv. $\mathfrak{R}$ is equivalence relation it is reflexive, symmetric and transitive.

Definition 1.29. [Multiset Topology] ([13, 14]) Let $\delta \in [\Delta]^{\omega}$ and $\Upsilon \subseteq P^{*}(\delta)$, then Υ is called a multiset topology if Υ satisfies the following properties.

i. Φ and δ are in Υ.

ii. The union of elements of any sub collection of Υ is in Υ.

iii. The intersection of any finite sub collection of Υ is in Υ.

A multiset topological space is the ordered pair (δ, Υ). Note that Υ is an ordinary set whose elements are msets and the multiset topology is abbreviated as M-topology.

Definition 1.30. ([13, 14]) Let (δ, Υ) be an M-topological space and $\xi \subseteq \delta$, then the interior of ξ is defined as the mset union of all open msets contained in ξ and is denoted by $\mathrm{Int}(\xi)$.
i.e. $\mathrm{Int}(\xi) = \cup\{G \subseteq \delta : G \text{ is an open mset and } G \subseteq \xi\}$ and $C_{\mathrm{Int}(\xi)}(x) = \mathrm{Max}\{C_G(\chi) : G \subseteq \xi\}$.

Definition 1.31. ([13, 14]) Let (δ, Υ) be an M-topological space and $\xi \subseteq \delta$, then the closure of ξ is defined as the mset intersection of all closed msets containing ξ and is denoted by $\mathrm{Cl}(\xi)$.
i.e. $\mathrm{Cl}(\xi) = \cap\{K \subseteq \delta : K \text{ is a closed mset and } \xi \subseteq K\}$ and $C_{\mathrm{Cl}(\xi)}(\chi) = \mathrm{Min}\{C_K(x) : \xi \subseteq K\}$.

Definition 1.32. ([12]) Suppose that $\mathfrak{R}$ is an arbitrary mset relation on δ. With respect to $\mathfrak{R}$, we can define a neighborhood of an element m/x in M as follows:
$(m/x)r = \{n/y : y \in^{n} M, (m/x)\mathfrak{R}(n/y)\}$.
The mset $(m/x)r$ is also called an $\mathfrak{R}$-relative mset.

Definition 1.33. ([12]) Let $\mathfrak{R}$ be an arbitrary mset relation on the universal set δ and let $(m/x)r$ be the $\mathfrak{R}$-relative mset (i.e. neighborhood of (m/x) in δ). For any $\xi \subseteq \delta$, a pair of lower and upper mset approximations:
$\mathfrak{R}_L(\xi) = \{m/x : (m/x)r \subseteq \xi\}$.
$\mathfrak{R}_U(\xi) = \{m/x : (m/x)r \cap \xi \neq \Phi\}$.

Definition 1.34. ([13]) Let $\mathfrak{R}$ be any binary mset relation on δ in $[\Delta]^{\omega}$, then the mset $\langle n/y\rangle r$ is defined as:

$$\langle n/y\rangle r = \begin{cases} \bigcap\limits_{y \in^{n} (m/x)r} (m/x)r & \text{if there exists } m/x \in \delta : y \in^{n} (m/x)r \\ \phi & \text{Otherwise} \end{cases}$$

Definition 1.35. ([3]) Let $\mathfrak{R}$ be any binary mset relation on a non-empty multiset δ. For any subset $\xi \subseteq \delta$, the lower and upper approximations of ξ according to $\mathfrak{R}$ are defined as:
$\underline{\mathfrak{R}}(A) = \{m/x : \langle m/x\rangle r \subseteq \xi\}$.
$\overline{\mathfrak{R}}(A) = \{m/x : \langle m/x\rangle r \cap \xi \neq \phi\}$.
An accuracy measure of the mset $\xi \subseteq \delta$ according to any mset relation $\mathfrak{R}$ is defined as:
$\mu_{\mathfrak{R}}(\xi) = \frac{|\underline{\mathfrak{R}}(\xi)|}{|\overline{\mathfrak{R}}(\xi)|}$.
$0 \leq \mu_{\mathfrak{R}}(A) \leq 1$ and if ξ is definable in δ then $\mu_{\mathfrak{R}}(\xi) = 1$.

2. Multiset neighborhoods

In this section, we generalize the two multiset neighborhoods of definitions 1.32 and 1.34 -which were studied in [12] and [3] respectively- to be eight types, we study the relations between these neighborhood as well as give an illustrative example.

Definition 2.1. [Mset Neighborhood] Let $\delta \in [\Delta]^m$ be an mset and $\mathfrak{R}$ be an mset relation on δ, then the mset j-neighborhoods of an element $x \in {}^w\delta$ denoted by $\eta_j(x)$ are defined as:
$j \in \{r, l, i, u, \langle r\rangle, \langle l\rangle, \langle i\rangle, \langle u\rangle\}$.

i. $\eta_r(w/x) = \{n/y : y \in {}^n\delta, (w/x)\mathfrak{R}(n/y)\}$

ii. $\eta_l(w/x) = \{n/y : y \in {}^n\delta, (n/y)\mathfrak{R}(w/x)\}$

iii. $\eta_i(w/x) = \eta_r(w/x) \cap \eta_l(w/x)$

iv. $\eta_u(w/x) = \eta_r(w/x) \cup \eta_l(w/x)$

v. $\eta_{\langle r\rangle}(w/x) = \begin{cases} \bigcap\limits_{x \in^w \eta_r(n/y)} \eta_r(n/y) & \text{if there exists } y \in {}^n\delta \text{ such that } x \in {}^w\eta_r(n/y) \\ \phi & \text{Otherwise} \end{cases}$

vi. $\eta_{\langle l\rangle}(w/x) = \begin{cases} \bigcap\limits_{x \in^w \eta_r(n/y)} \eta_l(n/y) & \text{if there exists } y \in {}^n\delta \text{ such that } x \in {}^w\eta_l(n/y) \\ \phi & \text{Otherwise} \end{cases}$

vii. $\eta_{\langle i\rangle}(w/x) = \eta_{\langle r\rangle}(w/x) \cap \eta_{\langle l\rangle}(w/x)$

viii. $\eta_{\langle u\rangle}(w/x) = \eta_{\langle r\rangle}(w/x) \cup \eta_{\langle l\rangle}(w/x)$

Remark 2.2. $\eta_r(w/x)$ is the same as $(w/x)r$ in Definition 1.32 and $\eta_{\langle r\rangle}(w/x)$ is the same as $\langle w/x\rangle r$ in Definition 1.34.

Definition 2.3. Let $\mathfrak{R}$ be an mset relation on δ and $\psi_j : \delta \longrightarrow P^*(\delta)$ be a mapping which assigns for each $x \in {}^w\delta$ its mset j-neighborhood in $P^*(\delta)$. then $(\delta, \mathfrak{R}, \psi_j)$ is called an mset j-neighborhood space (j-MNS).

Definition 2.4. Let $(\delta, \mathfrak{R}, \psi_j)$ be an mset j-neighborhood space (δ-ξ_jS) and $\chi \subseteq \delta$, then:

i. $(w/x) \in \chi$ means that $x \in {}^w\chi$.

ii. $\chi^c = \delta \ominus \chi$.

Proposition 2.5. *Let* $(\delta, \mathfrak{R}, \psi_j)$ *be a* j-MNS, $x \in {}^w\delta$ *then:*

i. $\eta_i(w/x) \subseteq \eta_r(w/x) \subseteq \eta_u(w/x)$.

ii. $\eta_i(w/x) \subseteq \eta_l(w/x) \subseteq \eta_u(w/x)$.

iii. $\eta_{\langle i\rangle}(w/x) \subseteq \eta_{\langle r\rangle}(w/x) \subseteq \eta_{\langle u\rangle(w/x)}$.

iv. $\eta_{\langle i\rangle}(w/x) \subseteq \eta_{\langle l\rangle}(w/x) \subseteq \eta_{\langle u\rangle(w/x)}$.

v. If $\mathfrak{R}$ *is symmetric then* $\eta_r(w/x) = \eta_l(w/x) = \eta_i(w/x) = \eta_u(w/x)$.

Proof. i. $\eta_i(w/x) = [\eta_r(w/x) \cap \eta_l(w/x)] \subseteq \eta_r(w/x) \subseteq [\eta_r(w/x) \cup \eta_l(w/x)] = \eta_u(w/x)$.
(ii-iv) similar to (i)
v. If $\mathfrak{R}$ *is symmetric, then* $(w/x)\mathfrak{R}(k/y) \Leftrightarrow (k/y)\mathfrak{R}(w/x)$. *Hence* $\eta_r(w/x) = \eta_l(w/x) = \eta_i(w/x) = \eta_u(w/x)$. □

Example 2.6. Let $\delta = \{2/a, 3/b\}$ be an mset and $\mathfrak{R}$ be an mset relation on δ defined by:
$\mathfrak{R} = \{(1/a, 2/a), (1/a, 3/b), (2/a, 2/a), (2/a, 1/b), (2/b, 2/a), (2/b, 2/b), (3/b, 1/a), (3/b, 3/b)\}$.
Then $P^*(\delta) = \{\{1/a\}, \{2/a\}, \{1/b\}, \{2/b\}, \{3/b\}, \{1/a, 1/b\}, \{1/a, 2/b\}, \{1/a, 3/b\}, \{2/a, 1/b\}, \{2/a, 2/b\}, \delta, \phi\}$ and the mset j-neighborhoods of each $x \in {}^{w}\delta$ are given by:

Table 2: Mset j-Neighborhoods

w/x	$\eta_r(w/x)$	$\eta_l(w/x)$	$\eta_i(w/x)$	$\eta_u(w/x)$	$\eta_{\langle r\rangle}(w/x)$	$\eta_{\langle l\rangle}(w/x)$	$\eta_{\langle i\rangle}(w/x)$	$\eta_{\langle u\rangle}(w/x)$
1/a	$\{2/a, 3/b\}$	$\{3/b\}$	$\{3/b\}$	$\{2/a, 3/b\}$	$\{1/a, 3/b\}$	$\{1/a, 3/b\}$	$\{1/a, 3/b\}$	$\{1/a, 3/b\}$
2/a	$\{2/a, 1/b\}$	$\{2/a, 2/b\}$	$\{2/a, 1/b\}$	$\{2/a, 2/b\}$	$\{2/a, 1/b\}$	$\{2/a\}$	$\{2/a\}$	$\{2/a, 1/b\}$
1/b	ϕ	$\{2/a\}$	ϕ	$\{2/a\}$	$\{2/a, 1/b\}$	ϕ	ϕ	$\{2/a, 1/b\}$
2/b	$\{2/a, 2/b\}$	$\{2/b\}$	$\{2/b\}$	$\{2/a, 2/b\}$	$\{2/a, 2/b\}$	$\{2/b\}$	$\{2/b\}$	$\{2/a, 2/b\}$
3/b	$\{1/a, 3/b\}$	$\{1/a, 3/b\}$	$\{1/a, 3/b\}$	$\{1/a, 3/b\}$	$\{1/a, 3/b\}$	$\{3/b\}$	$\{3/b\}$	$\{1/a, 3/b\}$

3. Multiset approximations based on multiset neighborhoods

In this section, we generalize the multiset approximations of definitions 1.33 and 1.35 -that were studied in [3], study their properties and give an illustrative example.

Definition 3.1. Let $(\delta, \mathfrak{R}, \psi_j)$ be aj-MNS, $\chi \subseteq \delta$. Then the mset lower approximation $\underline{\Gamma}^{\delta}_{\eta_j}(\chi)$, upper approximation $\overline{\Gamma}^{\delta}_{\eta_j}(A)$, boundary region $\beta^{\delta}_{\eta_j}(\chi)$, positive region $POS^{\delta}_{\eta_j}(\chi)$, negative region $NEG^{\delta}_{\eta_j}(\chi)$ and accuracy measure $\mu^{\delta}_{\eta_j}(\chi)$ are defined as:

i. $\underline{\Gamma}^{\delta}_{\eta_j}(\chi) = \{(w/x) \in \delta : \eta_j(w/x) \subseteq \chi\}$.

ii. $\overline{\Gamma}^{\delta}_{\eta_j}(\chi) = \{(w/x) \in \delta : \eta_j(w/x) \cap \chi \neq \phi\}$.

iii. $\beta^{\delta}_{\eta_j}(\chi) = \overline{\Gamma}^{\delta}_{\eta_j}(\chi) \ominus \underline{\Gamma}^{\delta}_{\eta_j}(\chi)$.

iv. $POS^{\delta}_{\eta_j}(\chi) = \underline{\Gamma}^{\delta}_{\eta_j}(\chi)$

v. $NEG^{\delta}_{\eta_j}(\chi) = \delta \ominus \overline{\Gamma}^{\delta}_{\eta_j}(\chi)$

vi. $\mu^{\delta}_{\eta_j}(\chi) = \dfrac{CARD[\underline{\Gamma}^{\delta}_{\eta_j}(\chi)]}{CARD[\overline{\Gamma}^{\delta}_{\eta_j}(\chi)]}$.

Remark 3.2. The lower approximation, upper approximation and accuracy measure of Definition 3.1 at j = r are the same as that of Definition 1.33 and at j = ⟨r⟩ they are the same as that of Definition 1.35.

Proposition 3.3. *Let $(\delta, \mathfrak{R}, \psi_j)$ be a*j-MNS, $A, B \subseteq \delta$, *then the mset approximations of* A *and* B *satisfy (L2)-(L5), (U2)-(U5) and (LU).*

Proof. *(L2)* $\underline{\Gamma}^{\delta}_{\eta_j}(\delta)\{(w/x) \in \delta : \eta_j(w/x) \subseteq \delta\} = \delta$.
(L3) $\underline{\Gamma}^{\delta}_{\eta_j}(A \cap B) = \{(w/x) : \eta_j(w/x) \subseteq (A \cap B)\} = \{(w/x) \in \delta : \eta_j(w/x) \subseteq A$ *and* $\eta_j(w/x) \subseteq B\} = \{(w/x) \in \delta : \eta_j(w/x) \subseteq A\} \cap \{(w/x) \in \delta : \eta_j(w/x) \subseteq B\} = \underline{\Gamma}^{\delta}_{\eta_j}(A) \cap \underline{\Gamma}^{\delta}_{\eta_j}(B)$.
(L4) Let $(w/x) \in \underline{\Gamma}^{\delta}_{\eta_j}(A) \cup \underline{\Gamma}^{\delta}_{\eta_j}(B)$, *then* $(w/x) \in \underline{\Gamma}^{\delta}_{\eta_j}(A)$ *or* $\underline{\Gamma}^{\delta}_{\eta_j}(B)$, *thus* $\eta_j(w/x) \subseteq A$ *or* $\eta_j(w/x) \subseteq B$, *therefore* $\eta_j(w/x) \subseteq (A \cup B)$, *so* $(w/x) \in \underline{\Gamma}^{\delta}_{\eta_j}(A \cup B)$. *Hence* $\underline{\Gamma}^{\delta}_{N_j}(A) \cup \underline{\Gamma}^{\delta}_{\eta_j}(B) \subseteq \underline{\Gamma}^{\delta}_{\eta_j}(A \cup B)$.

(L5) $A \subseteq B$, *then* $\underline{\Gamma}^{\delta}_{\eta_j}(A) = \{(w/x) \in \delta : \eta_j(w/x) \subseteq A\} \subseteq \{(w/x) \in \delta : \eta_j(w/x) \subseteq B\} = \underline{\Gamma}^{\delta}_{\eta_j}(B)$.

(U2) $\overline{\Gamma}^{\delta}_{\eta_j}(\phi) = \{(w/x) \in \delta : \eta_j(w/x) \cap \phi \neq \phi\} = \phi$.

(U3) Let $(w/x) \in \overline{\Gamma}^{\delta}_{\eta_j}(A \cap B)$, *thus* $\eta_j(w/x) \cap (A \cap B) \neq \phi$, *then* $\eta_j(w/x) \cap A \neq \phi$ *and* $\eta_j(w/x) \cap B \neq \phi$, *so* $(w/x) \in \overline{\Gamma}^{\delta}_{N_j}(A)$ *and* $(w/x) \in \overline{\Gamma}^{\delta}_{\eta_j}(B)$, *therefore* $(w/x) \in \overline{\Gamma}^{\delta}_{N_j}(A) \cap \overline{\Gamma}^{\delta}_{\eta_j}(B)$. *Hence* $\overline{\Gamma}^{\delta}_{N_j}(A \cap B) \subseteq \overline{\Gamma}^{\delta}_{\eta_j}(A) \cap \overline{\Gamma}^{\delta}_{N_j}(B)$.

(U4) $\overline{\Gamma}^{\delta}_{\eta_j}(A \cup B) = \{(w/x) \in \delta : \eta_j(w/x) \cap (A \cup B) \neq \phi\} = \{(w/x) \in \delta : \eta_j(w/x) \cap A \neq \phi$ *or* $\eta_j(w/x) \cap B \neq \phi\} = \{(w/x) \in \delta : \eta_j(w/x) \cap A \neq \phi\} \cup \{(w/x) \in \delta : \eta_j(w/x) \cap B \neq \phi\} = \overline{\Gamma}^{\delta}_{\eta_j}(A) \cup \overline{\Gamma}^{\delta}_{\eta_j}(B)$.

(U5) Let $A \subseteq B$, *then* $\overline{\Gamma}^{\delta}_{N_j}(A) = \{(w/x) \in \delta : \eta_j(w/x) \cap A \neq \phi\} \subseteq \{(w/x) \in \delta : \eta_j(w/x) \cap B \neq \phi\} = \overline{\Gamma}^{\delta}_{\eta_j}(B)$. *(LU)* $\underline{\Gamma}^{\delta}_{\eta_j}(A) = \{(w/x) \in \delta : \eta_j(w/x) \subseteq A\} \subseteq \{(w/x) \in \delta : \eta_j(w/x) \cap A \neq \phi\} = \overline{\Gamma}^{\delta}_{\eta_j}(A)$. □

Proposition 3.4. *Let* $(\delta, \mathfrak{R}, \psi_j)$ *be a* j-MNS, $\chi \subseteq \delta$, *then:*

i. $\underline{\Gamma}^{\delta}_{\eta_i}(\chi) \subseteq \underline{\Gamma}^{\delta}_{\eta_r}(\chi) \subseteq \underline{\Gamma}^{\delta}_{\eta_u}(\chi)$.

ii. $\underline{\Gamma}^{\delta}_{\eta_i}(A) \subseteq \underline{\Gamma}^{\delta}_{\eta_l}(\chi) \subseteq \underline{\Gamma}^{\delta}_{\eta_u}(\chi)$.

iii. $\underline{\Gamma}^{\delta}_{\eta_{\langle i\rangle}}(\chi) \subseteq \underline{\Gamma}^{\delta}_{\eta_{\langle r\rangle}}(\chi) \subseteq \underline{\Gamma}^{\delta}_{\eta_{\langle u\rangle}}(\chi)$

iv. $\underline{\Gamma}^{\delta}_{\eta_{\langle i\rangle}}(\chi) \subseteq \underline{\Gamma}^{\delta}_{\eta_{\langle l\rangle}}(\chi) \subseteq \underline{\Gamma}^{\delta}_{\eta_{\langle u\rangle}}(\chi)$.

v. $\overline{\Gamma}^{\delta}_{\eta_u}(\chi) \subseteq \overline{\Gamma}^{\delta}_{\eta_r}(\chi) \subseteq \overline{\Gamma}^{\delta}_{\eta_i}(\chi)$.

vi. $\overline{\Gamma}^{\delta}_{\eta_u}(\chi) \subseteq \overline{\Gamma}^{\delta}_{\eta_l}(\chi) \subseteq \overline{\Gamma}^{\delta}_{\eta_i}(\chi)$.

vii. $\overline{\Gamma}^{\delta}_{\eta_{\langle u\rangle}}(\chi) \subseteq \overline{\Gamma}^{\delta}_{\eta_{\langle r\rangle}}(A) \subseteq \overline{\Gamma}^{\delta}_{\eta_{\langle i\rangle}}(\chi)$.

viii. $\overline{\Gamma}^{\delta}_{\eta_{\langle u\rangle}}(\chi) \subseteq \overline{\Gamma}^{\delta}_{\eta_{\langle l\rangle}}(\chi) \subseteq \overline{\Gamma}^{\delta}_{\eta_{\langle i\rangle}}(\chi)$.

ix. If $\mathfrak{R}$ *is symmetric, then* $\underline{\Gamma}^{\delta}_{\eta_r}(\chi) = \underline{\Gamma}^{\delta}_{\eta_l}(\chi) = \underline{\Gamma}^{\delta}_{\eta_i}(\chi) = \underline{\Gamma}^{\delta}_{\eta_u}(\chi)$ *and* $\overline{\Gamma}^{\delta}_{\eta_r}(\chi) = \overline{\Gamma}^{\delta}_{\eta_l}(\chi) = \overline{\Gamma}^{\delta}_{\eta_i}(\chi) = \overline{\Gamma}^{\delta}_{\eta_u}(\chi)$.

Proof. *This follows directly from Proposition 2.5.* □

Corollary 3.5.

i. $\mu^{\delta}_{\eta_i}(\chi) \subseteq \mu^{\delta}_{\eta_r}(\chi) \subseteq \mu^{\delta}_{\eta_u}(\chi)$.

ii. $\mu^{\delta}_{\eta_i}(\chi) \subseteq \mu^{\delta}_{\eta_l}(\chi) \subseteq \mu^{\delta}_{\eta_u}(\chi)$.

iii. $\mu^{\delta}_{\eta_{\langle i\rangle}}(\chi) \subseteq \mu^{\delta}_{\eta_{\langle r\rangle}}(\chi) \subseteq \mu^{\delta}_{\eta_{\langle u\rangle}}(\chi)$.

iv. $\mu^{\delta}_{\eta_{\langle i\rangle}}(\chi) \subseteq \mu^{\delta}_{\eta_{\langle l\rangle}}(A) \subseteq \mu^{\delta}_{\eta_{\langle u\rangle}}(\chi)$.

v. If $\mathfrak{R}$ *is symmetric, then* $\mu^{\delta}_{\eta_r}(\chi) = \mu^{\delta}_{\eta_l}(\chi) = \mu^{\delta}_{\eta_i}(\chi) = \mu^{\delta}_{\eta_u}(\chi)$.

Example 3.6. Continued from Example 2.3. The mset lower approximations, upper approximations and accuracy measures of each mset in $P^*(\delta)$ are given by:

Table 3: Mset lower approximations

χ	$\underline{\Gamma}^{\delta}_{\eta_r}(\chi)$	$\underline{\Gamma}^{\delta}_{\eta_l}(\chi)$	$\underline{\Gamma}^{\delta}_{\eta_i}(\chi)$	$\underline{\Gamma}^{\delta}_{\eta_u}(\chi)$	$\underline{\Gamma}^{\delta}_{\eta_{\langle r\rangle}}(\chi)$	$\underline{\Gamma}^{\delta}_{\eta_{\langle l\rangle}}(\chi)$	$\underline{\Gamma}^{\delta}_{\eta_{\langle i\rangle}}(\chi)$	$\underline{\Gamma}^{\delta}_{\eta_{\langle u\rangle}}(\chi)$
{1/a}	{1/b}	ϕ	{1/b}	ϕ	ϕ	{1/b}	{1/b}	ϕ
{2/a}	{1/b}	{1/b}	{1/b}	{1/b}	ϕ	{2/a, 1/b}	{2/a, 1/b}	ϕ
{1/b}	{1/b}	ϕ	{1/b}	ϕ	ϕ	{1/b}	{1/b}	ϕ
{2/b}	{1/b}	{2/b}	{2/b}	ϕ	ϕ	{2/b}	{2/b}	ϕ
{3/b}	{1/b}	{1/a, 2/b}	{1/a, 2/b}	ϕ	ϕ	{3/b}	{3/b}	ϕ
{1/a, 1/b}	{1/b}	ϕ	{1/b}	ϕ	ϕ	{1/b}	{1/b}	ϕ
{1/a, 2/b}	{1/b}	{2/b}	{2/b}	ϕ	ϕ	{3/b}	{2/b}	ϕ
{1/a, 3/b}	{3/b}	{1/a, 3/b}	{1/a, 3/b}	{3/b}	{1/a, 3/b}	{1/a, 3/b}	{1/a, 3/b}	{1/a, 3/b}
{2/a, 1/b}	{2/a, 1/b}	{1/b}	{2/a, 1/b}	{1/b}	{2/a, 1/b}	{2/a, 1/b}	{2/a, 1/b}	{2/a, 1/b}
{2/a, 2/b}	{2/a, 2/b}	{2/a, 2/b}	{2/a, 2/b}	{2/a, 2/b}	{2/a, 2/b}	{2/a, 2/b}	{2/a, 2/b}	{2/a, 2/b}

Table 4: Mset upper approximations

χ	$\overline{\Gamma}^{\delta}_{\eta_r}(\chi)$	$\overline{\Gamma}^{\delta}_{\eta_l}(\chi)$	$\overline{\Gamma}^{\delta}_{\eta_i}(\chi)$	$\overline{\Gamma}^{\delta}_{N_u}(\chi)$	$\overline{\Gamma}^{\delta}_{\eta_{\langle r\rangle}}(\chi)$	$\overline{\Gamma}^{\delta}_{\eta_{\langle l\rangle}}(\chi)$	$\overline{\Gamma}^{\delta}_{\eta_{\langle i\rangle}}(\chi)$	$\overline{\Gamma}^{\delta}_{\eta_{\langle u\rangle}}(\chi)$
{1/a}	δ	δ	δ	δ	δ	δ	δ	δ
{2/a}	δ	δ	δ	δ	δ	δ	δ	δ
{1/b}	δ	δ	δ	δ	δ	δ	δ	δ
{2/b}	δ	δ	δ	δ	δ	δ	δ	δ
{3/b}	δ	δ	δ	δ	δ	δ	δ	δ
{1/a, 1/b}	δ	δ	δ	δ	δ	δ	δ	δ
{1/a, 2/b}	δ	δ	δ	δ	δ	δ	δ	δ
{1/a, 3/b}	δ	δ	δ	δ	δ	δ	δ	δ
{2/a, 1/b}	δ	δ	δ	δ	δ	δ	δ	δ
{2/a, 2/b}	δ	δ	δ	δ	δ	δ	δ	δ

Table 5: Mset accuracy measures

χ	$\mu^{\delta}_{\eta_r}(\chi)$	$\mu^{\delta}_{\eta_l}(\chi)$	$\mu^{\delta}_{\eta_i}(\chi)$	$\mu^{\delta}_{\eta_u}(\chi)$	$\mu^{\delta}_{\eta_{\langle r\rangle}}(\chi)$	$\mu^{\delta}_{\eta_{\langle l\rangle}}(\chi)$	$\mu^{\delta}_{\eta_{\langle i\rangle}}(\chi)$	$\mu^{\delta}_{\eta_{\langle u\rangle}}(\chi)$
{1/a}	$\frac{1}{5}$	0	$\frac{1}{5}$	0	0	$\frac{1}{5}$	$\frac{1}{5}$	0
{2/a}	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	0	$\frac{3}{5}$	$\frac{3}{5}$	0
{1/b}	$\frac{1}{5}$	0	$\frac{1}{5}$	0	0	$\frac{1}{5}$	$\frac{1}{5}$	0
{2/b}	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{2}{5}$	0	0	$\frac{2}{5}$	$\frac{2}{5}$	0
{3/b}	$\frac{1}{5}$	$\frac{3}{5}$	$\frac{3}{5}$	0	0	$\frac{3}{5}$	$\frac{3}{5}$	0
{1/a, 1/b}	$\frac{1}{5}$	0	$\frac{1}{5}$	0	0	$\frac{1}{5}$	$\frac{1}{5}$	0
{1/a, 2/b}	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{2}{5}$	0	0	$\frac{3}{5}$	$\frac{2}{5}$	0
{1/a, 3/b}	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{4}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{4}{5}$	$\frac{4}{5}$	$\frac{4}{5}$
{2/a, 1/b}	$\frac{3}{5}$	$\frac{1}{5}$	$\frac{3}{5}$	$\frac{1}{5}$	$\frac{3}{5}$	$\frac{3}{5}$	$\frac{3}{5}$	$\frac{3}{5}$
{2/a, 2/b}	$\frac{4}{5}$	$\frac{4}{5}$	$\frac{4}{5}$	$\frac{4}{5}$	$\frac{4}{5}$	$\frac{4}{5}$	$\frac{4}{5}$	$\frac{4}{5}$

Remark 3.7. In [3], the author claims that $\underline{\mathfrak{R}}(\chi)$ and $\overline{\mathfrak{R}}(\chi)$ which are the same as $\underline{\Gamma}^{\delta}_{\eta_{\langle r\rangle}}(\chi)$ and $\underline{\Gamma}^{\delta}_{\eta_{\langle r\rangle}}(\chi)$ respectively satisfy (L1) and (U1) but this is not true as shown from the previous example. From example 3.6

$\underline{\Gamma}^{\delta}_{\eta_{\langle r\rangle}}(\{1/a\}) = \phi$

$\overline{\Gamma}^{\delta}_{\eta_{\langle r\rangle}}(\{1/a\}) = \delta$

$\underline{\Gamma}^{\delta}_{\eta_{\langle r\rangle}}(\{1/a, 3/b\}) = \{1/a, 3/b\}$

$\overline{\Gamma}^{\delta}_{\eta_{\langle r\rangle}}(\{1/a, 3/b\}) = \delta$

$[\underline{\Gamma}^{\delta}_{\eta_{\langle r\rangle}}(\{1/a\}^c)]^c = [\underline{\Gamma}^{\delta}_{\eta_{\langle r\rangle}}(\{1/a, 3/b\})]^c = \{1/a, 3/b\}^c = \{1/a\}$

$[\overline{\Gamma}^{\delta}_{\eta_{\langle r\rangle}}(\{1/a, 3/b\}^c)]^c = [\overline{\Gamma}^{\delta}_{\eta_{\langle r\rangle}}(\{1/a\})]^c = \delta^c = \phi$

Clearly $\overline{\Gamma}^{\delta}_{\eta_{\langle r\rangle}}(\{1/a\}) \neq [\underline{\Gamma}^{\delta}_{\eta_{\langle r\rangle}}(\{1/a\}^c)]^c$ and $\underline{\Gamma}^{\delta}_{\eta_{\langle r\rangle}}(\{1/a, 3/b\}) \neq [\overline{\Gamma}^{\delta}_{\eta_{\langle r\rangle}}(\{1/a, 3/b\}^c)]^c$.

Therefore (L1) and (U1) are not satisfied for Definition 3.1

4. Multiset topologies generated by multiset neighborhoods

In this section, we generate new eight multiset topologies from the approximations that we construct in the previous chapter, define new multiset approximations and give an illustrative example.

Proposition 4.1. *Let* $(\delta, \mathfrak{R}, \psi_j)$ *be a* j-MNS, *then the collection* $\Upsilon^{\delta}_{\eta_j} = \{A \subseteq \delta : \Upsilon_j(w/x) \subseteq \chi$ *for all* $x \in {}^{w}\chi\}$ *is an M topology on* δ.

Proof. i. Obviously δ *and* $\phi \in \Upsilon^{\delta}_{\eta_j}$.
ii. Let $\chi_i \in \Upsilon^{\delta}_{\eta_j}$, $i \in I$, *then* $\eta_j(w/x) \subseteq A_i$ *for all* $x \in {}^{w}\chi_i$ *and for each* $i \in I$. *Let* $x \in {}^{w}\bigcup_{i\in I} A_i$, *thus there exists* $i_o \in I$ *such that* $x \in {}^{w}A_{i_o}$, *therefore* $\eta_j(w/x) \subseteq A_{i_o} \subseteq \bigcup_{i\in I} A_i$. *Hence* $\bigcup_{i\in I} A_i \in \Upsilon^{\delta}_{\eta_j}$.
iii. Let $\chi_1, \chi_2 \in \Upsilon^{\delta}_{\eta_j}$, *then* $\eta_j(w/x) \subseteq A_1$ *for each* $x \in {}^{w}\chi_1$ *and* $\eta_j(w/x) \subseteq \chi_2$ *for each* $x \in {}^{w}\chi_2$. *Let* $x \in {}^{w}(\chi_1 \cap \chi_2)$, *thus* $x \in {}^{w}\chi_1$ *and* $x \in {}^{w}\chi_2$, *therefore* $\eta_j(w/x) \subseteq \chi_1$ *and* $\eta_j(w/x) \subseteq \chi_2$, *thus* $\eta_j(w/x) \subseteq (\chi_1 \cap \chi_2)$. *Hence* $(\chi_1 \cap \chi_2) \in \Upsilon^{\delta}_{\eta_j}$. □

Proposition 4.2. *Let* $(\delta, \mathfrak{R}, \psi_j)$ *be a* j-MNS, $\Upsilon^{\delta}_{\eta_j}$ *be the generated M topologies, then:*

i. $\Upsilon^{\delta}_{\eta_i} \subseteq \Upsilon^{\delta}_{\eta_r} \subseteq \Upsilon^{\delta}_{\eta_u}$.

ii. $\Upsilon^{\delta}_{\eta_i} \subseteq \Upsilon^{\delta}_{\eta_l} \subseteq \Upsilon^{\delta}_{\eta_u}$.

iii. $\Upsilon^{\delta}_{\eta_{\langle i\rangle}} \subseteq \Upsilon^{\delta}_{\eta_{\langle r\rangle}} \subseteq \Upsilon^{\delta}_{\eta_{\langle u\rangle}}$.

iv. $\Upsilon^{\delta}_{\eta_{\langle i\rangle}} \subseteq \Upsilon^{\delta}_{\eta_{\langle l\rangle}} \subseteq \Upsilon^{\delta}_{\eta_{\langle u\rangle}}$.

v. If $\mathfrak{R}$ *is symmetric, then* $\Upsilon^{\delta}_{\eta_r} = \mu^{\delta}_{\eta_l} = \mu^{\delta}_{\eta_i} = \Upsilon^{\delta}_{\eta_u}$.

Definition 4.3. Let $(\delta, \mathfrak{R}, \psi_j)$ be a j-MNS, $\Upsilon^{\delta}_{\eta_j}$ be the generated M topologies and $\chi \subseteq \delta$ then the mset lower approximation $\underline{\Upsilon}^{\delta}_{\eta_j}(\chi)$, upper approximation $\overline{\Upsilon}^{\delta}_{\eta_j}(\chi)$, boundary region $B\Upsilon^{\delta}_{N_j}(A)$, positive region $POS\Upsilon^{\delta}_{\eta_j}(\chi)$, negative region $NEG\Upsilon^{\delta}_{N_j}(A)$ and accuracy measure $\mu\Upsilon^{\delta}_{\eta_j}(\chi)$ are defined by:

i. $\underline{\Upsilon}^{\delta}_{\eta_j}(\chi) = \mathrm{int}^{\delta}_{\eta_j}(\chi)$.

ii. $\overline{\Upsilon}^{\delta}_{\eta_j}(\chi) = \mathrm{cl}^{\delta}_{\eta_j}(\chi)$.

iii. $B\Upsilon^{\delta}_{\eta_j}(\chi) = \mathrm{cl}^{\delta}_{\eta_j}(\chi) \ominus \mathrm{int}^{\delta}_{\eta_j}(\chi)$.

iv. $POS\Upsilon^{\delta}_{\eta_j}(\chi) = \mathrm{int}^{\delta}_{\eta_j}(\chi)$.

v. $NEG\Upsilon^{\delta(\chi)}_{\eta_j} = \delta \ominus \mathrm{cl}^{\delta(\chi)}_{\Upsilon_{\eta_j}}$.

vi. $\mu\Upsilon^{\delta}_{\eta_j}(\chi) = \dfrac{\mathrm{Card}[\mathrm{int}^{\delta(\chi)]}_{\Upsilon_{\eta_j}}}{\mathrm{Card}[\mathrm{cl}\Upsilon^{M}_{\eta_j}(\chi)]}$.

Example 4.4. Continued from Example 2.3. The generated M topologies are given by:
$\Upsilon^{\delta}_{\eta_r} = \{\delta, \phi, \{1/b\}\}$.
$\Upsilon^{\delta}_{\eta_l} = \Upsilon^{\delta}_{\eta_u} = \Upsilon^{\delta}_{\eta\langle r\rangle} = \Upsilon^{\delta}_{\eta\langle u\rangle} = \{\delta, \phi\}$.
$\Upsilon^{\delta}_{\eta_i} = \{\delta, \phi, \{1/b\}, \{2/b\}, \{1/a, 3/b\}\}$.
$\Upsilon^{\delta}_{\eta\langle l\rangle} = \Upsilon^{\delta}_{\eta\langle i\rangle} = \{\delta, \phi, \{1/b\}, \{2/b\}, \{3/b\}\}$.
The mset lower approximations, upper approximations and accuracy measures of each set in $P^*(\delta)$ are given by:

Table 6: Mset lower approximations

χ	$\underline{\Upsilon}^{\delta}_{\eta_r}(\chi)$	$\underline{\Upsilon}^{\delta}_{\eta_l}(\chi)$	$\underline{\Upsilon}^{\delta}_{\eta_i}(\chi)$	$\underline{\Upsilon}^{\delta}_{\eta_u}(\chi)$	$\underline{\Upsilon}^{\delta}_{\eta\langle r\rangle}(\chi)$	$\underline{\Upsilon}^{\delta}_{\eta\langle l\rangle}(\chi)$	$\underline{\Upsilon}^{\delta}_{\eta\langle i\rangle}(\chi)$	$\underline{\Upsilon}^{\delta}_{\eta\langle u\rangle}(\chi)$
{1/a}	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ
{2/a}	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ
{1/b}	{1/b}	ϕ	{1/b}	ϕ	ϕ	{1/b}	{1/b}	ϕ
{2/b}	{1/b}	ϕ	{2/b}	ϕ	ϕ	{2/b}	{2/b}	ϕ
{3/b}	{1/b}	ϕ	{2/b}	ϕ	ϕ	{3/b}	{3/b}	ϕ
{1/a, 1/b}	{1/b}	ϕ	{1/b}	ϕ	ϕ	{1/b}	{1/b}	ϕ
{1/a, 2/b}	{1/b}	ϕ	{2/b}	ϕ	ϕ	{2/b}	{2/b}	ϕ
{1/a, 3/b}	{1/b}	ϕ	{1/a, 3/b}	ϕ	ϕ	{3/b}	{3/b}	ϕ
{2/a, 1/b}	{1/b}	ϕ	{1/b}	ϕ	ϕ	{1/b}	{1/b}	ϕ
{2/a, 2/b}	{1/b}	ϕ	{2/b}	ϕ	ϕ	{2/b}	{2/b}	ϕ

Table 7: Mset upper approximations

A	$\overline{\Upsilon}^{\delta}_{\eta_r}(\chi)$	$\overline{\Upsilon}^{\delta}_{\eta_l}(\chi)$	$\overline{\Upsilon}^{\delta}_{\eta_i}(\chi)$	$\overline{\Upsilon}^{\delta}_{\eta_u}(\chi)$	$\overline{\Upsilon}^{\delta}_{\eta\langle r\rangle}(\chi)$	$\overline{\Upsilon}^{\delta}_{\eta\langle l\rangle}(\chi)$	$\overline{\Upsilon}^{\delta}_{\eta\langle i\rangle}(\chi)$	$\overline{\Upsilon}^{\delta}_{\eta\langle u\rangle}(\chi)$
{1/a}	{2/a, 2/b}	δ	{1/a}	δ	δ	{2/a}	{2/a}	δ
{2/a}	{2/a, 2/b}	δ	{2/a, 1/b}	δ	δ	{2/a}	{2/a}	δ
{1/b}	{2/a, 2/b}	δ	{2/a, 1/b}	δ	δ	{2/a, 1/b}	{2/a, 1/b}	δ
{2/b}	{2/a, 2/b}	δ	{2/a, 2/b}	δ	δ	{2/a, 2/b}	{2/a, 2/b}	δ
{3/b}	δ	δ	δ	δ	δ	δ	δ	δ
{1/a, 1/b}	{2/a, 2/b}	δ	{2/a, 1/b}	δ	δ	{2/a, 1/b}	{2/a, 1/b}	δ
{1/a, 2/b}	{2/a, 2/b}	δ	{2/a, 2/b}	δ	δ	{2/a, 2/b}	{2/a, 2/b}	δ
{1/a, 3/b}	δ	δ	δ	δ	δ	δ	δ	δ
{2/a, 1/b}	{2/a, 2/b}	δ	{2/a, 1/b}	δ	δ	{2/a, 1/b}	{2/a, 1/b}	δ
{2/a, 2/b}	{2/a, 2/b}	δ	{2/a, 1/b}	δ	δ	{2/a, 2/b}	{2/a, 2/b}	δ

Table 8: Mset accuracy measures

χ	$\mu\Upsilon^{\delta}_{\eta_r}(\chi)$	$\mu\Upsilon^{\delta}_{\eta_l}(\chi)$	$\mu\Upsilon^{\delta}_{\eta_i}(\chi)$	$\mu\Upsilon^{\delta}_{\eta_u}(\chi)$	$\mu\Upsilon^{\delta}_{\eta_{\langle r\rangle}}(\chi)$	$\mu\Upsilon^{\delta}_{\eta_{\langle l\rangle}}(\chi)$	$\mu\Upsilon^{\delta}_{\eta_{\langle i\rangle}}(\chi)$	$\mu\Upsilon^{\delta}_{\eta_{\langle u\rangle}}(\chi)$
{1/a}	0	0	0	0	0	0	0	0
{2/a}	0	0	0	0	0	0	0	0
{1/b}	$\frac{1}{4}$	0	$\frac{1}{3}$	0	0	$\frac{1}{3}$	$\frac{1}{3}$	0
{2/b}	$\frac{1}{4}$	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0
{3/b}	$\frac{1}{5}$	0	$\frac{2}{5}$	0	0	$\frac{3}{5}$	$\frac{3}{5}$	0
{1/a, 1/b}	$\frac{1}{4}$	0	$\frac{1}{3}$	0	0	$\frac{1}{3}$	$\frac{1}{3}$	0
{1/a, 2/b}	$\frac{1}{4}$	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0
{1/a, 3/b}	$\frac{1}{5}$	0	$\frac{4}{5}$	0	0	$\frac{3}{5}$	$\frac{3}{5}$	0
{2/a, 1/b}	$\frac{1}{4}$	0	$\frac{1}{3}$	0	0	$\frac{1}{3}$	$\frac{1}{3}$	0
{2/a, 2/b}	$\frac{1}{4}$	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0

5. Multiset reflexive approximations

In this section, we generalize the three types of rough set approximations -that were studied in [8] based on reflexive relation- in multisets and we study their properties.

Definition 5.1. Let $\mathfrak{R}$ be an mset reflexive relation on δ and $\psi_j : \delta \longrightarrow P^*(\delta)$ be a mapping which assigns for each $x \in {}^w\delta$ its mset j-neighborhood in $P^*(\delta)$. Then $(\delta, \mathfrak{R}, \psi_j)$ is called an mset reflexive j-neighborhood space (j-MRNS).

Definition 5.2. Let $(\delta, \mathfrak{R}, \psi_j)$ be a j-MRNS then:

$$\eta_j^{\cap}(w/x) = \begin{cases} \bigcap\limits_{(w/x)\in\eta_j(n/y)} \eta_j(n/y), & \text{If there exists } y \in {}^n\delta \text{ such that } (w/x) \in \eta_j(n/y) \\ \phi & \text{Otherwise} \end{cases}.$$

Definition 5.3. Let $(\delta, \mathfrak{R}, \psi_j)$ be a j-MRNS, the mset reflexive type 1 j-neighborhood approximations of $\chi \subseteq \delta$ are defined by:
${}^{-1}\mathfrak{R}^{\delta}_{\eta_j}(\chi) = \{(w/x) \in \delta : \eta_j^{\cap}(w/x) \subseteq \chi\}$.
(Called the mset reflexive type 1 lower approximation of χ).
${}^{+1}\mathfrak{R}^{\delta}_{\eta_j}(\chi) = \{(w/x) \in \delta : \eta_j^{\cap}(w/x) \cap \chi \neq \phi\}$.
(Called the mset reflexive type 1 upper approximation of χ).

Lemma 5.4. *Let* $(n/y) \in \eta_j^{\cap}(w/x)$, *then* $\eta_j^{\cap}(n/y) \subseteq \eta_j^{\cap}(w/x)$.

Proof. Let $(k_1/z_1) \in \eta_j^{\cap}(n/y)$ *and let* $(k_2/z_2) \in \delta$ *be any element such that* $(w/x) \in \eta_j(k_2/z_2)$. $(w/x) \in \eta_j(k_2/z_2)$ *and* $(n/y) \in \eta_j^{\cap}(w/x)$ *imply* $(n/y) \in \eta_j(k_2/z_2)$. $(n/y) \in \eta_j(k_2/z_2)$ *and* $(k_1/z_1) \in \eta_j^{\cap}(n/y)$ *imply* $(k_1/z_1) \in \eta_j(k_2/z_2)$ *and since* $\eta_j(k_2/z_2)$ *is any neighborhood containing* (w/x), *therefore* $(k_1/z_1) \in \eta_j^{\cap}(w/x)$. *Hence* $\eta_j^{\cap}(n/y) \subseteq \eta_j^{\cap}(w/x)$. □

Proposition 5.5. *Let $(\delta, \mathfrak{R}, \psi_{\mathrm{j}})$ be a* j-MRNS, $\mathrm{A}, \mathrm{B} \subseteq \delta$, *then the mset reflexive type 1* j*-neighborhood approximations of* A *and* B *satisfy (L2)-(L7), (L9), (U2)-(U7) and (LU).*

Proof. *(L2)* $^{-1}\mathfrak{R}^{\delta}_{\eta_{\mathrm{j}}}(\delta) = \{(\mathrm{w/x}) \in \delta : \eta^{\cap}_{\mathrm{j}}(\mathrm{w/x}) \subseteq \delta\} = \delta$.

(L3) $^{-1}\mathfrak{R}^{\delta}_{\eta_{\mathrm{j}}}(\mathrm{A} \cap \mathrm{B}) = \{(\mathrm{w/x}) \in \delta : \eta^{\cap}_{\mathrm{j}}(\mathrm{w/x}) \subseteq (\mathrm{A} \cap \mathrm{B})\} = \{(\mathrm{w/x}) \in \delta : \eta^{\cap}_{\mathrm{j}}(\mathrm{w/x}) \subseteq \mathrm{A}$ *and* $\eta^{\cap}_{\mathrm{j}}(\mathrm{w/x}) \subseteq \mathrm{B}\} = \{(\mathrm{w/x}) : \eta^{\cap}_{\mathrm{j}}(\mathrm{w/x}) \subseteq \mathrm{A}\} \cap \{(\mathrm{w/x}) : \eta^{\cap}_{\mathrm{j}}(\mathrm{w/x}) \subseteq \mathrm{B}\} = {}^{-1}\mathfrak{R}^{\delta}_{\eta_{\mathrm{j}}}(\mathrm{A}) \cap {}^{-1}\mathfrak{R}^{\delta}_{\eta_{\mathrm{j}}}(\mathrm{B})$.

(L4) Let $(\mathrm{w/x}) \in [{}^{-1}\mathfrak{R}^{\delta}_{\eta_{\mathrm{j}}}(\mathrm{A}) \cup {}^{-1}\mathfrak{R}^{\delta}_{\eta_{\mathrm{j}}}(\mathrm{B})]$, *then* $(\mathrm{w/x}) \in {}^{-1}\mathfrak{R}^{\delta}_{\eta_{\mathrm{j}}}(\mathrm{A})$ *or* $(\mathrm{w/x}) \in {}^{-1}\mathfrak{R}^{\delta}_{\eta_{\mathrm{j}}}(\mathrm{B})$, *thus* $\eta^{\cap}_{\mathrm{j}}(\mathrm{w/x}) \subseteq \mathrm{A}$ *or* $\eta^{\cap}_{\mathrm{j}}(\mathrm{w/x}) \subseteq \mathrm{B}$, *therefore* $\eta^{\cap}_{\mathrm{j}}(\mathrm{w/x}) \subseteq (\mathrm{A} \cup \mathrm{B})$, *so* $(\mathrm{w/x}) \in {}^{-1}\mathfrak{R}^{\delta}_{\eta_{\mathrm{j}}}(\mathrm{A} \cup \mathrm{B})$. *Hence* ${}^{-1}\mathfrak{R}^{\delta}_{\eta_{\mathrm{j}}}(\mathrm{A}) \cup {}^{-1}\mathfrak{R}^{\delta}_{\eta_{\mathrm{j}}}(\mathrm{B}) \subseteq {}^{-1}\mathfrak{R}^{\delta}_{\eta_{\mathrm{j}}}(\mathrm{A} \cup \mathrm{B})$.

(L5) Let $\mathrm{A} \subseteq \mathrm{B}$ *and* $(\mathrm{w/x}) \in {}^{-1}\mathfrak{R}^{\delta}_{\mathrm{N_j}}(\mathrm{A})$, *then* $\eta^{\cap}_{1}(\mathrm{w/x}) \subseteq \mathrm{A}$, *thus* $\eta^{\cap}_{\mathrm{j}}(\mathrm{w/x}) \subseteq \mathrm{B}$, *therefore* $(\mathrm{w/x}) \in {}^{-1}\mathfrak{R}^{\delta}_{\eta_{\mathrm{j}}}(\mathrm{B})$. *Hence* ${}^{-1}\mathfrak{R}^{\delta}_{\eta_{\mathrm{j}}}(\mathrm{A}) \subseteq {}^{-1}\mathfrak{R}^{\delta}_{\eta_{\mathrm{j}}}(\mathrm{B})$.

(L6) ${}^{-1}\mathfrak{R}^{\delta}_{\eta_{\mathrm{j}}}(\phi) = \{(\mathrm{w/x}) \in \delta : \eta^{\cap}_{\mathrm{j}}(\mathrm{w/x}) \subseteq \phi\}$, *and since* $\mathfrak{R}$ *is reflexive, therefore* $\eta^{\cap}_{\mathrm{j}}(\mathrm{w/x}) \neq \phi$ *for all* $(\mathrm{w/x}) \in \delta$. *Hence* ${}^{-1}\mathfrak{R}^{\delta}_{\eta_{\mathrm{j}}}(\phi) = \phi$.

(L7) Let $(\mathrm{w/x}) \in {}^{-1}\mathfrak{R}^{\delta}_{\eta_{\mathrm{j}}}(\mathrm{A})$, *then* $\mathrm{N}^{\cap}_{\mathrm{j}}(\mathrm{w/x}) \subseteq \mathrm{A}$, *since* $\mathfrak{R}$ *is reflexive, then* $(\mathrm{w/x}) \in \eta^{\cap}_{\mathrm{j}}(\mathrm{w/x})$, *so* $(\mathrm{w/x}) \in \mathrm{A}$. *Hence* ${}^{-1}\mathfrak{R}^{\delta}_{\eta_{\mathrm{j}}}(\mathrm{A}) \subseteq \mathrm{A}$.

(L9) i. From (L7), we have ${}^{-1}\mathfrak{R}^{\delta}_{\eta_{\mathrm{j}}}(\mathrm{A}) \subseteq \mathrm{A}$, *then by using (L5),* ${}^{-1}\mathfrak{R}^{\delta}_{\eta_{\mathrm{j}}}[{}^{-1}\mathfrak{R}^{\delta}_{\eta_{\mathrm{j}}}(\mathrm{A})] \subseteq {}^{-1}\mathfrak{R}^{\delta}_{\eta_{\mathrm{j}}}(\mathrm{A})$.

ii. Let $(\mathrm{w/x}) \in {}^{-1}\mathfrak{R}^{\delta}_{\eta_{\mathrm{j}}}(\mathrm{A})$, *then* $\eta^{\cap}_{\mathrm{j}}(\mathrm{w/x}) \subseteq \mathrm{A}$. *Let* $(\mathrm{n/y}) \in \eta^{\cap}_{\mathrm{j}}(\mathrm{w/x})$, *then by Lemma 5.4,* $\eta^{\cap}_{\mathrm{j}}(\mathrm{n/y}) \subseteq \eta^{\cap}_{\mathrm{j}}(\mathrm{w/x})$, *thus* $\eta^{\cap}_{\mathrm{j}}(\mathrm{n/y}) \subseteq \mathrm{A}$. *Then* $(\mathrm{n/y}) \in {}^{-1}\mathfrak{R}^{\delta}_{\eta_{\mathrm{j}}}(\mathrm{A})$ *and so* $\eta^{\cap}_{\mathrm{j}}(\mathrm{w/x}) \subseteq {}^{-1}\mathfrak{R}^{\delta}_{\eta_{\mathrm{j}}}(\mathrm{A})$, *thus* $(\mathrm{w/x}) \in {}^{-1}\mathfrak{R}^{\delta}_{\eta_{\mathrm{j}}}[{}^{-1}\mathfrak{R}^{\delta}_{\eta_{\mathrm{j}}}(\mathrm{A})]$, *therefore* ${}^{-1}\mathfrak{R}^{\delta}_{\eta_{\mathrm{j}}}(\mathrm{A}) \subseteq {}^{-1}\mathfrak{R}^{\delta}_{\eta_{\mathrm{j}}}[{}^{-1}\mathfrak{R}^{\delta}_{\eta_{\mathrm{j}}}(\mathrm{A})]$.

Hence ${}^{-1}\mathfrak{R}^{\delta}_{\eta_{\mathrm{j}}}[{}^{-1}\mathfrak{R}^{\delta}_{\eta_{\mathrm{j}}}(\mathrm{A})] = {}^{-1}\mathfrak{R}^{\delta}_{\eta_{\mathrm{j}}}(\mathrm{A})$.

(U2) ${}^{+1}\mathfrak{R}^{\delta}_{\eta_{\mathrm{j}}}(\phi) = \{(\mathrm{w/x}) \in \delta : \mathrm{N}^{\cap}_{\mathrm{j}}(\mathrm{w/x}) \cap \phi \neq \phi\} = \phi$.

(U3) Let $(\mathrm{w/x}) \in {}^{+1}\mathfrak{R}^{\delta}_{\eta_{\mathrm{j}}}(\mathrm{A} \cap \mathrm{B})$, *then* $\eta^{\cap}_{\mathrm{j}}(\mathrm{w/x}) \cap (\mathrm{A} \cap \mathrm{B}) \neq \phi$, *thus* $\eta^{\cap}_{\mathrm{j}}(\mathrm{w/x}) \cap \mathrm{A} \neq \phi$ *and* $\eta^{\cap}_{\mathrm{j}}(\mathrm{w/x}) \cap \mathrm{B} \neq \phi$, *therefore* $(\mathrm{w/x}) \in {}^{+1}\mathfrak{R}^{\delta}_{\eta_{\mathrm{j}}}(\mathrm{A})$ *and* $(\mathrm{w/x}) \in {}^{+1}\mathfrak{R}^{\delta}_{\eta_{\mathrm{j}}}(\mathrm{B})$. *Hence* ${}^{+1}\mathfrak{R}^{\delta}_{\eta_{\mathrm{j}}}(\mathrm{A} \cap \mathrm{B}) \subseteq {}^{+1}\mathfrak{R}^{\delta}_{\eta_{\mathrm{j}}}(\mathrm{A}) \cap {}^{+1}\mathfrak{R}^{\delta}_{\eta_{\mathrm{j}}}(\mathrm{B})$.

(U4) ${}^{+1}\mathfrak{R}^{\delta}_{\eta_{\mathrm{j}}}(\mathrm{A} \cup \mathrm{B}) = \{(\mathrm{w/x}) \in \delta : \eta^{\cap}_{\mathrm{j}}(\mathrm{A} \cup \mathrm{B}) \neq \phi\} = \{(\mathrm{w/x}) \in \delta : \eta^{\cap}_{\mathrm{j}}(\mathrm{A}) \neq \phi$ *or* $\eta^{\cap}_{\mathrm{j}}(\mathrm{B}) \neq \phi\} = \{(\mathrm{w/x}) \in \delta : \eta^{\cap}_{\mathrm{j}}(\mathrm{A}) \neq \phi\} \cup \{(\mathrm{w/x}) \in \delta : \eta^{\cap}_{\mathrm{j}}(\mathrm{B}) \neq \phi\} = {}^{+1}\mathfrak{R}^{\delta}_{\eta_{\mathrm{j}}}(\mathrm{A}) \cup {}^{+1}\mathfrak{R}^{\delta}_{\eta_{\mathrm{j}}}(\mathrm{B})$.

(U5) Let $\mathrm{A} \subseteq \mathrm{B}$, *then* ${}^{+1}\mathfrak{R}^{\delta}_{\eta_{\mathrm{j}}}(\mathrm{A}) = \{(\mathrm{w/x}) \in \delta : \eta^{\cap}_{\mathrm{j}}(\mathrm{w/x}) \cap \mathrm{A} \neq \phi\} \subseteq \{(\mathrm{w/x}) : \eta^{\cap}_{\mathrm{j}}(\mathrm{w/x}) \cap \mathrm{B} \neq \phi\} = {}^{+1}\mathfrak{R}^{\delta}_{\eta_{\mathrm{j}}}(\mathrm{B})$.

(U6) ${}^{+1}\mathfrak{R}^{\delta}_{\eta_{\mathrm{j}}}(\delta) = \{(\mathrm{w/x}) : \eta^{\cap}_{\mathrm{j}}(\mathrm{w/x}) \cap \delta \neq \phi\}$ *and since* $\mathfrak{R}$ *is reflexive, therefore* $\eta^{\cap}_{\mathrm{j}}(\mathrm{w/x}) \cap \delta \neq \phi$ *for all* $(\mathrm{w/x}) \in \delta$. *Hence* ${}^{+1}\mathfrak{R}^{\delta}_{\eta_{\mathrm{j}}}(\delta) = \delta$.

(U7) Let $(\mathrm{w/x}) \in \mathrm{A}$, *since* $\mathfrak{R}$ *is reflexive, therefore* $(\mathrm{w/x}) \in \eta^{\cap}_{\mathrm{j}}(\mathrm{w/x})$, *so* $\eta^{\cap}_{\mathrm{j}}(\mathrm{w/x}) \cap \mathrm{A} \neq \phi$, *thus* $(\mathrm{w/x}) \in {}^{+1}\mathfrak{R}^{\delta}_{\eta_{\mathrm{j}}}(\mathrm{A})$. *Hence* $\mathrm{A} \subseteq {}^{+1}\mathfrak{R}^{\delta}_{\eta_{\mathrm{j}}}(\mathrm{A})$.

(LU) ${}^{-1}\mathfrak{R}^{\delta}_{\eta_{\mathrm{j}}}(\mathrm{A}) = \{(\mathrm{w/x}) \in \delta : \eta^{\cap}_{\mathrm{j}}(\mathrm{w/x}) \subseteq \mathrm{A}\} \subseteq \{(\mathrm{w/x}) : \eta_{\mathrm{j}} \cap (\mathrm{w/x}) \cap \mathrm{A} \neq \phi\} = {}^{+1}\mathfrak{R}^{\delta}_{\eta_{\mathrm{j}}}(\mathrm{A})$. □

Definition 5.6. Let $(\delta, \mathfrak{R}, \psi_{\mathrm{j}})$ be a j-MRNS, the mset reflexive type 2 j-neighborhood approximations of $\mathrm{A} \subseteq \delta$ are defined by:
${}^{-2}\mathfrak{R}^{\delta}_{\eta_{\mathrm{j}}}(\mathrm{A}) = \cup \{\eta^{\cap}_{\mathrm{j}}(\mathrm{w/x}) : \eta^{\cap}_{\mathrm{j}}(\mathrm{w/x}) \subseteq \mathrm{A}\}$.
(Called the mset reflexive type 2 lower approximation of A).
${}^{+2}\mathfrak{R}^{\delta}_{\eta_{\mathrm{j}}}(\mathrm{A}) = [{}^{-2}\mathfrak{R}^{\delta}_{\eta_{\mathrm{j}}}(\mathrm{A^c})]^{\mathrm{c}}$.
(Called the mset reflexive type 2 upper approximation of A).

Proposition 5.7. *Let $(\delta, \mathfrak{R}, \psi_{\mathrm{j}})$ be a* j-MRNS, $\mathrm{A}, \mathrm{B} \subseteq \delta$, *then the mset reflexive type 2* j*-neighborhood approximations of* A *and* B *satisfy (L1), (L2), (L4)-(L7), (L9), (U1), (U2), (U5)-(U7), (U9) and (LU).*

Proof. *(L1)* $[{}^{+2}\mathfrak{R}^{\delta}_{\eta_{\mathrm{j}}}(\mathrm{A^c})]^{\mathrm{c}} = ([{}^{-2}\mathfrak{R}^{\delta}_{\eta_{\mathrm{j}}}([\mathrm{A^c}]^{\mathrm{c}})]^{\mathrm{c}})^{\mathrm{c}} = {}^{-2}\mathfrak{R}^{\delta}_{\eta_{\mathrm{j}}}(\mathrm{A})$.

(L2) ${}^{-2}\mathfrak{R}^{\delta}_{\eta_j}(\delta) = \cup\{\eta_j^{\cap}(w/x) : \eta_j^{\cap}(w/x) \subseteq \delta\} = \delta$.
(L4) Let $(w/x) \in {}^{-2}\mathfrak{R}^{\delta}_{\eta_j}(A) \cup {}^{-2}\mathfrak{R}^{\delta}_{\eta_j}(B)$, *then* $(w/x) \in \cup\{\eta_j^{\cap}(n/y) : \eta_j^{\cap}(n/y) \subseteq A\}$ *or* $(w/x) \in \cup\{\eta_j^{\cap}(n/y) : \eta_j^{\cap}(n/y) \subseteq B\}$, *thus* $(w/x) \in \cup\{\eta_j^{\cap}(n/y) : \eta_j^{\cap}(n/y) \subseteq (A \cup B)\} = {}^{-2}\mathfrak{R}^{\delta}_{\eta_j}(A \cup B)$. *Therefore* ${}^{-2}\mathfrak{R}^{\delta}_{\eta_j}(A) \cup {}^{-2}\mathfrak{R}^{\delta}_{\eta_j}(B) \subseteq {}^{-2}\mathfrak{R}^{\delta}_{\eta_j}(A \cup B)$.
(L5) Let $A \subseteq B$ *and* $(w/x) \in {}^{-2}\mathfrak{R}^{\delta}_{\eta_j}(A)$, *then* $(w/x) \in \cup\{\eta_j^{\cap}(n/y) : \eta_j^{\cap}(n/y) \subseteq A\} \subseteq \cup\{\eta_j^{\cap}(n/y) : \eta_j^{\cap}(n/y) \subseteq B\}$, *therefore* $(w/x) \in {}^{-2}\mathfrak{R}^{\delta}_{\eta_j}(B)$. *Hence* ${}^{-2}\mathfrak{R}^{\delta}_{\eta_j}(A) \subseteq {}^{-2}\mathfrak{R}^{\delta}_{\eta_j}(B)$.
(L6) ${}^{-2}\mathfrak{R}^{\delta}_{\eta_j}(\phi) = \cup\{\eta_j^{\cap}(w/x) : \eta_j^{\cap}(w/x) \subseteq \phi\}$ *and since* $\mathfrak{R}$ *is reflexive, thus* $\eta_j^{\cap}(w/x) \neq \phi$ *for all* $(w/x) \in \delta$. *Therefore* ${}^{-2}\mathfrak{R}^{\delta}_{\eta_j}(\phi) = \phi$.
(L7) ${}^{-2}\mathfrak{R}^{\delta}_{\eta_j}(A) = \cup\{\eta_j^{\cap}(w/x) : (w/x) \subseteq A\} \subseteq A$.
(L9) i. From (L7), we have ${}^{-2}\mathfrak{R}^{\delta}_{\eta_j}(A) \subseteq A$, *then by using (L5),* ${}^{-2}\mathfrak{R}^{\delta}_{\eta_j}[{}^{-2}\mathfrak{R}^{\delta}_{\eta_j}(A)] \subseteq {}^{-2}\mathfrak{R}^{\delta}_{\eta_j}(A)$.
ii. Let $(w/x) \in {}^{-2}\mathfrak{R}^{\delta}_{\eta_j}(A) = \cup\{\eta_j^{\cap}(n/y) : \eta_j^{\cap}(n/y) \subseteq A\}$, *then there exists* $(n_1/y_1) \in \delta$ *such that* $(w/x) \in \eta_j^{\cap}(n_1/y_1) \subseteq A$, *thus* $\eta_j^{\cap}(n_1/y_1) \subseteq \cup\{\eta_j^{\cap}(n/y) : \eta_j^{\cap}(n/y) \subseteq A\} = {}^{-2}\mathfrak{R}^{\delta}_{\eta_j}(A)$, *so* $(w/x) \in \eta_j^{\cap}(n_1/y_1) \subseteq {}^{-2}\mathfrak{R}^{\delta}_{\eta_j}(A)$, *thus* $(w/x) \in {}^{-2}\mathfrak{R}^{\delta}_{\eta_j}[{}^{-2}\mathfrak{R}^{\delta}_{\eta_j}(A)]$ *and therefore* ${}^{-2}\mathfrak{R}^{\delta}_{\eta_j}(A) \subseteq {}^{-2}\mathfrak{R}^{\delta}_{\eta_j}[{}^{-2}\mathfrak{R}^{\delta}_{\eta_j}(A)]$. *Hence* ${}^{-2}\mathfrak{R}^{\delta}_{\eta_j}[{}^{-2}\mathfrak{R}^{\delta}_{\eta_j}(A)] = {}^{-2}\mathfrak{R}^{\delta}_{\eta_j}(A)$.
(U1) From definition 5.6, we have ${}^{+2}\mathfrak{R}^{\delta}_{\eta_j}(A) = [{}^{-2}\mathfrak{R}^{\delta}_{\eta_j}(A^c)]^c$.
(U2) ${}^{+2}\mathfrak{R}^{\delta}_{\eta_j}(\phi) = [{}^{-2}\mathfrak{R}^{\delta}_{\eta_j}(\phi)^c]^c = [{}^{-2}\mathfrak{R}^{\delta}_{\eta_j}(\delta)]^c = \delta^c = \phi$.
(U5) Let $A \subseteq B$, *thus* ${}^{-2}\mathfrak{R}^{\delta}_{\eta_j}(B^c) \subseteq {}^{-2}\mathfrak{R}^{\delta}_{\eta_j}(A^c)$, *so* $[{}^{-2}\mathfrak{R}^{\delta}_{\eta_j}(A^c)]^c = {}^{+2}\mathfrak{R}^{\delta}_{\eta_j}(A) \subseteq {}^{+2}\mathfrak{R}^{\delta}_{\eta_j}(B) = [{}^{-2}\mathfrak{R}^{\delta}_{\eta_j}(B^c)]^c$.
(U6) ${}^{+2}\mathfrak{R}^{\delta}_{\eta_j}(\delta) = [{}^{-2}\mathfrak{R}^{\delta}_{\eta_j}(\delta)^c]^c = [{}^{-2}\mathfrak{R}^{\delta}_{\eta_j}(\phi)]^c = \phi^c = \delta$.
(U7) $[{}^{+2}\mathfrak{R}^{\delta}_{\eta_j}(A)]^c = {}^{-2}\mathfrak{R}^{\delta}_{\eta_j}(A^c) \subseteq A^c$. *Thus* $A \subseteq {}^{+2}\mathfrak{R}^{\delta}_{\eta_j}(A)$.
(U9) ${}^{+2}\mathfrak{R}^{\delta}_{\eta_j}[{}^{+2}\mathfrak{R}^{\delta}_{\eta_j}(A)] = ({}^{-2}\mathfrak{R}^{\delta}_{\eta_j}[{}^{+2}\mathfrak{R}^{\delta}_{\eta_j}(A)]^c)^c = ({}^{-2}\mathfrak{R}^{\delta}_{\eta_j}[{}^{-2}\mathfrak{R}^{\delta}_{\eta_j}(A^c)])^c$ *and from (L9),* $({}^{-2}\mathfrak{R}^{\delta}_{\eta_j}[{}^{-2}\mathfrak{R}^{\delta}_{\eta_j}(A^c)])^c = [{}^{-2}\mathfrak{R}^{\delta}_{\eta_j}(A^c)]^c = {}^{+2}\mathfrak{R}^{\delta}_{\eta_j}(A)$. *Hence* ${}^{+2}\mathfrak{R}^{\delta}_{\eta_j}[{}^{+2}\mathfrak{R}^{\delta}_{\eta_j}(A)] = {}^{+2}\mathfrak{R}^{\delta}_{\eta_j}(A)$ *(LU) From (L7), we have* ${}^{-2}\mathfrak{R}^{\delta}_{\eta_j}(A) \subseteq A$. *From (U7), we have* $A \subseteq {}^{+2}\mathfrak{R}^{\delta}_{\eta_j}(A)$. *Therefore* ${}^{-2}\mathfrak{R}^{\delta}_{\eta_j}(A) \subseteq {}^{+2}\mathfrak{R}^{\delta}_{\eta_j}(A)$. □

Definition 5.8. Let $(\delta, \mathfrak{R}, \psi_j)$ be a j-MRNS, the reflexive reflexive type 3 j-neighborhood approximations of $A \subseteq \delta$ are defined by:
${}^{-3}\mathfrak{R}^{\delta}_{\eta_j}(A) = [{}^{+3}\mathfrak{R}^{\delta}_{\eta_j}(A^c)]^c$.
(Called the mset reflexive type 3 lower approximation of A).
${}^{+3}\mathfrak{R}^{\delta}_{\eta_j}(A) = \cup\, \{\eta_j^{\cap}(x/w) : \eta_j^{\cap}(w/x) \cap A \neq \phi\}$.
(Called the mset reflexive type 3 upper approximation of A).

Proposition 5.9. *Let* $(\delta, \mathfrak{R}, \psi_j)$ *be a* j-MRNS, $A, B \subseteq \delta$, *then the mset reflexive type* 3 j*-neighborhood approximations of* A *and* B *satisfy (L1), (L2), (L5)-(L7), (U1)-(U7) and (LU).*

Proof. (U1) $[{}^{-3}\mathfrak{R}^{\delta}_{\eta_j}(A^c)]^c = ([{}^{+3}\mathfrak{R}^{\delta}_{N_j}([A^c]^c)]^c)^c = {}^{+3}\mathfrak{R}^{\delta}_{\eta_j}(A)$.
(U2) ${}^{+3}\mathfrak{R}^{\delta}_{\eta_j}(\phi) = \cup\{\eta_j^{\cap}(w/x) : \eta_j^{\cap}(w/x) \cap \phi \neq \phi\} = \phi$.
(U3) Let $(w/x) \in {}^{+3}\mathfrak{R}^{\delta}_{\eta_j}(A \cap B)$, *then* $(w/x) \in \cup\{\eta_j^{\cap}(w/x) : \eta_j^{\cap}(w/x) \cap (A \cap B) \neq \phi\}$, *thus* $(w/x) \in \cup\{\eta_j^{\cap}(w/x) : \eta_j^{\cap}(w/x) \cap (A) \neq \phi$ *and* $N_j^{\cap}(w/x) \cap (B) \neq \phi\}$, *so* $(w/x) \in \cup\{\eta_j^{\cap}(w/x) : \eta_j^{\cap}(w/x) \cap (A) \neq \phi\} \bigcap \cup\{\eta_j^{\cap}(w/x) : \eta_j^{\cap}(w/x) \cap (B) \neq \phi\}$, *therefore* $(w/x) \in {}^{-3}\mathfrak{R}^{\delta}_{\eta_j}(A) \cap {}^{-3}\mathfrak{R}^{\delta}_{\eta_j}(B)$. *Hence* ${}^{+3}\mathfrak{R}^{\delta}_{\eta_j}(A \cap B) \subseteq {}^{-3}\mathfrak{R}^{\delta}_{\eta_j}(A) \cap {}^{-3}\mathfrak{R}^{\delta}_{\eta_j}(B)$
(U4) ${}^{+3}\mathfrak{R}^{\delta}_{\eta_j}(A \cup B) = \cup\{\eta_j^{\cap}(w/x) : \eta_j^{\cap}(w/x) \cap (A \cup B) \neq \phi\} = \cup\{\eta_j^{\cap}(w/x) : \eta_j^{\cap}(w/x) \cap A \neq \phi$ *or* $\eta_j^{\cap}(w/x) \cap B \neq \phi\} = \cup\{\eta_j^{\cap}(w/x) : \eta_j^{\cap}(w/x) \cap A \neq \phi\} \bigcup \cup\{\eta_j^{\cap}(w/x) : \eta_j^{\cap}(w/x) \cap B \neq \phi\} = {}^{+3}\mathfrak{R}^{\delta}_{\eta_j}(A) \cup {}^{+3}\mathfrak{R}^{\delta}_{N_j}(B)$.
(U5) Let $A \subseteq B$, *then* ${}^{+3}\mathfrak{R}^{\delta}_{\eta_j}(A) = \cup\{\eta_j^{\cap}(w/x) : \eta_j^{\cap}(w/x) \cap A \neq \phi\} \subseteq \cup\{\eta_j^{\cap}(w/x) : \eta_j^{\cap}(w/x) \cap B \neq \phi\} = {}^{+3}\mathfrak{R}^{\delta}_{\eta_j}(B)$.
(U6) Since $\mathfrak{R}$ *is reflexive, then* $(w/x) \in \eta_j^{\cap}(w/x)$ *for all* $(w/x) \in \delta$. *Thus* ${}^{+3}\mathfrak{R}^{\delta}_{\eta_j}(\delta) = \cup\{\eta_j^{\cap}(w/x) \cap \delta \neq \phi\} = \delta$.
(U7) Let $(w/x) \in A$, *since* $(w/x) \in \eta_j^{\cap}(w/x)$ *(from reflexivity), then* $\eta_j^{\cap}(w/x) \cap A \neq \phi$ *and so* $(w/x) \in {}^{+3}\mathfrak{R}^{\delta}_{\eta_j}(A)$.

Hence $A \subseteq {}^{+3}\mathfrak{R}^{\delta}_{\eta_j}(A)$ *(L1) From Definition 5.8, we have* ${}^{-3}\mathfrak{R}^{\delta}_{\eta_j}(A) = [{}^{+3}\mathfrak{R}^{\delta}_{\eta_j}(A^c)]^c$.
(L2) ${}^{-3}\mathfrak{R}^{\delta}_{\eta_j}(\delta) = [{}^{+3}\mathfrak{R}^{\delta}_{\eta_j}(\delta)^c]^c = [{}^{+3}\mathfrak{R}^{\delta}_{\eta_j}(\phi)]^c = \phi^c = \delta$.
(L5) Let $A \subseteq B$, *then* ${}^{+3}\mathfrak{R}^{\delta}_{\eta_j}(B^c) \subseteq {}^{+3}\mathfrak{R}^{\delta}_{\eta_j}(A^c)$, *thus* $[{}^{+3}\mathfrak{R}^{\delta}_{\eta_j}(A^c)]^c = {}^{-3}\mathfrak{R}^{\delta}_{\eta_j}(A) \subseteq {}^{-3}\mathfrak{R}^{\delta}_{\eta_j}(B) = [{}^{+3}\mathfrak{R}^{\delta}_{\eta_j}(B^c)]^c$.
(L6) ${}^{-3}\mathfrak{R}^{\delta}_{\eta_j}(\phi) = [{}^{+3}\mathfrak{R}^{\delta}_{\eta_j}(\phi)^c]^c = [{}^{+3}\mathfrak{R}^{\delta}_{\eta_j}(\delta)]^c = \delta^c = \phi$.
(L7) $A^c \subseteq {}^{+3}\mathfrak{R}^{\delta}_{\eta_j}(A^c)$, *then* $[{}^{+3}\mathfrak{R}^{\delta}_{\eta_j}(A^c)]^c = {}^{-3}\mathfrak{R}^{\delta}_{N_j}(A) \subseteq A$.
(LU) From (L7), we have ${}^{-3}\mathfrak{R}^{\delta}_{\eta_j}(A) \subseteq A$. *From (U7), we have* $A \subseteq {}^{+3}\mathfrak{R}^{\delta}_{\eta_j}(A)$. *Therefore* ${}^{-3}\mathfrak{R}^{\delta}_{\eta_j}(A) \subseteq {}^{+3}\mathfrak{R}^{\delta}_{\eta_j}(A)$ □

6. Multiset similarity approximations

In this section, we generalize the three types of rough set approximations -that were studied in [8] based on similarity relation- in multisets and we study their properties.

Definition 6.1. Let $\mathfrak{R}$ be an mset similarity relation on δ and $\psi_j : \delta \longrightarrow P^*(\delta)$ be a mapping which assigns for each $x \in {}^{w}\delta$ its mset j-neighborhood in $P^*(\delta)$. Then $(\delta, \mathfrak{R}, \psi_j)$ is called an mset similarity j-neighborhood space (j-MSNS).

Definition 6.2. Let $(\delta, \mathfrak{R}, \psi_j)$ be a j-MSNS, then:

$$\eta_j^{\cup}(w/x) = \begin{cases} \bigcup\limits_{(w/x)\in\eta_j(n/y)} \eta_j(n/y), & \text{If there exists } y \in {}^{n}\delta \text{ such that } (w/x) \in \eta_j(n/y) \\ \phi, & \text{Otherwise} \end{cases}.$$

Definition 6.3. Let $(\delta, \mathfrak{R}, \psi_j)$ be a j-MSNS, the mset similarity type 4 j-neighborhood approximations of $\chi \subseteq \delta$ are defined by:
${}^{-4}\mathfrak{R}^{\delta}_{\eta_j}(\chi) = \{(w/x) \in \delta : \eta_j^{\cup}(w/x) \subseteq \chi\}$.
(Called the mset similarity type 4 lower approximation of χ).
${}^{+4}\mathfrak{R}^{\delta}_{\eta_j}(\chi) = \{(w/x) \in \delta : \eta_j^{\cup}(w/x) \cap \chi \neq \phi\}$.
(Called the mset similarity type 4 upper approximation of χ).

Lemma 6.4. *Let* $(\delta, \mathfrak{R}, \psi_j)$ *be a* j-MSNS *and* $(w/x), (n/y) \in \delta$, *thus if* $(w/x) \in \eta_j^{\cup}(n/y)$ *then* $(n/y) \in N_j^{\cup}(w/x)$.

Proof. Let $(w/x) \in \eta_j^{\cup}(n/y)$, *then there exists* (k/z) *such that* $(n/y) \in \eta_j(k/z)$ *and* $(w/x) \in \eta_j(k/z)$, *thus* $(n/y) \in \eta_j(k/z) \subseteq \eta_j^{\cup}(w/x)$. *Hence* $(n/y) \in \eta_j^{\cup}(w/x)$. □

Proposition 6.5. *Let* $(\delta, \mathfrak{R}, \psi_j)$ *be a* j-MSNS, $A, B \subseteq \delta$, *then the mset similarity type 4* j*-neighborhood approximations of* A *and* B *satisfy (L2)-(L8), (U2)-(U8) and (LU).*

Proof. (L8) Let $(w/x) \in A$, *then for each* $(n/y) \in \eta_j^{\cup}(w/x)$ *we have* $(w/x) \in \eta_j^{\cup}(n/y)$. *Thus* $A \cap \eta_j^{\cup}(n/y) \neq \phi$, *so* $(n/y) \in {}^{+4}\mathfrak{R}^{\delta}_{\eta_j}(A)$. *Therefore* $\eta_j^{\cup}(w/x) \subseteq {}^{+4}\mathfrak{R}^{\delta}_{\eta_j}(A)$, *then* $(w/x) \in {}^{-4}\mathfrak{R}^{\delta}_{\eta_j}[{}^{+4}\mathfrak{R}^{\delta}_{\eta_j}(A)]$. *Hence* $A \subseteq {}^{-4}\mathfrak{R}^{\delta}_{\eta_j}[{}^{+4}\mathfrak{R}^{\delta}_{\eta_j}(A)]$.
(U8) Let $(w/x) \in {}^{+4}\mathfrak{R}^{\delta}_{\eta_j}[{}^{-4}\mathfrak{R}^{\delta}_{\eta_j}(A)]$, *then* $\eta_j^{\cup}(w/x) \cap {}^{-4}\mathfrak{R}^{\delta}_{\eta_j}(A) \neq \phi$. *Let* $(n/y) \in N_j^{\cup}(w/x) \cap {}^{-4}\mathfrak{R}^{\delta}_{\eta_j}(A)$. *Since* $(n/y) \in N_j^{\cup}(w/x)$ *thus* $(w/x) \in N_j^{\cup}(n/y)$ *and since* $(n/y) \in {}^{-4}\mathfrak{R}^{\delta}_{\eta_j}(A)$ *we have* $N_j^{\cup}(n/y) \subseteq A$. *Then* $(w/x) \in \eta_j^{\cup}(n/y) \subseteq A$, *therefore* $(w/x) \in A$. *Hence* ${}^{+4}\mathfrak{R}^{\delta}_{\eta_j}[{}^{-4}\mathfrak{R}^{\delta}_{\eta_j}(A)] \subseteq A$. *The proof of the other properties is similar to that of Proposition 5.5.* □

Definition 6.6. Let $(\delta, \mathfrak{R}, \psi_j)$ be a j-MSNS, the mset similarity type 5 j-neighborhood approximations of $A \subseteq \delta$ are defined by:
${}^{-5}\mathfrak{R}^{\delta}_{\eta_j}(A) = \{(w/x) \in \delta : \eta_j^{\cup}(w/x) \subseteq A\}$.
(Called the mset similarity type 5 lower approximation of A).
${}^{+5}\mathfrak{R}^{\delta}_{\eta_j}(A) = [{}^{-5}\mathfrak{R}^{\delta}_{\eta_j}(A^c)]^c$.
(Called the mset similarity type 5 upper approximation of A).

Proposition 6.7. *Let* $(\delta, \Re, \psi_j)$ *be a* j-MSNS, A, B $\subseteq \delta$*, then the mset similarity type 5* j*-neighborhood approximations of* A *and* B *satisfy (L1), (L2), (L4)-(L7), (L9), (U1), (U2), (U5)-(U7), (U9) and (LU).*

Proof. The proof is similar to that of Proposition 5.7. □

Definition 6.8. Let $(\delta, \Re, \psi_j)$ be a j-MSNS, the mset similarity type 6 j-neighborhood approximations of $\chi \subseteq \delta$ are defined by:
$^{-6}\Re^{\delta}_{\eta_j}(\chi) = [^{+6}\Re^{\delta}_{\eta_j}(\chi^c)]^c$.
(Called the mset similarity type 6 lower approximation of χ).
$^{+6}\Re^{\delta}_{\eta_j}(\chi) = \{(w/x) \in \delta : \eta^{\cup}_j(w/x) \cap \chi \neq \phi\}$.
(Called the mset similarity type 6 upper approximation of χ).

Proposition 6.9. *Let* $(\delta, \Re, \psi_j)$ *be a* j-MSNS, A, B $\subseteq \delta$*, then the mset similarity type 6* j*-neighborhood approximations of* A *and* B *satisfy (L1), (L2), (L5)-(L7), (U1)-(U7) and (LU)*

Proof. The proof is similar to that of Proposition 5.9. □

7. Conclusion

This research introduced a comprehensive generalization of multiset neighborhoods into eight distinct types and used them to define new forms of approximations. Building on these, eight multiset topologies were constructed. Furthermore, we extended six classical types of rough approximations to the multiset context and explored their logical properties. Through rigorous definitions, propositions, and illustrative examples, we have demonstrated the interrelations between these newly defined structures and their adherence to rough set axioms.

The proposed multiset rough approximations can be applied in information systems involving repeated observations, such as transaction databases, web logs, or sensor-generated data, where objects may occur multiple times and frequency carries semantic meaning. In such settings, multiset representations preserve quantitative information that is lost in classical rough set models. The framework is also suitable for decision-making modeling, where decisions depend not only on the presence of attributes but also on their repetition. Furthermore, the use of j-neighborhoods enables local approximation of data patterns, making the proposed approach relevant to pattern recognition and data classification problems involving noisy or redundant data. Finally, the induced topological structures provide a theoretical basis for further applications in granular computing and topological analysis of multiset-valued data, suggesting several directions for future research.

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