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Geometry of mixed geodesic warped product submanifolds in Kenmotsu space forms

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Abstract. This study investigates mixed geodesic warped product pseudo-slant submanifolds in Kenmotsu space forms, deriving several inequalities. We provide examples to demonstrate the existence of these submanifolds and verify the second fundamental inequality. Furthermore, we classify various physical quantities, including Dirichlet energy, Laplacian, Hamiltonian, and eigenvalues of the warping function, within a compact support.

1. Introductions and motivations

Nash's embedding theorem [22] states that every warped product can be immersed as a Riemannian submanifold in some Euclidean space with a sufficiently high codimension. Based on Nash's theorem, geometric obstructions, specifically *intrinsic* and *extrinsic invariants*, for warped products have been established in various ambient space forms [1, 13, 20]. Chen [11, 12] provided optimizations for the second fundamental form as a main intrinsic invariant, the constant holomorphic sectional curvature, and the Laplacian of the warping function as a main extrinsic invariant for CR-warped products in complex space form and complex projective spaces. He also completely classified the equality case of these inequalities. Significant research on warped product submanifolds has been conducted across various ambient space forms [3, 4, 14, 15, 21, 25], notably influenced by the work of Chen [11, 12]. Applications have also been derived for compact Riemannian submanifolds, particularly in cases where equality holds, and the boundary is empty. Chen [11] developed a novel technique to establish the relationship between extrinsic and intrinsic invariants for warped product submanifolds of Kähler manifolds and space forms, utilizing the Codazzi equation. From this point of view, by using the Gauss equation instead of the Codazzi equation in the sense of [11], Mustafa et al.[5] obtained the second fundamental inequality for contact CR-warped products in Kenmotsu space forms. There are not too many studies on the second fundamental inequality

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for warped product pseudo-slant submanifolds other than Sahin's, who obtained a mixed geodesic inequality for warped product pseudo-slant submanifolds in a Kaehler manifold [7]. Then, such inequalities for warped product pseudo-slant submanifolds in different ambient manifolds were derived, and many authors discussed conditions for submanifolds in [2, 17, 18, 28, 29, 35–39] recently. Taking this into consideration, use Gauss equations instead of Codazzi equations, as in the sense of [5]. The squared norm of the mean curvature can be calculated by using a warping function and the constant holomorphic sectional curvature in the spirit of [3, 4] and in light of the historical development in the study of a warping function of a warped product submanifold [13]. The primary objective of this study is to introduce a novel method for establishing inequalities related to the second fundamental form for mixed geodesic warped product pseudo-slant submanifolds and minimal mixed geodesic warped product pseudo-slant submanifolds that are isometrically immersed in complex space forms. This represents the first such inequality derived for mixed geodesic warped product pseudo-slant submanifolds in any space form.

2. Basic formulas

An almost contact structure (φ, ξ, η) on a smooth manifold $\widetilde{V}^{2d+1}$ includes a $(1,1)$-type tensor field φ, a vector field ξ and a 1-form η which satisfies the accompanying relation

$$\varphi^2 - \eta \otimes \xi + \mathcal{I} = 0, \ \eta(\xi) = 1. \tag{1}$$

As an immediate consequence of (1), we easily observe that the tensor field φ induces an almost complex structure $\mathcal{J}$ on a $2n$-dimensional horizontal distribution $\mathfrak{D}$, which is described as the kernel of a 1-form η, i.e. $\mathfrak{D} = ker(\eta)$. Such horizontal distribution $\mathfrak{D}$ can be represented as the orthogonal direct sum of the two n-dimensional eigen distributions $\mathfrak{D}^+$ and $\mathfrak{D}^-$ having eigenvalues $+i$ and $-i$, respectively. From this, the tangent bundle $T\widetilde{V}^{2d+1}$ can be decomposed as;

$$T\widetilde{V}^{2d+1} = \mathfrak{D}^{+i} \oplus \mathfrak{D}^{-i} \oplus \langle \xi \rangle. \tag{2}$$

An almost contact manifold is a smooth manifold $\widetilde{V}^{2d+1}$ consisting of an almost contact structure (φ, ξ, η). An almost contact manifold is an almost contact metric manifold if it is furnished with a Riemannian metric g that fulfills

$$g(\mathcal{Z}_1, \mathcal{Z}_2) = g(\varphi\mathcal{Z}_1, \varphi\mathcal{Z}_2) + \eta(\mathcal{Z}_1)\eta(\mathcal{Z}_2), \tag{3}$$

for every $U, V \in \Gamma(T\widetilde{V}^{2d+1})$. By the application of (1), we receive that

$$\varphi \circ \xi = 0, \ \eta \circ \varphi = 0, \ and \ rank(\varphi) = 2d. \tag{4}$$

The vector field ξ and η are allied to g by the following relation

$$\eta(\mathcal{Z}_1) = g(\mathcal{Z}_1, \xi). \tag{5}$$

By the utilization of (1)-(3), we attain

$$g(\mathcal{Z}_1, \varphi\mathcal{Z}_2) + g(\varphi\mathcal{Z}_1, \mathcal{Z}_2) = 0. \tag{6}$$

Definition 2.1. *Let $\widetilde{V}^{2d+1}$ be an almost contact metric manifold, then $\widetilde{V}^{2d+1}$ is Kenmotsu manifold if*

(1) φ satisfies

$$(\widetilde{\nabla}_{\mathcal{Z}_1}\varphi)\mathcal{Z}_2 = g(\varphi\mathcal{Z}_1, \mathcal{Z}_2)\xi - \eta(\mathcal{Z}_2)\varphi\mathcal{Z}_1. \tag{7}$$

(ii) ξ satisfies

$$\widetilde{\nabla}_{\mathcal{Z}_1}\xi = -\varphi^2\mathcal{Z}_1. \tag{8}$$

The following relation expresses the Riemann tensor for the Kenmotsu space form

$$\begin{aligned}
\widetilde{R}(\mathcal{Z}_1,\mathcal{Z}_2,\mathcal{Z}_3,\mathcal{Z}_4)) =& \frac{c-3}{4}(g(\mathcal{Z}_2,\mathcal{Z}_3)g(\mathcal{Z}_1,\mathcal{Z}_4)) - g(\mathcal{Z}_1,\mathcal{Z}_3)g(\mathcal{Z}_2,\mathcal{Z}_4))) \\
&+ \frac{c+1}{4}(\eta(\mathcal{Z}_1)\eta(\mathcal{Z}_3)g(\mathcal{Z}_2,\mathcal{Z}_4)) + \eta(\mathcal{Z}_4))\eta(\mathcal{Z}_2)g(\mathcal{Z}_1,\mathcal{Z}_3) \\
&- \eta(\mathcal{Z}_2)\eta(\mathcal{Z}_3)g(\mathcal{Z}_1,\mathcal{Z}_4) - \eta(\mathcal{Z}_1)g(\mathcal{Z}_2,\mathcal{Z}_3)\eta(\mathcal{Z}_4)) \\
&+ g(\varphi\mathcal{Z}_2,\mathcal{Z}_3)g(\varphi\mathcal{Z}_1,\mathcal{Z}_4) \\
&- g(\varphi\mathcal{Z}_1,\mathcal{Z}_3)g(\varphi\mathcal{Z}_2,\mathcal{Z}_4)) + 2g(\mathcal{Z}_1,\varphi\mathcal{Z}_2)g(\varphi\mathcal{Z}_3,\mathcal{Z}_4)).
\end{aligned} \tag{9}$$

Let $\mathcal{V}$ be a paracompact, connected, and smooth Riemannian manifold, $\widetilde{V}^{2d+1}$ be a Kenmotsu manifold, and $\psi : \mathcal{V} \to \widetilde{V}^{2d+1}$ be an isometric immersion.

The hyperbolic space $\mathbb{H}^{2n+1} = \{x_1 \cdots x_{2n+1} \in \mathbb{R}^{2n+1} | x_1 > 0\}$ equipped with the almost contact structure $((\varphi, \xi, \eta, g)$ constructed by Chinea and Gonzales in[16] is a Kenmotsu manifold with sectional curvature $c = -1$. Then $\psi(\mathcal{V}) \subset \widetilde{V}^{2d+1}$ be an isometrically immersed submanifold of the Kenmotsu manifold. The induced metric on $\mathcal{V}$ is denoted by the same symbol g. For any $\widetilde{\mathcal{Z}_1} \in \Gamma(T\widetilde{V}^{2d+1})$, we have the following representation:

$$\widetilde{\mathcal{Z}_1} = tan\widetilde{\mathcal{Z}_1} + nor\widetilde{\mathcal{Z}_1}, \tag{10}$$

where $tan : T\widetilde{V}^{2d+1} \to T\mathcal{V}$ and $nor : T\widetilde{V}^{2d+1} \to T\mathcal{V}^{\perp}$ are orthogonal projections on the tangent bundle $T\mathcal{V}$ and normal bundle $T\mathcal{V}^{\perp}$. Let us denotes $\Gamma(T\mathcal{V})$ and $\Gamma(T\mathcal{V}^{\perp})$ for the section of tangent and normal bundle. Then for any $\mathcal{Z}_1 \in \Gamma(T\mathcal{V})$ and $\zeta \in \Gamma(T\mathcal{V}^{\perp})$, we obtain

$$\varphi\mathcal{Z}_1 = t\mathcal{Z}_1 + n\mathcal{Z}_1, \tag{11}$$

$$\varphi\zeta = t'\zeta + n'\zeta, \tag{12}$$

where $t\mathcal{Z}_1 = tan\mathcal{Z}_1$ (resp. $n\mathcal{Z}_1 = nor\mathcal{Z}_1$) of $\varphi\mathcal{Z}_1$ and $t'\zeta = tan\zeta$ (resp. $n'\zeta = nor\zeta$) of $\varphi\mathcal{Z}_1$. In view of (3) and (11), we receive that

$$g(t\mathcal{Z}_1,\mathcal{Z}_2) = -g(\mathcal{Z}_1, t\mathcal{Z}_2). \tag{13}$$

By the utilization of (3), (11) and (12), we concede that

$$g(n'\zeta_1,\zeta_2) = -g(\zeta_1, n'\zeta_2),\ g(\mathcal{Z}_1, t'\zeta_1) = -g(n\mathcal{Z}_1,\zeta_1), \tag{14}$$

for every $\zeta_1, \zeta_2 \in \Gamma(T\mathcal{V}^{\perp})$. If we denote ∇ and $\nabla^{\perp}$ for the induced Levi-Civita connection in $\mathcal{V}$. Then, Gauss and Weingarten's formulas are expressed by the following relation

$$\overline{\nabla}_{\mathcal{Z}_1}\mathcal{Z}_2 = \nabla_{\mathcal{Z}_1}\mathcal{Z}_2 + h(\mathcal{Z}_1,\mathcal{Z}_2), \tag{15}$$

$$\overline{\nabla}_{\mathcal{Z}_1}\xi = -A_{\xi}\mathcal{Z}_1 + \nabla^{\perp}_{\mathcal{Z}_1}\xi, \tag{16}$$

$\forall\ \mathcal{Z}_1,\mathcal{Z}_2 \in \Gamma(T\mathcal{V})$ and $\xi \in \Gamma(T\mathcal{V}^{\perp})$, where h and A_{ξ} are the second fundamental form and shape operator, connected by the following relation

$$g(A_{\xi}\mathcal{Z}_1,\mathcal{Z}_2) = g(h(\mathcal{Z}_1,\mathcal{Z}_2),\xi). \tag{17}$$

The mean curvature vector H is defined by $H = \frac{1}{m}trace(h)$. Now, we recall some important definitions for later use.

An isometric immersion ψ is called totally geodesic if $h \equiv 0$, and it is minimal if $H = 0$. A normal vector field ξ is an umbilical section if A_ξ is proportional to ξ i.e., $A_\xi = k\mathcal{I}$, for $k \in C^\infty(\mathcal{V})$. If every normal vector field is umbilical, then ψ turns totally umbilical. This is equivalent to the following expression:

$$h(\mathcal{Z}_1, \mathcal{Z}_2) = g(\mathcal{Z}_1, \mathcal{Z}_2)H. \tag{18}$$

for all $\mathcal{Z}_1, \mathcal{Z}_2 \in \Gamma(T\mathcal{V})$ (for more details, [1, 27, 32, 34]. Further, the covariant derivative of $(\nabla_{\mathcal{Z}_1}\varphi)\mathcal{Z}_2$, $(\nabla_{\mathcal{Z}_1}t)\mathcal{Z}_2$ and $(\nabla_{\mathcal{Z}_1}n)\mathcal{Z}_2$ are characterized by

$$(\nabla_{\mathcal{Z}_1}\varphi)\mathcal{Z}_2 = \nabla_{\mathcal{Z}_1}\varphi\mathcal{Z}_2 - \varphi\nabla_{\mathcal{Z}_1}\mathcal{Z}_2, \tag{19}$$
$$(\nabla_{\mathcal{Z}_1}t)\mathcal{Z}_2 = \nabla_{\mathcal{Z}_1}t\mathcal{Z}_2 - t\nabla_{\mathcal{Z}_1}\mathcal{Z}_2, \tag{20}$$
$$(\nabla_{\mathcal{Z}_1}n)\mathcal{Z}_2 = \nabla^\perp_{\mathcal{Z}_1}n\mathcal{Z}_2 - n\nabla_{\mathcal{Z}_1}\mathcal{Z}_2. \tag{21}$$

Given Eqs. (7) and (12), we achieve

$$(\nabla_{\mathcal{Z}_1}t)\mathcal{Z}_2 = t' h(\mathcal{Z}_1, \mathcal{Z}_2) + A_{n\mathcal{Z}_2}\mathcal{Z}_1, \tag{22}$$
$$(\nabla_{\mathcal{Z}_1}n)\mathcal{Z}_2 = n' h(\mathcal{Z}_1, \mathcal{Z}_2) - h(\mathcal{Z}_1, t\mathcal{Z}_2). \tag{23}$$

The covariant derivative of h is given by

$$(\nabla_{\mathcal{Z}_3}h)(\mathcal{Z}_1, \mathcal{Z}_2) = \nabla^\perp_{\mathcal{Z}_3}h(\mathcal{Z}_1, \mathcal{Z}_2) - h(\nabla_{\mathcal{Z}_3}\mathcal{Z}_1, \mathcal{Z}_2) - h(\mathcal{Z}_1, \nabla_{\mathcal{Z}_3}\mathcal{Z}_2). \tag{24}$$

Further, the equation of Gauss and Codazzi is defined by

$$\overline{R}(\mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_3, \mathcal{Z}_4) = R(\mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_3, \mathcal{Z}_4) + g(h(\mathcal{Z}_1, \mathcal{Z}_3), h(\mathcal{Z}_2, \mathcal{Z}_4)) - g(h(\mathcal{Z}_1, \mathcal{Z}_4), h(\mathcal{Z}_2, \mathcal{Z}_3)), \tag{25}$$
$$(\overline{R}(\mathcal{Z}_1, \mathcal{Z}_2)\mathcal{Z}_3)^\perp = (\nabla_{\mathcal{Z}_1}h)(\mathcal{Z}_2, \mathcal{Z}_3) - (\nabla_{\mathcal{Z}_2}h)(\mathcal{Z}_1, \mathcal{Z}_3), \tag{26}$$

$\forall\ \mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_3, \mathcal{Z}_4 \in \Gamma(T\mathcal{V})$. A submanifold $\mathcal{V}$ is *totally real submanifold* (or anti-invariant) if for every $p \in \mathcal{V}$, $\mathcal{V}$ satisfies $\varphi(T_p\mathcal{V}) \subset T_p\mathcal{V}^\perp$ and *holomorphic* (or invariant) if for every $p \in \mathcal{V}$, $\mathcal{V}$ satisfies $\varphi(T_p\mathcal{V}) \subset T_p\mathcal{V}$. An isometric immersion $\varphi : M \to \widetilde{V}^{2d+1}$ is slant if and only if

$$t^2 = \lambda(\mathcal{I} - \eta \otimes \xi), \tag{27}$$

for some $\lambda \in [0, 1]$ (see [8]). Moreover, for a slant immersion $\lambda = \cos^2\theta$.

Lemma 2.2. *For a slant submanifold $\mathcal{V}$ of a Kenmotsu manifold $\widetilde{V}^{2d+1}$, we have*

$$g(t\mathcal{Z}_1, t\mathcal{Z}_2) = \cos^2\theta g(\varphi\mathcal{Z}_1, \varphi\mathcal{Z}_2), \tag{28}$$

$$g(n\mathcal{Z}_1, n\mathcal{Z}_2) = \sin^2\theta g(\varphi\mathcal{Z}_1, \varphi\mathcal{Z}_2), \tag{29}$$

for $\mathcal{Z}_1, \mathcal{Z}_2 \in \Gamma(D^\perp)$.

Definition 2.3. *An isometric immersion is said to be a pseudo-slant warped product submanifold $\mathcal{V}$ in a Kenmotsu manifold $\widetilde{V}^{2d+1}$ if there exists a pair of orthogonal distributions $(\mathfrak{B}_\theta, \mathfrak{B}_\perp)$ such that the tangent bundle is expressed as*

$$T\mathcal{V} = \mathfrak{B}_\theta \oplus \mathfrak{B}_\perp \oplus \langle\xi\rangle, \tag{30}$$

where $\mathfrak{B}_\theta$ is a slant distribution, $\mathfrak{B}_\perp$ is an anti-invariant distribution, and $\langle\xi\rangle$ is a 1-dimensional distribution of $T\mathcal{V}$.

The mean curvature vector H in terms of $\{E_1, E_2 \cdots, E_m\}$ which is called an orthonormal frame of tangent space $T\mathcal{V}$ and it is described as:

$$H = \frac{1}{n}trace(h) = \frac{1}{m}\sum_{a_1=1}^{m} h(E_{a_1}, E_{a_1}), \tag{31}$$

where $m = \dim \mathcal{V}$. Also, we have

$$||h||^2 = \sum_{a_1,a_2=1}^{m} g(h(E_{a_1}, E_{a_2}), h(E_{a_1}, E_{a_2})) \quad and \quad h^{b_1}_{a_1a_2} = g(h(E_{a_1}, E_{a_2}), E_{b_1}), \tag{32}$$

for $a_1, b_1 \in \{1, \cdots m\}$ and $b_1 \in \{1, \cdots 2m\}$ are orthonormal frames tangent to $\mathcal{V}$ and normal to $\mathcal{V}$, respectively.

The concept of curvature is a fundamental tool in the theory of general relativity and black holes. Ricci curvature and scalar curvature are often more helpful than the full Riemann curvature tensor. These curvatures also appear in the Einstein field equations, which connect the geometric properties of spacetime with its matter content. Ricci curvature has been significantly utilized in Ricci-Flow, initiated by R. S. Hamilton, and was instrumental in Perelman's resolution of the geometrization conjecture proposed by W. Thurston. We now recall some basic definitions and results related to curvatures:
The Ricci-curvature and scalar curvature denoted by Ric and τ, respectively, is represented by the following relation

$$Ric = \sum_{q} R(X, E_q)E_q, \tag{33}$$

$$\tau(T\mathcal{V}) = \sum_{1\leq q\neq s\leq m} \mathcal{K}(E_q \wedge E_s), \tag{34}$$

where $\mathcal{K}(E_q \wedge E_s)$ indicates the sectional curvature of the plane spanned by E_q and E_s. The gradient and Laplacian of smooth function $f : M \longrightarrow \mathbb{R}$ are described by the following expression

$$||\nabla f||^2 = \sum_{q=1}^{m} (E_q(f))^2 \; and \; g(\nabla f, \mathcal{Z}_1) = \mathcal{Z}_1 f, \tag{35}$$

$$\Delta f = \sum_{q=1}^{m} (\nabla_{E_q} E_q)f - E_q(E_q(f), \tag{36}$$

for all $\mathcal{Z}_1 \in \Gamma(T\mathcal{V})$. In local coordinates, the relation (36) can be written as

$$\Delta f = -\sum_{q=1}^{m} g(\nabla_{E_q} grad(f), E_q). \tag{37}$$

The Laplacian is related with Hessian H^f by the following relation

$$\Delta f = -Trace H^f = -\sum_{q=1}^{m} H^f(E_q, E_q). \tag{38}$$

For a smooth function $f : \mathcal{V} \longrightarrow \mathbb{R}$, the Dirichlet energy is defined by

$$E(f) = \frac{1}{2}\int_{\mathcal{V}} ||\nabla f||^2 dV, \tag{39}$$

where $E(f)$ and dV indicate Dirichlet energy and volume element, respectively. The Sigma model can estimate the Dirichlet energy if $\mathbb{R}$ is replaced with a smooth manifold. Solutions to the Lagrange equation for the sigma model produce extremely high Dirichlet energies. In Lagrangian mechanics, a mechanical system's Lagrangian L is expressed as $T - V$, where T denotes the system's kinetic energy and V is its potential energy. The Lagrangian of the smooth function f is given by the following

$$L = \frac{1}{2}||f||^2. \tag{40}$$

The Euler-Lagrange equation for a Lagrangian L is $\Delta f = 0$.

Lemma 2.4. *[1] Let $\mathcal{V}$ be a compact, connected Riemannian manifold without boundary and f be a smooth function on $\mathcal{V}$ such that $\nabla \geq 0$ ($\nabla \leq 0$), then f is a constant function.*

Moreover, by the consequence of the Green theorem, the following relation holds on a compact orientable Riemannian manifold without boundary

$$\int_{\mathcal{V}} \Delta f dV = 0, \tag{41}$$

since $\Delta\rho = div(X)$, the above relation immediately taking the following form

$$\int_{\mathcal{V}} div(X) dV = 0, \tag{42}$$

where $div(X)$ is the divergence of smooth vector field X tangent to $\mathcal{V}$ [1]. The well-known Hopf lemma is stated as follows:

Theorem 2.5. *[1] Let $\mathcal{V}$ be a compact, connected Riemannian manifold with boundary and f be a smooth function on $\mathcal{V}$ such that $\Delta f = 0$ on $\mathcal{V}$ and $f_{\partial\mathcal{V}} = 0$, then $f = 0$.*

The Hamiltonian [1] $\mathcal{H}$ is defined as the total energy of the mechanical system. It induces the symplectic structure on a smooth even-dimensional manifold. The following formulas compute it

$$\mathcal{H}(df, x) = \frac{1}{2}\sum_{q=1}^{m}(df(E_q))^2 = \frac{1}{2}\sum_{q=1}^{m}(E_{a_1}(f))^2 = \frac{1}{2}\|\nabla f\|^2. \tag{43}$$

3. Curvature Inequality

Let $(\mathcal{V}_1, g_{\mathcal{V}_1})$ and $(\mathcal{V}_2, g_{\mathcal{V}_2})$ are two Riemannian manifolds. Then $\mathcal{V} = \mathcal{V}_1 \times \mathcal{V}_2$ is a Riemannian manifold. There are natural projections $\pi : \mathcal{V}_1 \times \mathcal{V}_2 \longrightarrow \mathcal{V}_1$ and $\pi' : \mathcal{V}_1 \times \mathcal{V}_2 \longrightarrow \mathcal{V}_2$.

Definition 3.1. *A product manifold $\mathcal{V} = \mathcal{V}_1 \times \mathcal{V}_2$ is called warped product [13, 31] (denoted by $\mathcal{V}_1 \times_\rho \mathcal{V}_2$) if it is furnished a Riemannian metric (called warping metric) g fulfills*

$$g(U, U) = g(\pi_* U, \pi_* U) + (\rho \circ \pi)^2 g(\pi'_* U, \pi'_* U), \tag{44}$$

for any tangent vector U to $\mathcal{V}$, where $f : \mathcal{V}_1 \longrightarrow (0, 1)$ be a smooth positive function, called warping function and $*$ denotes the push-foreword map (or differential map).

For a non-constant f, $\mathcal{V}$ is a proper (or non-trivial) warped product; otherwise, it is trivial. Moreover, the above relation is equivalent to:

$$g = g_{\mathcal{V}_1} + \rho^2 g_{\mathcal{V}_2}. \tag{45}$$

In [9, 11], Chen derived the general inequality for a $\mathcal{CR}$-warped product in complex space form, which establishes a connection between intrinsic and extrinsic invariants of the $\mathcal{CR}$-warped product, such as the second fundamental form and the warping function. By the immediate consequence of [11] he presents all the classification of $\mathcal{CR}$-warped product in $\mathbb{C}^K$ which fulfills the relation (46), which locally a Segre $\mathcal{CR}$-warped product defined by partial Segre embedding up to rigid motion as $\phi_a^{pk} : \mathbb{C}_*^k \times \mathbb{S}^p \longrightarrow \mathbb{C}^{ap+k}$, where $\mathbb{C}_*^F = \mathbb{C} \sim \{0\}$ and $a, p, k \in \mathbb{N}$. By the application of Hopf fibration, he determined some conditions under which the equality sign of (46) holds for $\mathcal{CR}$-warped products in complex projective space $\mathbb{CP}^m(4)$ and in complex hyperbolic space $\mathbb{CH}^m(-4)$ (for more details, see Theorem 5.1 and Theorem 5.1 [11]). Later, Arslan et al. [23] derived a functional inequality for a $\mathcal{CR}$-warped product $\mathcal{V}_T \times_\rho \mathcal{V}_\perp$ in a Kenmotsu space form that connects curvature, the second fundamental form, Laplacian, and the gradient of the warping function. This inequality is represented as follows in the form of a theorem:

Theorem 3.2. *[23] Let $\mathcal{V}$ be a $\mathcal{CR}$-warped product of type $\mathcal{V}_T \times_\rho \mathcal{V}_\perp$ in a Kentmotsu space form $\widetilde{V}^{2n+1}(c)$ with β dimensional on anti-invariant submanifold, then we have*

$$||h||^2 \geq 2\beta\Big\{||\nabla(\ln\rho)||^2 - \beta\Delta(\ln\rho) + 1\Big\} + \alpha\beta(c+1). \tag{46}$$

After that, numerous geometers studied $\mathcal{CR}$-warped products in different ambient spaces. Motivated by Theorem 3.2, we construct such inequalities for pseudo-slant warped product submanifolds in a Kentmotsu space form. Before deriving these inequalities, one can recall some important facts related to pseudo-slant warped products.

Lemma 3.3. *[18] For a pseudo-slant warped product submanifold $\mathcal{V} = \mathcal{V}_\theta \times_\rho \mathcal{V}_\perp$ of a Kenmotsu manifold $\widetilde{V}^{2n+1}$, we have*

$$g(h(\mathcal{X}_1, \mathcal{X}_2), \varphi\mathcal{Z}_1) = g(h(\mathcal{X}_1, \mathcal{Z}_1), n\mathcal{X}_1) \tag{47}$$

$$g(h(\mathcal{Z}_1, \mathcal{X}_1), \varphi\mathcal{Z}_2) = g(h(\mathcal{Z}_2, \mathcal{X}_1), \varphi\mathcal{Z}_1), \tag{48}$$

for any $\mathcal{X}_1, \mathcal{X}_2 \in T\mathcal{V}_\theta$ and $\mathcal{Z}_1, \mathcal{Z}_2 \in T\mathcal{V}_\perp$.

Lemma 3.4. *[18] For a pseudo-slant warped product submanifold $\mathcal{V} = \mathcal{V}_\theta \times_\rho \mathcal{V}_\perp$ of a Kenmotsu manifold $\widetilde{V}^{2n+1}$, we have*
(i) $\xi \ln f = 1$
(ii) $g(h(\mathcal{Z}_1, \mathcal{Z}_2), n\mathcal{X}_1) = g(h(\mathcal{X}_1, \mathcal{Z}_2), \varphi\mathcal{Z}_1) + (t\mathcal{X}_1 \ln f)g(\mathcal{Z}_1, \mathcal{Z}_2)$
(iii) $g(h(\mathcal{Z}_1, \mathcal{Z}_2), nt\mathcal{X}_1) = g(h(t\mathcal{X}_1, W), \varphi Z) - \cos^2\theta\{(\mathcal{X}_1 \ln f) - \eta(\mathcal{X}_1)\}g(\mathcal{Z}_1, \mathcal{Z}_2)$
for any $\mathcal{X}_1 \in T\mathcal{V}_\theta$ and $\mathcal{Z}_1, \mathcal{Z}_2 \in T\mathcal{V}_\perp$.

It is well known that the following results are satisfied for a warped product submanifold, we have

$$K(U_2 \wedge W_1) = g(\nabla_W \nabla_{\mathcal{Z}_1} U_1 - \nabla_{U_1}\nabla_{W_1} U_1, W_1) = \frac{1}{\rho}\Big\{(\nabla_{U_1} U_1)f - U_1^2 f\Big\}, \tag{49}$$

$$\tau(T\mathcal{V}) = \widetilde{\tau}(T\mathcal{V}) + \sum_{b_1=1}^{2d+1} \sum_{1\leq a_1\neq a_2\leq n} (h^{b_1}_{a_1a_1} h^{b_1}_{a_2a_2} - (h^{b_1}_{a_1a_2})^2), \tag{50}$$

from (34), we have

$$K(E_{a_1} \wedge E_{a_2}) = \widetilde{K}(E_{a_1} \wedge E_{a_2}) + \sum_{b_1=1}^{2d+1} \sum_{1\leq a_1\neq a_2\leq n} (h^{b_1}_{a_1a_1} h^{b_1}_{a_2a_2} - (h^{b_1}_{a_1a_2})^2) \tag{51}$$

for each $a_2 = \beta_1 + 1 \cdots m$ and from (49), we have

$$\sum_{a_1=1}^{\beta_1} \sum_{a_2=\beta_1+1}^{m} K(E_{a_1} \wedge E_{a_2}) = \frac{\beta_2 \Delta\rho}{\rho}. \tag{52}$$

For mixed totally geodesic warped product pseudo-slant submanifolds in Kenmotsu space forms, we estimate the squared norm of the second fundamental form as follows;

Theorem 3.5. *Let $\mathcal{V} = \mathcal{V}_\theta \times_\rho \mathcal{V}_\perp$ be a mixed totally geodesic pseudo-slant warped product submanifold in Kenmotsu space form $\mathcal{K}(c)$, then*

(i) the length of h fulfills

$$||h||^2 \geq \beta_2\left(\frac{(c-3)\beta_1}{2} + \frac{c+1}{2}\right) - \frac{2\beta_2\Delta\rho}{\rho} - m^2||H||^2 + 2\beta_2\cot^2\theta\Big(||\nabla(\ln\rho)||^2 - 1\Big). \tag{53}$$

(ii) the equality holds in (53) *if and only if* $\mathcal{V}_\theta$ *is totally geodesic,* $\mathcal{V}_\perp^{\beta_2}$ *is totally umbilical in* $\mathcal{K}(c)$*.*

Proof. In view of Eq. (25), (31) and (32), we find that

$$\|h\|^2 = 2\widetilde{\tau}(T_x\mathcal{V}) - 2\tau(T_x\mathcal{V}) + m^2\|H\|^2. \tag{54}$$

By the consequence of (34), we obtain

$$\begin{aligned}\|h\|^2 =& 2\widetilde{\tau}(T_x\mathcal{V}) - 2\tau(T_x\mathcal{V}) + m^2\|H\|^2 - 2\sum_{A=1}^{\beta_1}\sum_{B=\beta_1+1}^{m} K(e_A \wedge e_B)\\ &- 2\sum_{1\leq A\neq C\leq\beta_1} K(e_A \wedge e_C) - \sum_{\beta_1+1\leq B\neq D\leq m} K(e_B \wedge e_D) + m^2\|H\|^2.\end{aligned} \tag{55}$$

From (52) and (34), we derive as follows

$$\|h\|^2 = 2\widetilde{\tau}(T_x\mathcal{V}) - \frac{2\beta_2\Delta\rho}{\rho} - 2\tau(T_x\mathcal{V}_\theta) - 2\tau(T_x\mathcal{V}_\perp) + m^2\|H\|^2. \tag{56}$$

Thus from (50) and (56), we derive

$$\begin{aligned}\|h\|^2 =& 2\widetilde{\tau}(T_x\mathcal{V}) - 2\Big(\sum_{r=n+1}^{2d+1}\sum_{1\leq a\neq b\leq\beta_1}\Big(h^r_{aa}h^r_{bb} - (h^r_{ab})^2\Big)\Big) - 2\Big(\sum_{r=n+1}^{2d+1}\sum_{\beta_1+1\leq s_1\neq s_2\leq n}\Big(h^r_{s_1s_1}h^r_{s_2s_2} - (h^r_{s_1s_2})^2\Big)\Big)\\ &- \frac{2\beta_2\Delta\rho}{\rho} - 2\tau(T_x{\mathcal{V}_\theta}^{\beta_1}) - 2\tau(T_x{\mathcal{V}_\perp}^{\beta_2}) + m^2\|H\|^2.\end{aligned} \tag{57}$$

Now, consider

$$\Big(\sum_{r=n+1}^{2d+1}\sum_{1\leq a\neq b\leq\beta_1}\Big(h^r_{aa}h^r_{bb} - (h^r_{ab})^2\Big)\Big) = \sum_{r=n+1}^{2d+1}\left(\sum_{1\leq a\neq b\leq\beta_1} h^r_{aa}h^r_{bb} - \sum_{1\leq a\neq b\leq\beta_1}(h^r_{ab})^2\right).$$

Adding and subtracting the same term in the above equation, we get

$$\begin{aligned}\Big(\sum_{r=n+1}^{2d+1}\sum_{1\leq a\neq b\leq\beta_1}\Big(h^r_{aa}h^r_{bb} - (h^r_{ab})^2\Big)\Big) =& \sum_{r=n+1}^{2d+1}\sum_{1\leq a\neq b\leq\beta_1} h^r_{aa}h^r_{bb} + \sum_{r=n+1}^{2d+1}\sum_{1\leq l_1\neq k_1\leq\beta_1}\Big(h^r_{l_1k_1}\Big)^2\\ &- \sum_{r=n+1}^{2d+1}\sum_{1\leq a\neq b\leq\beta_1}(h^r_{ab})^2 - \sum_{r=n+1}^{2d+1}\sum_{1\leq l_1\neq k_1\leq\beta_1}\Big(h^r_{l_1k_1}\Big)^2.\end{aligned}$$

Now, using the binomial theorem in the last two terms of the right-hand side in the last equation and computing other terms, also, then

$$\Big(\sum_{r=n+1}^{2d+1}\sum_{1\leq a\neq b\leq\beta_1}\Big(h^r_{aa}h^r_{bb} - (h^r_{ab})^2\Big)\Big) = \sum_{r=n+1}^{2d+1}\Big(h^r_{11} + \cdots + h^r_{\beta_1\beta_1}\Big)^2 - \sum_{r=n+1}^{2d+1}\sum_{a,b=}^{\beta_1}\Big(h^r_{ab}\Big)^2. \tag{58}$$

Similarly, we derive the following results.

$$\Big(\sum_{r=n+1}^{2d+1}\sum_{\beta_1+1\leq s_1\neq s_2\leq n}\Big(h^r_{s_1s_1}h^r_{s_2s_2} - (h^r_{s_1s_2})^2\Big)\Big) = \sum_{r=n+1}^{2d+1}\Big(h^r_{\beta_1+1\beta_1+1} + \cdots + h^r_{mm}\Big)^2 - \sum_{r=n+1}^{2d+1}\sum_{s,t=\beta_1+1}^{m}\Big(h^r_{s_1s_2}\Big)^2. \tag{59}$$

By the consequence of (57), (58), and (59), one can easily derive the following.

$$\|h\|^2 = 2\widetilde{\tau}(T_x\mathcal{V}) - \frac{2\beta_2\Delta\rho}{\rho} - 2\widetilde{\tau}(T_x\mathcal{V}_\theta) - 2\widetilde{\tau}(T_x\mathcal{V}_\perp) + m^2\|H\|^2 + 2\sum_{r=n+1}^{2d+1}\sum_{a,b=}^{\beta_1}\left(h^r_{ab}\right)^2 + 2\sum_{r=n+1}^{2d+1}\sum_{s,t=\beta_1+1}^{m}\left(h^r_{s_1s_2}\right)^2$$
$$-2\sum_{r=n+1}^{2d+1}\left(h^r_{11}+\cdots+h^r_{\beta_1\beta_1}\right)^2 - 2\sum_{r=m+1}^{2d+1}\left(h^r_{\beta_1+1\beta_1+1}+\cdots+h^r_{mm}\right)^2. \tag{60}$$

Now, utilizing $x^2+y^2=(x+y)^2-2xy$ into the above relation by taking $x=h_{ii}$ and $y=h_{ss}$, we compute

$$\|h\|^2 = 2\widetilde{\tau}(T_x\mathcal{V}) - \frac{2\beta_2\Delta\rho}{\rho} - 2\widetilde{\tau}(T_x\mathcal{V}_\theta) - 2\widetilde{\tau}(T_x\mathcal{V}_\perp) - m^2\|H\|^2$$
$$+2\sum_{r=n+1}^{2d+1}\sum_{a,b=}^{\beta_1}\left(h^r_{ab}\right)^2 + 2\sum_{r=n+1}^{2d+1}\sum_{s,t=\beta_1+1}^{m}\left(h^r_{s_1s_2}\right)^2 + 4\sum_{b_1=m+1}^{2d+1}\sum_{a_1=1}^{\beta_1}\sum_{a_2=1}^{\beta_2}h^r_{aa}h^r_{bb}. \tag{61}$$

Putting $\mathcal{Z}_1=\mathcal{Z}_4=E_i, \mathcal{Z}_2=\mathcal{Z}_3=E_j$ in the above equation, we derive

$$\begin{aligned}\widetilde{R}\left(E_i,E_j,E_j,E_i\right) = {} & \frac{c+3}{4}\left(g\left(E_i,E_i\right)g\left(E_j,E_j\right)-g\left(E_i,E_j\right)g\left(E_j,E_i\right)\right)\\ & +\frac{c+1}{4}\Big(\eta\left(E_i\right)\eta\left(E_j\right)g\left(E_i,E_j\right)-\eta\left(E_j\right)\eta\left(E_j\right)g\left(E_i,E_i\right)\\ & +\eta\left(E_i\right)\eta\left(E_j\right)g\left(E_i,E_j\right)-\eta\left(E_i\right)\eta\left(E_i\right)g\left(E_j,E_j\right)\\ & +g\left(\varphi E_j,E_j\right)g\left(\varphi E_i,E_i\right)-g\left(\varphi E_i,E_j\right)g\left(\varphi E_j,E_i\right)+2g^2\left(E_i,\varphi E_j\right)\Big).\end{aligned} \tag{62}$$

Summing up over basis vectors of $T\mathcal{V}$ such that $1\le i\ne j\le n$, we have

$$2\widetilde{\tau}(TM) = \frac{c+3}{4}n(n-1) + \frac{c-1}{4}\left(3\sum_{1\le i\ne j\le n}g^2\left(\varphi E_i,E_j\right)-2(n-1)\right). \tag{63}$$

Since $\mathcal{V}$ is a pseudo-slant warped product submanifold in $\mathcal{K}(c)$. For this, we consider the orthonormal frame for $T\mathcal{N}$ as follows:

$$\begin{aligned}\mathfrak{B}_\theta &= span\{E_1=\hat{E}_1,\cdots,E_s=\hat{E}_s,E_{s+1}=\sec\theta t\hat{E}_1,\cdots,E_{2s}=\sec\theta t\hat{E}_s,\xi\},\\ \mathfrak{D}_\perp &= span\{E_{2s+2}=E^*_1,e_{2s+3}=E^*_2,\cdots,e_{2s+\beta_2}=E^*_{\beta_2}\},\end{aligned}$$

Additionally, the orthonormal frame for $T\mathcal{N}^\perp$ defined as follows:

$$\begin{aligned}n\mathfrak{D}_\perp &= span\{\tilde{E}_1=\mathcal{J}E^*_1,\tilde{E}_2=\mathcal{J}E^*_2,\cdots,\tilde{E}_{\beta_2}=\mathcal{J}E^*_{\beta_2}\},\\ n\mathfrak{B}_\theta &= span\{\tilde{E}_{\beta_2+1}=\csc\theta n\hat{E}_1,\cdots,\tilde{E}_{\beta_2+s}=\csc\theta\hat{E}_s,\tilde{E}_{\beta_2+s+1}=\csc\theta\sec\theta nt\hat{E}_1,\\ &\qquad\cdots,\tilde{E}_{\beta_2+2s}=\csc\theta\sec\theta nt\hat{E}_s\},\\ \nu &= \{\tilde{e}_{\beta_2+2s}=\overline{E}_1,\cdots,\tilde{E}_{\beta_2+2s+k}=\overline{E}_k,\tilde{E}_{q+2s+k+1}=\overline{E}_{k+1},\cdots,\tilde{E}_{\beta_2+2s+2k}=\overline{E}_{2k}\}.\end{aligned}$$

$$\begin{aligned}g^2\left(\varphi E_i,E_{i+1}\right) &= 0,\ \text{for } i\in\{1,\ldots,2\alpha\}\\ &= \cos^2\theta \text{ for } \mathrm{i}\in\{2\alpha+1,\ldots,2\alpha+2\beta-1\}.\end{aligned}$$

Then one obtains

$$\sum_{i,j=1}^{n}g^2\left(tE_i,E_j\right) = \beta_1\cos^2\theta. \tag{64}$$

From (63) and (64), it follows that

$$2\widetilde{\rho}(T\mathcal{V}) = \frac{c-3}{4}n(n-1) + \frac{c+1}{4}\Big(3\beta_1\cos^2\theta - 2(n-1)\Big). \tag{65}$$

Summing up over the vector fields on $T\mathcal{V}_\perp$ in the above equation, one obtains

$$2\tilde{\tau}\left(T\mathcal{V}_\perp\right) = \frac{c-3}{4}\beta_2\left(\beta_2 - 1\right). \tag{66}$$

Similarly, for a slant submanifold, we have

$$\tilde{\tau}\left(T\mathcal{V}_\theta\right) = \frac{c-3}{4}\left\{\beta_1\Big\{(\beta_1-1)\Big\}\right\} + \frac{c+1}{4}\Big(3\beta_1\cos^2\theta - 2(\beta_1-1)\Big). \tag{67}$$

$$\begin{aligned} 2\sum_{r=n+1}^{2d+1}\sum_{a,b=}^{\beta_1}\left(h^r_{ab}\right)^2 + 2\sum_{r=n+1}^{2d+1}\sum_{s,t=\beta_1+1}^{m}\left(h^r_{s_1s_2}\right)^2 + 4\sum_{b_1=m+1}^{2d+1}\sum_{a_1=1}^{\beta_1}\sum_{a_2=1}^{\beta_2}h^r_{aa}h^r_{bb} \\ = 2\sum_{a,b=1}^{\beta_1}\sum_{r=1}^{2d+1}g(h(E_a,E_b),E_r)^2 + 2\sum_{a,b=1}^{\beta_2}\sum_{r=1}^{2d+1}g(h(E_a,E_b),E_r)^2 \\ + 4\sum_{r=m+1}^{2d+1}\sum_{a=1}^{\beta_1}\sum_{b=1}^{\beta_2}g(h(E_a,E_a),E_r)g(h(E_b,E_b),E_r). \end{aligned}$$

By using the adopted frame, we compute

$$\begin{aligned} 2\sum_{r=n+1}^{2d+1}\sum_{a,b=}^{\beta_1}\left(h^r_{ab}\right)^2 + 2\sum_{r=n+1}^{2d+1}\sum_{s,t=\beta_1+1}^{m}\left(h^r_{s_1s_2}\right)^2 + 4\sum_{b_1=m+1}^{2d+1}\sum_{a_1=1}^{\beta_1}\sum_{a_2=1}^{\beta_2}h^r_{aa}h^r_{bb} \\ = 2\sum_{a,b=1}^{\beta_1}\sum_{r=1}^{\beta_2}g(h(E_a,E_b),E_r)^2 + 2\sum_{a,b=1}^{\beta_1}\sum_{r=1}^{k}g(h(E_a,E_b),E_r)^2 \\ + 2\sum_{a,b,r=1}^{\beta_1}g(h(E_a,E_b),E_r)^2 + 2\sum_{a,b=1}^{\beta_2}\sum_{r=1}^{\beta_1}g(h(E_a,E_b),E_r)^2 \\ + 2\sum_{a,b=1}^{\beta_2}\sum_{r=1}^{k}g(h(E_a,E_b),E_r)^2 + 2\sum_{a,b,r=1}^{\beta_2}g(h(E_a,E_b),E_r)^2 \\ + 4\sum_{r=m+1}^{k}\sum_{a=1}^{\beta_1}\sum_{b=1}^{\beta_2}g(h(E_a,E_a),E_r)g(h(E_b,E_b),E_r) \\ + 4\sum_{a,r=1}^{\beta_1}\sum_{b=1}^{\beta_2}g(h(E_a,E_a),E_r)g(h(E_b,E_b),E_r) \\ + 4\sum_{a=1}^{\beta_1}\sum_{b,r=1}^{\beta_2}g(h(E_a,E_a),E_r)g(h(E_b,E_b),E_r). \end{aligned} \tag{68}$$

We did not find any relation for the ν-component, so leave these quantities

$$2\sum_{r=n+1}^{2d+1}\sum_{a,b=}^{\beta_1}\left(h^r_{ab}\right)^2 + 2\sum_{r=n+1}^{2d+1}\sum_{s,t=\beta_1+1}^{m}\left(h^r_{s_1s_2}\right)^2 + 4\sum_{b_1=m+1}^{2d+1}\sum_{a_1=1}^{\beta_1}\sum_{a_2=1}^{\beta_2}h^r_{aa}h^r_{bb}$$

$$= 2\sum_{a,b=1}^{\beta_1}\sum_{r=1}^{\beta_2} g(h(E_a,E_b),E_r)^2 + 2\sum_{a,b,r=1}^{\beta_1} g(h(E_a,E_b),E_r)^2$$

$$+ 2\sum_{a,b=1}^{\beta_2}\sum_{r=1}^{\beta_1} g(h(E_a,E_b),E_r)^2 + 2\sum_{a,b,r=1}^{\beta_2} g(h(E_a,E_b),E_r)^2$$

$$+ 4\sum_{a,r=1}^{\beta_1}\sum_{b=1}^{\beta_2} g(h(E_a,E_a),E_r)g(h(E_b,E_b),E_r)$$

$$+ 4\sum_{a=1}^{\beta_1}\sum_{b,r=1}^{\beta_2} g(h(E_a,E_a),E_r)g(h(E_b,E_b),E_r).$$

Applying lemma 3.3, we have

$$2\sum_{r=n+1}^{2d+1}\sum_{a,b=}^{\beta_1}\left(h^r_{ab}\right)^2 + 2\sum_{r=n+1}^{2d+1}\sum_{s,t=\beta_1+1}^{m}\left(h^r_{s_1s_2}\right)^2 + 4\sum_{b_1=m+1}^{2d+1}\sum_{a_1=1}^{\beta_1}\sum_{a_2=1}^{\beta_2} h^r_{aa}h^r_{bb}$$

$$= 2\sum_{a,b,r=1}^{\beta_1} g(h(E_a,E_b),E_r)^2 + 2\sum_{a,b=1}^{\beta_2}\sum_{r=1}^{\beta_1} g(h(E_a,E_b),E_r)^2$$

$$+ 2\sum_{a,b,r=1}^{\beta_2} g(h(E_a,E_b),E_r)^2 + 4\sum_{a,r=1}^{\beta_1}\sum_{b=1}^{\beta_2} g(h(E_a,E_a),E_r)g(h(E_b,E_b),E_r). \tag{69}$$

We leave 1^{st}, 3^{rd} and 4^{th} of above expression

$$2\sum_{r=n+1}^{2d+1}\sum_{a,b=}^{\beta_1}\left(h^r_{ab}\right)^2 + 2\sum_{r=n+1}^{2d+1}\sum_{s,t=\beta_1+1}^{m}\left(h^r_{s_1s_2}\right)^2 + 4\sum_{b_1=m+1}^{2d+1}\sum_{a_1=1}^{\beta_1}\sum_{a_2=1}^{\beta_2} h^r_{aa}h^r_{bb}$$

$$\geq 2\sum_{a,b=1}^{\beta_2}\sum_{r=1}^{\beta_1} g(h(E_a,E_b),E_r)^2 + 4\sum_{a,r=1}^{\beta_1}\sum_{b=1}^{\beta_2} g(h(E_a,E_a),E_r)g(h(E_b,E_b),E_r).$$

By using the adopted frame, we compute

$$2\sum_{r=n+1}^{2d+1}\sum_{a,b=}^{\beta_1}\left(h^r_{ab}\right)^2 + 2\sum_{r=n+1}^{2d+1}\sum_{s,t=\beta_1+1}^{m}\left(h^r_{s_1s_2}\right)^2 + 4\sum_{b_1=m+1}^{2d+1}\sum_{a_1=1}^{\beta_1}\sum_{a_2=1}^{\beta_2} h^r_{aa}h^r_{bb}$$

$$\geq 2\csc^2\theta\sum_{a,b=1}^{\beta_2}\sum_{r=1}^{s} g(h(E^*_a,E^*_b),n\hat{E}_r)^2 + 2\csc^2\theta\sec^2\theta\sum_{a,b=1}^{\beta_2}\sum_{r=1}^{s} g(h(E^*_a,E^*_b),nt\hat{E}_r)^2.$$

By the application of Lemma 3.4, we concede

$$2\sum_{r=n+1}^{2d+1}\sum_{a,b=}^{\beta_1}\left(h^r_{ab}\right)^2 + 2\sum_{r=n+1}^{2d+1}\sum_{s,t=\beta_1+1}^{m}\left(h^r_{s_1s_2}\right)^2 + 4\sum_{b_1=m+1}^{2d+1}\sum_{a_1=1}^{\beta_1}\sum_{a_2=1}^{\beta_2} h^r_{aa}h^r_{bb}$$

$$\geq 2\csc^2\theta\sum_{a,b=1}^{\beta_2}\sum_{r=1}^{s}(t\hat{E}_r(\ln f))^2 g(E^*_a,E^*_b)^2$$

$$+ 2\csc^2\theta\cos^2\theta\sum_{a,b=1}^{\beta_2}\sum_{r=1}^{s}(\hat{E}_r(\ln f))^2 g(E^*_a,E^*_b)^2.$$

By adding and subtracting the same quantity from the above expression, we achieve

$$\begin{aligned}
&2\sum_{r=n+1}^{2d+1}\sum_{a,b=}^{\beta_1}\left(h^r_{ab}\right)^2+2\sum_{r=n+1}^{2d+1}\sum_{s,t=\beta_1+1}^{m}\left(h^r_{s_1s_2}\right)^2+4\sum_{b_1=m+1}^{2d+1}\sum_{a_1=1}^{\beta_1}\sum_{a_2=1}^{\beta_2}h^r_{aa}h^r_{bb}\\
&\quad\geq 2\csc^2\theta\sum_{a,b=1}^{\beta_2}\sum_{r=1}^{\beta_1}g(\hat{E}_r,t\nabla(\ln f))^2g(E^*_a,E^*_b)^2\\
&\quad\quad+2\csc^2\theta\cos^2\theta\sum_{a,b=1}^{\beta_2}\sum_{r=1}^{s}(\hat{E}_r(\ln f))^2g(E^*_a,E^*_b)^2\\
&\quad\quad-2\csc^2\theta\sum_{a,b=1}^{\beta_2}\left(\sum_{r=1}^{s}g(\hat{E}_r,t\nabla(\ln f))^2g(E^*_a,E^*_b)^2-(\xi(\ln f))^2g(E^*_a,E^*_b)^2\right).
\end{aligned}$$

This implies the following inequality

$$2\sum_{r=n+1}^{2d+1}\sum_{a,b=}^{\beta_1}\left(h^r_{ab}\right)^2+2\sum_{r=n+1}^{2d+1}\sum_{s,t=\beta_1+1}^{m}\left(h^r_{s_1s_2}\right)^2+4\sum_{b_1=m+1}^{2d+1}\sum_{a_1=1}^{\beta_1}\sum_{a_2=1}^{\beta_2}h^r_{aa}h^r_{bb}\ \geq 2\beta_2\cot^2\theta\|\nabla(\ln f)-1)\|^2. \tag{70}$$

Combining (61), (63), (66), (67) and (70), we obtain (53). If the equality holds in (53), then from (68), we have

$$g\Big(h(\mathfrak{B}_\theta,\mathfrak{B}_\theta),F\mathfrak{B}_\theta\Big)=0,\quad\Longrightarrow\quad h(\mathfrak{B}_\theta,\mathfrak{B}_\theta)\subset J\mathfrak{B}_\perp\oplus\nu, \tag{71}$$

and

$$g\Big(h(\mathfrak{B}_\theta,\mathfrak{B}_\theta),\nu\Big)=0,\quad\Longrightarrow\quad h(\mathfrak{B}_\theta,\mathfrak{B}_\theta)\subset\mathcal{J}\mathfrak{B}_\perp\oplus F\mathfrak{B}_\theta. \tag{72}$$

It concluded from (71) and (72), we get $h(\mathfrak{B}_\theta,\mathfrak{B}_\theta)\subset\mathcal{J}\mathfrak{B}_\perp$. It is noticed that $h(\mathfrak{B}_\theta,\mathfrak{B}_\theta)\perp\mathcal{J}\mathfrak{B}_\perp$ from Lemma 3.3 for mixed total geodesic warped product pseudo-slant submanifolds. Then, finally, we have the following

$$h(\mathfrak{B}_\theta,\mathfrak{B}_\theta)=0. \tag{73}$$

As we know that $\mathcal{V}_\theta$ is totally geodesic in $\mathcal{V}$ with the fact that (73), it concludes that $\mathcal{V}_\theta$ is totally geodesic in $\widetilde{V}$. From (69), we derive

$$g\Big(h(\mathfrak{B}_\perp,\mathfrak{B}_\perp),\mathcal{J}\mathfrak{B}_\perp\Big)=0,\quad\Longrightarrow\quad h(\mathfrak{B}_\perp,\mathfrak{B}_\perp)\subset n\mathfrak{B}_\theta\oplus\nu, \tag{74}$$

and

$$g\Big(h(\mathfrak{B}_\perp,\mathfrak{B}_\perp),\nu\Big)=0,\quad\Longrightarrow\quad h(\mathfrak{B}_\perp,\mathfrak{B}_\perp)\subset\mathcal{J}\mathfrak{B}_\perp\oplus F\mathfrak{B}_\theta. \tag{75}$$

This implies that $h(\mathfrak{B}_\perp,\mathfrak{B}_\perp)\subset n\mathfrak{B}_\theta$. Now replacing X by tX in Lemma 3.4 with mixed totally geodesic case. then we get

$$g(h(Z_1,W_1),ntU_1)=\cos^2\theta(U_1\ln\rho)g(Z_1,W_1), \tag{76}$$

for any $U_1\in\Gamma(\mathfrak{B}_\theta)$ and $Z_1,W_1\in\Gamma(\mathfrak{B}_\perp)$. From the last two above facts and $\mathcal{V}_\perp$ is totally umbilical in $\mathcal{V}$, therefore, $\mathcal{V}_\perp$ is totally umbilical in $\widetilde{V}$. This completes the proof of the theorem. □

Theorem 3.6. *Let $\mathcal{K}(c)$ be a Kenmotsu space form and $\mathcal{V}$ be a compact orientable pseudo-slant warped product submanifold without boundary in $\mathcal{K}(c)$ such that ρ satisfies the Euler-Lagrange equation. If $\psi:\mathcal{V}=\mathcal{V}_\theta\times_\rho\mathcal{V}_\perp\longrightarrow\mathcal{K}(c)$ is an isometric immersion with mixed totally geodesic, then the necessary condition of $\mathcal{V}$ to be a pseudo-slant product is*

$$\|h\|^2\leq\beta_2\left(\frac{(c-3)\beta_1}{2}+\frac{c+1}{2}\right)-m^2\|H\|^2-2\beta_2\cot^2\theta. \tag{77}$$

Proof. By the relation (53), we have

$$\|h\|^2 \geq \beta_2\left(\frac{(c-3)\beta_1}{2}+\frac{c+1}{2}\right)-m^2\|H\|^2+2\beta_2\cot^2\theta\left(\|\nabla(\ln\rho)\|^2-1\right).$$

From above relation, we easily visualise that the relation (77) holds if and only if $2\beta_2\cot^2\|\nabla(\ln\rho)\|^2 \leq 0$. Since both the quantities of the last expression are positive, then by doing some computation, one can easily obtain $\|\nabla(\ln\rho)\| = 0$. □

Corollary 3.7. *Let $\mathcal{K}(c)$ be a Kenmotsu space form and $\mathcal{V}$ be a compact orientable pseudo-slant warped product submanifold without boundary in $\mathcal{K}(c)$ such that ρ satisfies the Euler-Lagrange equation. If $\psi : \mathcal{V} = \mathcal{V}_\theta \times_\rho \mathcal{V}_\perp \longrightarrow \mathcal{K}(c)$ is a minimal isometric immersion with mixed totally geodesic, then the necessary condition of $\mathcal{V}$ to be a pseudo-slant product is*

$$\|h\|^2 \leq \beta_2\left(\frac{(c-3)\beta_1}{2}+\frac{c+1}{2}\right)-2\beta_2\cot^2\theta.$$

Theorem 3.8. *Let $\mathcal{K}(c)$ be a Kenmotsu space form and $\mathcal{V}$ be a compact orientable pseudo-slant warped product submanifold without boundary in $\mathcal{K}(c)$. If $\psi : \mathcal{V} = \mathcal{V}_\theta \times_\rho \mathcal{V}_\perp \longrightarrow \mathcal{K}(c)$ is an isometric immersion with mixed totally geodesic, then the necessary condition of $\mathcal{V}$ to be a pseudo-slant product is*

$$\|h\|^2 \leq \beta_2\left(\frac{(c-3)\beta_1}{2}+\frac{c+1}{2}\right)-2\beta_2\cot^2\theta-m^2\|H\|^2. \tag{78}$$

Proof. Now taking integration on both sides of the relation (53), we find

$$\begin{aligned}\int_{\mathcal{V}_\theta\times\{q\}}\|h\|^2dV \geq& \int_{\mathcal{V}_\theta\times\{q\}}2\beta_2\cot^2\theta\left(\|\nabla(\ln\rho)\|^2-1\right)dV-\int_{\mathcal{V}_\theta\times\{q\}}\frac{2\beta_2\Delta\rho}{\rho}dV\\ &+\int_{\mathcal{V}_\theta\times\{q\}}\left(\beta_2\left(\frac{(c-3)\beta_1}{2}+\frac{c+1}{2}\right)-m^2\|H\|^2\right)dV.\end{aligned} \tag{79}$$

By doing simple computations, we receive

$$\begin{aligned}\int_{\mathcal{V}_\theta\times\{q\}}2\beta_2\Delta(\ln\rho)dV \geq& \int_{\mathcal{V}_\theta\times\{q\}}\left(\beta_2\left(\frac{(c-3)\beta_1}{2}+\frac{c+1}{2}\right)-m^2\|H\|^2-\|h\|^2\right)dV\\ &+2\beta_2\int_{\mathcal{V}_\theta\times\{q\}}\left((1+\cot^2\theta)\|\nabla(\ln\rho)\|^2dV-\cot^2\theta\right)dV.\end{aligned} \tag{80}$$

By the application of (41), we have

$$\begin{aligned}0 \geq& \int_{\mathcal{V}_\theta\times\{q\}}\left(\beta_2\left(\frac{(c-3)\beta_1}{2}+\frac{c+1}{2}\right)-m^2\|H\|^2-\|h\|^2\right)dV\\ &+2\beta_2\int_{\mathcal{V}_\theta\times\{q\}}\left((1+\cot^2\theta)\|\nabla(\ln\rho)\|^2dV-\cot^2\theta\right)dV.\end{aligned} \tag{81}$$

From (78) and (81), we have

$$\beta_2\int_{\mathcal{V}_\theta\times\{q\}}\left(1+\cot^2\theta\right)\|\nabla(\ln\rho)\|^2dV \leq 0.$$

Above expression holds if and only if $\left(1+2\cot^2\theta\right)\|\nabla(\ln\rho)\|^2 \leq 0$. Due the positiveness of $\left(1+2\cot^2\theta\right)$ and $\|\nabla(\ln\rho)\|^2$, we have $\|\nabla(\ln\rho)\| = 0$. From the above discussion, we conclude that ρ is constant. □

Corollary 3.9. *Let $\mathcal{K}(c)$ be a Kenmotsu space form and $\mathcal{V}$ be a compact orientable pseudo-slant warped product submanifold without boundary in $\mathcal{K}(c)$. If $\psi : \mathcal{V} = \mathcal{V}_\theta \times_\rho \mathcal{V}_\perp \longrightarrow \mathcal{K}(c)$ is a minimal isometric immersion with mixed totally geodesic, then the necessary condition of $\mathcal{V}$ to be a pseudo-slant product is*

$$\|h\|^2 \leq \beta_2\left(\frac{(c-3)\beta_1}{2} + \frac{c+1}{2}\right) - 2\beta_2 \cot^2\theta. \tag{82}$$

Theorem 3.10. *Let $\mathcal{K}(c)$ be a Kenmotsu space form and $\mathcal{V}$ be a compact orientable pseudo-slant warped product submanifold with boundary in $\mathcal{K}(c)$. If $\psi : \mathcal{V} = \mathcal{V}_\theta \times_\rho \mathcal{V}_\perp \longrightarrow \mathcal{K}(c)$ is an isometric immersion with mixed totally geodesic, then the necessary condition of $\mathcal{V}$ to be a pseudo-slant product is*

$$\begin{aligned} E(\ln\rho) \leq & \frac{\beta_2}{4\beta_2\left(1+\cot^2\theta\right)} \int_{\mathcal{V}_\theta\times\{q\}} \left(\frac{(c-3)\beta_1}{2} + \frac{c+1}{2}\right) dV \\ & + \frac{1}{4\beta_2\left(1+\cot^2\theta\right)} \int_{\mathcal{V}_\theta\times\{q\}} \left(m^2\|H\|^2 + \|h\|^2 + 2\beta_2\cot^2\theta\right) dV. \end{aligned} \tag{83}$$

Proof. Taking integration on both sides of the relation (53), we receive

$$\begin{aligned} \int_{\mathcal{V}_\theta\times\{q\}} \|h\|^2 dV \geq & \int_{\mathcal{V}_\theta\times\{q\}} \left(\beta_2\left(\frac{(c-3)\beta_1}{2} + \frac{c+1}{2}\right) - m^2\|H\|^2 - 2\beta_2\Delta(\ln\rho)\right) dV \\ & + 2\beta_2 \int_{\mathcal{V}_\theta\times\{q\}} \left((1+\cot^2\theta)\|\nabla(\ln\rho)\|^2 dV - \cot^2\theta\right) dV. \end{aligned} \tag{84}$$

By doing some simple computations, we achieve that

$$\begin{aligned} \int_{\mathcal{V}_\theta\times\{q\}} \|h\|^2 dV \geq & \int_{\mathcal{V}_\theta\times\{q\}} \left(\beta_2\left(\frac{(c-3)\beta_1}{2} + \frac{c+1}{2}\right) - m^2\|H\|^2 - 2\beta_2\Delta(\ln\rho)\right) dV \\ & + 2\beta_2\left(1+\cot^2\theta\right) \int_{\mathcal{V}_\theta\times\{q\}} \|\nabla(\ln\rho)\|^2 dV - 2\beta_2 \int_{\mathcal{V}_\theta\times\{q\}} \cot^2\theta dV \\ & + \int_{\mathcal{V}_\theta\times\{q\}} \left(4\beta_2\cot\theta\csc^2\theta\left(\frac{d\theta}{dV}\right) \int_{\mathcal{V}_\theta\times\{q\}} \|\nabla(\ln\rho)\|^2 dV\right) dV. \end{aligned} \tag{85}$$

By the utilization of (39), we have

$$\begin{aligned} \int_{\mathcal{V}_\theta\times\{q\}} \|h\|^2 + 2\beta_2\cot\theta dV \geq & \int_{\mathcal{V}_\theta\times\{q\}} \left(\beta_2\left(\frac{(c-3)\beta_1}{2} + \frac{c+1}{2}\right) - m^2\|H\|^2 - 2\beta_2\Delta(\ln\rho)\right) dV \\ & + 4\beta_2\left(1+\cot^2\theta\right) E(\ln\rho) + \int_{\mathcal{V}_\theta\times\{q\}} \left(8\beta_2\cot\theta\csc^2\theta\left(\frac{d\theta}{dV}\right) E(\ln\rho)\right) dV. \end{aligned} \tag{86}$$

From (83) and (86), we have

$$\beta_2 \int_{\mathcal{V}_\theta\times\{q\}} \Delta(\ln\rho) dV = 0.$$

By the immediate consequence of the Hopf lemma, we have $\ln\rho = 0$. This implies that $f = 1$, i.e. ρ is constant. □

Corollary 3.11. *Let $\mathcal{K}(c)$ be a Kenmotsu space form and $\mathcal{V}$ be a compact orientable pseudo-slant warped product submanifold without boundary in $\mathcal{K}(c)$. If $\psi : \mathcal{V} = \mathcal{V}_\theta \times_\rho \mathcal{V}_\perp \longrightarrow \mathcal{K}(c)$ is a minimal isometric immersion with mixed totally geodesic, then the necessary condition of $\mathcal{V}$ to be a pseudo-slant product is*

$$E(\ln\rho) \leq \frac{\beta_2}{4\beta_2\left(1+\cot^2\theta\right)} \int_{\mathcal{V}_\theta\times\{q\}} \left(\frac{(c-3)\beta_1}{2} + \frac{c+1}{2}\right) dV + \frac{1}{4\beta_2\left(1+\cot^2\theta\right)} \int_{\mathcal{V}_\theta\times\{q\}} \left(\|h\|^2 + 2\beta_2\cot^2\theta\right) dV. \tag{87}$$

Theorem 3.12. *Let $\mathcal{K}(c)$ be a Kenmotsu space form and $\mathcal{V}$ be a compact orientable pseudo-slant warped product submanifold with boundary in $\mathcal{K}(c)$. If $\psi : \mathcal{V} = \mathcal{V}_\theta \times_\rho \mathcal{V}_\perp \longrightarrow \mathcal{K}(c)$ is an isometric immersion with mixed totally geodesic, then the necessary condition of $\mathcal{V}$ to be a pseudo-slant product is*

$$\mathcal{H}(df,p) \leq \frac{\beta_2\left((c-3)\beta_1 + c + 1\right) + 2m^2\|H\|^2 + 2\|h\|^2 + 4\beta_2 \cot^2\theta}{8\beta_2\left(1+\cot^2\theta\right)}. \tag{88}$$

Proof. By the direct consequence of (43) and (53), we have

$$\|h\|^2 \geq \beta_2\left(\frac{(c-3)\beta_1}{2} + \frac{c+1}{2}\right) - 2\beta_2\Delta(\ln\rho) - m^2\|H\|^2 + 2\beta_2((1+\cot^2\theta)\mathcal{H}(df,p) - \cot^2\theta. \tag{89}$$

By the virtue of (88) and (89), we receive that $\beta_2\Delta(\ln\rho) = 0$. By the immediate consequence of the Hopf lemma, we have $\ln\rho = 0$. This implies to $f = 1$, i.e., ρ is constant. □

Corollary 3.13. *Let $\mathcal{K}(c)$ be a Kenmotsu space form and $\mathcal{V}$ be a compact orientable pseudo-slant warped product submanifold without boundary in $\mathcal{K}(c)$. If $\psi : \mathcal{V} = \mathcal{V}_\theta \times_\rho \mathcal{V}_\perp \longrightarrow \mathcal{K}(c)$ is a minimal isometric immersion with mixed totally geodesic, then the necessary condition of $\mathcal{V}$ to be a pseudo-slant product is*

$$\mathcal{H}(df,p) \leq \frac{\beta_2\left((c-3)\beta_1 + c + 1\right) + 2\|h\|^2 + 4\beta_2 \cot^2\theta}{8\beta_2\left(1+\cot^2\theta\right)}.$$

Theorem 3.14. *Let $\mathcal{K}(c)$ be a complex space form and $\mathcal{V}$ be a pseudo-slant warped product submanifold in $\mathcal{K}(c)$. If $\varphi : \mathcal{V} = \mathcal{V}_\theta^{\beta_1} \times_\rho \mathcal{V}_\perp^{\beta_2} \longrightarrow \mathcal{K}(c)$ is An isometric immersion with mixed totally geodesic, then the first eigenvalue satisfies*

$$\lambda_1 \leq \left\{\frac{\int_{\mathcal{V}_\theta\times\{q\}}\left(\|h\|^2 + 2\beta_2\cot^2\theta + m^2\|H\|^2 - \beta_2\left(\frac{(c-3)\beta_1}{2} + \frac{c+1}{2}\right)\right)dV}{2\beta_2(1+\cot^2\theta)\int_{\mathcal{V}_\theta\times\{q\}}(\ln\rho)^2 dV - 2\beta_2\int_{\mathcal{V}_\theta\times\{q\}}(\ln\rho)dV}\right\}. \tag{90}$$

Proof. From the minimum principle on the first eigenvalue λ_1, then we obtains

$$\int_{\mathcal{V}_\theta\times\{q\}}\|\nabla(\ln\rho)\|^2 dV \geq \int_{\mathcal{V}_\theta\times\{q\}}(\ln\rho)^2 dV. \tag{91}$$

By the consequence of (80) and (91), we concede that

$$\begin{aligned}\int_{\mathcal{V}_\theta\times\{q\}}&\left(\|h\|^2 + 2\beta_2\cot^2\theta + m^2\|H\|^2 - \beta_2\left(\frac{(c-3)\beta_1}{2} + \frac{c+1}{2}\right)\right)dV\\ &\geq 2\beta_2\lambda_1(1+\cot^2\theta)\int_{\mathcal{V}_\theta\times\{q\}}(\ln\rho)^2 dV - 2\beta_2\lambda_1\int_{\mathcal{V}_\theta\times\{q\}}(\ln\rho)dV.\end{aligned} \tag{92}$$

□

Corollary 3.15. *Let $\mathcal{K}(c)$ be a complex space form and $\mathcal{V}$ be a pseudo-slant warped product submanifold in $\mathcal{K}(c)$. If $\varphi : \mathcal{V} = \mathcal{V}_\theta^{\beta_1} \times_\rho \mathcal{V}_\perp^{\beta_2} \longrightarrow \mathcal{K}(c)$ is a minimal isometric immersion with mixed totally geodesic, then the first eigenvalue satisfies*

$$\lambda_1 \leq \frac{\int_{\mathcal{V}_\theta\times\{q\}}\left(\|h\|^2 + 2\beta_2\cot^2\theta - \beta_2\left(\frac{(c-3)\beta_1}{2} + \frac{c+1}{2}\right)\right)dV}{2\beta_2(1+\cot^2\theta)\int_{\mathcal{V}_\theta\times\{q\}}(\ln\rho)^2 dV - 2\beta_2\int_{\mathcal{V}_\theta\times\{q\}}(\ln\rho)dV}.$$

4. Results for $\mathcal{CR}$-warped product submanifolds in Kenmotsu space forms

It is well known that the geometry of $\mathcal{CR}$-warped product submanifolds, which is a special case of pseudo-slant warped product submanifolds. This section is devoted to $\mathcal{CR}$-warped product submanifolds in Kenmotsu space forms; we compute inequalities directly from the inequality (53). In the pseudo-slant $\mathcal{V} = \mathcal{V}_{\theta}^{\beta_1} \times_{\rho} \mathcal{V}_{\perp}^{\beta_2}$, we take $\theta = 0$ then it become $\mathcal{CR}$-warped product submanifolds. It is well known that $\cot^2\theta \geq \cos^2\theta$; so the inequality (53) reduces into the following form

$$\|h\|^2 \geq \beta_2\left(\frac{(c-3)\beta_1}{2} + \frac{c+1}{2}\right) - \frac{2\beta_2\Delta\rho}{\rho} - m^2\|H\|^2 + 2\beta_2\cos^2\theta\left(\|\nabla(\ln\rho)\|^2 - 1\right). \tag{93}$$

To achieve the inequality for $\mathcal{CR}$-warped product submanifolds in Kenmotsu space forms; we just substitute $\theta = 0$ in (93); which taking the following form

$$\|h\|^2 \geq \beta_2\left(\frac{(c-3)\beta_1}{2} + \frac{c+1}{2}\right) - \frac{2\beta_2\Delta\rho}{\rho} - m^2\|H\|^2 + 2\beta_2\left(\|\nabla(\ln\rho)\|^2 - 1\right). \tag{94}$$

Similarly, we compute other inequalities for $\mathcal{CR}$-warped product submanifolds in Kenmotsu space forms.

Corollary 4.1. *Let $\mathcal{K}(c)$ be a Kenmotsu space form and $\mathcal{V}$ be a compact orientable $\mathcal{CR}$-warped product submanifold without boundary in $\mathcal{K}(c)$ such that ρ satisfies the Euler-Lagrange equation. If $\psi : \mathcal{V} = \mathcal{V}_T \times_\rho \mathcal{V}_\perp \longrightarrow \mathcal{K}(c)$ is a minimal isometric immersion with mixed totally geodesic, then the necessary condition of $\mathcal{V}$ to be a $\mathcal{CR}$-product is*

$$\|h\|^2 \leq \beta_2\left(\frac{(c-3)\beta_1}{2} + \frac{c+1}{2}\right) - 2\beta_2. \tag{95}$$

Theorem 4.2. *Let $\mathcal{K}(c)$ be a Kenmotsu space form and $\mathcal{V}$ be a compact orientable $\mathcal{CR}$-warped product submanifold without boundary in $\mathcal{K}(c)$ such that ρ satisfies the Euler-Lagrange equation. If $\psi : \mathcal{V} = \mathcal{V}_T \times_\rho \mathcal{V}_\perp \longrightarrow \mathcal{K}(c)$ is an isometric immersion with mixed totally geodesic, then the necessary condition of $\mathcal{V}$ to be a $\mathcal{CR}$-product is*

$$\|h\|^2 \leq \beta_2\left(\frac{(c-3)\beta_1}{2} + \frac{c+1}{2}\right) - m^2\|H\|^2 - 2\beta_2. \tag{96}$$

Theorem 4.3. *Let $\mathcal{K}(c)$ be a Kenmotsu space form and $\mathcal{V}$ be a compact orientable $\mathcal{CR}$-warped product submanifold without boundary in $\mathcal{K}(c)$. If $\psi : \mathcal{V} = \mathcal{V}_T \times_\rho \mathcal{V}_\perp \longrightarrow \mathcal{K}(c)$ is an isometric immersion with mixed totally geodesic, then the necessary condition of $\mathcal{V}$ to be a $\mathcal{CR}$-product is*

$$\|h\|^2 \leq \beta_2\left(\frac{(c-3)\beta_1}{2} + \frac{c+1}{2}\right) - 2\beta_2 - m^2\|H\|^2. \tag{97}$$

Theorem 4.4. *Let $\mathcal{K}(c)$ be a Kenmotsu space form and $\mathcal{V}$ be a compact orientable $\mathcal{CR}$-warped product submanifold with boundary in $\mathcal{K}(c)$. If $\psi : \mathcal{V} = \mathcal{V}_T \times_\rho \mathcal{V}_\perp \longrightarrow \mathcal{K}(c)$ is an isometric immersion with mixed totally geodesic, then the necessary condition of $\mathcal{V}$ to be a $\mathcal{CR}$-product is*

$$E(\ln\rho) \leq \frac{1}{8}\int_{\mathcal{V}_T\times\{q\}}\left(\frac{(c-3)\beta_1}{2} + \frac{c+1}{2}\right)dV + \frac{1}{8\beta_2}\int_{\mathcal{V}_T\times\{q\}}\left(m^2\|H\|^2 + \|h\|^2 + 2\beta_2\right)dV. \tag{98}$$

Corollary 4.5. *Let $\mathcal{K}(c)$ be a Kenmotsu space form and $\mathcal{V}$ be a compact orientable $\mathcal{CR}$-warped product submanifold without boundary in $\mathcal{K}(c)$. If $\psi : \mathcal{V} = \mathcal{V}_T \times_\rho \mathcal{V}_\perp \longrightarrow \mathcal{K}(c)$ is a minimal isometric immersion with mixed totally geodesic, then the necessary condition of $\mathcal{V}$ to be a $\mathcal{CR}$-product is*

$$E(\ln\rho) \leq \frac{1}{8}\int_{\mathcal{V}_T\times\{q\}}\left(\frac{(c-3)\beta_1}{2} + \frac{c+1}{2}\right)dV + \frac{1}{8\beta_2}\int_{\mathcal{V}_T\times\{q\}}\left(\|h\|^2 + 2\beta_2\right)dV. \tag{99}$$

Theorem 4.6. *Let $\mathcal{K}(c)$ be a Kenmotsu space form and $\mathcal{V}$ be a compact orientable $\mathcal{CR}$-warped product submanifold with boundary in $\mathcal{K}(c)$. If $\psi : \mathcal{V} = \mathcal{V}_T \times_\rho \mathcal{V}_\perp \longrightarrow \mathcal{K}(c)$ is an isometric immersion with mixed totally geodesic, then the necessary condition of $\mathcal{V}$ to be a $\mathcal{CR}$-product is*

$$\mathcal{H}(df, p) \leq \frac{\beta_2\left((c-3)\beta_1 + c + 1\right) + 2m^2\|H\|^2 + 2\|h\|^2 + 4\beta_2}{16\beta_2}. \tag{100}$$

Corollary 4.7. *Let $\mathcal{K}(c)$ be a Kenmotsu space form and $\mathcal{V}$ be a compact orientable $\mathcal{CR}$-warped product submanifold without boundary in $\mathcal{K}(c)$. If $\psi : \mathcal{V} = \mathcal{V}_T \times_\rho \mathcal{V}_\perp \longrightarrow \mathcal{K}(c)$ is a minimal isometric immersion with mixed totally geodesic, then the necessary condition of $\mathcal{V}$ to be a $\mathcal{CR}$-product is*

$$\mathcal{H}(df, p) \leq \frac{\beta_2\left((c-3)\beta_1 + c + 1\right) + 2\|h\|^2 + 4\beta_2}{16\beta_2}. \tag{101}$$

Theorem 4.8. *Let $\mathcal{K}(c)$ be a complex space form and $\mathcal{V}$ be a $\mathcal{CR}$-warped product submanifolds in $\mathcal{K}(c)$. If $\varphi : \mathcal{V} = \mathcal{V}_T \times_\rho \mathcal{V}_\perp \longrightarrow \mathcal{K}(c)$ is An isometric immersion with mixed totally geodesic, then the first eigenvalue satisfies*

$$\lambda_1 \leq \left\{ \frac{\int_{\mathcal{V}_T \times \{q\}} \left(\|h\|^2 + 2\beta_2 + m^2\|H\|^2 - \beta_2\left(\frac{(c-3)\beta_1}{2} + \frac{c+1}{2}\right)\right) dV}{4\beta_2 \int_{\mathcal{V}_T \times \{q\}} (\ln \rho)^2 dV - 2\beta_2 \int_{\mathcal{V}_T \times \{q\}} (\ln \rho) dV} \right\}. \tag{102}$$

Corollary 4.9. *Let $\mathcal{K}(c)$ be a complex space form and $\mathcal{V}$ be a $\mathcal{CR}$-warped product submanifolds in $\mathcal{K}(c)$. If $\varphi : \mathcal{V} = \mathcal{V}_T \times_\rho \mathcal{V}_\perp \longrightarrow \mathcal{K}(c)$ is a minimal isometric immersion with mixed totally geodesic, then the first eigenvalue satisfies*

$$\lambda_1 \leq \left\{ \frac{\int_{\mathcal{V}_T \times \{q\}} \left(\|h\|^2 + 2\beta_2 - \beta_2\left(\frac{(c-3)\beta_1}{2} + \frac{c+1}{2}\right)\right) dV}{4\beta_2 \int_{\mathcal{V}_T \times \{q\}} (\ln \rho)^2 dV - 2\beta_2 \int_{\mathcal{V}_T \times \{q\}} (\ln \rho) dV} \right\}. \tag{103}$$

5. Examples

Example 5.1. *Let us consider $\mathbb{R}^7$ with the coordinate $(u_1, \cdots, u_3, v_1, \cdots, v_3, t)$. The almost contact metric structure (φ, ξ, η, g); defined by the following relations*

$$\varphi\left(\frac{\partial}{\partial u_i}\right) = \frac{\partial}{\partial v_i},\ \varphi\left(\frac{\partial}{\partial v_i}\right) = -\frac{\partial}{\partial u_i},\ \varphi\left(\frac{\partial}{\partial t}\right) = 0,$$
$$\xi = \frac{\partial}{\partial t},\ \eta = dt,\ g = dt^2 + e^{2t}\left(\sum_{i=1}^{3} du_i^2 + \sum_{i=1}^{3} dv_i^2\right).$$

By doing simple computation, we easily achieved that the $\mathbb{R}^7$ with (φ, ξ, η, g) forms a Kenmotsu manifold. Let $\mathcal{V}$ be a submanifold of the Kenmotsu manifold, which is defined by the following immersion

$$u_1 = y\cos x,\ u_2 = k,\ u_3 = y,\ v_1 = k,\ v_2 = y\sin x,\ v_3 = z.$$

The tangent bundle of $\mathcal{V}$ is expressed by the following vectors

$$D_x = -y\sin x\frac{\partial}{\partial u_1} + y\cos x\frac{\partial}{\partial v_2},$$
$$D_y = \cos x\frac{\partial}{\partial u_1} + \sin x\frac{\partial}{\partial v_2} + \frac{\partial}{\partial v_3},$$
$$D_z = \frac{\partial}{\partial u_3},\ D_t = \frac{\partial}{\partial t}.$$

The following vector fields express the transformed tangent bundle

$$\varphi D_x = y\cos x\frac{\partial}{\partial u_2} - y\sin x\frac{\partial}{\partial v_1},$$
$$\varphi D_y = \sin x\frac{\partial}{\partial u_2} + \frac{\partial}{\partial u_3} + \cos x\frac{\partial}{\partial v_1},$$
$$\varphi D_z = \frac{\partial}{\partial v_3},\ \varphi D_t = 0.$$

It is cleared that $\mathfrak{D}^\perp = span\{D_x\}$ *and* $\mathfrak{D}^\theta = \{D_y, D_z, D_t\}$, *where* $\mathfrak{D}^\perp$ *and* $\mathfrak{D}^\theta$ *are total real and slant distributions with slant angle* $\frac{\pi}{4}$. *Now it easily checks that a submanifold* $\mathcal{V}$ *from the above calculation turned into a pseudo-slant submanifold. Both distributions are integrals. The metric is calculated as*

$$g_{\mathcal{V}} = dt^2 + dx^2e^{2t}y^2 + 2dy^2e^{2t} + dz^2e^{2t}.$$

Therefore, $\mathcal{V}$ *is a warped product pseudo-slant submanifold in Kenmotsu manifold* $\mathbb{R}^7$ *including warping function* $\rho^2 = e^{2t}y^2$.

Example 5.2. *Let us consider* $\mathbb{R}^{11}$ *with the coordinate* $(u_1, \cdots, u_5, v_1, \cdots, v_5, t)$. *The almost contact metric structure* (φ, ξ, η, g); *defined by the following relations*

$$\varphi\left(\frac{\partial}{\partial u_i}\right) = \frac{\partial}{\partial v_i},\ \varphi\left(\frac{\partial}{\partial v_i}\right) = -\frac{\partial}{\partial u_i},\ \varphi\left(\frac{\partial}{\partial t}\right) = 0,\ \xi = \frac{\partial}{\partial t},\ \eta = dt,\ g = dt^2 + e^{2t}\left(\sum_{i=1}^{5} du_i^2 + \sum_{i=1}^{5} dv_i^2\right).$$

By doing simple computation, we easily achieved that the $\mathbb{R}^{11}$ *with* (φ, ξ, η, g) *forms a Kenmotsu manifold. Let* $\mathcal{V}$ *be a submanifold of the Kenmotsu manifold, which is defined by the following immersion*

$$u_1 = y\sin x,\ u_2 = k,\ u_3 = z\cos x,\ u_4 = k,\ u_5 = 3y,\ v_1 = k,$$
$$v_2 = y\cos x,\ v_3 = k,\ v_4 = z\sin x,\ v_5 = z,\ t = t.$$

The tangent bundle of $\mathcal{V}$ *is expressed by the following vectors*

$$D_x = y\cos x\frac{\partial}{\partial u_1} - z\sin x\frac{\partial}{\partial u_3} - y\sin x\frac{\partial}{\partial v_2} + z\cos x\frac{\partial}{\partial v_4},$$
$$D_y = \sin x\frac{\partial}{\partial u_1} + \frac{3\partial}{\partial u_5} + \cos x\frac{\partial}{\partial v_2},$$
$$D_z = \cos x\frac{\partial}{\partial u_3} + \sin x\frac{\partial}{\partial v_4} + \frac{v\partial}{\partial v_5},\ D_t = \frac{\partial}{\partial t}.$$

The following vector fields express the transformed tangent bundle

$$D_x = -y\sin x\frac{\partial}{\partial u_2} + z\cos x\frac{\partial}{\partial u_4} + y\cos x\frac{\partial}{\partial v_1} - z\sin x\frac{\partial}{\partial v_3},$$
$$D_y = \cos x\frac{\partial}{\partial u_2} + \sin x\frac{\partial}{\partial v_1} + \frac{3\partial}{\partial v_5},$$
$$D_z = \sin x\frac{\partial}{\partial u_4} + \frac{v\partial}{\partial u_5} + \cos x\frac{\partial}{\partial v_3},\ D_t = \frac{\partial}{\partial t}.$$

It is cleared that $\mathfrak{D}^\perp = span\{D_x\}$ *and* $\mathfrak{D}^\theta = \{D_y, D_z, D_t\}$, *where* $\mathfrak{D}^\perp$ *and* $\mathfrak{D}^\theta$ *are total real and slant distributions with slant angle* $\cos^{-1}\left(\frac{3}{2\sqrt{5}}\right)$. *Now it easily checks that a submanifold* $\mathcal{V}$ *from the above calculation turned into a pseudo-slant submanifold. Both distributions are integrals. The metric is calculated as*

$$g_{\mathcal{V}} = dt^2 + 10dy^2e^{2t} + 2dz^2e^{2t} + \left(e^{2t}(y^2 + z^2)\right)dx^2.$$

Therefore, $\mathcal{V}$ is a warped product pseudo-slant submanifold in the Kenmotsu manifold $\mathbb{R}^{11}$ including the warping function $\rho^2 = e^{2t}\left(y^2 + z^2\right)$.

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