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Approximation by generalized Kantorovich forms of exponential sampling series

Sadettin Kursun[a,b]

[a]*Izmir Katip Celebi University, Department of Information Security Technology, Cigli, 35620, Izmir, Türkiye*
[b]*Selcuk University, Constructive Mathematical Analysis Research Laboratory (CMARL), Selcuklu, 42003, Konya, Türkiye*

Abstract. In this article, we establish several new results concerning the generalized Kantorovich forms of exponential sampling operators. Especially, we enhance the rate of approximation by constructing suitable linear combinations of these operators. Finally, we illustrate the applicability of the proposed theory by presenting examples of kernel functions, accompanied by numerical computations and graphical representations.

1. Introduction

The concept of sampling-type series emerged as a solution to address the key limitations of the renowned Whittaker–Kotelnikov–Shannon (WKS) (see, [35, 41, 44]) sampling theorem. Let $f : \mathbb{R} \to \mathbb{R}$ be a signal that is both band-limited and of finite energy; the WKS sampling theorem asserts the interpolation formula given below:

$$f(y) = \sum_{k\in\mathbb{Z}} f\left(\frac{k}{w}\right) \operatorname{sinc}(wy-k), \ y \in \mathbb{R}, w > 0. \tag{1}$$

In addition to the conditions imposed on f, as a consequence of the Paley-Wiener theorem [40], this implies that the signal is restricted to the real axis of an exponential-type function, meaning that, in practice, only very regular signals can be reconstructed. These considerations have inspired numerous follow-up studies with the same objective: to develop families of sampling-type formulas capable of reconstructing signals that lack such high levels of regularity.

For these reasons, Butzer and his school at RWTH Aachen [25] introduced generalized sampling series as follows, replacing the sinc function with continuous kernels that satisfy the typical assumptions of approximate identities, in order to provide interpolation formulas for functions that do not need to be band-limited:

$$\left(G_w^{\varphi} f\right)(y) := \sum_{k\in\mathbb{Z}} f\left(\frac{k}{w}\right) \varphi(wy-k), \ y \in \mathbb{R}, w > 0, \tag{2}$$

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Communicated by Calogero Vetro
Email address: sadettinkursun@yahoo.com (Sadettin Kursun)
ORCID iD: https://orcid.org/0000-0001-6697-9627 (Sadettin Kursun)

where $f:\mathbb{R}\to\mathbb{R}$ is a continuous function for which the series is absolutely convergent and $\varphi:\mathbb{R}\to\mathbb{R}$ represents a suitable kernel function. In the same study, the authors examined the approximation properties of the operators in equation (2) within the space of continuous functions on $\mathbb{R}$.

It is well-established that the operators G_w^{φ} are not sufficiently effective for approximating integrable functions $f:\mathbb{R}\to\mathbb{R}$ that are not continuous. Since Kantorovic's idea allows the approximation of functions that need not be continuous, in 2007 Bardaro et al. [20] constructed Kantorovich forms of the series (2) as follows:

$$\left(K_w^{\varphi}f\right)(y) := \sum_{k\in\mathbb{Z}}\left[w\int_{k/w}^{(k+1)/w} f(t)\,dt\right]\varphi(wy-k),\ y\in\mathbb{R}, w>0 \tag{3}$$

for locally Lebesgue integrable functions on $\mathbb{R}$.

The families of sampling type operators play an important role in signal analysis and image processing (see, e.g. [7, 30, 43]). Very recently, the approximation properties of the series (2) and its different forms have been studied in various function spaces. For this, we refer the reader to the papers [1, 2, 5, 26, 28, 29, 31–33, 42].

During the 1980s, a group of physicists and engineers introduced the exponential sampling series. One of the main benefits of the exponential sampling operator (series), which uses exponentially spaced samples, is that it offers valuable insights for solving specific inverse problems found in fields like light scattering, optical physics, radio astronomy, and Fraunhofer diffraction (see, [21, 27, 34, 39]). For a given function $f:\mathbb{R}^+\to\mathbb{C}$, the exponential sampling series is given by

$$\left(E_{\mathrm{lin}_c}f\right)(x) := \sum_{k\in\mathbb{Z}} f\left(e^{\frac{k}{w}}\right)\mathrm{lin}_{\frac{c}{w}}\left(e^{-k}x^{w}\right),\ w>0, c\in\mathbb{R}, x\in\mathbb{R}^+, \tag{4}$$

where $\mathrm{lin}_c:\mathbb{R}^+\to\mathbb{R}$ function is defined by

$$\mathrm{lin}_c(x) := x^{-c}\,\mathrm{sinc}(\log x)$$

and for Mellin band-limited functions, the exponential sampling theorem states

$$\left(E_{\mathrm{lin}_c}f\right)(x) = f(x) \tag{5}$$

at every point $x\in\mathbb{R}^+$ and the series is absolutely and uniformly convergent on every compact interval of $\mathbb{R}^+$ (see, [23]).

In 2017, Bardaro and colleagues [12] constructed a generalized form of the series in (4) by replacing the lin_c function with a continuous χ function, which satisfies the typical assumptions of approximate identities, as in classical sampling series, in order to provide an approach for functions that are not necessarily Mellin band-limited. The generalization of exponential sampling series is given by

$$\left(E_w^{\chi}f\right)(x) := \sum_{k\in\mathbb{Z}} f\left(e^{\frac{k}{w}}\right)\chi\left(e^{-k}x^{w}\right),\ w>0, x\in\mathbb{R}^+, \tag{6}$$

for any function $f:\mathbb{R}^+\to\mathbb{R}$ such that the above series is absolutely convergent on $\mathbb{R}^+$ and $\chi:\mathbb{R}^+\to\mathbb{R}$ represent an appropriate kernel function. In the same paper, the authors investigated the approximation properties of the operators in equation (6) within the space of (log) continuous functions on $\mathbb{R}^+$.

As mentioned above, we know that Kantorovich's idea allows the approximation of integrable (not necessarily continuous) functions, and the E_w^{χ} operators are not an approximation method for non-continuous functions. Therefore, similar to the construction of operators (3), in 2020 Anagamuthu and Bajpeyi [6] considered the Kantorovich form of operators (6) as follows:

$$\left(K_w^{\chi}f\right)(x) := \sum_{k\in\mathbb{Z}}\left[w\int_{k/w}^{(k+1)/w} f(e^{u})\,du\right]\chi\left(e^{-k}x^{w}\right),\ w>0, x\in\mathbb{R}^+ \tag{7}$$

for locally integrable signal on $\mathbb{R}^+$.

Later on, Aral et al. [8] constructed a generalization form of (7) using Mellin-Gauss-Weierstrass singular integrals defined by

$$I_w^{\mathcal{G}} f(x) := \int_0^\infty \mathcal{G}_w(z) f(zx) \frac{dz}{z},\ w > 0, x \in \mathbb{R}^+$$

where $\mathcal{G}_w$ is the well-known Mellin–Gauss–Weierstrass kernel in the form of

$$\mathcal{G}_w(z) = \frac{w}{2\sqrt{\pi}} \exp\left(-\left(\frac{w}{2}\log z\right)^2\right),\ z \in \mathbb{R}^+.$$

The generalizaton of the Kantorovich exponential sampling series is defined by

$$\left(\mathcal{K}_w^{\chi,\mathcal{G}} f\right)(x) := \sum_{k\in\mathbb{Z}} \chi\left(e^{-k}x^w\right) \int_0^\infty \mathcal{G}_w\left(ze^{-\frac{k}{w}}\right) f(z) \frac{dz}{z},\ w > 0; z, x \in \mathbb{R}^+, \tag{8}$$

where $f : \mathbb{R}^+ \to \mathbb{R}$ is an integrable function on $\mathbb{R}^+$ such that the above series is convergent for every $x \in \mathbb{R}^+$.

The families of exponential-type sampling operators play an important role in modelling seismic waves (see, e.g. [10, 19]). Very recently, the approximation properties of the series (6) and its different forms have been studied in various function spaces. For this, we refer the reader to the papers [3, 4, 9, 16–19, 36, 37].

In 2012, with the aim of enhancing the approximation order, Bardaro and Mantellini [13] introduced linear combinations of Mellin–Gauss–Weierstrass operators of the form

$$\left(I_{w,\ell}^{\mathcal{G}} f\right)(x) := \int_0^\infty \sum_{j=1}^{\ell} \alpha_j \mathcal{G}_{wj}(z) f(zx) \frac{dz}{z}, \quad w \geq 1; z, x \in \mathbb{R}^+, \tag{9}$$

where $\alpha_j,\ j = 1, \ldots, \ell$ are real numbers satisfying the condition $\sum_{j=1}^{\ell} \alpha_j = 1$. It is worth noting that these operators are, in general, not positive, a feature that allows one to achieve a higher order of approximation.

In this study, firstly, we establish a quantitative version of a Voronovskaya-type theorem for these operators by employing the Mellin K-functional. Subsequently, in order to improve the rate of approximation, we construct suitable linear combinations of the existing operators, taking into account the operators given in (9). Finally, we demonstrate the applicability of the proposed theory by presenting several examples of kernel functions and by providing numerical results and graphical representations based on these kernels. It should be noted that the results presented here are, in fact, an extension of those given in [8].

2. Preliminaries and Basic Notations

By $\mathbb{N}$ and $\mathbb{Z}$, we denote the sets of all positive integers and integers, respectively. Further, by $\mathbb{R}$ and $\mathbb{R}^+$, we shall denote the sets of all real and positive real numbers, respectively.

We denote by $C_B(\mathbb{R}^+)$ the space of all continuous and bounded functions on $\mathbb{R}^+$ equipped with the supremum norm. We say that a (log) continuous function $f : \mathbb{R}^+ \to \mathbb{C}$ is log-uniformly contiunous (see, [38]) on $\mathbb{R}^+$ if for any given $\varepsilon > 0$ there exists $\delta > 0$ such that $\left|f(x) - f(y)\right| < \varepsilon$, whenever $\left|\log x - \log y\right| \leq \delta$ for any $x, y \in \mathbb{R}^+$. We denote $UC_{\log}(\mathbb{R}^+)$ the space comprising all log-uniformly continuous and bounded functions defined on $\mathbb{R}^+$. Similarly, by $UC_{\log}^{(r)}(\mathbb{R}^+), r \in \mathbb{N}$, we will denote the space of functions which are $r-$times continuously Mellin differentiable and $\Theta^r f \in UC_{\log}(\mathbb{R}^+)$. For any function $f \in C_B(\mathbb{R}^+)$, we will that f belongs to $C^{(r)}$ locally at $x \in \mathbb{R}^+$, if there exists a neighborhood of x in which f is $(r-1)$-times continuously differentiable and $f^{(n)}$ exists. Further, $C^{(r)}(\mathbb{R}^+)$ denotes the subspace of $C_B(\mathbb{R}^+)$ such that each $f^{(m)} \in C_B(\mathbb{R}^+)$ for $m \leq r$.

For $1 \leq \mathcal{P} < \infty$, by $L^{\mathcal{P}}(\mathbb{R}^+)$, we denote the usual Lebesgue spaces comprising all Lebesgue measurable functions such that

$$\left\|f\right\|_{\mathcal{P}} := \left[\int_0^\infty \left|f(x)\right|^{\mathcal{P}} dx\right]^{\frac{1}{\mathcal{P}}} < \infty$$

and for $\mathcal{P} = \infty$, we consider $L^{\infty}(\mathbb{R}^{+})$ as the space of all the essential bounded functions defined on $\mathbb{R}^{+}$ with usual the norm $\left\|f\right\|_{\infty} := \operatorname{esssup}_{x\in\mathbb{R}^{+}} \left|f(x)\right|$. We know that $C(\mathbb{R}^{+}) \subset L^{\infty}(\mathbb{R}^{+})$ and the norm of two spaces is the same.

Now, let us fix $c \in \mathbb{R}$. For $1 \leq \mathcal{P} < \infty$, by $X_c^{\mathcal{P}}$, we will denote the Mellin-Lebesgue spaces given by

$$X_c^{\mathcal{P}} = \left\{ f : \mathbb{R}^{+} \to \mathbb{C} : f(\cdot)(\cdot)^{c-\frac{1}{\mathcal{P}}} \in L^{\mathcal{P}}(\mathbb{R}^{+}) \right\}$$

endowed with the norm

$$\left\|f\right\|_{X_c^{\mathcal{P}}} = \left[\int_0^{\infty} \left|f(x)\right|^{\mathcal{P}} x^{c\mathcal{P}} \frac{dx}{x} \right]^{\frac{1}{\mathcal{P}}} < \infty.$$

In an equivalent manner, $X_c^{\mathcal{P}}$ is the space of all functions f such that $(\cdot)^c f(\cdot) \in L_{\mu}^{\mathcal{P}}(\mathbb{R}^{+})$, where $L_{\mu}^{\mathcal{P}}(\mathbb{R}^{+})$ denotes the Lebesgue space endowed with the Haar (logarithmic/invariant) measure $\mu(A) = \displaystyle\int_A \frac{dx}{x}$ for any measurable set $A \subset \mathbb{R}^{+}$. For more details, see [22].

Let $f \in X_c := X_c^1$. Then the Mellin transform of f is defined by (see, e.g. [22, 38])

$$[f]_M(s) := \int_0^{\infty} f(x) x^s \frac{dx}{x}, \ s = c + it; t \in \mathbb{R}.$$

We know that the Mellin transform is well-defined in X_c as a Lebesgue integral.

Let $m \in \mathbb{R}^{+}$ and let $f : \mathbb{R}^{+} \to \mathbb{C}$ be a function. The Mellin translation operator τ_m is defined by

$$(\tau_m f)(x) := f(mx), \ x \in \mathbb{R}^{+}.$$

Then, the first pointwise Mellin derivative is given by the following limit:

$$\lim_{m\to 1} \frac{(\tau_m f)(x) - f(x)}{m-1} = x f'(x),$$

provided f' exists at the point $x \in \mathbb{R}^{+}$. Thus, the first pointwise derivative Θ of any function $f : \mathbb{R}^{+} \to \mathbb{C}$ in the Mellin sense is defined by

$$\Theta f(x) := x f'(x), \ x \in \mathbb{R}^{+},$$

provided f' exists. Subsequently, the $r-$th order Mellin differential operator is defined by

$$\Theta^r := \Theta\left(\Theta^{r-1}\right).$$

For example, the first three Mellin derivatives of f are given by

$$\begin{aligned} \Theta f(x) &= x f'(x), \\ \Theta^2 f(x) &= x^2 f''(x) + x f'(x), \\ \Theta^3 f(x) &= x^3 f'''(x) + 3x^2 f''(x) + x f'(x) \end{aligned}$$

for more information see [22]. For any $f \in C^{(r)}$, locally at $x \in \mathbb{R}^{+}$, we have the Mellin-Taylor formula based Mellin derivatives (see, [38]) as follows:

$$f(tx) := \sum_{i=0}^{r} \frac{\left(\Theta^i f\right)(x) \log^i t}{i!} + h(t) \log^r t, \tag{10}$$

where $h : \mathbb{R}^{+} \to \mathbb{R}$ is a bounded function on $\mathbb{R}^{+}$ such that $h(t) \to 0$ as $t \to 1$.

Throughout the rest of the paper, a contiunous function $\chi : \mathbb{R}^{+} \to \mathbb{R}$ will be called a kernel if it satisfies the following assumptions:

χ_1) for every $x \in \mathbb{R}^+$,

$$\sum_{k\in\mathbb{Z}} \chi\left(e^{-k}x\right) = 1$$

and the convergence of the series $\sum_{k\in\mathbb{Z}} \left|\chi\left(e^{-k}x\right)\right|$ is uniform on the compact subset of $\mathbb{R}^+$;

χ_2)

$$M_0(\chi) := \sup_{x\in\mathbb{R}^+} \sum_{k\in\mathbb{Z}} \left|\chi\left(e^{-k}x\right)\right| < \infty.$$

We will denote by Φ the class of all functions of χ satisfying the assumption χ_1) and χ_2).

Let $\nu \in \mathbb{N} \cup \{0\}$. The algebraic moment of order ν of $\chi \in \Phi$ is defined by

$$m_\nu(\chi, x) := \sum_{k\in\mathbb{Z}} \chi\left(e^{-k}x\right)(k - \log x)^\nu, \ x \in \mathbb{R}^+.$$

Also, the absolute moment of order ν of $\chi \in \Phi$ is given by

$$M_\nu(\chi) := \sup_{x\in\mathbb{R}^+} \sum_{k\in\mathbb{Z}} \left|\chi\left(e^{-k}x\right)\right| \left|k - \log x\right|^\alpha.$$

Remark 2.1. *We note that if $M_\alpha(\chi) < \infty$, then we have $M_\beta(\chi) < \infty$ for $0 \le \beta < \alpha$ (see, [18]).*

3. SOME NEW RESULTS

In this section, we present new results in addition to the results given in the paper [8].

3.1. A Quantitative Form of Voronovskaja-type Theorem

Here, we present a quantitative estimate of the Voronovskaja-type formula in terms of Mellin K-functional.

First, we recall the following qualitative Voronovskaja-type formula for the family of operators $\left(\mathcal{K}_w^{\chi,\mathcal{G}}\right)_{w>0}$ (see, [8, Theorem 3]).

Theorem 3.1. *Let $\chi \in \Phi$ be a kernel such that $m_1(\chi, u) = 0, m_2(\chi, u) \neq 0 < \infty$ at a point $u \in \mathbb{R}^+$ and let $f \in C^{(2)}$ locally at $x \in \mathbb{R}^+$. Then the following equality holds:*

$$\left(\mathcal{K}_w^{\chi,\mathcal{G}} f\right)(x) - f(x) = \frac{\left(\Theta^2 f\right)(x)}{2}\left(\frac{m_2(\chi) + 2}{w^2}\right) + R_2^w(x),$$

where

$$R_2^w(x) = \sum_{k\in\mathbb{Z}} \chi\left(e^{-k}x^w\right) \int_0^\infty \mathcal{G}_w\left(ze^{-\frac{k}{w}}\right) h\left(\frac{z}{x}\right)(\log z - \log x)^2 \frac{dz}{z}$$

and $h : \mathbb{R}^+ \to \mathbb{R}$ is a bounded function on $\mathbb{R}^+$ such that $h\left(\frac{z}{x}\right) \to 0$ as $z \to x$.

For $f \in UC_{\log}^{(r)}(\mathbb{R}^+)$, the Mellin K-functional is defined (see, [15]) by

$$K(f, \varepsilon) := \inf\left\{\left\|(\Theta^r)(f - g)\right\|_\infty + \varepsilon \left\|\left(\Theta^{r+1} g\right)\right\|_\infty : \ g \in UC_{\log}^{(r+1)}(\mathbb{R}^+), \varepsilon \ge 0\right\}. \tag{11}$$

Now, we have the following theorem.

Theorem 3.2. *Let $\chi \in \Phi$ be a kernel such that $m_2(\chi, u) \neq 0 < \infty, M_3(\chi) < \infty$ and let $f \in UC^{(2)}_{\log}(\mathbb{R}^+)$ be a function. Then we have*

$$\left|\left[\left(\mathcal{K}_w^{\chi,\mathcal{G}} f\right)(x) - f(x)\right] - \frac{\left(\Theta^2 f\right)(x)}{2}\left(\frac{m_2(\chi)+2}{w^2}\right)\right|$$
$$\leq \frac{2}{w^2}\left(2M_0(\chi) + M_2(\chi)\right) K\left(\left(\Theta^2 f\right), \frac{1}{3w}\left(\frac{\frac{8}{\sqrt{\pi}}M_0(\chi) + M_3(\chi)}{2M_0(\chi)+M_2(\chi)}\right)\right).$$

Proof. Thanks to the Theorem 3.1, we get

$$\left|\left[\left(\mathcal{K}_w^{\chi,\mathcal{G}} f\right)(x) - f(x)\right] - \frac{\left(\Theta^2 f\right)(x)}{2}\left(\frac{m_2(\chi)+2}{w^2}\right)\right|$$
$$= \left|\sum_{k\in\mathbb{Z}} \chi\left(e^{-k}x^w\right)\int_0^\infty \mathcal{G}_w\left(ze^{-\frac{k}{w}}\right) h\left(\frac{z}{x}\right)(\log z - \log x)^2 \frac{dz}{z}\right|.$$

Now, we consider $R_2(f,x,z) := h\left(\frac{z}{x}\right)(\log z - \log x)^2$. If we use the following estimate in Mellin-Taylor formula (see, [15])

$$\left|R_2(f,x,z)\right| \leq \left|\log z - \log x\right|^2 K\left(\left(\Theta^2 f\right), \frac{\left|\log z - \log x\right|}{6}\right), \tag{12}$$

we can write

$$\left|\left[\left(\mathcal{K}_w^{\chi,\mathcal{G}} f\right)(x) - f(x)\right] - \frac{\left(\Theta^2 f\right)(x)}{2}\left(\frac{m_2(\chi)+2}{w^2}\right)\right|$$
$$\leq \sum_{k\in\mathbb{Z}} \left|\chi\left(e^{-k}x^w\right)\right| \int_0^\infty \mathcal{G}_w\left(ze^{-\frac{k}{w}}\right)\left|\log z - \log x\right|^2 K\left(\left(\Theta^2 f\right), \frac{\left|\log z - \log x\right|}{6}\right)\frac{dz}{z}$$
$$\leq \sum_{k\in\mathbb{Z}} \left|\chi\left(e^{-k}x^w\right)\right| \int_0^\infty \mathcal{G}_w\left(ze^{-\frac{k}{w}}\right)\left|\log z - \log x\right|^2 \left(\left\|\left(\Theta^2\right)(f-g)\right\|_\infty + \frac{\left|\log z - \log x\right|}{6}\left\|\left(\Theta^3 g\right)\right\|_\infty\right)$$
$$= \left\|\left(\Theta^2\right)(f-g)\right\|_\infty \sum_{k\in\mathbb{Z}} \left|\chi\left(e^{-k}x^w\right)\right| \int_0^\infty \mathcal{G}_w\left(ze^{-\frac{k}{w}}\right)\left|\log z - \log x\right|^2 \frac{dz}{z}$$
$$+ \frac{\left\|\left(\Theta^3 g\right)\right\|_\infty}{6} \sum_{k\in\mathbb{Z}} \left|\chi\left(e^{-k}x^w\right)\right| \int_0^\infty \mathcal{G}_w\left(ze^{-\frac{k}{w}}\right)\left|\log z - \log x\right|^3 \frac{dz}{z}$$
$$=: I_1 + I_2.$$

Now, using the inequality

$$|a+b|^r \leq 2^{r-1}\left(|a|^r + |b|^r\right),\ a,b \in \mathbb{R}, 1 \leq r < \infty$$

and considering the change of variable $z = ye^{\frac{k}{w}}$, we evaluate I_1 as

$$\begin{aligned} I_1 &= \left\|\left(\Theta^2\right)(f-g)\right\|_\infty \sum_{k\in\mathbb{Z}} \left|\chi\left(e^{-k}x^w\right)\right| \int_0^\infty \mathcal{G}_w(y)\left|\log y + \frac{k}{w} - \log x\right|^2 \frac{dy}{y} \\ &\leq 2\left\|\left(\Theta^2\right)(f-g)\right\|_\infty \sum_{k\in\mathbb{Z}} \left|\chi\left(e^{-k}x^w\right)\right| \int_0^\infty \mathcal{G}_w(y)\left|\log y\right|^2 \frac{dy}{y} \end{aligned}$$

$$
\begin{aligned}
&+ \quad \frac{2}{w^2}\left\|\left(\Theta^2\right)(f-g)\right\|_\infty \sum_{k\in\mathbb{Z}}\left|\chi\left(e^{-k}x^w\right)\right|\left|k-w\log x\right|^2\int_0^\infty \mathcal{G}_w(y)\frac{dy}{y}\\
&\leq \quad \frac{2}{w^2}\left\|\left(\Theta^2\right)(f-g)\right\|_\infty\left(2M_0(\chi)+M_2(\chi)\right).
\end{aligned}
$$

Similarly, we evaluate I_2 as

$$
\begin{aligned}
I_2 \quad &= \quad \frac{\left\|\left(\Theta^3 g\right)\right\|_\infty}{6}\sum_{k\in\mathbb{Z}}\left|\chi\left(e^{-k}x^w\right)\right|\int_0^\infty \mathcal{G}_w(y)\left|\log y+\frac{k}{w}-\log x\right|^3\frac{dy}{y}\\
&\leq \quad \frac{2}{3}\left\|\left(\Theta^3 g\right)\right\|_\infty\sum_{k\in\mathbb{Z}}\left|\chi\left(e^{-k}x^w\right)\right|\int_0^\infty \mathcal{G}_w(y)\left|\log y\right|^3\frac{dy}{y}\\
&+ \quad \frac{2}{3w^3}\left\|\left(\Theta^3 g\right)\right\|_\infty\sum_{k\in\mathbb{Z}}\left|\chi\left(e^{-k}x^w\right)\right|\left|k-w\log x\right|^3\int_0^\infty \mathcal{G}_w(y)\frac{dy}{y}\\
&\leq \quad \frac{2}{3w^3}\left\|\left(\Theta^3 g\right)\right\|_\infty\left(\frac{8}{\sqrt{\pi}}M_0(\chi)+M_3(\chi)\right).
\end{aligned}
$$

Finally, combining the estimates I_1 and I_2 and taking the infimum over g, we obtain

$$
\begin{aligned}
&\left|\left[\left(\mathcal{K}_w^{\chi,\mathcal{G}}f\right)(x)-f(x)\right]-\frac{\left(\Theta^2 f\right)(x)}{2}\left(\frac{m_2(\chi)+2}{w^2}\right)\right|\\
&\leq \frac{2}{w^2}\left(2M_0(\chi)+M_2(\chi)\right)\left\{\left\|\left(\Theta^2\right)(f-g)\right\|_\infty+\frac{1}{3w}\left\|\left(\Theta^3 g\right)\right\|_\infty\left(\frac{\frac{8}{\sqrt{\pi}}M_0(\chi)+M_3(\chi)}{2M_0(\chi)+M_2(\chi)}\right)\right\}\\
&\leq \frac{2}{w^2}\left(2M_0(\chi)+M_2(\chi)\right)K\left(\left(\Theta^2 f\right),\frac{1}{3w}\left(\frac{\frac{8}{\sqrt{\pi}}M_0(\chi)+M_3(\chi)}{2M_0(\chi)+M_2(\chi)}\right)\right).
\end{aligned}
$$

So, the proof is completed. □

3.2. *Linear Combinations of the Operatos*

The purpose of this subsection is to introduce specific linear combinations of the series $\left(\mathcal{K}_w^{\chi,\mathcal{G}}f\right)$ to achieve an improved rate of convergence in the asymptotic formula for functions $f\in C^{(r)}(\mathbb{R}^+)$. Using the operators (9), we form a linear combination of the series $\left(\mathcal{K}_w^{\chi,\mathcal{G}}f\right)$ in the following manner:

Let $\alpha_j,\ j=1,\ldots,\ell$ be non-zero real numbers such that $\sum_{j=1}^{\ell}\alpha_j=1$. For $x\in\mathbb{R}^+$ and $w\geq 1$, we define the linear combination of the operators (8) as

$$
\left(\mathcal{K}_{w,\ell}^{\chi,\mathcal{G}}f\right)(x):=\sum_{j=1}^{\ell}\alpha_j\sum_{k\in\mathbb{Z}}\chi\left(e^{-k}x^{jw}\right)\int_0^\infty \mathcal{G}_{wj}\left(ze^{-\frac{k}{w}}\right)f(z)\frac{dz}{z},\ w\geq 1;z,x\in\mathbb{R}^+. \tag{13}
$$

Remark 3.3. *One readily verifies that, for all $w\geq 1$ and $\ell\in\mathbb{N}$, there holds:*

$$
\int_0^\infty\sum_{j=1}^{\ell}\alpha_j\mathcal{G}_{wj}(z)\frac{dz}{z}=1. \tag{14}
$$

Now, we derive the Voronovskaja-type formula for the family of operators defined in (13). To do this, here we assume the following two conditions:

χ_3) For any $0\leq v\leq 2$, $m_v(\chi):=m_v\left(\chi,x^{jw}\right)$ are independent of x;

χ_4) $M_2(\chi) < \infty$ and

$$\lim_{w\to\infty} \sum_{\left|k-j\log x\right|>w} \left|\chi\left(e^{-k}x^{j}\right)\right| \left|k-j\log x\right|^2 = 0.$$

Theorem 3.4. *Let $f \in C^{(2)}(\mathbb{R}^+)$ be a function and let χ be a kernel satisfying assumptions χ_1), χ_3) and χ_4). If we suppose that $m_1\left(\chi, x^{jw}\right) = 0$ and $m_2\left(\chi, x^{jw}\right) < \infty$ at a point $x \in \mathbb{R}^+$, then we have*

$$\lim_{w\to\infty} w^2 \left[\left(\mathcal{K}_{w,\ell}^{\chi,\mathcal{G}} f\right)(x) - f(x)\right] = \frac{\left(\Theta^2 f\right)(x)}{2} \sum_{j=1}^{\ell} \frac{\alpha_j}{j^2} \left(m_2(\chi) + 2\right).$$

Proof. From the condition $\sum_{j=1}^{\ell} \alpha_j = 1$ and the equality (14), we can write

$$\left(\mathcal{K}_{w,\ell}^{\chi,\mathcal{G}} f\right)(x) - f(x) = \sum_{j=1}^{\ell} \alpha_j \sum_{k\in\mathbb{Z}} \chi\left(e^{-k}x^{jw}\right) \int_0^\infty \mathcal{G}_{wj}\left(ze^{-\frac{k}{w}}\right)\left(f(z) - f(x)\right) \frac{dz}{z}.$$

Now, since f'' exists at the point x, using the 2^{nd} order Mellin-Taylor's expansion, we obtain

$$\begin{aligned}
\left(\mathcal{K}_{w,\ell}^{\chi,\mathcal{G}} f\right)(x) - f(x) &= (\Theta f)(x) \sum_{j=1}^{\ell} \alpha_j \sum_{k\in\mathbb{Z}} \chi\left(e^{-k}x^{jw}\right) \int_0^\infty \mathcal{G}_{wj}\left(ze^{-\frac{k}{w}}\right) \log\left(\frac{z}{x}\right) \frac{dz}{z} \\
&= \frac{\left(\Theta^2 f\right)(x)}{2} \sum_{j=1}^{\ell} \alpha_j \sum_{k\in\mathbb{Z}} \chi\left(e^{-k}x^{jw}\right) \int_0^\infty \mathcal{G}_{wj}\left(ze^{-\frac{k}{w}}\right) \log^2\left(\frac{z}{x}\right) \frac{dz}{z} \\
&+ \sum_{j=1}^{\ell} \alpha_j \sum_{k\in\mathbb{Z}} \chi\left(e^{-k}x^{jw}\right) \int_0^\infty \mathcal{G}_{wj}\left(ze^{-\frac{k}{w}}\right) h\left(\frac{z}{x}\right) \log^2\left(\frac{z}{x}\right) \frac{dz}{z}
\end{aligned}$$

which can be stated in the form

$$\begin{aligned}
\left(\mathcal{K}_{w,\ell}^{\chi,\mathcal{G}} f\right)(x) - f(x) &= (\Theta f)(x) \sum_{j=1}^{\ell} \frac{\alpha_j}{jw} \sum_{k\in\mathbb{Z}} \chi\left(e^{-k}x^{jw}\right)\left(k - \log x^{jw}\right) \\
&= \frac{\left(\Theta^2 f\right)(x)}{2} \sum_{j=1}^{\ell} \frac{\alpha_j}{j^2w^2} \left(\sum_{k\in\mathbb{Z}} \chi\left(e^{-k}x^{jw}\right)\left(k - \log x^{jw}\right)^2 + 2\right) \\
&+ J,
\end{aligned}$$

where

$$J = \sum_{j=1}^{\ell} \alpha_j \sum_{k\in\mathbb{Z}} \chi\left(e^{-k}x^{jw}\right) \int_0^\infty \mathcal{G}_{wj}\left(ze^{-\frac{k}{w}}\right) h\left(\frac{z}{x}\right) \log^2\left(\frac{z}{x}\right) \frac{dz}{z}.$$

Even, thanks to definitions of the algebraic moments, we can state it as

$$\left(\mathcal{K}_{w,\ell}^{\chi,\mathcal{G}} f\right)(x) - f(x) = \frac{\left(\Theta^2 f\right)(x)}{2} \sum_{j=1}^{\ell} \frac{\alpha_j}{j^2w^2} \left(m_2(\chi) + 2\right) + J.$$

On the other hand, for a given $\varepsilon > 0$, let $\delta > 1$ such that $\left|h\left(\frac{z}{x}\right)\right| < \varepsilon$ for $z \in U_\delta = \left(\frac{e^{\frac{k}{wj}}}{\delta}, e^{\frac{k}{wj}}\delta\right)$ and for any $1 \leq j \leq \ell$. Then

$$\begin{aligned} w^2|J| &\leq w^2\sum_{j=1}^{\ell}\alpha_j\sum_{k\in\mathbb{Z}}\left|\chi\left(e^{-k}x^{jw}\right)\right|\int_0^{\infty}\mathcal{G}_{wj}\left(ze^{-\frac{k}{w}}\right)\left|h\left(\frac{z}{x}\right)\right|\log^2\left(\frac{z}{x}\right)\frac{dz}{z} \\ &\leq \varepsilon w^2\sum_{j=1}^{\ell}\alpha_j\sum_{k\in\mathbb{Z}}\left|\chi\left(e^{-k}x^{jw}\right)\right|\int_{U_\delta}\mathcal{G}_{wj}\left(ze^{-\frac{k}{w}}\right)\log^2\left(\frac{z}{x}\right)\frac{dz}{z} \\ &+ w^2\sum_{j=1}^{\ell}\alpha_j\sum_{k\in\mathbb{Z}}\left|\chi\left(e^{-k}x^{jw}\right)\right|\int_{\mathbb{R}^+\setminus U_\delta}\mathcal{G}_{wj}\left(ze^{-\frac{k}{w}}\right)\left|h\left(\frac{z}{x}\right)\right|\log^2\left(\frac{z}{x}\right)\frac{dz}{z} \\ &=: J_1 + J_2. \end{aligned}$$

As to J_1, since $\mathcal{G}_w(z) = w\mathcal{G}(z^w)$, we get

$$\begin{aligned} J_1 &\leq \varepsilon\sum_{j=1}^{\ell}\alpha_j\sum_{k\in\mathbb{Z}}\left|\chi\left(e^{-k}x^{jw}\right)\right|\int_{\frac{1}{\delta}<z<\delta}\mathcal{G}_{wj}(z)\log^2 z\frac{dz}{z} \\ &+ w^2\sum_{j=1}^{\ell}\alpha_j j^2\sum_{k\in\mathbb{Z}}\left|\chi\left(e^{-k}x^{jw}\right)\right|\log^2\left(\frac{e^{\frac{k}{wj}}}{x}\right)\int_{\frac{1}{\delta}<z<\delta}\mathcal{G}_{wj}(z)\frac{dz}{z} \\ &= \varepsilon\sum_{j=1}^{\ell}\alpha_j\sum_{k\in\mathbb{Z}}\left|\chi\left(e^{-k}x^{jw}\right)\right|\int_{\frac{1}{\delta^{wj}}<z<\delta^{wj}}\mathcal{G}(z)\log^2 z\frac{dz}{z} \\ &+ \sum_{j=1}^{\ell}\alpha_j j^2\sum_{k\in\mathbb{Z}}\left|\chi\left(e^{-k}x^{jw}\right)\right|\log^2\left(\frac{e^{\frac{k}{wj}}}{x}\right)\int_{\frac{1}{\delta^{wj}}<z<\delta^{wj}}\mathcal{G}(z)\frac{dz}{z} \\ &\leq \varepsilon\sum_{j=1}^{\ell}\alpha_j\left(2M_0(\chi) + j^2M_2(\chi)\right). \end{aligned}$$

As regards J_2, since the function $h\left(\frac{z}{x}\right)$ is bounded on $\mathbb{R}^+$ with $M > 0$, we have

$$\begin{aligned} J_2 &\leq M\sum_{j=1}^{\ell}\alpha_j\sum_{k\in\mathbb{Z}}\left|\chi\left(e^{-k}x^{jw}\right)\right|\int_{\mathbb{R}^+\setminus\left(\delta^{-wj},\delta^{wj}\right)}\mathcal{G}(z)\log^2 z\frac{dz}{z} \\ &+ Mw^2\sum_{j=1}^{\ell}\alpha_j j^2\sum_{k\in\mathbb{Z}}\left|\chi\left(e^{-k}x^{jw}\right)\right|\log^2\left(\frac{e^{\frac{k}{wj}}}{x}\right)\int_{\mathbb{R}^+\setminus\left(\delta^{-wj},\delta^{wj}\right)}\mathcal{G}(z)\frac{dz}{z} \\ &\leq M\sum_{j=1}^{\ell}\alpha_j M_0(\chi)\int_{\mathbb{R}^+\setminus\left(\delta^{-wj},\delta^{wj}\right)}\mathcal{G}(z)\log^2 z\frac{dz}{z} + M_2(\chi)\,\varepsilon M\sum_{j=1}^{\ell}\alpha_j j^2 \\ &\leq M\sum_{j=1}^{\ell}\alpha_j\left(\frac{2M_0(\chi)}{w^2} + \varepsilon j^2M_2(\chi)\right). \end{aligned}$$

for sufficiently large w. Therefore, as J_1 and J_2 vanish in the limit $w \to \infty$, it follows that $\lim_{w\to\infty} w^2|J| = 0$. Then, the proof is completed. □

Now, for $l = 2$, we obtain the following linear system:

$$\alpha_1 + \alpha_2 = 1$$

$$\alpha_1 + \frac{\alpha_2}{4} = 0,$$

whose solution is $\alpha_1 = -\frac{1}{3}$ and $\alpha_2 = \frac{4}{3}$. Utilizing these coefficients, we define the linear combination:

$$\left(\mathcal{K}_{w,2}^{\chi,\mathcal{G}} f\right)(x) = -\frac{1}{3}\left(\mathcal{K}_{w}^{\chi,\mathcal{G}} f\right)(x) + \frac{4}{3}\left(\mathcal{K}_{2w}^{\chi,\mathcal{G}} f\right)(x).$$

Finally, we establish a quantitative estimate for the rate of convergence of the operator (13) in terms of the Mellin K-functional given in (11).

Theorem 3.5. *Let χ be a kernel satisfying assumptions $\chi_1)$, $\chi_3)$ and $\chi_4)$ and let $f \in UC_{\log}^{(2)}(\mathbb{R}^+)$ be a function. If we suppose that $m_2\left(\chi, x^{jw}\right) := m_2(\chi) \neq 0$ and $M_3(\chi) < \infty$, then we have*

$$\left|\left[\left(\mathcal{K}_{w,\ell}^{\chi,\mathcal{G}} f\right)(x) - f(x)\right] - \frac{\left(\Theta^2 f\right)(x)}{2w^2}\sum_{j=1}^{\ell}\frac{\alpha_j}{j^2}\left(m_2(\chi)+2\right)\right|$$
$$\leq \frac{2}{w^2}\sum_{j=1}^{\ell}\alpha_j\left(\frac{2}{j^2}M_0(\chi) + j^2 M_2(\chi)\right) K\left(\left(\Theta^2 f\right), \frac{1}{3w}\left(\frac{\frac{8}{\sqrt{\pi}j^3}M_0(\chi) + j^3 M_3(\chi)}{\frac{2}{j^2}M_0(\chi) + j^2 M_2(\chi)}\right)\right).$$

Proof. In view of Theorem 3.4, we have

$$\left|\left[\left(\mathcal{K}_{w,\ell}^{\chi,\mathcal{G}} f\right)(x) - f(x)\right] - \frac{\left(\Theta^2 f\right)(x)}{2w^2}\sum_{j=1}^{\ell}\frac{\alpha_j}{j^2}\left(m_2(\chi)+2\right)\right|$$
$$= \left|\sum_{j=1}^{\ell}\alpha_j\sum_{k\in\mathbb{Z}}\chi\left(e^{-k}x^{jw}\right)\int_0^\infty \mathcal{G}_{wj}\left(ze^{-\frac{k}{w}}\right) h\left(\frac{z}{x}\right)\log^2\left(\frac{z}{x}\right)\frac{dz}{z}\right|.$$

Now, we set $R_2(f;x,z) := h\left(\frac{z}{x}\right)\log^2\left(\frac{z}{x}\right)$. Then by using the estimate (12), we can write

$$\left|\left[\left(\mathcal{K}_{w,\ell}^{\chi,\mathcal{G}} f\right)(x) - f(x)\right] - \frac{\left(\Theta^2 f\right)(x)}{2w^2}\sum_{j=1}^{\ell}\frac{\alpha_j}{j^2}\left(m_2(\chi)+2\right)\right|$$
$$\leq \sum_{j=1}^{\ell}\alpha_j\sum_{k\in\mathbb{Z}}\left|\chi\left(e^{-k}x^{jw}\right)\right|\int_0^\infty \mathcal{G}_{wj}\left(ze^{-\frac{k}{w}}\right)\left|\log z - \log x\right|^2 K\left(\left(\Theta^2 f\right), \frac{\left|\log z - \log x\right|}{6}\right)\frac{dz}{z}$$
$$\leq \sum_{j=1}^{\ell}\alpha_j\sum_{k\in\mathbb{Z}}\left|\chi\left(e^{-k}x^{jw}\right)\right|\int_0^\infty \mathcal{G}_{wj}\left(ze^{-\frac{k}{w}}\right)\left|\log z - \log x\right|^2$$
$$\times\left(\left\|\left(\Theta^2\right)(f-g)\right\|_\infty + \frac{\left|\log z - \log x\right|}{6}\left\|\left(\Theta^3 g\right)\right\|_\infty\right)\frac{dz}{z}$$
$$= \left\|\left(\Theta^2\right)(f-g)\right\|_\infty\sum_{j=1}^{\ell}\alpha_j\sum_{k\in\mathbb{Z}}\left|\chi\left(e^{-k}x^{jw}\right)\right|\int_0^\infty \mathcal{G}_{wj}\left(ze^{-\frac{k}{w}}\right)\left|\log z - \log x\right|^2\frac{dz}{z}$$
$$+ \frac{\left\|\left(\Theta^3 g\right)\right\|_\infty}{6}\sum_{j=1}^{\ell}\alpha_j\sum_{k\in\mathbb{Z}}\left|\chi\left(e^{-k}x^{jw}\right)\right|\int_0^\infty \mathcal{G}_{wj}\left(ze^{-\frac{k}{w}}\right)\left|\log z - \log x\right|^3\frac{dz}{z}$$
$$=: S_1 + S_2.$$

Analogously to the computations carried out for I_1 in Theorem 3.2, we can evaluate S_1 as

$$\begin{aligned}
S_1 &= \left\|\left(\Theta^2\right)(f-g)\right\|_\infty \sum_{j=1}^{\ell} \alpha_j \sum_{k\in\mathbb{Z}} \left|\chi\left(e^{-k}x^{jw}\right)\right| \int_0^\infty \mathcal{G}_{wj}(y) \left|\log y + \frac{k}{w} - \log x\right|^2 \frac{dy}{y} \\
&\leq 2\left\|\left(\Theta^2\right)(f-g)\right\|_\infty \sum_{j=1}^{\ell} \alpha_j \sum_{k\in\mathbb{Z}} \left|\chi\left(e^{-k}x^{jw}\right)\right| \int_0^\infty \mathcal{G}_{wj}(y) \left|\log y\right|^2 \frac{dy}{y} \\
&+ \frac{2}{w^2}\left\|\left(\Theta^2\right)(f-g)\right\|_\infty \sum_{j=1}^{\ell} \alpha_j j^2 \sum_{k\in\mathbb{Z}} \left|\chi\left(e^{-k}x^{jw}\right)\right| \left|k - \log x^{wj}\right|^2 \int_0^\infty \mathcal{G}_{wj}(y) \frac{dy}{y} \\
&\leq \frac{2}{w^2}\left\|\left(\Theta^2\right)(f-g)\right\|_\infty \sum_{j=1}^{\ell} \alpha_j \left(\frac{2}{j^2} M_0(\chi) + j^2 M_2(\chi)\right).
\end{aligned}$$

Similarly, we can evaluate S_2 as

$$\begin{aligned}
S_2 &= \frac{\left\|\left(\Theta^3 g\right)\right\|_\infty}{6} \sum_{j=1}^{\ell} \alpha_j \sum_{k\in\mathbb{Z}} \left|\chi\left(e^{-k}x^{jw}\right)\right| \int_0^\infty \mathcal{G}_{wj}(y) \left|\log y + \frac{k}{w} - \log x\right|^3 \frac{dy}{y} \\
&\leq \frac{2}{3}\left\|\left(\Theta^3 g\right)\right\|_\infty \sum_{j=1}^{\ell} \alpha_j \sum_{k\in\mathbb{Z}} \left|\chi\left(e^{-k}x^{jw}\right)\right| \int_0^\infty \mathcal{G}_{wj}(y) \left|\log y\right|^3 \frac{dy}{y} \\
&+ \frac{2}{3w^3}\left\|\left(\Theta^3 g\right)\right\|_\infty \sum_{j=1}^{\ell} \alpha_j j^3 \sum_{k\in\mathbb{Z}} \left|\chi\left(e^{-k}x^{jw}\right)\right| \left|k - \log x^{wj}\right|^3 \int_0^\infty \mathcal{G}_{wj}(y) \frac{dy}{y} \\
&\leq \frac{2}{3w^3}\left\|\left(\Theta^3 g\right)\right\|_\infty \sum_{j=1}^{\ell} \alpha_j \left(\frac{8}{\sqrt{\pi} j^3} M_0(\chi) + j^3 M_3(\chi)\right).
\end{aligned}$$

Finally, combining the estimates S_1 and S_2 and taking the infimum over g, we obtain desired result. Thus, the proof is completed. □

4. Some Examples of Certain Kernels

In this section, we give some examples of certain kernels $\chi \in \Phi$ which satisfy the assumptions employed in our results.

Mellin-Spline Kernel

We first consider the well-known Mellin-Spline kernel. For $n \in \mathbb{N}$, the Mellin-Spline kernel of order n is given by

$$B_n(x) := \frac{1}{(n-1)!} \sum_{i=0}^{n} (-1)^i \binom{n}{i} \left(\frac{n}{2} + \log x - i\right)_+^{n-1},\ x \in \mathbb{R}^+,$$

where r_+ denotes the positive part of the number $r \in \mathbb{R}$. For $x \in \mathbb{R}^+$, we observe that $\tilde{B}_n(\log x) = B_n(x)$, where

$$\tilde{B}_n(u) := \frac{1}{(n-1)!} \sum_{i=0}^{n} (-1)^i \binom{n}{i} \left(\frac{n}{2} + u - i\right)_+^{n-1},\ u \in \mathbb{R}$$

denotes the classical B-Splines kernel of order $n \in \mathbb{N}$. By using the definition of the Mellin transform, we get

$$[B_n]_M(it) = \int_0^\infty B_n(v)\, v^{it} \frac{dv}{v},\ t \in \mathbb{R}.$$

Now, substituting $y = \log v$, we easily obtain

$$[B_n]_M(it) = \int_{-\infty}^{\infty} \tilde{B}_n(y)\, e^{ity} dy = \hat{\tilde{B}}_n(-y),$$

where $\hat{f}(y) = \int_{-\infty}^{\infty} f(u)\, e^{-iyu} du$, $y \in \mathbb{R}$ denotes the Fourier transform of f. Then, the Mellin transform of B_n is given by (see, [12])

$$[B_n]_M(it) = \left(\frac{\sin(\mathrm{t}/2)}{t/2}\right)^n,\ t \in \mathbb{R} \backslash \{0\}.$$

Now, we set

$$g(x) := \sum_{k \in \mathbb{Z}} B_n\left(e^k x\right).$$

Then the function g is a recurrent function with fundamental interval $[1, e]$, that is, $g(ex) = g(x)$ for every $x \in \mathbb{R}^+$. Thus, the Mellin-Fourier series of g can be written as

$$g \equiv \sum_{k \in \mathbb{Z}} g_{k,m} x^{-2k\pi i},\ k \in \mathbb{Z}$$

where

$$g_{k,m} := \int_{\frac{1}{e}}^{e} g(x)\, x^{2k\pi i} \frac{dx}{x}$$

are the Mellin-Fourier coefficient of g. From here, we have $g_{k,m} = [B_n]_M(2k\pi i)$. Therefore, we obtain the following Mellin-Poisson summation formula (see, [24])

$$\sum_{k \in \mathbb{Z}} B_n\left(e^k x\right) = \sum_{k \in \mathbb{Z}} [B_n]_M(2k\pi i)\, x^{-2k\pi i}. \tag{15}$$

We recall the following equivalence.

Lemma 4.1. *The condition* $\sum_{k \in \mathbb{Z}} \chi\left(e^{-k} x\right) = 1$ *is equivalent to the condition*

$$[\chi]_M(2k\pi i) = \begin{cases} 1, & k = 0 \\ 0, & otherwise \end{cases}. \tag{16}$$

Further, the condition $m_j(\chi, u) = 0$ *for* $j = 1, 2, ..., n$ *is equivalent to the condition* $[\chi]_M^{(j)}(2k\pi i) = 0$ *for* $j = 1, 2, ..., n$ *and for every* $k \in \mathbb{Z}$*, see, [6].*

Thanks to the Lemma 4.1 and the equality (15), we can easily calculate that

$$\sum_{k \in \mathbb{Z}} B_n\left(e^k x\right) = 1$$

for every $x \in \mathbb{R}^+$.

Using the definition of Mellin transform and by differentiating under the sign of the integral, we have

$$\begin{aligned} \frac{d}{dt}\left([B_n]_M(it)\right) &= \int_0^\infty B_n(v)\frac{d}{dt}\left(v^{it}\right)\frac{dv}{v} \\ &= i\int_0^\infty B_n(v)\,v^{it}\log v\frac{dv}{v}. \end{aligned}$$

Similarly, we can easily obtain

$$\frac{d^j}{dt^j}\left([B_n]_M(it)\right) = i^j\int_0^\infty B_n(v)\,v^{it}\log^j v\frac{dv}{v}.$$

Thus, we get the following Mellin-Poisson summation formula (see, [24])

$$(-i)^j\sum_{k\in\mathbb{Z}} B_n\left(e^{-k}x\right)(k-\log x)^j = \sum_{k\in\mathbb{Z}}\frac{d^j}{dt^j}[B_n]_M(2k\pi i)\,x^{-2k\pi i}. \tag{17}$$

Since B_n has compact support on $[e^{-a}, e^{a}]\,, a>0$, all the absolute moments of χ are finite (indeed all absolute moments have a finite numbers of non-zero terms). Finally, we note that all algebraic moments $m_j(B_n, x)$ are independent of x for every $j = 1, 2, ..., n$, see e.g. [14]. As a particular case, 3^{rd} order Mellin-Spline kernel is given by

$$B_3(x) := \begin{cases} \frac{1}{2}\left(\frac{3}{2}+\log x\right)^2 &, \quad e^{-3/2} < x \le e^{-1/2} \\ \frac{3}{4}-\log^2 x &, \quad e^{-1/2} < x \le e^{1/2} \\ \frac{1}{2}\left(\frac{3}{2}-\log x\right)^2 &, \quad e^{1/2} < x < e^{3/2} \\ 0 &, \quad \text{otherwise} \end{cases}.$$

Using the formula (17), values of $m_j(B_3)\,, j = 0, 1, 2,$ can be calculated as the following

$$m_0(B_3) = 1,\ m_1(B_3) = 0,\ m_2(B_3) = \frac{1}{4}.$$

Now, let us show that $M_3(B_3)$ is finite. Since $\operatorname{supp}(B_3) = \left[e^{-\frac{3}{2}}, e^{\frac{3}{2}}\right]$, we have

$$\left|\left\{k : e^{-\frac{3}{2}} \le e^{-k}x \le e^{\frac{3}{2}}\right\}\right| \le 3.$$

Thus, we have

$$\begin{aligned} M_3(B_3) &= \sup_{x\in\mathbb{R}^+}\sum_{k\in\mathbb{Z}}\left|B_3\left(e^{-k}x\right)\right|\left|k-\log x\right|^3 \\ &\le 3\left(\sup_{x\in\mathbb{R}^+} B_3(x)\right)\left(\frac{3}{2}\right)^3. \end{aligned}$$

Since $\sup_{x\in\mathbb{R}^+} B_3(x) = B_3(1) = \frac{3}{4}$, then we have $M_3(B_3) \le 7.59375 < \infty$. From Remark 2.1, we have that $M_0(B_3)\,, M_1(B_3)$ and $M_2(B_3)$ are also finite.

Mellin-Jackson Kernel

We finally present the Mellin-Jackson kernel which is given, for $c \in \mathbb{R}$, by

$$J_{\alpha,n}(x) := d_{\alpha,n}x^{-c}\operatorname{sinc}^{2n}\left(\frac{\log x}{2\alpha n\pi}\right),$$

where $x \in \mathbb{R}^+, n \in \mathbb{N}, \alpha \geq 1, d_{\alpha,n}$ is a normalization constant given by

$$d_{\alpha,n}^{-1} := \int_0^\infty \operatorname{sinc}^{2n}\left(\frac{\log u}{2\alpha n\pi}\right)\frac{du}{u}$$

(see, [12]). We can easily see that $J_{\alpha,n} \in X_c$ and $\left\|J_{\alpha,n}\right\|_{X_c} = 1$. Moreover, $J_{\alpha,n}$ is a Mellin band-limited kernel, that is $[J_{\alpha,n}]_M (c + it) = 0$ for $|t| \geq \frac{1}{\alpha}$. For the sake of simplicity, we consider $c = 0$ (the general case is handled in the same manner). We observe that $[J_{\alpha,n}]_M (0) = 1$ and $[J_{\alpha,n}]_M (2k\pi i) = 0$ for $k \in \mathbb{Z} \setminus \{0\}$. Then, using again Mellin-Poisson summation formula in the form

$$\sum_{k\in\mathbb{Z}} J_{\alpha,n}\left(e^k x\right) = \sum_{k\in\mathbb{Z}} [J_{\alpha,n}]_M (2k\pi i)\, x^{-2k\pi i}$$

and thanks to Lemma 4.1, we have

$$\sum_{k\in\mathbb{Z}} J_{\alpha,n}\left(e^k x\right) = 1.$$

Now, considering the definition of Mellin transform and by differentiating under the sign of the integral, one has

$$\frac{d^j}{dt^j}[J_{\alpha,n}]_M (0) = d_{\alpha,n}\int_0^\infty \operatorname{sinc}^{2n}\left(\frac{\log u}{2\alpha n\pi}\right)\log^j u\frac{du}{u}.$$

Then we have the following Mellin-Poisson summation formula:

$$\sum_{k\in\mathbb{Z}} J_{\alpha,n}\left(e^k x\right)(k - \log x)^j = \sum_{k\in\mathbb{Z}} \frac{d^j}{dt^j}[J_{\alpha,n}]_M (0)\, x^{-2k\pi i}. \tag{18}$$

For $j = 1$, we have $m_1 (J_{\alpha,n}) = 0$. Now, if we consider the cases $\alpha = 1$ and $n = 2$, then thanks to (18), we get $m_2 (J_{1,2}) = A_{1,2} < \infty$. Further, we have $M_2 (J_{1,2}) < \infty$. For more details, see [12].

5. Numerical Calculations and Graphical Representations

In this section, we present numerical examinations and graphical representations.

Firstly, let $f : \mathbb{R}^+ \to \mathbb{R}, f(x) = \sin(2\log x)$. Then, using the 3rd order Mellin-Spline kernel, we obtain the following Tables 1 and 2:

w	$\left\|\left(\mathcal{K}_w^{B_3,\mathcal{G}} f\right)(1.4) - f(1.4)\right\|$	$\left\|\left(\mathcal{K}_w^{B_3,\mathcal{G}} f\right)(3) - f(3)\right\|$
10	~ 0.02747248	~ 0.03564699
50	~ 0.00112059	~ 0.00145702
100	~ 0.00028037	~ 0.00036450

Table 1: Error bounds for the approximation of $f(x)$ via $\left(\mathcal{K}_w^{B_3,\mathcal{G}} f\right)(x)$ when $w = 10, 50, 100$.

Finally, let $g : \mathbb{R}^+ \to \mathbb{R}, g(x) = \frac{1+\log x}{1+\log^2 x}$. For $\alpha = 1$, $n = 2$ and $c = 0$, considering the Mellin-Jackson kernel, we get the following Table 3:

We present in Figure 1 the action of the operator $\mathcal{K}_w^{B_3,\mathcal{G}} f$ on f for increasing values of the parameter w namely $w = 10, 20$.

Finally, we present in Figure 2 the action of the operator $\mathcal{K}_{w,2}^{B_3,\mathcal{G}} g$ on g for increasing values of the parameter w namely $w = 3, 8$.

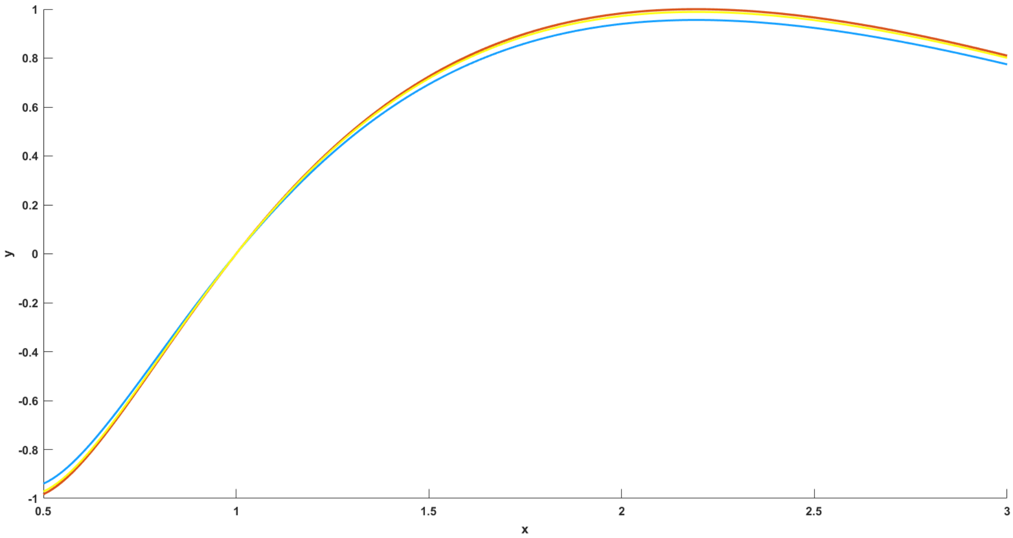

Figure 1: Details of the plots of f given by $f(x) = \sin(2\log x)$ (in red), $\mathcal{K}_{10}^{B_3,\mathcal{G}} f$ (in blue), $\mathcal{K}_{20}^{B_3,\mathcal{G}} f$ (in yellow).

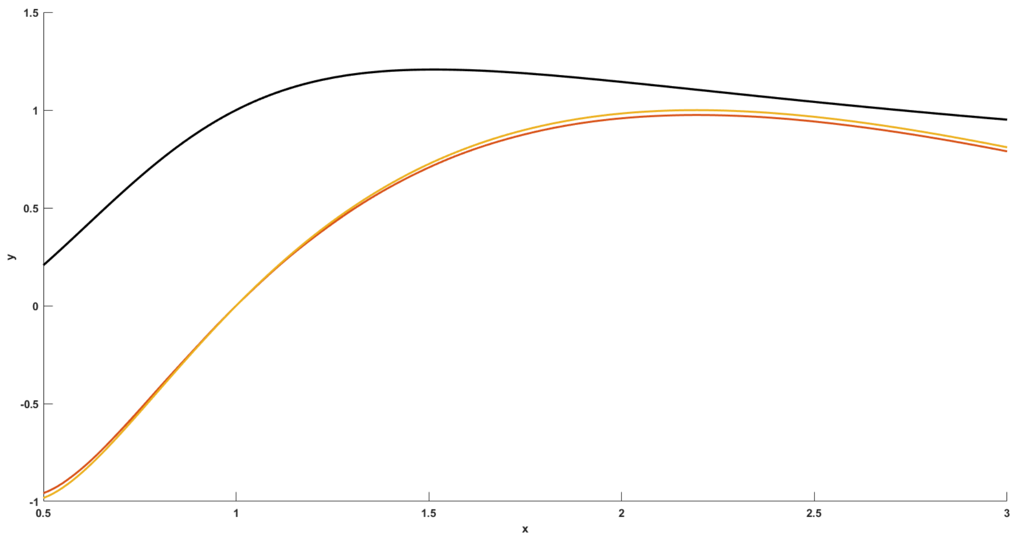

Figure 2: Details of the plots of g given by $g(x) = \frac{1+\log x}{1+\log^2 x}$ (in black), $\mathcal{K}_{3,2}^{B_3,\mathcal{G}} g$ (in red), $\mathcal{K}_{8,2}^{B_3,\mathcal{G}} g$ (in orange).

w	$\left\|\left(\mathcal{K}_{w,2}^{B_3,\mathcal{G}} f\right)(1.4) - f(1.4)\right\|$	$\left\|\left(\mathcal{K}_{w,2}^{B_3,\mathcal{G}} f\right)(3) - f(3)\right\|$
10	~ 0.00013136	~ 0.00020209
50	~ 0.00000030	~ 0.00000033
100	~ 0.00000004	~ 0.00000002

Table 2: Error bounds for the approximation of $f(x)$ via $\left(\mathcal{K}_{w,2}^{B_3,\mathcal{G}} f\right)(x)$ when $w = 10, 50, 100$.

w	$\left\|\left(\mathcal{K}_{w}^{J_{1,2},\mathcal{G}} g\right)(2) - g(2)\right\|$	$\left\|\left(\mathcal{K}_{w}^{J_{1,2},\mathcal{G}} g\right)(4.45) - g(4.45)\right\|$
20	~ 0.56569505	~ 0.68086825
60	~ 0.21957546	~ 0.62624025
180	~ 0.16769062	~ 0.61787312

Table 3: Error bounds for the approximation of $g(x)$ via $\left(\mathcal{K}_{w}^{J_{1,2},\mathcal{G}} g\right)(x)$ when $w = 20, 60, 180$.

6. Conclusion

In this paper, we have studied generalized Kantorovich forms of exponential sampling series and established new quantitative and constructive results that improve both the theoretical understanding and practical approximation performance of these operators. In particular, we derived a quantitative Voronovskaja-type formula expressed via the Mellin K-functional (Theorem 3.2), which refines the asymptotic behaviour of the Kantorovich exponential sampling operators and provides an explicit, usable error estimate. Building on this, we introduced carefully chosen linear combinations of Mellin–Gauss–Weierstrass type operators (Section 3.2) and proved that these combinations raise the convergence order in the asymptotic expansion (Theorems 3.4 and 3.5). An explicit two-term combination ($\ell = 2$) was displayed and its improved error behaviour was confirmed.

To illustrate the theory we examined concrete kernel families (Mellin–Spline and Mellin–Jackson kernels) and provided numerical computations and plots that validate the theoretical rates and show substantial practical improvement when linear combinations are used (Section 5). These examples demonstrate that, under the moment and absolute-moment conditions imposed on the kernels, the proposed approach produces both sharper asymptotic formulae and significantly smaller numerical errors.

Overall, the results unify and extend several strands in exponential sampling theory and provide both rigorous asymptotics and constructive tools for improving approximation by Kantorovich-type exponential sampling operators.

Data Availability. No datasets were used for the research described in the article.

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