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Spectra of graphs with loops with respect to some unary graph operations and beyond

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Abstract. Characteristic polynomials of certain self-loop graphs of a given graph with respect to some unary graph operations are derived by use of the relations connected with the incidence matrix and its transpose of both simple graphs and graphs with loops. The rank of the incidence matrix of a self-loop graph of a connected graph is determined, and it is proved that the multiplicity of the eigenvalue -2 in the spectrum of the line graph $L(G_S)$ of the self-loop graph G_S of a connected graph G is equal to the multiplicity of the same eigenvalue in the spectrum of the line graph $L(G)$ of the graph G. The nullity of certain connected bipartite graphs with pendant vertices is determined, and some structural perturbations of these graphs, which preserve the nullity, or which increase (decrease) it by 2, are considered. A part of the presented results is applied for considering some other problems in the domain of graph theory. Precisely, an example of self-loop graphs whose energy is strictly greater than the energy of their underlying bipartite graphs is found, in which way a recently posed conjecture is partially confirmed; an alternative proof to the existing proofs in the literature, which concerns the nullity set of the set of all connected bipartite graphs, is proposed; and finally, based on computational results for trees up to 9 vertices, it is conjectured in which way all trees of the given order and nullity can be constructed.

1. Introduction

Let $G = (V(G), E(G))$ be a finite simple graph of order n and size m, where $V(G) = \{v_1, v_2, \ldots, v_n\}$ stands for the set of vertices of G, while $E(G) = \{e_1, e_2, \ldots, e_m\}$ is the set of its edges.

The degree of the vertex v_i will be denoted by $d_G(v_i)$, $1 \leq i \leq n$, while the diagonal matrix of vertex degrees of G will be labeled $D(G)$. A vertex of degree 1 is called a *pendant vertex*, while an edge that contains a vertex of degree 1 is a *pendant edge*. A vertex of G is called *quasi-pendant* if it is adjacent to a pendant vertex. A *petal* in a graph means that a pendant edge in this graph is duplicated to form a pendant 2-cycle. If $d_G(v_i) = r$ holds for each $1 \leq i \leq n$, we say that G is an *r-regular graph*. In that case, $D(G) = r\,I_n$, where I_n is the identity matrix of order n.

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The *adjacency matrix* of G is a matrix $A = A(G) = [a_{ij}]$ of order n given by $a_{ij} = 1$, if the vertex v_i is adjacent to the vertex v_j, and $a_{ij} = 0$, otherwise. The *characteristic polynomial* $P_G(\lambda) = \det(\lambda I_n - A)$ of G is the characteristic polynomial of its adjacency matrix $A(G)$. The (adjacency) *eigenvalues* $\lambda_1(G) \geq \cdots \geq \lambda_n(G)$ of G are actually the eigenvalues of $A(G)$, and we say that they form the *(adjacency) spectrum* of G. By the *index* of G we mean its largest eigenvalue $\lambda_1(G)$. If $\lambda_i(G)$, for some i, is the eigenvalue of the multiplicity k, we will write $[\lambda_i(G)]^k$. The multiplicity of the eigenvalue zero in the spectrum of G, denoted by $\eta = \eta(G)$, is the *nullity* of the graph G.

Let $r(A(G))$ be the rank of the matrix $A = A(G)$. Clearly, $\eta(G) = n - r(A(G))$. The *rank of the graph G* is the rank of its adjacency matrix $A(G)$, denoted by $r(G)$. Then, $\eta(G) = n - r(G)$. Each of $\eta(G)$ and $r(G)$ determines the other, once n is specified. Graph G is called *singular* (or *non-singular*) if $A(G)$ is singular, i.e. $\eta(G) > 0$ (or $A(G)$ is non-singular, i.e. $\eta(G) = 0$).

Let $S \subseteq V(G)$ and $|S| = \sigma$, where $1 \leq \sigma \leq n$. The graph G_S obtained from G by attaching a self-loop (or simply a loop) at each vertex from the set S, is the *self-loop graph* of G, with respect to S. The adjacency matrix $A(G_S)$ of G_S is of the form $A(G_S) = A(G) + \Im_S$, where $\Im_S$ is the "almost" identity matrix, with exactly σ ones on the main diagonal corresponding to S, and all other entries equal to zero. The (adjacency) eigenvalues $\lambda_1(G_S) \geq \lambda_2(G_S) \geq \cdots \geq \lambda_n(G_S)$ of G_S are the eigenvalues of the matrix $A(G_S)$, and since $A(G_S)$ is square and symmetric, these eigenvalues are reals. If at some vertices from the set S more than one loop is attached, the resulting graph with multiple loops will be denoted by G_S^M. In that case, the adjacency matrix $A(G_S^M)$ of G_S^M takes the form $A(G_S^M) = A(G) + \Im_S^M$, where $\Im_S^M$ is the matrix with exactly σ non-zero entries corresponding to S on the main diagonal, which represent the numbers of loops at particular vertices from the set S. Lately, some research problems in the domain of chemical graph theory connected with graphs with loops appeared in the literature, so this type of graphs become again interesting to study [1], [2], [3], [14], [16], [22].

The goal of the paper is not only to develop the theory of self-loop graphs as such, but also to consider some new as well as some traditional problems from the theory of both simple and looped graphs by examining certain well-established concepts from the theory of simple graphs on graphs with loops. These concepts are primarily connected with the incidence matrix of a graph and some unary graph operations, in the first place line graph and subdivision graph.

The paper is organized as follows. In Section 2, additional notation is given and some results needed in the sequel are presented. Section 3 is devoted to the incidence matrix of a self-loop graph. Relations which are analogous to the relations connected with the incidence matrix of simple graphs and its transpose are elaborated, and the rank of the incidence matrix of a self-loop graph of a connected graph is determined. Consequently, it is proved that the multiplicity of the eigenvalue -2 in the spectrum of the line graph $L(G_S)$ of the self-loop graph G_S of a connected graph G is equal to the multiplicity of the same eigenvalue in the spectrum of the line graph $L(G)$ of the graph G. Characteristic polynomials of certain self-loop graphs of a given graph G with respect to some unary graph operations are exposed in Section 4. Due to these results, the nullity of some connected bipartite graphs with pendant vertices is determined in Section 5. Perturbations of these graphs, which preserve their nullity, or which increase (decrease) it by 2, are considered in this section, as well. Section 6 describes how results exposed in the previous sections can be applied to solving other problems connected with graphs. In that way, an example of self-loop graphs whose energy is strictly greater than the energy of their underlying bipartite graphs is found, in which way a conjecture posed in [1] is partially confirmed. An alternative proof to the existing proofs in the literature (see [9] and [17]), which concerns the nullity set of the set of all connected bipartite graphs is proposed. Finally, based on computational results for trees up to 9 vertices, it is conjectured in which way all trees of the given order and nullity can be constructed.

2. Preliminaries

In this section, we will recall some basic notation and fundamental statements of spectral graph theory which will be used in the paper.

A *bipartite graph* G is one whose vertex set $V(G)$ can be partitioned into two subsets, say X and Y, so that each edge has one end in X and one in Y; such a partition (X, Y) is called a *bipartition* of the graph. A *complete bipartite graph* is a bipartite graph with bipartition (X, Y) in which each vertex of X is adjacent to each vertex of Y. A complete bipartite graph where one of the sets of the bipartition is of the cardinality 1 is called the *star*. The star of order $n+1$ will be denoted by $K_{1,n}$, while for the *complete graph* on n vertices the label K_n will be used. The graph $G \cup H$ means the *disjoint union* of the graphs G and H, while the disjoint union of k copies of the same graph G is labeled kG. The *line graph* $L(G)$ of a graph G is the graph whose vertices are the edges of G, with two vertices in $L(G)$ adjacent whenever the corresponding edges in G have exactly one vertex in common. For the remaining notation and terminology, the reader is referred to [7] and [8].

As for the eigenvalues of G, the following theorems will be useful in the subsequent sections of the paper:

Theorem 2.1. *[8, Theorem 1.3.15] Let A and B be $n \times n$ Hermitian matrices. Then,*

$$\lambda_i(A+B) \leq \lambda_j(A) + \lambda_{i-j+1}(B),\ 1 \leq j \leq i \leq n, \tag{1}$$

$$\lambda_i(A+B) \geq \lambda_j(A) + \lambda_{i-j+n}(B),\ 1 \leq i \leq j \leq n. \tag{2}$$

Theorem 2.2. *[7, Theorem 0.13] Let G be an arbitrary graph of order n with at least one edge, and with the eigenvalues $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$. Then the following holds:*

$$1 \leq \lambda_1(G) \leq n-1, \tag{3}$$

$$-\lambda_1(G) \leq \lambda_n(G) \leq -1. \tag{4}$$

In (3), the upper bound is attained if and only if G is a complete graph, while the lower bound is reached if and only if the components of G consists of graphs K_2 and possibly K_1. In (4), the upper bound is reached if and only if the components of G are complete graphs, and the lower bound if and only if a component of G having the greatest index is a bipartite graph. If G is connected, the lower bound in (3) is replaced with $2\cos\frac{\pi}{n+1}$. Then equality holds if and only if G is a path.

Theorem 2.3. *[8, Theorem 3.2.1] Let $\lambda_1(G)$ be the index of the graph G, and let $\overline{d}$ and Δ be its average degree and maximum degree, respectively. Then*

$$\overline{d} \leq \lambda_1(G) \leq \Delta. \tag{5}$$

Moreover, $\overline{d} = \lambda_1(G)$ if and only if G is regular. For a connected graph G, $\lambda_1(G) = \Delta$ if and only if G is regular.

It is worth to recall two lemmas from Linear Algebra which we will refer to in the proofs of some statements.

Lemma 2.1. *[8, Equation (2.29)] Let M be a non-singular square matrix. If Q is also a square matrix, then,*

$$\det\begin{pmatrix} M & N \\ P & Q \end{pmatrix} = \det(M)\det(Q - PM^{-1}N).$$

Lemma 2.2. *[7, Lemma 2.1] If A is an $m \times n$ matrix, and if $P_X(\lambda)$ denotes the characteristic polynomial of the square matrix X, then*

$$\lambda^n P_{AA^T}(\lambda) = \lambda^m P_{A^TA}(\lambda).$$

3. On the incidence matrix of self-loop graphs

Let G be a graph of order n and size m, with the vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$, and the set of edges $E(G) = \{e_1, e_2, \ldots, e_m\}$. Let $S \subseteq V(G)$, such that $|S| = \sigma$, $1 \leq \sigma \leq n$, and let G_S be the graph obtained from G by attaching a loop at each vertex in the set S. Let us denote by $\mathcal{L}(G_S) = \{e_{m+1}, e_{m+2}, \ldots, e_{m+\sigma}\}$ the set of loops of G_S.

Recall, the vertex-edge *incidence matrix* [8] of G is the matrix $B(G)$ whose rows and columns are indexed by the vertices and edges of G, respectively. The (i, j)-th entry of $B(G)$ is equal to 1 if the vertex v_i is incident with the edge e_j, and to 0 otherwise. In the case G is connected, the rank $r(B(G))$ of the matrix $B(G)$ is given by the following statement:

Lemma 3.1. *[8, Lemma 3.4.8] Let $B(G)$ be the vertex-edge incidence matrix of a connected graph G with $n > 1$ vertices. Then,*

$$r(B(G)) = \begin{cases} n-1, & \text{if } G \text{ is bipartite;} \\ n, & \text{if } G \text{ is non-bipartite.} \end{cases}$$

The subsequent relations are well-known (see, for example, Relation (1) and Relation (2) in [6]):

$$B(G)B(G)^T = A(G) + D(G), \tag{6}$$

$$B(G)^T B(G) = A(L(G)) + 2I_m, \tag{7}$$

where $A(L(G))$ stands for the adjacency matrix of the line graph $L(G)$ of G. Below, we will consider and elaborate relations for the incidence matrix of a graph with loops and its transpose which are analogous to the relations (6) and (7).

The *incidence matrix* $B(G_S)$ of G_S has already been considered in [2]. The incidence matrix $B(G_S) = [b^S_{ij}]$ of G_S is a $n \times (m + \sigma)$ matrix defined as

$$b^S_{ij} = \begin{cases} 1, & \text{if } v_i \text{ and } e_j \text{ are incident;} \\ 0, & \text{otherwise.} \end{cases}$$

In fact, $B(G_S) = (B(G)\,|\,N)$, where N can be interpreted as the vertex-loop incidence matrix of G_S. In more detail, N is a $n \times \sigma$ matrix, such that in each column of N there is exactly one 1, and all other entries are equal to 0.

By analogy with the definition of the line graph $L(G)$ of a graph G, in [2], the line graph $L(G_S)$ of the self-loop graph G_S of G was introduced:

Definition 3.1. *[2] The* line graph $L(G_S)$ *of G_S is a graph whose vertices are the edges of G_S, with two vertices in $L(G_S)$ adjacent whenever the corresponding edges in G_S have exactly one vertex in common. Each loop attached at the vertex v_i in G_S is the vertex with a loop in $L(G_S)$, and this vertex is adjacent to those vertices in $L(G_S)$ which correspond to the edges of G_S incident with v_i in G_S.*

It was also observed [2] that the adjacency matrix $A(L(G_S))$ of $L(G_S)$ is of the following form

$$A(L(G_S)) = \begin{pmatrix} A(L(G)) & M^T \\ M & I_\sigma \end{pmatrix}, \tag{8}$$

where M stands for the loop-edge adjacency matrix. Precisely, $M = [m_{ij}]_{\sigma\times m}$, where $m_{ij} = 1$ if the loop, i.e. vertex e_{m+i} is adjacent to the edge, i.e. vertex e_j in $L(G_S)$, and $m_{ij} = 0$, otherwise. In that way, in each column of M, there can be at most two entries equal to 1, since at most two loops can be adjacent with an edge (precisely, in each of the two vertices of an edge there can be at most one loop). When $\sigma = n$, $M = B(G)$, while in the proof of Theorem 2.6 in [2], it was confirmed that $M = N^T B(G)$.

An analogue of Relation (7) has been reported in [2] (see the proof of Theorem 2.6), where it was noted that $B(G_S)^T B(G_S)$ is a positive semi-definite matrix, and that, by taking into account (7), the following holds

$$B(G_S)^T B(G_S) = \begin{pmatrix} B(G)^T B(G) & B(G)^T N \\ N^T B(G) & N^T N \end{pmatrix} = \begin{pmatrix} A(L(G)) + 2I_m & M^T \\ M & I_\sigma \end{pmatrix},$$

that is, considering (8),

$$B(G_S)^T B(G_S) = A(L(G_S)) + 2\,\mathfrak{I}_m. \tag{9}$$

Here, $A(L(G_S))$ is the adjacency matrix of the line graph $L(G_S)$ of G_S, while $\mathfrak{I}_m$ is the square matrix of order $m + \sigma$ with exactly m ones on the main diagonal, and all other entries equal to zero, i.e.

$$\mathfrak{I}_m = \begin{pmatrix} I_m & O^T \\ O & O \end{pmatrix}.$$

As for the line graph $L(G_S)$ of the self-loop graph G_S of the given graph G, the following statement has also been proved in [2]:

Theorem 3.1. *[2, Theorem 2.6] For every eigenvalue λ of $L(G_S)$, $\lambda \geq -2$.*

The multiplicity $m_{L(G)}(-2)$ of the eigenvalue -2 in the spectrum of the line graph $L(G)$ of a connected graph G is known:

Theorem 3.2. *[8, Theorem 3.4.9] Let G be a connected graph with n vertices and m edges. Then*

$$m_{L(G)}(-2) = \begin{cases} m - n + 1, & \text{if } G \text{ is bipartite;} \\ m - n, & \text{if } G \text{ is non-bipartite.} \end{cases}$$

As an addition to the two above results, we state the following theorem:

Theorem 3.3. *Let G be a connected graph of order n and size m, and let $S \subseteq V(G)$, such that $|S| = \sigma \geq 1$. Then*

$$m_{L(G)}(-2) = m_{L(G_S)}(-2),$$

where $m_H(\lambda)$ denotes the multiplicity of λ as an eigenvalue of the graph (with loops) H.

Proof. We distinguish two cases, depending on the number of loops in G.

If $|S| = n$, then the proof follows from Theorem 2.8 in [2].

So, let us suppose that $|S| = \sigma < n$. Considering Theorem 3.1, -2 is the least eigenvalue of $L(G_S)$ if and only if

$$A(L(G_S))\,\mathbf{x} = -2\,\mathbf{x}, \tag{10}$$

for some non-zero vector $\mathbf{x} = (x_1, \ldots, x_m, x_{m+1}, \ldots, x_{m+\sigma})^T$. Let $\mathbf{x}_E = (x_1, \ldots, x_m)^T$ be the eigenvector for $L(G)$ corresponding to the eigenvalue -2, and let $\mathbf{x}_S = (x_{m+1}, \ldots, x_{m+\sigma})^T$. Then, $\mathbf{x} = (\mathbf{x}_E, \mathbf{x}_S)^T$.

According to (8), the relation (10) is equivalent to

$$\begin{pmatrix} A(L(G))\,\mathbf{x}_E + M^T\,\mathbf{x}_S \\ M\,\mathbf{x}_E + I_\sigma\,\mathbf{x}_S \end{pmatrix} = \begin{pmatrix} -2\,\mathbf{x}_E \\ -2\,\mathbf{x}_S \end{pmatrix},$$

i.e.

$$A(L(G))\,\mathbf{x}_E + M^T\,\mathbf{x}_S = -2\,\mathbf{x}_E, \tag{11}$$

$$M\,\mathbf{x}_E + I_\sigma\,\mathbf{x}_S = -2\,\mathbf{x}_S. \tag{12}$$

From Equation (11), we obtain

$$M^T \mathbf{x}_S = \mathbf{0}. \tag{13}$$

In order to determine the column vector $M^T \mathbf{x}_S$, i.e. to solve the system of linear equations (13), considering any edge $e_i = v_j v_k$, $i \in \{1, 2, \ldots, m\}$, for some $j, k \in \{1, 2, \ldots, n\}$, $j \neq k$, we identify three possibilities:

(1) neither of the vertices v_j nor v_k has a loop attached;

(2) there is the loop e_{m+p}, $p \in \{1, 2, \ldots, \sigma\}$, at the vertex v_j;

(3) there are two loops e_{m+p} and e_{m+q}, $p, q \in \{1, 2, \ldots, \sigma\}$, $p \neq q$, at vertices v_j and v_k, respectively.

Case **(1)** is irrelevant, since the i-th row of M^T is zero-row. The i-th entry of the column vector $M^T \mathbf{x}_S$ in Case **(2)**, is equal to x_{m+p}, and therefore, in view of (13), it follows $x_{m+p} = 0$. In Case **(3)**, i-th entry of $M^T \mathbf{x}_S$ is equal to $x_{m+p} + x_{m+q}$, and hence $x_{m+p} + x_{m+q} = 0$. Since G has at least one vertex without loop attached, we find $x_{m+1} = \cdots = x_{m+\sigma} = 0$, i.e. $\mathbf{x}_S = \mathbf{0}^T$.

So, Equation (12) reduces to $M \mathbf{x}_E = \mathbf{0}$, which is equivalent to $N^T B(G) \mathbf{x}_E = \mathbf{0}$. Since $\mathbf{x}_E$ is the eigenvector for $L(G)$ corresponding to the eigenvalue -2, this means that the multiplicity of -2 as an eigenvalue of $L(G_S)$ is equal to the nullity of the matrix $B(G)$, i.e. that it is the same as in the case of the graph $L(G)$ (see Lemma 3.1 and Theorem 3.2). □

Remark 3.1. *From the previous proof it follows that if $(x_1, \ldots, x_m)^T$ is an eigenvector for $L(G)$ corresponding to the eigenvalue -2, then $(x_1, \ldots, x_m, \underbrace{0, 0, \ldots, 0}_{\sigma})^T$ is an eigenvector for $L(G_S)$ corresponding to the same eigenvalue.*

In a similar way, we can obtain an analogue of Relation (6) (see [2]). Namely, bearing in mind this relation, we have

$$B(G_S)B(G_S)^T = \begin{pmatrix} B(G) & N \end{pmatrix} \begin{pmatrix} B(G)^T \\ N^T \end{pmatrix} = A(G) + D(G) + \mathfrak{I}_S,$$

i.e.

$$B(G_S)B(G_S)^T = A(G_S) + D(G), \tag{14}$$

that is,

$$B(G_S)B(G_S)^T = Q(G) + \mathfrak{I}_S = Q(G_S),$$

where $Q(G) = A(G) + D(G)$ is the signless Laplacian matrix of G, while $Q(G_S) = [q_{ij}]$ is the *signless Laplacian matrix of* G_S. Precisely (see [2]),

$$q_{ij} = \begin{cases} 1, & \text{if vertices } v_i \text{ and } v_j \text{ are adjacent,} \\ 0, & \text{if vertices } v_i \text{ and } v_j \text{ are not adjacent,} \\ d_G(v_i) + 1, & \text{if } i = j \text{ and } v_i \in S, \\ d_G(v_i), & \text{if } i = j \text{ and } v_i \notin S. \end{cases}$$

Obviously, $B(G_S)B(G_S)^T$, i.e. $Q(G_S)$ is a positive semi-definite matrix. Even more, if G is a connected graph, $|S| \geq 1$, and $\mathbf{x} \in \mathbb{R}^n$ is an arbitrary column vector, then

$$\mathbf{x}^T B(G_S)B(G_S)^T \mathbf{x} = \mathbf{0}$$

is equivalent to

$$B(G_S)^T \mathbf{x} = \mathbf{0}.$$

The later holds if and only if $\mathbf{x} = \mathbf{0}$, which means that $B(G_S)B(G_S)^T$ is a positive definite matrix, and therefore

Theorem 3.4. *The signless Laplacian eigenvalues of a connected self-loop graph with at least one loop are positive reals.*

In this way, the following statement is also proved:

Proposition 3.1. *Let G be a connected graph of order n, and $S \subseteq V(G)$, such that $|S| \geq 1$. Then $r(B(G_S)) = n$, where $r(B(G_S))$ is the rank of the matrix $B(G_S)$.*

The following theorem was proved in [2]:

Theorem 3.5. *[2, Theorem 3.1] Let G be a graph of order n, whose eigenvalues with respect to the adjacency matrix are $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$. Let G_S be the self-loop graph of G, such that $S \subseteq V(G)$ and $|S| = \sigma$ ($0 \leq \sigma \leq n$). Then, for the eigenvalues $\lambda_i(G_S)$, $1 \leq i \leq n$, of G_S the following holds*

$$\lambda_i(G) \leq \lambda_i(G_S) \leq \lambda_i(G) + 1, \ 1 \leq i \leq n. \tag{15}$$

The left-hand side inequality in (15) is attained for $\sigma = 0$, while the right-hand side inequality for $\sigma = n$.

If G contains at least one edge, by use of Inequality (4) and Inequality (5), the left-hand side inequality in (15) for $i = n$ reduces to

$$-\Delta \leq -\lambda_1(G) \leq \lambda_n(G) \leq \lambda_n(G_S),$$

which means that $\lambda_n(G_S) \geq -\Delta$, where Δ is the maximum degree of G. But, if in addition, G is connected, while $|S| \geq 1$, the strict inequality holds here, which is proved by the following statement:

Proposition 3.2. *Let G be a connected graph of order n, $n \geq 2$, and let $S \subseteq V(G)$, such that $|S| \geq 1$. Then $\lambda_n(G_S) > -\Delta$.*

Proof. By applying Inequality (2) for $i = j = n$ to Equation (14), we obtain

$$\lambda_n(A(G_S)) \geq \lambda_n(B(G_S)B(G_S)^T) + \lambda_n(-D(G)) > -\Delta.$$

□

4. Some unary graph operations and self-loop graphs

In this section, we will expose the characteristic polynomials of certain self-loop graphs of a given graph G with respect to some unary graph operations.

Theorem 4.1. *Let G be an r-regular graph of order n and size $m\left(= \frac{nr}{2}\right)$, and let $S \subseteq V(G)$, such that $|S| = \sigma$, and $\sigma \geq 1$. Then*

$$P_{\mathcal{D}L(G_S)}(\lambda) = \lambda^{m+\sigma-n} \, P_{G_S}(\lambda - r),$$

where $\mathcal{D}L(G_S)$ is the graph obtained from $L(G_S)$ by attaching a double loop at each vertex of $L(G_S)$ which has no loop attached (i.e. which corresponds to an edge of G_S).

Proof. For the adjacency matrix $A(\mathcal{D}L(G_S))$ of $\mathcal{D}L(G_S)$ the following holds:

$$A(\mathcal{D}L(G_S)) = B(G_S)^T B(G_S). \tag{16}$$

By applying Lemma 2.2 to the matrix $B(G_S)$ and its transpose, we obtain

$$\lambda^n \det\left(\lambda I_{m+\sigma} - B(G_S)^T B(G_S)\right) = \lambda^{m+\sigma} \det\left(\lambda I_n - B(G_S)B(G_S)^T\right).$$

Using Relation (14) and Relation (16), the proof follows. □

Let us, now, recall the definition of one well-known and frequently used unary graph operation.

Definition 4.1. *[6] The* subdivision graph $\mathcal{S}(G)$ *of a graph G is the graph obtained from G by forming a new vertex on every edge of G.*

The following result is also exposed in [6]:

Theorem 4.2. *[6, Theorem 3] If G is a regular graph of degree r with n vertices, and $m\left(=\frac{nr}{2}\right)$ edges, then*

$$P_{\mathcal{S}(G)}(\lambda) = \lambda^{m-n}\, P_G(\lambda^2 - r).$$

Theorem 4.3. *Let G be an r-regular graph of order n and size $m\left(=\frac{nr}{2}\right)$. Then*

$$P_{\Sigma_E(\mathcal{S}(G))}(\lambda) = (\lambda-1)^{m-n}\, P_G(\lambda^2 - \lambda - r),$$

where $\Sigma_E(\mathcal{S}(G))$ is the self-loop graph of the subdivision graph $\mathcal{S}(G)$ of G, with respect to the set of vertices $V(\mathcal{S}(G)) \setminus V(G)$.

Proof. The adjacency matrix $A(\Sigma_E(\mathcal{S}(G)))$ of the graph $\Sigma_E(\mathcal{S}(G))$ is of the following form

$$A(\Sigma_E(\mathcal{S}(G))) = \begin{pmatrix} I_m & B(G)^T \\ B(G) & O_n \end{pmatrix}.$$

Using Lemma 2.1 and Formula (6), we obtain

$$\begin{aligned}
P_{\Sigma_E(\mathcal{S}(G))}(\lambda) &= \det(\lambda I_{m+n} - A(\Sigma_E(\mathcal{S}(G)))) \\
&= \det((\lambda-1)I_m)\, \det\left(\lambda I_n - B(G)\,((\lambda-1)I_m)^{-1}\, B(G)^T\right) \\
&= (\lambda-1)^m\, \det\left(\lambda I_n - \frac{1}{\lambda-1} B(G)B(G)^T\right) \\
&= (\lambda-1)^{m-n}\, \det(\lambda(\lambda-1)I_n - A(G) - r\,I_n) \\
&= (\lambda-1)^{m-n}\, P_G(\lambda^2 - \lambda - r).
\end{aligned}$$

□

In a similar way, the following statement can be proved :

Theorem 4.4. *Let G be an r-regular graph of order n and size $m\left(=\frac{nr}{2}\right)$. Then*

$$P_{\Sigma_V(\mathcal{S}(G))}(\lambda) = \lambda^{m-n}\, P_G(\lambda^2 - \lambda - r),$$

where $\Sigma_V(\mathcal{S}(G))$ is the self-loop graph of the subdivision graph $\mathcal{S}(G)$ of G, with respect to the set of vertices $V(G)$.

Let G be a graph of order n, size m, and let $S \subseteq V(G)$, such that $|S| = \sigma > 0$. Let us define the *subdivision graph $\mathcal{S}(G_S)$ of the self-loop graph G_S* of the given graph G in the way it was done for simple graphs, i.e.

Definition 4.2. *The* subdivision graph $\mathcal{S}(G_S)$ of the self-loop graph G_S *is the graph obtained by inserting a new vertex on every edge and every loop of G_S (see Figure 1).*

From the definition, we can notice that by subdividing a loop, we actually obtain a petal, so $\mathcal{S}(G_S)$ can be considered as $\mathcal{S}(G)$ with petals (i.e. 2-cycles) at vertices from the set S. The adjacency matrix $A(\mathcal{S}(G_S))$ of $\mathcal{S}(G_S)$ is of the following form

$$A(\mathcal{S}(G_S)) = \begin{pmatrix} O & \mathcal{B}_S^T \\ \mathcal{B}_S & O \end{pmatrix},$$

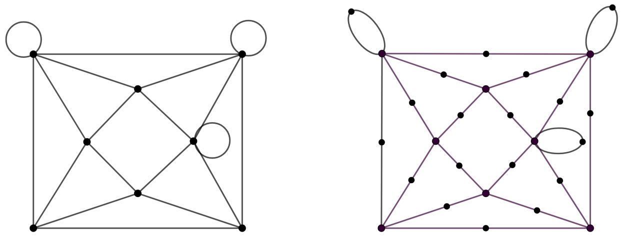


Figure 1: Graph with loops G_S and its subdivision graph $\mathcal{S}(G_S)$

where $\mathcal{B}_S = (B(G)\,|\,2N)$. It can be easily verified that

$$\mathcal{B}_S\,\mathcal{B}_S^T = A(G) + D(G) + 4\,\mathfrak{J}_S = A(G_S^4) + D(G), \tag{17}$$

where $A(G_S^4)$ is the adjacency matrix of the graph G_S^4 with multiple loops which is obtained from G by attaching 4 loops at each vertex from the set S. So, in case G is a regular graph, the following statement can be proved:

Theorem 4.5. *Let G be an r-regular graph of order n and size $m\left(=\frac{nr}{2}\right)$, and let $S \subseteq V(G)$, where $|S| = \sigma > 0$. Then*

$$P_{\mathcal{S}(G_S)}(\lambda) = \lambda^{m-n+\sigma}\,P_{G_S^4}(\lambda^2 - r),$$

where G_S^4 is the graph with multiple loops obtained from G by attaching 4 loops at each vertex from the set S.

Proof. By applying Lemma 2.1 and by use of Equality (17), we get

$$\begin{aligned}
P_{\mathcal{S}(G_S)}(\lambda) &= \det\left(\lambda I_{m+n+\sigma} - A(\mathcal{S}(G_S))\right)\\
&= \det(\lambda I_{m+\sigma})\ \det\left(\lambda I_n - \mathcal{B}_S(\lambda I_{m+\sigma})^{-1}\mathcal{B}_S^T\right)\\
&= \lambda^{m+\sigma-n}\ \det\left(\lambda^2 I_n - \mathcal{B}_S\mathcal{B}_S^T\right)\\
&= \lambda^{m+\sigma-n}\ \det\left((\lambda^2 - r)I_n - A(G_S^4)\right)\\
&= \lambda^{m+\sigma-n}\,P_{G_S^4}(\lambda^2 - r).
\end{aligned}$$

□

Definition 4.3. *[6] The* graph $R(G)$ *of a graph G is the graph obtained from G by associating a new vertex to each edge of G and by joining the ends of each edge of G with the vertex associated to this edge.*

Theorem 4.6. *Let G be an r-regular graph of order n and size $m\left(=\frac{nr}{2}\right)$. Then*

$$P_{\Sigma_E(R(G))}(\lambda) = \lambda^n(\lambda - 1)^{m-n}P_G\left(\frac{\lambda^2 - \lambda - r}{\lambda}\right),$$

where $\Sigma_E(R(G))$ is the self-loop graph of the graph $R(G)$ of G, with respect to the set of vertices $V(R(G)) \setminus V(G)$.

Proof. The adjacency matrix $A(\Sigma_E(R(G)))$ of the graph $\Sigma_E(R(G))$ is

$$A(\Sigma_E(R(G))) = \begin{pmatrix} I_m & B(G)^T \\ B(G) & A(G) \end{pmatrix},$$

so, using Lemma 2.1 and Relation (6), we obtain its characteristic polynomial

$$\begin{aligned}P_{\Sigma_E(R(G))}(\lambda) =& \det\left((\lambda-1)I_m\right)\,\det\left(\lambda I_n - A(G) - B(G)\left((\lambda-1)I_m\right)^{-1}B(G)^T\right)\\ =&(\lambda-1)^m\,\det\left(\lambda I_n - A(G) - \frac{1}{\lambda-1}B(G)B(G)^T\right)\\ =&(\lambda-1)^{m-n}\,\det\left(\lambda\,(\lambda-1)I_n - (\lambda-1)A(G) - A(G) - rI_n\right)\\ =&(\lambda-1)^{m-n}\,\lambda^n\,\det\left(\frac{\lambda^2-\lambda-r}{\lambda}I_n - A(G)\right)\\ =&\lambda^n\,(\lambda-1)^{m-n}\,P_G\left(\frac{\lambda^2-\lambda-r}{\lambda}\right).\end{aligned}$$

□

In a similar way, we can prove the following theorem:

Theorem 4.7. *Let G be an r-regular graph of order n and size $m\left(=\frac{nr}{2}\right)$. Then*

$$P_{\Sigma_V(R(G))}(\lambda) = \lambda^{m-n}\,(\lambda+1)^n\,P_G\left(\frac{\lambda^2-\lambda-r}{\lambda+1}\right),$$

where $\Sigma_V(R(G))$ is the self-loop graph of the graph $R(G)$ of G, with respect to the set of vertices $V(G)$.

Let G be a graph of order n, size m, and let $S \subseteq V(G)$, such that $|S| = \sigma > 0$. By analogy with Definition 4.3, we can introduce the so called *R-graph* of the self-loop graph G_S of the given graph G.

Definition 4.4. *The* graph $R(G_S)$ of the self-loop graph G_S *of a given graph G is the graph obtained from G_S by associating a new vertex to each edge and to each loop of G_S, and by joining by edges the ends of each edge of G_S with the vertex associated to this edge, and by joining by an edge the vertex associated to a loop with the vertex in which that loop is attached in G_S.*

Theorem 4.8. *Let G be an r-regular graph of order n and size $m\left(=\frac{nr}{2}\right)$, and $S \subseteq V(G)$, where $|S| = \sigma > 0$. Then*

$$P_{R(G_S)}(\lambda) = \lambda^{m+\sigma-n}\,(\lambda+1)^n\,P_{G_S}\left(\frac{\lambda^2-r}{\lambda+1}\right),$$

where $R(G_S)$ is the graph given by Definition 4.4.

Proof. Since the adjacency matrix $A(R(G_S))$ of the graph $R(G_S)$ is

$$A(R(G_S)) = \begin{pmatrix} O_{m+\sigma} & B(G_S)^T \\ B(G_S) & A(G_S) \end{pmatrix},$$

the proof follows by use of Lemma 2.1 and Relation (14). □

Now, let us suppose that $G = (V(G), E(G))$ is an arbitrary connected graph of order n, $n \geq 1$, and size m, $m \geq 0$, and let $\emptyset \neq S \subseteq V(G)$. Here, we assume that an isolated vertex, i.e. K_1, is a connected graph.

Definition 4.5. *The* graph $\mathcal{S}_p^S(G)$ *(or, the* graph of type $\mathcal{S}_p^S(G)$*) of a connected graph G, with respect to the set S, is the graph obtained from the subdivision graph $\mathcal{S}(G)$ of G by attaching p, where $p \geq |S|$, pendant vertices at vertices from the set S, at least one pendant vertex at each vertex from S, so that p is the total number of added pendant vertices.*

Remark 4.1. *In case G already has pendant vertices (i.e. pendant edges), we will use the label $\mathcal{SS}_p^S(G)$ to denote the graph $\mathcal{S}_p^S(G)$ where none of the p pendant vertices are attached at any of the pendant vertices in G, while the label $\overline{\mathcal{SS}}_p^S(G)$ will be used to indicate that the set S in $\mathcal{S}_p^S(G)$ must contain at least one pendant vertex of G.*

Whenever it does not cause confusion or misunderstanding (for example, when it does not important exactly which vertices have been chosen for the set S, or if we consider graphs $\mathcal{S}_p^S(G)$ for all possible non-empty sets $S \subseteq V(G)$ and some fixed p), we will omit the label S in the previously stated definition and the corresponding remark.

Remark 4.2. *In the case $n = 1$, i.e. $G \cong K_1$, we assume that $S = V(G)$, so $\mathcal{S}(K_1) \cong K_1$, and therefore $\mathcal{S}_p(K_1) \cong K_{1,p}$, for $p \geq 1$.*

It is obvious that $\mathcal{S}_p^S(G)$ is a connected bipartite, i.e. 2-colorable graph whose order is greater than or equal to 2. Let us denote the sets, i.e. classes of its bipartition by X and Y, such that the vertices of G are in the class X, that is $X = V(G)$, while all remaining vertices are in the class Y. We will also assume that the vertices in the class X are white colored, while the vertices in the class Y are black colored. In that way, for example, for any set $S \subseteq V(T_n)$, where T_n is an arbitrary tree of order n, whose cardinality is equal to 1, $\mathcal{S}_1(T_n)$ has n white vertices and also n black vertices, among which there is exactly one black pendant vertex (see Figure 2).

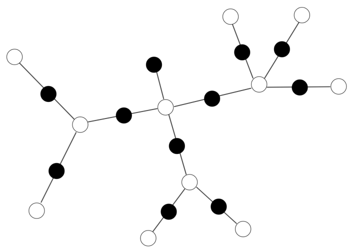


Figure 2: Graph of type $\mathcal{S}_1(T_{11})$

Let us notice that depending of the choice of the set S, and depending of how many pendant vertices are attached to each particular vertex from the set S, various non-isomorphic graphs of the same order could be generated. For example, if $G \cong K_2$, $S = V(G)$ and $p = 4$, we distinguish two non-isomorphic graphs of type $\mathcal{S}_4^S(K_2)$ (see Figure 3). All non-isomorphic graphs of type $\mathcal{S}_4(K_2)$, for all possible sets $S \subseteq V(G)$ and $p = 4$, are depicted on Figure 4.

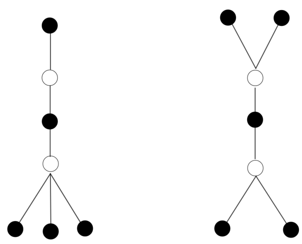


Figure 3: Two non-isomorphic graphs of type $\mathcal{S}_4^S(K_2)$, for $S = V(K_2)$

Let us associate a graph with multiple loops G_S^M to the graph $\mathcal{S}_p^S(G)$. The graph G_S^M is obtained from G by attaching multiple loops at vertices from the set S by the following rule: if there are p_v ($p_v \geq 1$) newly added (black) pendant vertices at the vertex v in $\mathcal{S}_p^S(G)$, then there are p_v loops at the vertex v in G_S^M (see Figure 5).

The adjacency matrix $A(\mathcal{S}_p^S(G))$ of $\mathcal{S}_p^S(G)$ is of the following form

$$A(\mathcal{S}_p^S(G)) = \begin{pmatrix} O & B(G_S^M)^T \\ B(G_S^M) & O \end{pmatrix},$$

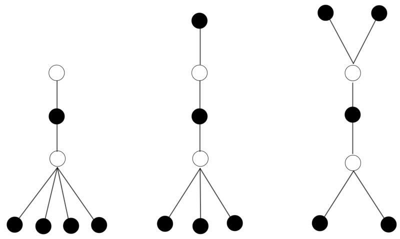


Figure 4: All non-isomorphic graphs of type $\mathcal{S}_4(K_2)$

where $B(G_S^M)$ is the incidence matrix of G_S^M. Precisely,

$$B(G_S^M) = (B(G)\,|\,\mathcal{N}),$$

where $\mathcal{N}$ is the $n \times p$ vertex-loop incidence matrix of G_S^M, such that in each of its columns there is exactly one entry equal to 1 (since each of p loops can be attached to at most one vertex) and all other entries are equal to zero.

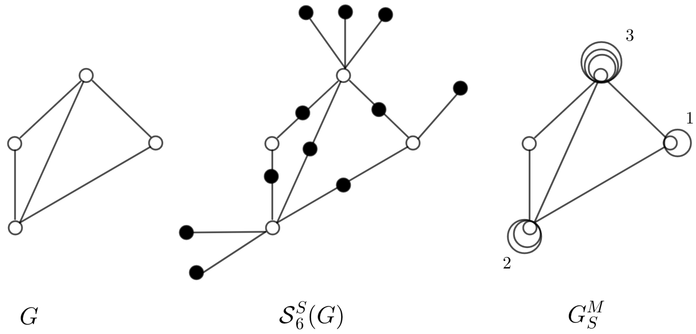


Figure 5: Graph G, graph $\mathcal{S}_6^S(G)$, for $p = 6$ and $|S| = 3$, and the corresponding G_S^M graph with 6 loops (the number of loops at some particular vertex is indicated by the number next to the loop)

If G is a connected regular graph, the following statement can be proved:

Theorem 4.9. *Let G be a connected r-regular graph of order n, size $m\left(= \frac{nr}{2}\right)$, and $S \subseteq V(G)$, such that $|S| = \sigma > 0$. Then*

$$P_{\mathcal{S}_p^S(G)}(\lambda) = \lambda^{m-n+p}\, P_{G_S^M}(\lambda^2 - r).$$

Proof. It can be easily verified that

$$B(G_S^M)B(G_S^M)^T = A(G) + D(G) + \mathfrak{I}_S^M = A(G_S^M) + D(G) = A(G_S^M) + rI_n.$$

Then by applying Lemma 2.1, we obtain the proof. $\square$

5. On the nullity of some bipartite graphs

More than sixty years ago, Collatz and Sinogowitz [5] first posed the problem of characterizing all graphs which have the number zero in their spectrum. So far, problems related to the nullity have been studied mainly by considering some particular graph classes, such as trees, unicyclic graphs, bicyclic graphs, graphs with pendant vertices, and so on (see, for example, [4], [10], [11], [13], [15], [17], [18], [20], [21], [23]).

Using some of the results stated in the previous sections, we are able to compute the nullity of bipartite graphs given by Definition 4.5. But, before we do that, let us remind that if G is a simple graph on n vertices and G is not isomorphic to $n\,K_1$, i.e. it contains at least one edge, then $0 \leq \eta(G) \leq n-2$. Precisely, the following statement holds:

Lemma 5.1. *[13, Lemma 1] Let G be a graph on n vertices. Then $\eta(G) = n$ if and only if G is a graph without edges, i.e. an empty graph.*

In the following, we list two well-known results needed in the rest of the paper.

Lemma 5.2. *[13, Lemma 3] Let $G = G_1 \cup G_2 \cup \cdots \cup G_t$, where $G_1, G_2, \dots, G_t$ are connected components of G. Then, $r(G) = \sum_{i=1}^{t} r(G_i)$, i.e. $\eta(G) = \sum_{i=1}^{t} \eta(G_i)$.*

Corollary 5.1. *[13, Corollary 6] If the bipartite graph G contains a pendant vertex, and if the induced subgraph H of G is obtained by deleting this vertex together with the vertex adjacent to it, then $\eta(G) = \eta(H)$.*

If G is a bipartite graph with the bipartition (X, Y), then its vertices may be numbered so that the adjacency matrix $A(G)$ has the following form:

$$A(G) = \begin{pmatrix} O & \mathcal{B} \\ \mathcal{B}^T & O \end{pmatrix}.$$

Here, O is the null matrix, while $\mathcal{B}$ is the "incidence matrix" between the two sets X and Y of vertices of G. The following theorem is also of importance:

Theorem 5.1. *[19] For the bipartite graph G with n vertices and the "incidence matrix" $\mathcal{B}$ it holds: $\eta(G) = n - 2r(\mathcal{B})$, where $r(\mathcal{B})$ is the rank of the matrix $\mathcal{B}$.*

Theorem 5.2. *Let G be a connected graph of order n, $n \geq 1$, and size m, $m \geq 0$, and let $\emptyset \neq S \subseteq V(G)$. Then,*

$$\eta(\mathcal{S}_p^S(G)) = m - n + p, \tag{18}$$

where $p \geq |S|$, while $\mathcal{S}_p^S(G)$ is the graph given by Definition 4.5.

Proof. As we have already noted, the adjacency matrix $A(\mathcal{S}_p^S(G))$ of the bipartite graph $\mathcal{S}_p^S(G)$ is of the form

$$A(\mathcal{S}_p^S(G)) = \begin{pmatrix} O & B(G_S^M)^T \\ B(G_S^M) & O \end{pmatrix},$$

where $B(G_S^M) = (B(G)\,|\,\mathcal{N})$ is the incidence matrix of the graph G_S^M with multiple loops associated to $\mathcal{S}_p^S(G)$, as previously described. We can consider that G_S^M is derived from G_S by attaching $p_v - 1$, $p_v \geq 1$, loops at vertices v in which there are p_v newly added pendant vertices in $\mathcal{S}_p^S(G)$. Therefore,

$$B(G_S^M) = (B(G)\,|\,\mathcal{N}) = (B(G_S)\,|\,\mathcal{N}^*) = (B(G)\,|\,N\,|\,\mathcal{N}^*).$$

Since each column of the matrix $\mathcal{N}^*$, where $\mathcal{N} = (N\,|\,\mathcal{N}^*)$, is linearly dependent with exactly one of the columns of N, from Proposition 3.1 it follows $r(B(G_S^M)) = n$. Then by use of Theorem 5.1, we obtain (18). □

Remark 5.1. *Since K_1 is the unique graph of order $n = 1$, $\mathcal{S}_p(K_1) = K_{1,p}$, $p \geq 1$, is the unique graph of type $\mathcal{S}_p(K_1)$, whose nullity, according to Theorem 5.2, is:*

$$\eta(K_{1,p}) = 0 - 1 + p = (p+1) - 2 = (\text{order of } \mathrm{K}_{1,\mathrm{p}}) - 2.$$

Remark 5.2. *According to the previous observations, we have*

$$r(\mathcal{S}_p^S(G)) = (n+m+p) - \eta(\mathcal{S}_p^S(G)) = 2n.$$

Below, we will consider some perturbations of certain bipartite graphs (mainly of graphs of type $\mathcal{S}_p^S(G)$, for some connected graph G, non-empty set $S \subseteq V(G)$ and $p \geq |S|$), which preserve the nullity of these graphs, or which increase (decrease) it by 2. By *graph perturbation* we mean a small structural alteration, such as the relocation of vertices and/or edges. But, before that, we state a lemma needed in the proofs of the corresponding statements.

Lemma 5.3. *Let G be a connected graph of order $n > 1$ and size $m \geq 1$. Then,*

$$\eta(\mathcal{S}(G)) = \begin{cases} m-n+2, & \text{if } G \text{ is bipartite;} \\ m-n, & \text{if } G \text{ is non-bipartite.} \end{cases}$$

Proof. The adjacency matrix $A(\mathcal{S}(G))$ of $\mathcal{S}(G)$ is of the following form:

$$A(\mathcal{S}(G)) = \begin{pmatrix} O & B(G)^T \\ B(G) & O \end{pmatrix},$$

where $B(G)$ is the vertex-edge incidence matrix of G, while O is the null matrix, as usual. Since $\mathcal{S}(G)$ is a bipartite graph, according to Theorem 5.1 and Lemma 3.1, we find $\eta(\mathcal{S}(G)) = (m+n) - 2r(B(G))$, i.e.

$$\eta(\mathcal{S}(G)) = m+n-2\cdot\begin{cases} (n-1), & \text{if } G \text{ is bipartite;} \\ n, & \text{if } G \text{ is non-bipartite,} \end{cases}$$

wherefrom the proof follows. □

Remark 5.3. *In accordance with Remark 4.2, we assume that $\eta(K_1) = 1$.*

Remark 5.4. *Starting from an arbitrary tree T_n of order $n > 1$, by use of Lemma 5.3, we can obtain the tree of order $2n-1$ whose nullity is equal to 1.*

Proposition 5.1. *Let G be a connected bipartite graph of order n, $n > 1$, and size m, $m \geq 1$, and let $\mathcal{G} = \mathcal{S}_p(G)$, for $p \geq 2$. Let v be one of the black pendant vertices of $\mathcal{G}$, and let $\mathcal{G}^\star$ be the graph obtained from $\mathcal{G}$ by relocating all $p-1$ remaining black pendant vertices of $\mathcal{G}$, different from v, to be the neighbours of v (i.e. by connecting them to v)* [1)] *(see Figure 6). Then, $\eta(\mathcal{G}^\star) = \eta(\mathcal{G})$.*

Remark 5.5. *Graph perturbation presented in Proposition 5.1, will be named $\mathcal{P}_1$, and therefore, we will write $\mathcal{G}^\star = \mathcal{P}_1(\mathcal{G}) = \mathcal{P}_1(\mathcal{S}_p(G))$, where G is a connected bipartite graph of order $n > 1$ and size $m \geq 1$, and $p \geq 2$.*

Proof. Let u be one of the white pendant vertices adjacent to the vertex v in $\mathcal{G}^\star$. Since $\mathcal{G}^\star$ is a bipartite graph, by use of Corollary 5.1, we have:

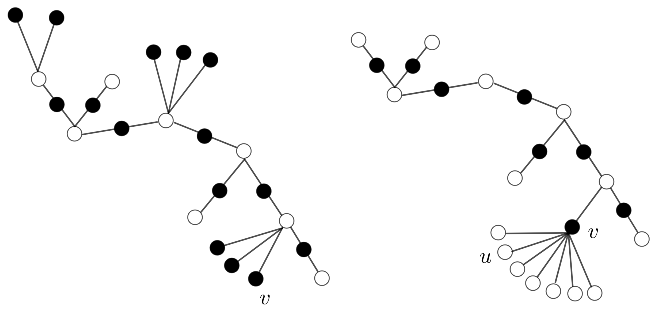
$$\eta(\mathcal{G}^\star) = \eta(\mathcal{G}^\star - \{u,v\}) = \eta((p-2)K_1 \cup \mathcal{S}(G)).$$

Then, by applying Lemma 5.2 and Lemma 5.3, we get

$$\eta(\mathcal{G}^\star) = \eta((p-2)K_1) + \eta(\mathcal{S}(G)) = (p-2) + (m-n+2) = m-n+p = \eta(\mathcal{G}),$$

which is true according to Theorem 5.2. □

[1)]Clearly, these vertices become white vertices in $\mathcal{G}^\star$.



$\mathcal{G} = \mathcal{S}_8(T_8)$ $\quad\quad$ $\mathcal{G}^\star = \mathcal{P}_1(\mathcal{S}_8(T_8))$

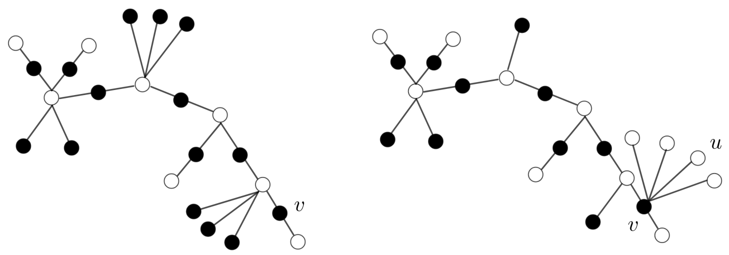
Figure 6: Example of applying graph perturbation $\mathcal{P}_1$ from Proposition 5.1

Remark 5.6. *In accordance with Remark 4.2 and Remark 5.3, the statement of Proposition 5.1 also holds for $G \cong K_1$.*

Remark 5.7. *If G is not a bipartite graph, then $\eta(\mathcal{G}^\star) = \eta(\mathcal{G}) - 2$.*

Proposition 5.2. *Let T_n be an arbitrary tree of order n, $n \geq 3$, and let $\mathcal{T} = \mathcal{SS}_p(T_n)$, for $p \geq 2$. Let v be a quasi-pendant black vertex of $\mathcal{T}$, and let $\mathcal{T}^\star$ be the tree obtained from $\mathcal{T}$ by relocating x, $1 \leq x \leq p-1$ black pendant vertices to be the neighbours of v (i.e. by connecting them to v) [2)] (see Figure 7). Then, $\eta(\mathcal{T}^\star) = \eta(\mathcal{T})$.*

Remark 5.8. *Graph perturbation presented in Proposition 5.2, will be referred to as $\mathcal{P}_2$, and therefore, we will write $\mathcal{T}^\star = \mathcal{P}_2(\mathcal{T}) = \mathcal{P}_2(\mathcal{SS}_p(T_n))$, where T_n is an arbitrary tree of order n, $n \geq 3$, and $p \geq 2$.*



$\mathcal{T} = \mathcal{SS}_8(T_8)$ $\quad\quad$ $\mathcal{T}^\star = \mathcal{P}_2(\mathcal{SS}_8(T_8)),\ for\ x = 4$

Figure 7: Example of applying graph perturbation $\mathcal{P}_2$ from Proposition 5.2

Proof. Let us denote by u one of the white pendant vertices adjacent to the vertex v in $\mathcal{T}^\star$. By use of Corollary 5.1 and Lemma 5.2, we have

$$\eta(\mathcal{T}^\star) = \eta(\mathcal{T}^\star - \{u, v\}) = \eta(x K_1 \cup \mathcal{S}_{p-x}(T_{n-1})) = x + \eta(\mathcal{S}_{p-x}(T_{n-1})),$$

where $\mathcal{S}_{p-x}(T_{n-1})$ is an induced subgraph of $\mathcal{T}$, and T_{n-1} is obtained from T_n by deleting the white pendant vertex adjacent to v in $\mathcal{T}$. By applying Theorem 5.2, it follows

$$\eta(\mathcal{T}^\star) = x + (n-2) - (n-1) + (p-x) = p - 1 = \eta(\mathcal{T}).$$

□

[2)] It is clear that these vertices become white in $\mathcal{T}^\star$.

Remark 5.9. *If $x = p$ and $p \geq 1$, then by use of Lemma 5.3, it follows $\eta(\mathcal{T}^\star) = \eta(\mathcal{T}) + 2$.*

Proposition 5.3. *Let G be an arbitrary connected graph of order n, $n \geq 1$, and size m, $m \geq 0$, and let $\mathcal{G} = \mathcal{S}_p(G)$, for $p \geq 3$. Let $\mathcal{G}^\star$ be the graph obtained from $\mathcal{G}$ by relocating x, $1 \leq x \leq p-2$, black pendant vertices to be the neighbours of one, say v, of the remaining $(p-x)$ black pendant vertices of $\mathcal{G}$ (i.e. by connecting them to v)* [3)] *(see Figure 8). Then, $\eta(\mathcal{G}^\star) = \eta(\mathcal{G}) - 2$.*

Remark 5.10. *Graph perturbation described by Proposition 5.3, will be named $\mathcal{P}_3$, and therefore, we will write $\mathcal{G}^\star = \mathcal{P}_3(\mathcal{G}) = \mathcal{P}_3(\mathcal{S}_p(G))$, where G is an arbitrary connected graph of order n, $n \geq 1$, and size m, $m \geq 0$, and $p \geq 3$.*

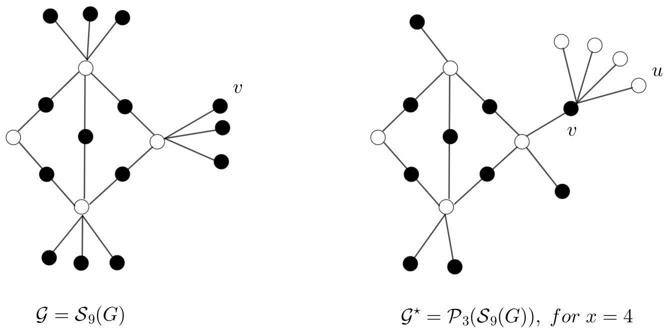


Figure 8: Example of applying graph perturbation $\mathcal{P}_3$ from Proposition 5.3

Proof. Let u be a white pendant vertex adjacent to v in $\mathcal{G}^\star$. By applying Corollary 5.1 and Lemma 5.2, we find

$$\eta(\mathcal{G}^\star) = \eta(\mathcal{G}^\star - \{u, v\}) = \eta((x-1)\,K_1 \cup \mathcal{S}_{p-x-1}(G)) = x - 1 + \eta(\mathcal{S}_{p-x-1}(G)),$$

where $\mathcal{S}_{p-x-1}(G)$ is the induced subgraph of $\mathcal{G}$. Now, by use of Theorem 5.2, we get

$$\eta(\mathcal{G}^\star) = x - 1 + m - n + (p - x - 1) = \eta(\mathcal{G}) - 2.$$

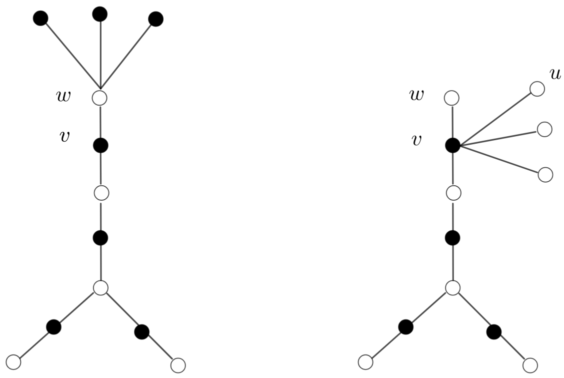
□

Remark 5.11. *If G is a connected non-bipartite graph of order $n \geq 3$ and size $m \geq 3$, and if $x = p-1$, for $p \geq 3$, it holds $\eta(\mathcal{G}^\star) = \eta(\mathcal{G}) - 2$.*

Proposition 5.4. *Let T_n be an arbitrary tree of order n, $n \geq 2$, and let w be one of its pendant vertices. Let $\mathcal{T} = \overline{\mathcal{SS}}_p(T_n)$, for $p \geq 1$, where q, $1 \leq q \leq p$, black pendant vertices are connected to w, and let v be the black neighbour of w of degree 2 (i.e. the quasi-pendant vertex adjacent to w in $\mathcal{S}(T_n)$). Let $\mathcal{T}^\star$ be the tree obtained from $\mathcal{T}$ by relocating x, $1 \leq x \leq q$, black pendant neighbours of w to be the neighbours of v (i.e. by connecting them to v)* [4)] *(see Figure 9 and Figure 10). Then,*

1. *if $q = p$ and $x = q$, then $\eta(\mathcal{T}^\star) = \eta(\mathcal{T}) + 2$;*
2. *if $p \geq 2$, $q = p$ and $1 \leq x \leq q-1$, then $\eta(\mathcal{T}^\star) = \eta(\mathcal{T})$;*
3. *if $p \geq 2$, $1 \leq q \leq p-1$ and $x = q$, then $\eta(\mathcal{T}^\star) = \eta(\mathcal{T})$;*
4. *if $p \geq 3$, $2 \leq q \leq p-1$ and $1 \leq x \leq q-1$, then $\eta(\mathcal{T}^\star) = \eta(\mathcal{T}) - 2$.*

[3)]It is clear that these vertices become white in $\mathcal{G}^\star$.
[4)]It is clear that these vertices become white in $\mathcal{T}^\star$.



$\mathcal{T} = \overline{\mathcal{SS}}_3(T_5)$ $\mathcal{T}^\star = \mathcal{P}_4(\overline{\mathcal{SS}}_3(T_5))$

Figure 9: Example of applying graph perturbation $\mathcal{P}_4$ from Proposition 5.4, case (1)

Remark 5.12. *Graph perturbation exposed in Proposition 5.4, will be denoted by $\mathcal{P}_4$, and therefore, we will write $\mathcal{T}^\star = \mathcal{P}_4(\mathcal{T}) = \mathcal{P}_4(\overline{\mathcal{SS}}_p(T_n))$, under the assumptions given in Proposition 5.4.*

Proof. (1) Let us denote by u one of the white pendant neighbours of v different from w in $\mathcal{T}^\star$. From Corollary 5.1 and Lemma 5.2, it follows

$$\eta(\mathcal{T}^\star) = \eta(\mathcal{T}^\star - \{u, v\}) = \eta(q\,K_1 \cup \mathcal{S}(T_{n-1})) = q + \eta(\mathcal{S}(T_{n-1})),$$

where T_{n-1} is obtained from T_n by deleting pendant vertex w. By use of Lemma 5.3 and Theorem 5.2, we finally obtain

$$\eta(\mathcal{T}^\star) = q + (n-2) - (n-1) + 2 = q + 1 = (p-1) + 2 = \eta(\mathcal{T}) + 2.$$

(2) Let us denote by u one of the white pendant neighbours of v in $\mathcal{T}^\star$ (in case $x = 1$, u will be the unique such a vertex). Similarly as before, by applying Corollary 5.1, Lemma 5.2 and Lemma 5.3, we find

$$\begin{aligned}\eta(\mathcal{T}^\star) &= \eta(\mathcal{T}^\star - \{u, v\}) = \eta((x-1)K_1 \cup K_{1,q-x} \cup \mathcal{S}(T_{n-1}))\\ &= (x-1) + (q - x + 1 - 2) + \eta(\mathcal{S}(T_{n-1}))\\ &= q - 2 + (n-2) - (n-1) + 2 = p - 1 = \eta(\mathcal{T}).\end{aligned}$$

In the upper relations, T_{n-1} denotes the tree which is obtained from T_n by deleting pendant vertex w.

(3) Let u be one of the white pendant neighbours of v different from w in $\mathcal{T}^\star$. Using Corollary 5.1, Lemma 5.2, and Theorem 5.2, we have:

$$\begin{aligned}\eta(\mathcal{T}^\star) &= \eta(\mathcal{T}^\star - \{u, v\}) = \eta(qK_1 \cup \mathcal{S}_{p-q}(T_{n-1}))\\ &= q + \eta(\mathcal{S}_{p-q}(T_{n-1})) = q + (n-2) - (n-1) + (p-q) = \eta(\mathcal{T}),\end{aligned}$$

where T_{n-1} is obtained from T_n by deleting pendant vertex w. Precisely, $\mathcal{S}_{p-q}(T_{n-1})$ is an induced subgraph of $\mathcal{T}$, which is obtained from $\mathcal{T}$ by deleting vertex v, vertex w, and all q neighbours of w.

(4) Let u be a white pendant neighbour of v in $\mathcal{T}^\star$. In the same way as before, we find

$$\eta(\mathcal{T}^\star) = \eta(\mathcal{T}^\star - \{u, v\}) = \eta((x-1)K_1 \cup K_{1,q-x} \cup \mathcal{S}_{p-q}(T_{n-1})),$$

where $\mathcal{S}_{p-q}(T_{n-1})$ is an induced subgraph of $\mathcal{T}$, and T_{n-1} is obtained from T_n by deleting pendant vertex w. Further we have

$$\eta(\mathcal{T}^\star) = (x-1) + (q + 1 - x - 2) + (n-2) - (n-1) + (p-q) = \eta(\mathcal{T}) - 2.$$

□

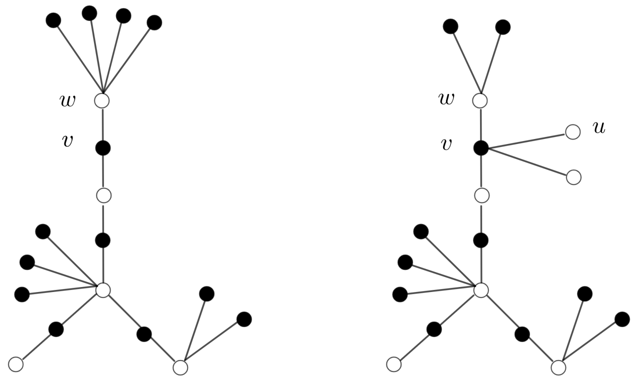


Figure 10: Example of applying graph perturbation $\mathcal{P}_4$ from Proposition 5.4, case (4)

Proposition 5.5. *Let T_n be a tree of order n, $n \geq 3$, with k, $k \geq 2$, pendant vertices $w_1, w_2, \ldots, w_k$. Let $\mathcal{T}_n = \mathcal{SS}_p(T_n)$, for $p \geq k$, and let u_i be the black quasi-pendant neighbour of w_i in $\mathcal{S}(T_n)$, i.e. $\mathcal{T}_n$, for $1 \leq i \leq k$. Let $\mathcal{T}_n^\star$ be the tree obtained from $\mathcal{T}_n$ by relocating x, $k \leq x \leq p$, black pendant vertices to be the neighbours of u_i, such that each u_i is connected with x_i pendant vertices, where $x_i \geq 1$, $1 \leq i \leq k$, and $\sum_{i=1}^{k} x_i = x$ [5)] (see Figure 11). It holds:*

1. *if $k \leq x < p$, then $\eta(\mathcal{T}_n^\star) = \eta(\mathcal{T}_n)$,*
2. *if $x = p$, then $\eta(\mathcal{T}_n^\star) = \eta(\mathcal{T}_n) + 2$.*

Remark 5.13. *Graph perturbation described in Proposition 5.5, will be named $\mathcal{P}_5$, and therefore, we will write $\mathcal{T}_n^\star = \mathcal{P}_5(\mathcal{T}_n) = \mathcal{P}_5(\mathcal{SS}_p(T_n))$, under the assumptions given in Proposition 5.5.*

Proof. Let us denote by v_i one of the white pendant neighbours of u_i different from w_i in $\mathcal{T}_n^\star$, for $1 \leq i \leq k$.

If $x < p$, by use of Corollary 5.1, Lemma 5.2 and Theorem 5.2, we have

$$\begin{aligned}
\eta(\mathcal{T}_n^\star) &= \eta(\mathcal{T}_n^\star - \{u_1, v_1\}) = \eta(x_1 K_1 \cup \mathcal{T}_{n-1}^\star) \\
&= x_1 + \eta(\mathcal{T}_{n-1}^\star) = x_1 + \eta(\mathcal{T}_{n-1}^\star - \{u_2, v_2\}) = x_1 + \eta(x_2 K_1 \cup \mathcal{T}_{n-2}^\star) \\
&= x_1 + x_2 + \eta(\mathcal{T}_{n-2}^\star) = \cdots = x_1 + x_2 + \cdots + x_k + \eta(\mathcal{S}_{p-x}(T_{n-k})) \\
&= x + (n-k-1) - (n-k) + (p-x) = p - 1 = \eta(\mathcal{T}_n).
\end{aligned}$$

In the previous relations, $\mathcal{T}_{n-i}^\star$ is an induced subgraph of $\mathcal{T}_{n-i+1}^\star$, for $i = 1, 2, \ldots, k-1$.

If $x = p$, in a similar way, by use of Lemma 5.3, we find

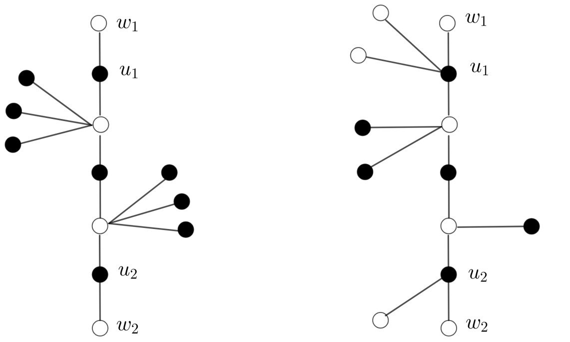
$$\eta(\mathcal{T}_n^\star) = \sum_{i=1}^{k} x_i + \eta(\mathcal{S}(T_{n-k})) = x + (n-k-1) - (n-k) + 2 = \eta(\mathcal{T}_n) + 2.$$

□

6. Some applications

In this section, we will consider certain problems connected with graphs using some of the results stated in the previous sections.

5) Clearly, these relocated vertices become white in $\mathcal{T}_n^\star$.



$\mathcal{T}_4 = \mathcal{SS}_6(T_4)$ $\mathcal{T}_4^\star = \mathcal{P}_5(\mathcal{SS}_6(T_4)),\ for\ x = 3$

Figure 11: Example of applying graph perturbation $\mathcal{P}_5$ from Proposition 5.5

6.1. Comparison of graph energies

One of the widely studied graph invariants is certainly graph energy, which was introduced by Ivan Gutman [12]. Precisely, the *energy* $\mathcal{E}(G)$ *of a graph* G of order n is defined to be the sum of all absolute eigenvalues of G, i.e.

$$\mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i(G)|,$$

while the *energy* $\mathcal{E}(G_S)$ *of the self-loop graph* G_S of G, with respect to the set of vertices $S \subseteq V(G)$, such that $|S| = \sigma$, $0 \le \sigma \le n$, was given by [14]:

$$\mathcal{E}(G_S) = \sum_{i=1}^{n} \left|\lambda_i(G_S) - \frac{\sigma}{n}\right|.$$

The authors of [1] (Theorem 3.7) proved that for a bipartite graph G and an arbitrary set $S \subseteq V(G)$, $\mathcal{E}(G_S) \ge \mathcal{E}(G)$ holds. In addition, in the same paper, they posed a conjecture, which claims that for every simple graph G, there is a set $S \subseteq V(G)$ such that $\mathcal{E}(G_S) > \mathcal{E}(G)$.

Since the subdivision graph $\mathcal{S}(G)$ of a graph G is a bipartite graph, from Theorem 4.2 and Theorem 4.3, that is, Theorem 4.4, it follows that for the bipartite graph $\mathcal{S}(G)$ of a regular graph G there is a self-loop graph of $\mathcal{S}(G)$ whose energy is strictly greater than the energy of $\mathcal{S}(G)$.

Namely, let G be an r-regular graph, $r \ge 2$, of order n and size $m\left(= \frac{nr}{2}\right)$, with the adjacency eigenvalues $\lambda_i(G)$, $i = 1, 2, \dots, n$. From Theorem 4.2, it follows that the spectrum of the subdivision graph $\mathcal{S}(G)$ of G consists of the following eigenvalues: $[0]^{m-n}$ and $\pm\sqrt{\lambda_i(G) + r}$, $i = 1, 2, \dots, n$. Therefore, the energy of this graph is equal to

$$\mathcal{E}(\mathcal{S}(G)) = \sum_{i=1}^{m+n} |\lambda_i(\mathcal{S}(G))| = \sum_{i=1}^{n} 2\sqrt{\lambda_i(G) + r}.$$

According to Theorem 4.3, the eigenvalues of the graph $\Sigma_E(\mathcal{S}(G))$ of order $m + n$ and with m loops, are: $[1]^{m-n}$ and $\frac{1}{2}\left(1 \pm \sqrt{1 + 4\,(r + \lambda_i(G))}\right)$, $i = 1, 2, \dots, n$. Since G is a r-regular graph, and so $-r \le \lambda_i(G) \le r$, $i = 1, 2, \dots, n$, it holds: $\frac{1}{2}\left(1 + \sqrt{1 + 4\,(r + \lambda_i(G))}\right) \ge 1$, and $\frac{1}{2}\left(1 - \sqrt{1 + 4\,(r + \lambda_i(G))}\right) \le 0$. The quotient of the number of loops m in $\Sigma_E(\mathcal{S}(G))$ and the order $m + n$ of this graph reduces to $\frac{\frac{nr}{2}}{\frac{nr}{2}+n} = \frac{r}{r+2} < 1$, so we find that the energy of the self-loop graph $\Sigma_E(\mathcal{S}(G))$ is:

$$\begin{aligned}\mathcal{E}(\Sigma_E(\mathcal{S}(G))) &= \sum_{i=1}^{n+m}\left|\lambda_i(\Sigma_E(\mathcal{S}(G))) - \frac{r}{r+2}\right| \\ &= \left(\frac{nr}{2} - n\right)\left(1 - \frac{r}{r+2}\right) + \sum_{i=1}^{n}\left(\frac{1+\sqrt{1+4\,(r+\lambda_i(G))}}{2} - \frac{r}{r+2}\right) \\ &+ \sum_{i=1}^{n}\left(-\frac{1}{2}\left(1-\sqrt{1+4\,(r+\lambda_i(G))}\right) + \frac{r}{r+2}\right),\end{aligned}$$

i.e.

$$\mathcal{E}(\Sigma_E(\mathcal{S}(G))) = \frac{n\,(r-2)}{r+2} + \sum_{i=1}^{n}\sqrt{1+4(r+\lambda_i(G))}.$$

As $\sqrt{1+4(r+\lambda_i(G))} > 2\sqrt{r+\lambda_i(G)}$, for $i = 1,2,\ldots,n$, and $\frac{n\,(r-2)}{r+2} \geq 0$, it follows

$$\begin{aligned}\mathcal{E}(\Sigma_E(\mathcal{S}(G))) &= \frac{n\,(r-2)}{r+2} + \sum_{i=1}^{n}\sqrt{1+4(r+\lambda_i(G))} > \frac{n\,(r-2)}{r+2} + \sum_{i=1}^{n} 2\sqrt{\lambda_i(G)+r} \\ &= \frac{n\,(r-2)}{r+2} + \mathcal{E}(\mathcal{S}(G)) \geq \mathcal{E}(\mathcal{S}(G)),\end{aligned}$$

i.e.

$$\mathcal{E}(\Sigma_E(\mathcal{S}(G))) > \mathcal{E}(\mathcal{S}(G)).$$

Since, according to Corollary 3.5 from [1], $\mathcal{E}(\Sigma_E(\mathcal{S}(G))) = \mathcal{E}(\Sigma_V(\mathcal{S}(G)))$, we also have $\mathcal{E}(\Sigma_V(\mathcal{S}(G))) > \mathcal{E}(\mathcal{S}(G))$. In that way, the conjecture posed in [1] is partially confirmed.

6.2. Nullity sets

Let us denote by $\mathfrak{B}_n$ the set of all connected bipartite graphs of order n, where $n > 1$.

A subset N of the set $\{0,1,2,\ldots,n\}$ is said to be the *nullity set* of $\mathfrak{B}_n$ provided that for any $k \in N$ there exists at least one graph $G \in \mathfrak{B}_n$ such that $\eta(G) = k$, and for any graph $G \in \mathfrak{B}_n$, $\eta(G) \in N$.

Using Theorem 5.2, we can propose an alternative proof, in the case of connected bipartite graphs, of the statement which is in [9] marked as Theorem 3.1.

Theorem 6.1. *The nullity set of $\mathfrak{B}_n$ is $N = \{n-2k : k = 1,2,\ldots,\lfloor\frac{n}{2}\rfloor\}$, i.e. $N = \{0,2,4,\ldots,n-2\}$, if n is even, and $N = \{1,3,5,\ldots,n-2\}$, if n is odd.*

Proof. Let $G \in \mathfrak{B}_n$ be an arbitrary graph. Since G contains at least one edge, according to Lemma 5.1, $0 \leq \eta(G) \leq n-2$, i.e. $\eta(G) \in N$.

Let H be an arbitrary connected graph of order k, $k \in \{1,2,\ldots,\lfloor\frac{n}{2}\rfloor\}$, with m_H edges, such that $m_H \geq 0$ and $k + m_H \leq n-1$. Such a graph certainly exists; for example, H can be a tree of order k. Here, we consider K_1 as a connected graph. Let us form the graph $\mathcal{S}_p(H)$, where $p = n-(k+m_H)$, and then $p \geq 1$. The graph $\mathcal{S}_p(H)$ is the connected bipartite graph of order n, and the proof now follows from Theorem 5.2. □

Remark 6.1. *Since $\mathcal{S}_p(T)$, $p \geq 1$, for an arbitrary tree T, is also tree, the proof of Theorem 6.1 is another way to prove Corollary 3.1 in [17].*

6.3. Construction of trees with given nullity

Let $\mathfrak{T}_n$ be the set of all trees of order n, $n > 1$. According to Theorem 6.1, the nullity set of $\mathfrak{T}_n$ is equal to $N = \{n - 2k : k = 1, 2, \dots, \lfloor \frac{n}{2} \rfloor\}$, i.e. $N = \{0, 2, 4, \dots, n-2\}$, if n is even, and $N = \{1, 3, 5, \dots, n-2\}$, if n is odd.

Based on some of the statements exposed in the previous sections, we are able to identify, i.e. to construct all trees of order n, for $2 \leq n \leq 9$, whose nullity is equal $n - 2k$, for $k = 1, 2, \dots, \lfloor \frac{n}{2} \rfloor$, and to conjecture a generalization of such a construction for $n > 9$.

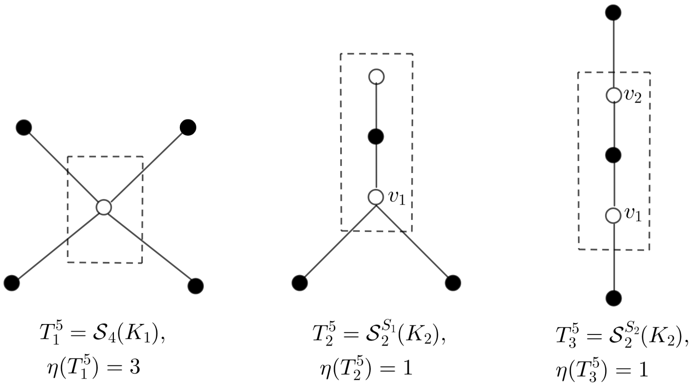


Figure 12: All non-isomorphic trees of order 5

Namely, we start from all non-isomorphic trees of order k, $k = 1, 2, \dots, \lfloor \frac{n}{2} \rfloor$. In this case, we assume that an isolated vertex (i.e. the case $k = 1$) is also a (trivial) tree. We can name these trees *base trees*, and denote them by B_j^k, where $j = 1, 2, \dots, \tau$, and τ is the number of such trees. After that, we generate all non-isomorphic trees $\mathcal{S}_p(B_j^k)$, for $p = n - 2k + 1$, where it holds $p \geq 1$. According to Theorem 5.2, the nullity of the graph $\mathcal{S}_p(B_j^k)$ is $\eta(\mathcal{S}_p(B_j^k)) = p - 1 = n - 2k$. All remaining non-isomorphic trees, if such exist, of order n and of nullity $n - 2k$, for $k = 1, 2, \dots, \lfloor \frac{n}{2} \rfloor$, can be obtained by applying graph perturbations $\mathcal{P}_i$, for $1 \leq i \leq 5$, on trees $\mathcal{S}_p(B_j^k)$, and their special cases $\mathcal{SS}_p(B_j^k)$ and $\overline{\mathcal{SS}}_p(B_j^k)$, in accordance with the following statements: Proposition 5.1, Proposition 5.2, Proposition 5.3, Proposition 5.4 and Proposition 5.5, which guarantee the preservation of the considered nullity.

Let us consider all trees up to seven vertices.

If $n = 2$, there is exactly one base tree B_1^1 of order 1, i.e. $B_1^1 \cong K_1$, and there is exactly one possibility, in accordance with Definition 4.5, to add a pendant vertex to $\mathcal{S}(B_1^1) \cong K_1$. Therefore, $\mathfrak{T}_2 = \{T_1^2\}$, where $T_1^2 = \mathcal{S}_1(K_1) \cong K_2$, and $\eta(T_1^2) = 0$.

For $n = 3$, there is, again, only one base tree B_1^1 of order 1, i.e. $B_1^1 \cong K_1$, and there is exactly one possibility to add $p = 3 - 2 \cdot 1 + 1 = 2$ pendant vertices to $\mathcal{S}(B_1^1)$. Therefore, $\mathfrak{T}_3 = \{T_1^3\}$, where $T_1^3 = \mathcal{S}_2(K_1) \cong K_{1,2}$, and $\eta(T_1^3) = 1$.

It is known that if $n = 4$, there are two non-isomorphic trees in the set $\mathfrak{T}_4$, i.e. $\mathfrak{T}_4 = \{T_1^4, T_2^4\}$, where $T_1^4 \cong K_{1,3}$ and $T_2^4 \cong P_4$. It holds $k \in \{1, 2\}$, so the nullity set of $\mathfrak{T}_4$ is $\{0, 2\}$. For $k = 1$, there is only one base tree, $B_1^1 \cong K_1$, and, since there is only one way, in accordance with Definition 4.5, to add $p = 4 - 2 \cdot 1 + 1 = 3$ pendant vertices to $\mathcal{S}(B_1^1) \cong K_1$, we have $T_1^4 = \mathcal{S}_3(K_1) \cong K_{1,3}$. For $k = 2$, there is only one base tree, too, $B_1^2 \cong K_2$, and since there is only one possibility to add $p = 4 - 2 \cdot 2 + 1 = 1$ pendant vertex to $\mathcal{S}(B_1^2)$ (otherwise we would get isomorphic trees), we have $T_2^4 = \mathcal{S}_1(K_2) \cong P_4$.

Let us now consider the set $\mathfrak{T}_5 = \{T_1^5, T_2^5, T_3^5\}$ of all non-isomorphic trees of order 5 (see Figure 12). It holds $k \in \{1, 2\}$, so the nullity set of $\mathfrak{T}_5$ is $\{1, 3\}$. For $k = 1$, there is only one base tree, $B_1^1 \cong K_1$, so $T_1^5 = \mathcal{S}_{5-2\cdot1+1}(K_1) = \mathcal{S}_4(K_1) \cong K_{1,4}$. For $k = 2$, there is only one base tree, $B_1^2 \cong K_2$. Let us denote the set of its vertices $V(K_2) = \{v_1, v_2\}$. Since there are two possibilities to add $5 - 2 \cdot 2 + 1 = 2$ black pendant vertices to the graph $\mathcal{S}(K_2)$ in order to obtain non-isomorphic trees of the same order and nullity, we find that $T_2^5 = \mathcal{S}_2^{S_1}(K_2)$, where $S_1 = \{v_1\}$, and $T_3^5 = \mathcal{S}_2^{S_2}(K_2)$, where $S_2 = \{v_1, v_2\}$ (see Figure 12). Base trees, precisely their subdivisions, in Figure 12, as well as in all subsequent figures, are framed.

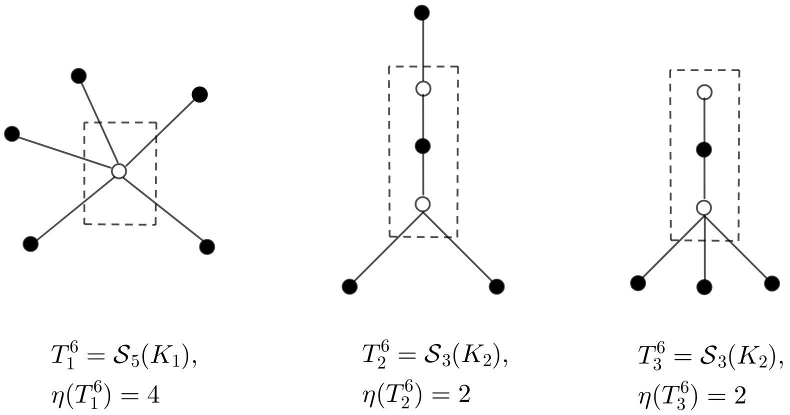


Figure 13: Non-isomorphic trees of order 6, part 1

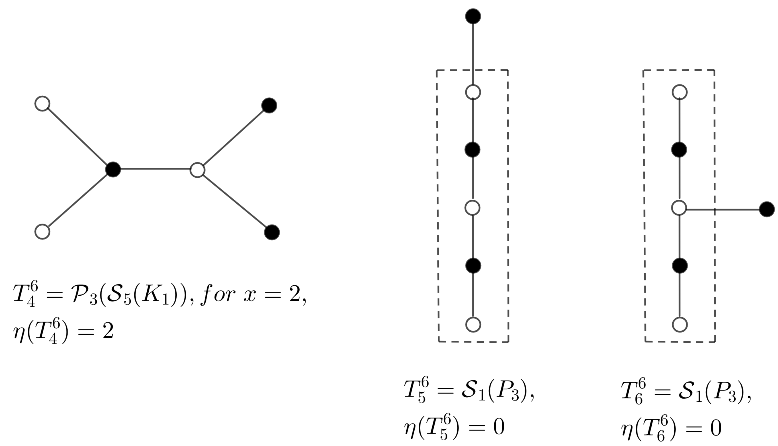


Figure 14: Non-isomorphic trees of order 6, part 2

For $n = 6$, there are six non-isomorphic trees, i.e. $\mathfrak{T}_6 = \{T_i^6, i = 1, 2, \ldots, 6\}$. Since $k \in \{1, 2, 3\}$, we can identify tree non-isomorphic base trees - one of order 1, $B_1^1 \cong K_1$; one of order 2, $B_1^2 \cong K_2$; and one of order 3, $B_1^3 \cong P_3$. There is only one way to add $p = 7 - 2 \cdot 1 = 5$ black pendant vertices to the graph B_1^1; $p = 7 - 2 \cdot 2 = 3$ black pendant vertices can be added to the graph B_1^2, according to Definition 4.5, in two different ways in order to obtain non-isomorphic trees of the same nullity; and, there are two ways to add $p = 7 - 2 \cdot 3 = 1$ pendant vertex to the graph B_1^3 in order to obtain non-isomorphic trees of the same nullity. All trees from the set $\mathcal{T}_6$ with respect to Definition 4.5, Theorem 5.2, and graph perturbations $\mathcal{P}_i$, for $1 \leq i \leq 5$, are depicted in Figure 13 and Figure 14.

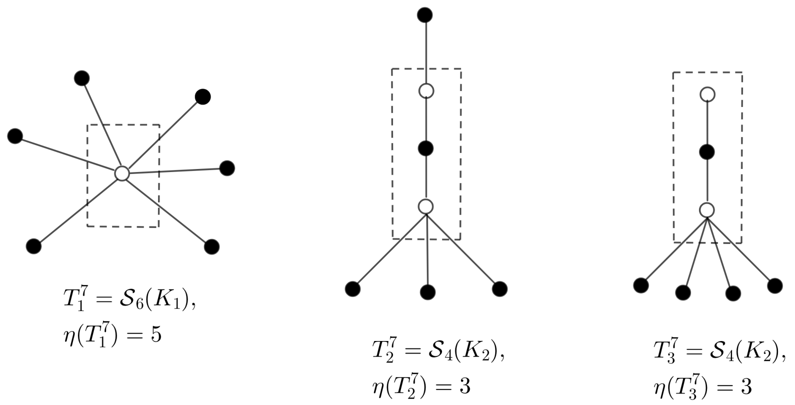


Figure 15: Non-isomorphic trees of order 7, part 1

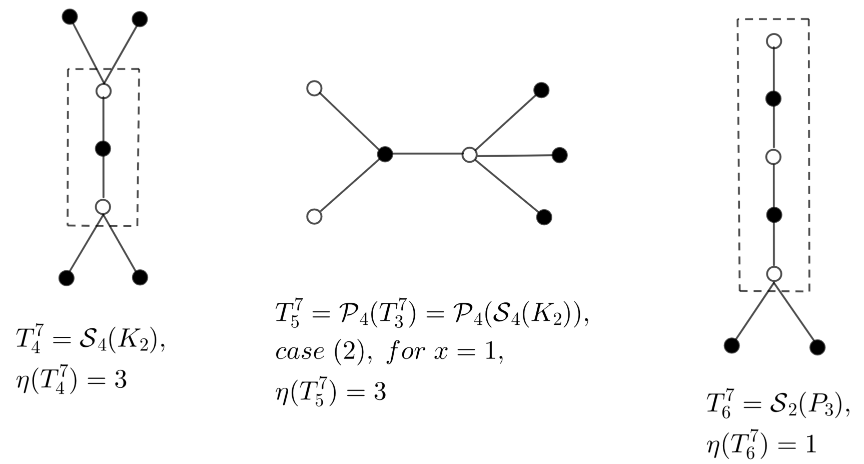


Figure 16: Non-isomorphic trees of order 7, part 2

There are eleven non-isomorphic trees of order 7, i.e. $\mathfrak{T}_7 = \{T_i^7, i = 1, 2, 3, \ldots, 11\}$. Since $k \in \{1, 2, 3\}$, similarly to the previous case, we can identify three non-isomorphic base trees - one of order 1, $B_1^1 \cong K_1$; one of order 2, $B_1^2 \cong K_2$; and one of order 3, $B_1^3 \cong P_3$. All trees from the set $\mathfrak{T}_7$ with respect to Definition 4.5, Theorem 5.2, and graph perturbations $\mathcal{P}_i$, for $1 \le i \le 5$, are depicted in Figure 15, Figure 16, Figure 17 and Figure 18.

Hand checking shows that all non-isomorphic trees of order $n = 8$ and $n = 9$ can also be obtained in the previously described way. Therefore, we can pose the following conjecture:

Conjecture 6.1. *All trees of order n, $n \ge 2$, whose nullity is equal to $n - 2k$, where $k = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor$, are trees $\mathcal{S}_p(B_j^k)$, where B_j^k, $j = 1, 2, \ldots, \tau$, are non-isomorphic base trees of order k, while τ is the number of such trees, and trees obtained by applying graph perturbations $\mathcal{P}_i$, for $1 \le i \le 5$, on trees $\mathcal{S}_p(B_j^k)$, and their special cases $\mathcal{SS}_p(B_j^k)$ and $\overline{\mathcal{SS}}_p(B_j^k)$, in accordance with the following statements: Proposition 5.1, Proposition 5.2, Proposition 5.3, Proposition 5.4 and Proposition 5.5.*

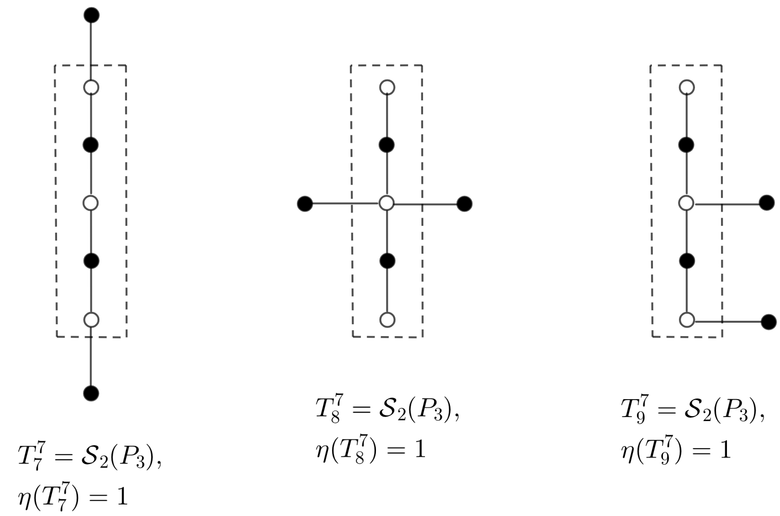


Figure 17: Non-isomorphic trees of order 7, part 3

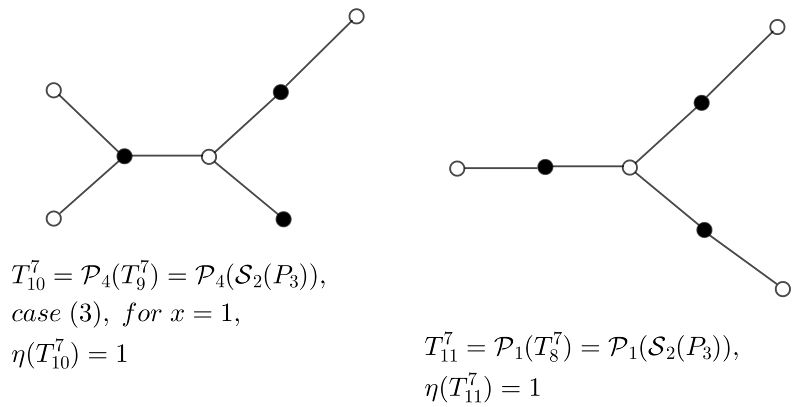


Figure 18: Non-isomorphic trees of order 7, part 4

References

[1] S. Akbari, H. Al Menderj, M. H. Ang, J. Lim, Z. C. Ng, *Some results on spectrum and energy of graphs with loops*, Bull. Malays. Math. Sci. Soc. (2023), 46:94.

[2] S. Akbari, I. M. Jovanović, J. Lim, *Line graphs and Nordhaus-Gaddum type bounds for self-loop graphs*, Bull. Malays. Math. Sci. Soc. (2024), 47:117.

[3] D. V. Anchan, S. D'Souza, H. J. Gowtham, P. G. Bhat, *Laplacian energy of a graph with self-loops*, MATCH Commun. Math. Comput. Chem. **90** (1) (2023), 247–258.

[4] B. Cheng, B. Liu, *On the nullity of graphs*, Electron. J. Linear Algebra **16** (2007), 60–67.

[5] L. Collatz, U. Sinogowitz, *Spektren endlicher grafen*, Abh. Math. Sem. Univ. Hamburg **21** (1957), 63–77.

[6] D. Cvetković, *Spectra of graphs formed by some unary operations*, Publ. Inst. Math. (Beograd) (N.S.) **19** (33) (1975), 37–41.

[7] D. M. Cvetković, M. Doob, H. Sachs, *Spectra of Graphs: theory and application*, Johann Ambrosius Barth Verlag, Heidelberg, Leipzig, 1995.

[8] D. Cvetković, P. Rowlinson, S. Simić, *An introduction to the theory of graph spectra.* London Mathematical Society Student Texts 75, Cambrige University Press, 2010.

[9] Y. Z. Fan, K. S. Qian, *On the nullity of bipartite graphs*, Linear Algebra Appl. **430** (2009), 2943–2949.

[10] S. Fiorini, I. Gutman, I. Sciriha, *Trees with maximum nullity*, Linear Algebra Appl. **397** (2005), 245–251.

[11] S. Gong, Y. Fan, Z. Yin, *On the nullity of graphs with pendant trees*, Linear Algebra Appl. **433** (2010), 1374–1380.
[12] I. Gutman, *The energy of a graph*, Ber. Math. -Statist. Sekt. Forschungsz. Graz **103** (1978), 1–22.
[13] I. Gutman, B. Borovićanin, *Nullity of Graphs: An Updated Survey*, Zbornik Radova (2011), 137–154.
[14] I. Gutman, I. Redžepović, B. Furtula, A. Sahal, *Energy of graphs with self-loops*, MATCH Commun. Math. Comput. Chem. **87** (2022), 645–652.
[15] I. Gutman, I. Sciriha, *On the nullity of line graphs of trees*, Discrete Math. **232** (2001), 35–45.
[16] I. M. Jovanović, E. Zogić, E. Glogić, *On the conjecture related to the energy of graphs with self-loops*, MATCH Commun. Math. Comput. Chem. **89** (2023), 479–488.
[17] S. Li, *On the nullity of graphs with pendant vertices*, Linear Algebra Appl. **429** (2008), 1619–1628.
[18] W. Li, A. Chang, *On the trees with maximum nullity*, MATCH Commun. Math. Comput. Chem. **56** (3) (2006), 501–508.
[19] H. C. Longuet-Higgins, *Resonance structures and molecular orbitals in unsaturated hydrocarbons*, J. Chem. Phys. **18** (1950), 265–274.
[20] G. R. Omidi, *On the nullity of bipartite graphs*, Graphs Combin. **25** (2009), 111–114.
[21] I. Sciriha, *On the construction of graphs of nullity one*, Discrete Math. **181** (1998), 193–211.
[22] S. S. Shetty, A. K. Bhat, *On the first Zagreb index of graphs with self-loops*, AKCE Int. J. Graphs Combin. **20** (3) (2023), 326–331.
[23] X. Tan, B. Liu, *On the nullity of unicyclic graphs*, Linear Algebra Appl. **408** (2005), 212–220.