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Strong convergence of an A-stable parameterized split-step Milstein method for nonlinear stochastic differential equations

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Abstract. In this paper, we consider general nonlinear stochastic differential equations of Itô type, and propose a split-step numerical method, in which parameterized balancing techniques are designed in both steps. The resulting numerical method is called parameterized split-step Milstein method. We present the bounded second moment of the numerical solution, and prove the first order of strong convergence. The mean-square stability of the new method is well investigated, and the range of the proper parameters are given to guarantee the A-stability of the proposed method. Compared with the balanced Milstein method with satisfied stability, the new method has better stability. Several numerical experiments are attached to exhibit the performance of the new method.

1. Introduction

Stochastic differential equations (SDEs) are important tools to model systems influenced by random disturbances, such as asset price changes in financial markets, noise effects in physical systems, population dynamics in biology, random vibrations in engineering and so on [1, 2]. Unlike classical deterministic differential equations, SDEs incorporate randomness to more accurately simulate dynamic processes in the real world by describing the random fluctuations or uncertainties in systems [3]. However, it is usually infeasible to get the analytical solution of SDEs, especially when they model the real-world phenomena that include noise and other stochastic elements. Therefore, the numerical method has been an important tool to solve SDEs.

Euler-Maruyama (EM) method is one of the most common used numerical methods for SDEs [4], which is explicit and of half order of strong convergence, and Milstein method with first order of strong convergence is also a popular numerical method for SDEs [5]. Milstein et al. [6] introduced a balanced implicit method for stiff SDEs. Tian and Burrage [7] presented the implicit Taylor method for stiff SDEs to

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improve the stability. Buckwar and Sickenberger [8] analyzed the mean-square stability of the θ-Maruyama and θ-Milstein method for SDE. Wang and Gan [9] proposed a family of fully implicit Milstein methods, and investigated the mean-square (MS) and almost surely asymptotic stability. Higham et al. [10] introduced a non-negative double implicit Milstein scheme for SDE in finance . Tocino et al. [11] gave a complete analytical study for the linear MS stability of a two-parameter family of Euler predictor-corrector schemes for scalar SDEs. Zhang and Tretyakov [12] presented two classes of theta-Milstein schemes for SDEs. There are also many other numerical methods, such as split-step Adams Moulton Milstein method [13], stochastic theta method [14, 15], strong predictor-corrector EM method [16].

During these methods, split-step method is an efficient numerical method. It decomposes complex SDEs into simple subproblems and each of them can be solved easily. By handing different types of dynamics separately, splitting methods can better control errors and enhance stability [17–20]. Wang and Liu [21] discussed the convergence, stability and pathwise positivity of several drift split-step backward balanced Milstein methods, and then constructed a class of fully explicit methods based on EM method and Milstein method [22]. Haghighi and Rler [23] constructed a class of split-step double balanced approximation methods for solving stiff Itô SDEs with multi-dimensional noise. Wu and Gan [24] analyzed mean-square convergence rates of split-step theta Milstein methods for SDEs with non-globally Lipschitz diffusion coefficients. Xiao et al. [25] applied the split-step method to solve Volterra integral equations.

The numerical methods mentioned above are generally described as explicit or implicit schemes. The explicit schemes compute the numerical solution directly [22], which are suitable for nonlinear problems, but they often require very small time steps to maintain numerical stability, especially for stiff problems. Implicit schemes exhibit good numerical stability when dealing with stiff problems, allowing for larger time steps without compromising stability, and this makes them suitable for systems with high-frequency oscillations or rapid changes [21, 26]. However, if the diffusion terms in SDEs are handed implicitly, some additional numerical errors and instability are probably to be introduced [9, 27], and such a property is not analogous to the corresponding deterministic equation. Both the explicit and implicit schemes have their own advantages and disadvantages. It is interesting to construct a numerical scheme that combines the advantages of both strategies, which will be explored in this paper.

Balancing technique can be used to increase the implicit weight, resulting in better stability. Ranjbar et al. [28] designed several families of balanced Euler approximations for stiff SDEs. He also introduced an exponential split-step double balanced Milstein scheme, and investigated the moment and the strong convergence [29]. Amiri and Hosseini [30] proposed a class of balanced stochastic Runge-Kutta methods for stiff SDEs. Ren and Bai [26] introduced balanced implicit methods with strong order 1.5 for SDEs.

Despite these advances, there remains a notable gap in the development of methods that can efficiently integrate high-order accuracy, strong stability, and computational efficiency, even under the standard assumption of globally Lipschitz continuous coefficients. Motivated by these considerations, we propose a novel parameterized split-step Milstein method that employs a double linearized balancing technique in both the predictor and corrector steps. The balancing parameters are carefully chosen to introduce an appropriate level of implicitness, which enhances stability while preserving the first-order of strong convergence. This linearized approach ensures that only two linear systems need to be solved per time step, making the method both efficient and suitable for long-time simulations. To the best of our knowledge, this is the first work to systematically incorporate double linearized balancing into a split-step Milstein framework for general nonlinear SDEs with Lipschitz coefficients. Our method bridges the gap between fully explicit and fully implicit schemes, offering a balanced compromise that maintains convergence order while achieving favorable stability, without compromising computational tractability.

The outline of the paper is as follows. In Section 2 we elaborate the nonlinear SDEs and propose the numerical scheme. In Section 3 we analyze uniform boundedness of the second moment and the strong order of convergence. In Section 4 we investigate the MS stability and the balancing parameters of this scheme. Several numerical experiments are introduced to confirm the theoretical results in Section 5. In Section 6 we conclude the paper.

2. Equations and Discretization

In this paper, we consider the numerical method for solving SDEs of Itô type [9],

$$\begin{cases} dy(t) = f(t, y(t))dt + \sum_{i=1}^{m} g_i(t, y(t))dW^i(t), \quad t \in [t_0, T], \quad y \in \mathbb{R}^d \\ y(t_0) = y_0. \end{cases} \tag{1}$$

The Borel measurable function $f : [t_0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ is the drift coefficient, and $g_i : [t_0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ for $i = 1, 2, ..., m$ are the diffusion coefficients, where $g_i(t, y) = [g_{1,i}(t, y), ..., g_{d,i}(t, y)]^T$, $W(t) = [W^1(t), W^2(t), ..., W^m(t)]^T$ is a m-dimensional Wiener process defined on the probability space $(\Omega, \mathcal{F}, P)$. For convenience, we denote $g = [g_1, ..., g_m]$, and denote by $\|\cdot\|$ the Euclidean norm for vectors in d-dimensional Euclidean space $\mathbb{R}^d$.

We define an operator $L^i = \sum_{j=1}^{d} g_{j,i}(y)\dfrac{\partial}{\partial y_j}$, and assume the noise term to be commutative, that is $L^{i_1} g_{j,i_2} = L^{i_2} g_{j,i_1}$, for $i_1, i_2 = 1, ..., m, j = 1, ..., d$. We suppose the initial value $y_0 \in \mathbb{R}^d$ is $\mathcal{F}_0$-measurable, the functions f, g and $L^i g_j$ in Eq. (1) satisfy the following global Lipschitz condition and linear growth condition:
(1) There exists a constant $L_1 > 0$, such that for any $x, y \in \mathbb{R}^d$, and $i, j = 1, ..., m$,

$$\|f(t, x) - f(t, y)\| + \|g(t, x) - g(t, y)\| + \|L^i g_j(t, x) - L^i g_j(t, y)\| \leq L_1\|x - y\|, \quad t \in [t_0, T]. \tag{2}$$

(2) There exists a constant $L_2 > 0$, such that for any $y \in \mathbb{R}^d$, and $i, j = 1, ..., m$,

$$\|f(t, y)\|^2 + \|g(t, y)\|^2 + \|L^i g_j(t, y)\|^2 \leq L_2(1 + \|y\|^2), \quad t \in [t_0, T]. \tag{3}$$

Based on the conditions above, SDE (1) has a unique strong solution, and the solution has following properties.

Theorem 2.1. *[5] Suppose $y_0 \in \mathbb{R}^d$, and functions f, g and $L^i g_j$ in Eq. (1) satisfy conditions (2)-(3). Then SDE (1) has a unique strong solution, and satisfies*

$$\mathbb{E}\Big[\sup_{t\in[0,T]} \int_0^t \|y(s)\|^2 ds\Big] < +\infty. \tag{4}$$

In order to propose the numerical discretization for Eq. (1), we divide the time interval $[t_0, T]$ uniformly with equidistant mesh points $t_n = t_0 + nh$, for $n = 0, 1, 2, ..., N$, where N is a positive integer, and the time step size $h = \frac{T-t_0}{N}$. We denote by y_n the approximation of $y(t)$ at the time instant $t = t_n$.

The numerical time-stepping scheme from t_n to t_{n+1} consists of two time steps. In the first step, we take the following balanced EM scheme to obtain a predicted value Y_n,

$$Y_n = y_n + hf(t_n, y_n) + \sum_{i=1}^{m} \Delta W_n^i \, g_i(t_n, y_n) + C_{n,1}(y_n - Y_n), \tag{5}$$

where $\Delta W_n^i = W^i(t_{n+1}) - W^i(t_n)$, and $C_{n,1}$ is chosen as

$$C_{n,1} = \beta h J_f(t_n, y_n), \tag{6}$$

and β is a parameter, $J_f(t_n, y_n)$ represents the Jacobian of the nonlinear function f at t_n. In the second step, the approximation at t_{n+1} is improved by

$$y_{n+1} = Y_n + \sum_{i,j=1}^{m} L^i g_j(t_n, Y_n) I_{i,j}^{t_n,t_{n+1}} + \sum_{i=1}^{m} C_{n,2}^i (Y_n - y_{n+1}), \tag{7}$$

where $I_{i,j}^{t_n,t_{n+1}} = \int_{t_n}^{t_{n+1}}\int_{t_n}^{s} dW^i(u)dW^j(s)$. Note that $I_{i,j}^{t_n,t_{n+1}} + I_{j,i}^{t_n,t_{n+1}} = \Delta W_n^i \Delta W_n^j - \delta_{i,j}h$ with the indicator $\delta_{i,j}$ (when $i = j, \delta_{i,j} = 1$ and $i \neq j, \delta_{i,j} = 0$), and specially we have $I_{i,i}^{t_n,t_{n+1}} = \frac{1}{2}\Big[(\Delta W_n^i)^2 - h\Big]$. The coefficient $C_{n,2}^i$ in the last term of scheme (7) is defined as $C_{n,2}^i = \frac{\alpha h}{2} J_{gg'}(t_n, y_n)$, where α is a parameter. The scheme (7) can be regarded as the Milstein supplementation of scheme (5) with additional balancing term. Due to the two balancing terms, schemes (5) and (7) are termed as the parameterized split-step Milstein (PSM) scheme. Taking scheme (5) into scheme (7), we can obtain the following equivalent form of the PSM scheme,

$$y_{n+1} = y_n + hf(t_n, y_n) + \sum_{i=1}^{m} \Delta W_n^i\, g_i(t_n, y_n) + \sum_{i=1}^{m} L^i g_j(t_n, Y_n) I_{i,j}^{t_n,t_{n+1}} + C_{n,1}(y_n - Y_n) + \sum_{i=1}^{m} C_{n,2}^i (Y_n - y_{n+1}). \quad (8)$$

We can see that, one just needs to solve two linear equations in the PSM method (5) and (7) at each time step. Next, we will analyze the PSM method, including the convergence and stability.

3. Strong convergence of the PSM method

For simplicity, we only consider the case $m = d = 1$ in the convergence and stability analysis. In the one-dimensional single-noise setting, the Milstein correction term simplifies to $\frac{1}{2}gg'(t_n, Y_n)\Big((\Delta W_n)^2 - h\Big)$, avoiding the complexities of cross-noise integrals (e.g., $I_{i,j}$ for $i \neq j$). This allows a clearer exposition of the core ideas behind the double balancing technique. Then, the PSM method can be degenerated as

$$Y_n = y_n + hf(t_n, y_n) + \Delta W_n g(t_n, y_n) + C_{n,1}(y_n - Y_n), \quad (9)$$

and

$$y_{n+1} = Y_n + \frac{1}{2}gg'(t_n, Y_n)\Big[(\Delta W_n)^2 - h\Big] + C_{n,2}(Y_n - y_{n+1}). \quad (10)$$

where $g'(t, y)$ denotes the derivative of $g(t, y)$ with respect to the second argument, and then the balanced parameters are degenerated as

$$C_{n,1} = \beta h f'(t_n, y_n), \quad C_{n,2} = \frac{\alpha}{2}(L^1 g)'(t_n, y_n)h. \quad (11)$$

In this section, we first investigate the bound of the second moment of the numerical solution, and then investigate the strong convergence order of the PSM method.

3.1. Uniform boundedness of the second moment

Based on the PSM method (9) and (10), the numerical solution $\{y_n, n = 0, 1, ..., N\}$ satisfies

$$\begin{aligned} y_{n+1} =& y_n + \int_{t_n}^{t_{n+1}} \frac{1}{1 + C_{n,1}} f(t_n, y_n)ds + \int_{t_n}^{t_{n+1}} \frac{1}{1 + C_{n,1}} g(t_n, y_n)dW_n(s) \\ &+ \int_{t_n}^{t_{n+1}}\int_{t_n}^{s} \frac{1}{1 + C_{n,2}} gg'(t_n, y_n)dW_n(z)dW_n(s) \\ &+ \int_{t_n}^{t_{n+1}}\int_{t_n}^{s} \frac{1}{1 + C_{n,2}} \Big[gg'(t_n, Y_n) - gg'(t_n, y_n)\Big]dW_n(z)dW_n(s). \end{aligned} \quad (12)$$

For $s \in [t_n, t_{n+1})$, we denote $\hat{s} = t_n$, $\hat{y}(s) = y_n$, $\hat{Y}(s) = Y_n$, and $\hat{C}_{n,i}(s) = C_{n,i}$ for $i = 1, 2$, then we can define a continuous approximation $\bar{y}(t)$ on the interval $[t_0, T]$ by

$$\begin{aligned}\bar{y}(t) =& y_0 + \int_{t_0}^{t} \frac{1}{1+\hat{C}_{n,1}(s)} f(\hat{s}, \hat{y}(s))ds + \int_{t_0}^{t} \frac{1}{1+\hat{C}_{n,1}(s)} g(\hat{s}, \hat{y}(s))dW(s)\\ &+ \sum_{j=0}^{n} \int_{t_j}^{\min(t_{j+1},t)} \int_{t_j}^{s} \frac{1}{1+\hat{C}_{n,2}(s)} gg'(\hat{s}, \hat{y}(s))dW_j(z)dW_j(s)\\ &+ \sum_{j=0}^{n} \int_{t_j}^{\min(t_{j+1},t)} \int_{t_j}^{s} \frac{1}{1+\hat{C}_{n,2}(s)} \Big[gg'(\hat{s}, \hat{Y}(s)) - gg'(\hat{s}, \hat{y}(s))\Big]dW_j(z)dW_j(s).\end{aligned} \tag{13}$$

We denote $L = \max\{L_1, L_2\}$, and the continuous approximation $\bar{y}(t)$ has the following property.

Theorem 3.1. *With conditions (2) and (3), the continuous approximation $\bar{y}(t)$ of SDE (1) has bounded second moment, and satisfies*

$$\mathbb{E}\Big[\sup_{t_0 \le t \le T} \big|\bar{y}(t)\big|^2\Big] \le \Big(1 + 5\big|y_0\big|^2\Big)e^{MT}, \tag{14}$$

where $M = 10(1+Ch)^2TL + 20(1+Ch)^2L + 5(1+Ch)^4TL(h+h^2)$, and C is a constant.

Proof. By Jensen's inequality $|\sum_{i=1}^{n} x_i|^2 \le n\sum_{i=1}^{n} x_i^2$, we can get from Eq. (13) that

$$\begin{aligned}&\mathbb{E}\Big[\sup_{t_0 \le t \le T} \big|\bar{y}(t)\big|^2\Big]\\ \le& 5|y_0|^2 + 5\mathbb{E}\Big[\sup_{t_0 \le t \le T} \Big|\int_{t_0}^{t} \frac{1}{1+\hat{C}_{n,1}(s)} f(\hat{s}, \hat{y}(s))ds\Big|^2\Big] + 5\mathbb{E}\Big[\sup_{t_0 \le t \le T} \Big|\int_{t_0}^{t} \frac{1}{1+\hat{C}_{n,1}(s)} g(\hat{s}, \hat{y}(s))dW(s)\Big|^2\Big]\\ &+ 5\mathbb{E}\Big[\sup_{t_0 \le t \le T} \Big|\sum_{j=0}^{n} \int_{t_j}^{\min(t_{j+1},t)} \int_{t_j}^{s} \frac{1}{1+\hat{C}_{n,2}(s)} gg'(\hat{s}, \hat{y}(s))dW_j(z)dW_j(s)\Big|^2\Big]\\ &+ 5\mathbb{E}\Big[\sup_{t_0 \le t \le T} \Big|\sum_{j=0}^{n} \int_{t_j}^{\min(t_{j+1},t)} \int_{t_j}^{s} \frac{1}{1+\hat{C}_{n,2}(s)} \Big[gg'(\hat{s}, \hat{Y}(s)) - gg'(\hat{s}, \hat{y}(s))\Big]dW_j(z)dW_j(s)\Big|^2\Big]\\ :=& 5|y_0|^2 + 5\sum_{i=2}^{5} H_i.\end{aligned} \tag{15}$$

Then we will analyze these terms separately.

According to expression (11), there exists a constant C, such that

$$\frac{1}{|1+\hat{C}_{n,1}(s)|} \le 1 + Ch, \quad \frac{1}{|1+\hat{C}_{n,2}(s)|} \le 1 + Ch.$$

By Cauchy-Schwarz inequality and condition (3), the term H_2 in Eq. (15) can be estimated by

$$\begin{aligned}H_2 \le& (1+Ch)^2\mathbb{E}\Big[\sup_{t_0 \le t \le T} \Big|\int_{t_0}^{t} f(\hat{s}, \hat{y}(s))ds\Big|^2\Big]\\ \le& (1+Ch)^2T\mathbb{E}\Big[\sup_{t_0 \le t \le T} \int_{t_0}^{t} \big|f(\hat{s}, \hat{y}(s))\big|^2 ds\Big]\\ \le& (1+Ch)^2TL \int_{t_0}^{t} \Big(1 + \mathbb{E}\Big[\sup_{t_0 \le s \le u} \big|\hat{y}(s)\big|^2\Big]\Big)du\\ \le& (1+Ch)^2TL \int_{t_0}^{T} \Big(1 + \mathbb{E}\Big[\sup_{t_0 \le s \le u} \big|\bar{y}(s)\big|^2\Big]\Big)du.\end{aligned} \tag{16}$$

For the term H_3 in Eq. (15), by the Burkholder-Davis-Gundy inequality,

$$H_3 \leq 4\mathbb{E}\Big[\sup_{t_0\leq t\leq T}\int_{t_0}^{t}\Big|\frac{1}{1+\hat{C}_{n,1}(s)}g(\hat{s},\hat{y}(s))\Big|^2 ds\Big]. \tag{17}$$

Similar to the estimation (16) for H_2, we have

$$H_3 \leq 4(1+Ch)^2 L\int_{t_0}^{T}\Big(1+\mathbb{E}\Big[\sup_{t_0\leq s\leq u}\big|\bar{y}(s)\big|^2\Big]\Big)du. \tag{18}$$

For the term H_4 in Eq. (15), by the Jensen's inequality, we have

$$\begin{aligned} H_4 =&\mathbb{E}\Big[\sup_{t_0\leq t\leq T}\Big|\sum_{j=0}^{n}\int_{t_j}^{\min(t_{j+1},t)}\int_{t_j}^{s}\frac{1}{1+\hat{C}_{n,2}(s)}gg'(\hat{s},\hat{y}(s))dW_j(z)dW_j(s)\Big|^2\Big]\\ \leq&(n+1)(1+Ch)^2\mathbb{E}\Big[\sup_{t_0\leq t\leq T}\sum_{j=0}^{n}\Big|\int_{t_j}^{\min(t_{j+1},t)}gg'(\hat{s},\hat{y}(s))\int_{t_j}^{s}dW_j(z)dW_j(s)\Big|^2\Big]. \end{aligned} \tag{19}$$

With the Burkholder-Davis-Gundy inequality, we can obtain

$$\begin{aligned} H_4 \leq&(n+1)(1+Ch)^2\sup_{t_0\leq t\leq T}\sum_{j=0}^{n}\int_{t_j}^{\min(t_{j+1},t)}\mathbb{E}\Big[\Big|gg'(\hat{s},\hat{y}(s))\int_{t_j}^{s}dW_j(z)\Big|^2\Big]ds\\ \leq&(n+1)(1+Ch)^2 h\int_{t_0}^{T}\mathbb{E}\Big[gg'(\hat{s},\hat{y}(s))\Big]^2 ds, \end{aligned} \tag{20}$$

similar to the estimation (16),

$$H_4 \leq (1+Ch)^2 TL\int_{t_0}^{T}\Big(1+\mathbb{E}\Big[\sup_{t_0\leq s\leq u}\big|\bar{y}(s)\big|^2\Big]\Big)du. \tag{21}$$

Similarly, the term H_5 can be estimated as

$$\begin{aligned} H_5 \leq&(n+1)h\mathbb{E}\Big[\sup_{t_0\leq t\leq T}\sum_{j=0}^{n}\Big|\int_{t_j}^{\min(t_{j+1},t)}\frac{1}{1+\hat{C}_{n,2}(s)}\Big[gg'(\hat{s},\hat{Y}(s))-gg'(\hat{s},\hat{y}(s))\Big]dW_j(s)\Big|^2\Big]\\ \leq&(1+Ch)^2 T\mathbb{E}\Big[\sup_{t_0\leq t\leq T}\sum_{j=0}^{n}\int_{t_j}^{\min(t_{j+1},t)}\Big|gg'(\hat{s},\hat{Y}(s))-gg'(\hat{s},\hat{y}(s))\Big|^2 ds\Big]. \end{aligned} \tag{22}$$

With condition (2), we have

$$H_5 \leq (1+Ch)^2 TL\mathbb{E}\Big[\sup_{t_0\leq t\leq T}\sum_{j=0}^{n}\int_{t_j}^{\min(t_{j+1},t)}\Big|\hat{Y}(s)-\hat{y}(s)\Big|^2 ds\Big]. \tag{23}$$

According to Eq. (9), for any $s\in[t_n,t_{n+1}]$, it can be obtained that

$$\hat{Y}(s)-\hat{y}(s)=\frac{1}{1+\hat{C}_{n,1}(s)}\Big[hf(\hat{s},\hat{y}(s))+\Delta W_n g(\hat{s},\hat{y}(s))\Big], \tag{24}$$

then we have

$$\begin{aligned} H_5 \leq&(1+Ch)^4 TL(h+h^2)\sum_{j=0}^{n}\int_{t_j}^{\min(t_{j+1},t)}\Big(1+\mathbb{E}\Big[\sup_{t_0\leq s\leq u}\big|\hat{y}(s)\big|^2\Big]\Big)du\\ \leq&(1+Ch)^4 TL(h+h^2)\int_{t_0}^{T}\Big(1+\mathbb{E}\Big[\sup_{t_0\leq s\leq u}\big|\bar{y}(s)\big|^2\Big]\Big)du. \end{aligned} \tag{25}$$

Substituting Eqs. (16), (18), (21), (25) into Eq. (15), we can get

$$1 + \mathbb{E}\Big[\sup_{t_0 \le t \le T} \big|\bar{y}(t)\big|^2\Big] \le 1 + 5\big|y_0\big|^2 + M \int_{t_0}^{T} \Big(1 + \mathbb{E}\Big[\sup_{t_0 \le s \le u} \big|\bar{y}(s)\big|^2\Big]\Big) du, \tag{26}$$

where

$$M = 10(1+Ch)^2 TL + 20(1+Ch)^2 L + 5(1+Ch)^4 TL(h+h^2). \tag{27}$$

By the Gronwall's inequality, we obtain

$$\mathbb{E}\Big[\sup_{t_0 \le t \le T} \big|\bar{y}(t)\big|^2\Big] \le \Big(1 + 5\big|y_0\big|^2\Big) e^{MT},$$

which completes the proof. □

3.2. *Strong convergence*

In order to measure the strong convergence order of the PSM method, we will employ the following convergence theorem.

Theorem 3.2. *[31] For a one-step discrete time approximation y_n to $y(t_n)$ for $n = 0, 1, ..., N$, assume the local mean error and MS error for $n = 0, 1, \dots, N-1$ satisfy the estimates,*

$$\Big|\mathbb{E}\Big[\Big(y_{n+1} - y(t_{n+1})\Big)\Big| y(t_n) = y_n\Big]\Big| \le K\Big(1 + |y_n|^2\Big)^{\frac{1}{2}} h^{p_1} \tag{28}$$

and

$$\Big(\mathbb{E}\Big[\Big(y_{n+1} - y(t_{n+1})\Big)^2 \Big| y(t_n) = y_n\Big]\Big)^{\frac{1}{2}} \le K\Big(1 + |y_n|^2\Big)^{\frac{1}{2}} h^{p_2} \tag{29}$$

with $p_2 \ge \dfrac{1}{2}$ and $p_1 \ge p_2 + \dfrac{1}{2}$. Then

$$\Big(\mathbb{E}\Big[\Big(y_n - y(t_n)\Big)^2 \Big| y_0 = y(t_0)\Big]\Big)^{\frac{1}{2}} \le K\Big(1 + |y_0|^2\Big)^{\frac{1}{2}} h^{p_2 - \frac{1}{2}} \tag{30}$$

holds for each $n = 0, 1, 2, \dots, N$. Here K is a constant independent of h, but dependent on the length $T - t_0$ of the time interval.

The convergence analysis of the PSM method needs the balanced Milstein method as the reference method, so we present its strong convergence properties here.

Lemma 3.3. *For the following balanced Milstein method for Eq. (1) with $d = m = 1$,*

$$y_{n+1}^B = y_n^B + hf(t_n, y_n^B) + \Delta W_n g(t_n, y_n^B) + \frac{1}{2} g g'(t_n, y_n^B)\Big[(\Delta W_n)^2 - h\Big] + C_n\Big(y_n^B - y_{n+1}^B\Big), \tag{31}$$

where $C_n = -f'(t_n, y_n^B)h$, the error satisfies

$$\begin{aligned} &\Big|\mathbb{E}\Big[y(t_{n+1}) - y_{n+1}^B \Big| y(t_n) = y_n^B\Big]\Big| \le K\Big(1 + \big|y_n^B\big|^2\Big)^{\frac{1}{2}} h^2, \\ &\Big(\mathbb{E}\Big[\Big(y(t_{n+1}) - y_{n+1}^B\Big)^2 \Big| y(t_n) = y_n^B\Big]\Big)^{\frac{1}{2}} \le K\Big(1 + \big|y_n^B\big|^2\Big)^{\frac{1}{2}} h^{\frac{3}{2}}, \end{aligned} \tag{32}$$

where K is a constant independent of y_n^B.

Proof. First the Itô-Taylor expansion of the exact solution at $t = t_{n+1}$ can be written as

$$y(t_{n+1}) = y(t_n) + hf(t_n, y(t_n)) + \Delta W_n g(t_n, y(t_n)) + \frac{1}{2}gg'(t_n, y(t_n))\Big[(\Delta W_n)^2 - h\Big] + R_n^B, \tag{33}$$

where

$$R_n^B = \int_{t_n}^{t_{n+1}}\int_{t_n}^{s} L^1 f(t_n, y_u)dW_u ds + \int_{t_n}^{t_{n+1}}\int_{t_n}^{s} L^0 g(t_n, y_u)du dW_s + \int_{t_n}^{t_{n+1}}\int_{t_n}^{s} L^0 f(t_n, y_u)du ds, \tag{34}$$

and $L^0 = f(t, y)\frac{\partial}{\partial x} + \frac{1}{2}g^2(t, y)\frac{\partial^2}{\partial x^2}$, $L^1 = g(t, y)\frac{\partial}{\partial x}$. Based on Ref. [32], we have $E(R_n^B) = O(h^2)$, $E\Big((R_n^B)^2\Big) = O(h^3)$.

With the assumption $y(t_n) = y_n^B$, the error of the balanced Milstein method can be expressed

$$y(t_{n+1}) - y_{n+1}^B = R_n^B - C_n\Big(y_n^B - y_{n+1}^B\Big) = R_n^B - C_n\Big[y_n^B - y(t_{n+1}) + y(t_{n+1}) - y_{n+1}^B\Big]. \tag{35}$$

Then we have

$$\begin{aligned} y(t_{n+1}) - y_{n+1}^B &= \frac{1}{1 + C_n}\Big[R_n^B + C_n\Big(y(t_{n+1}) - y_n^B\Big)\Big] \\ &= R_n^B + \frac{C_n}{1 + C_n}\Big(hf(t_n, y_n^B) + \Delta W_n g(t_n, y_n^B) + \frac{1}{2}gg'(t_n, y_n^B)\Big[(\Delta W_n)^2 - h\Big]\Big). \end{aligned} \tag{36}$$

According to inequality (3) and Eq. (33), we have

$$\Big|\mathbb{E}\Big[y(t_{n+1}) - y_{n+1}^B\Big|y(t_n) = y_n^B\Big]\Big| \le \Big|\mathbb{E}\Big[R_n^B + Kh(1 + Ch)h|f(t_n, y_n^B)|\Big|y(t_n) = y_n^B\Big]\Big| \le K\Big(1 + |y_n^B|^2\Big)^{\frac{1}{2}}h^2 \tag{37}$$

and

$$\begin{aligned} &\Big(\mathbb{E}\Big[\Big(y(t_{n+1}) - y_{n+1}^B\Big)^2\Big|y(t_n) = y_n^B\Big]\Big)^{\frac{1}{2}} \\ \le& K\Big(\mathbb{E}\Big[(R_n^B)^2 + \frac{C_n^2}{(1 + C_n)^2}\Big(hf(t_n, y_n^B) + \Delta W_n g(t_n, y_n^B) + \frac{1}{2}gg'(t_n, y_n^B)\Big[(\Delta W_n)^2 - h\Big)\Big]\Big)^2\Big|y(t_n) = y_n^B\Big]\Big)^{\frac{1}{2}} \\ \le& K\Big(\mathbb{E}\Big[\Big(R_n^B\Big)^2 + h^2(1 + Ch)^2(h^2 + h)\Big(1 + |y_n^B|^2\Big)\Big|y(t_n) = y_n^B\Big]\Big)^{\frac{1}{2}} \\ \le& K\Big(1 + |y_n^B|^2\Big)^{\frac{1}{2}}h^{\frac{3}{2}}. \end{aligned} \tag{38}$$

This completes the proof. □

The strong order of convergence of the PSM method can be found in the following theorem.

Theorem 3.4. *Suppose $y(t_n)$ is the true solution of SDE (1) at time instant $t = t_n$, and y_n is the numerical approximation of $y(t_n)$ obtained by the PSM method with uniform time step size $h = \frac{T}{N}$, then the error satisfies*

$$\Big(\mathbb{E}\Big[\Big(y_n - y(t_n)\Big)^2\Big|y_0 = y(t_0)\Big]\Big)^{1/2} = O(h). \tag{39}$$

Proof. With the referenced balanced Milstein method (31), the error at $t = t_{n+1}$ can be expressed as

$$y(t_{n+1}) - y_{n+1} = \Big[y(t_{n+1}) - y_{n+1}^B\Big] + \Big[y_{n+1}^B - y_{n+1}\Big] := I_1 + I_2. \tag{40}$$

The estimation of I_1 can be found in Lemma 3.3, and we only need to estimate the term I_2 here, which can be obtained by subtracting Eq. (31) from Eqs. (9) and (10). With the assumption $y_n = y_n^B = y(t_n)$, the term I_2 can be expressed as

$$I_2 = \frac{1}{2}\Big(gg'(y_n) - gg'(Y_n)\Big)\Big[(\Delta W_n)^2 - h\Big] + C_n(y_n - y_{n+1}^B) + C_{n,1}(Y_n - y_n) + C_{n,2}(y_{n+1} - Y_n). \tag{41}$$

Based on Eqs. (9), (10) and (31), we have

$$\begin{aligned} Y_n - y_n &= \frac{1}{1+C_{n,1}}\Big[hf(t_n,y_n) + \Delta W_n g(t_n,y_n)\Big], \\ y_n - y^B_{n+1} &= \frac{-1}{1+C_n}\Big[hf(t_n,y_n) + \Delta W_n g(t_n,y_n) + \frac{1}{2}gg'(y_n)[(\Delta W_n)^2 - h]\Big], \\ Y_n - y_{n+1} &= \frac{-1}{1+C_{n,2}}\Big[\frac{1}{2}gg'(t_n,Y_n)[(\Delta W_n)^2 - h]\Big]. \end{aligned} \tag{42}$$

By inequality (2), the term I_2 can be estimated by

$$\begin{aligned} &\Big|\mathbb{E}\Big[y^B_{n+1} - y_{n+1}\Big|y(t_n) = y_n\Big]\Big| \\ \leq& K\Big|\mathbb{E}\Big[\frac{1}{2}\Big(gg'(y_n) - gg'(Y_n)\Big)\Big[(\Delta W_n)^2 - h\Big]\Big|y(t_n) = y_n\Big]\Big| \\ &+ K\Big|\mathbb{E}\Big[C_n(y_n - y^B_{n+1}) + C_{n,1}(Y_n - y_n) + C_{n,2}(y_{n+1} - Y_n)\Big|y(t_n) = y_n\Big]\Big| \\ :=& H_1 + H_2, \end{aligned} \tag{43}$$

where

$$H_1 \leq K\Big|\mathbb{E}\Big[\frac{1}{(1+C_{n,1})}\Big|hf(t_n,y_n) + \Delta W_n g(t_n,y_n)\Big|\Big[(\Delta W_n)^2 - h\Big]\Big|y(t_n) = y_n\Big]\Big| \leq K\Big(1+|y_n|^2\Big)^{\frac{1}{2}}h^2 \tag{44}$$

and

$$\begin{aligned} H_2 \leq& K\Big|\mathbb{E}\Big[C_n(y_n - y^B_{n+1}) + C_{n,1}(Y_n - y_n) + C_{n,2}(y_{n+1} - Y_n)\Big|y(t_n) = y_n\Big]\Big| \\ \leq& K\Big|\mathbb{E}\Big[\frac{-C_n}{1+C_n}\Big|hf(t_n,y_n) + \Delta W_n g(t_n,y_n) + \frac{1}{2}gg'(y_n)\Big[(\Delta W_n)^2 - h\Big]\Big|\Big|y(t_n) = y_n\Big]\Big| \\ &+ \Big|\mathbb{E}\Big[\frac{C_{n,1}}{1+C_{n,1}}\Big(hf(t_n,y_n) + \Delta W_n g(t_n,y_n)\Big)\Big|y(t_n) = y_n\Big]\Big| \\ &+ \Big|\mathbb{E}\Big[\frac{C_{n,2}}{1+C_{n,2}}\Big(\frac{1}{2}gg'(Y_n)\Big)[(\Delta W_n)^2 - h\Big]\Big|y(t_n) = y_n\Big]\Big| \\ \leq& K(1+Ch)h^2(1+|y_n|^2)^{\frac{1}{2}} \\ \leq& K\Big(1+|y_n|^2\Big)^{\frac{1}{2}}h^2. \end{aligned} \tag{45}$$

Therefore,

$$\Big|\mathbb{E}\Big[y^B_{n+1} - y_{n+1}\Big|y(t_n) = y_n\Big]\Big| \leq K\Big(1+|y_n|^2\Big)^{\frac{1}{2}}h^2. \tag{46}$$

In addition,

$$\begin{aligned} &\Big(\mathbb{E}\Big[\Big(y^B_{n+1} - y_{n+1}\Big)^2\Big|y(t_n) = y_n\Big]\Big)^{\frac{1}{2}} \\ \leq& K\Big(\mathbb{E}\Big[\frac{1}{4}\Big[(\Delta W_n)^2 - h\Big]^2\Big(gg'(y_n) - gg'(Y_n)\Big)^2\Big|y(t_n) = y_n\Big]\Big)^{\frac{1}{2}} \\ &+ K\Big(\mathbb{E}\Big[\Big(C_n(y_n - y^B_{n+1}) + C_{n,1}(Y_n - y_n) + C_{n,2}(y_{n+1} - Y_n)\Big)^2\Big|y(t_n) = y_n\Big]\Big)^{\frac{1}{2}} \\ :=& H_3 + H_4, \end{aligned} \tag{47}$$

where

$$\begin{aligned} H_3 \leq & K\Big(\mathbb{E}\Big[\big[(\Delta W_n)^2 - h\big]^2 (Y_n - y_n)^2 \Big| y(t_n) = y_n\Big]\Big)^{\frac{1}{2}} \\ \leq & K\Big(\mathbb{E}\Big[\frac{1}{(1+C_{n,1})^2}\big[(\Delta W_n)^2 - h\big]^2 \Big[\big(hf(t_n, y_n)\big)^2 + \big(\Delta W_n g(t_n, y_n)\big)^2\Big] \Big| y(t_n) = y_n\Big]\Big)^{\frac{1}{2}} \\ \leq & K\big(1 + |y_n|^2\big)^{\frac{1}{2}} h^{\frac{3}{2}} \end{aligned} \tag{48}$$

and

$$\begin{aligned} H_4 \leq & 3\Big(\mathbb{E}\Big[C_n^2 (y_n - y_{n+1}^B)^2 \Big| y(t_n) = y_n\Big]\Big)^{\frac{1}{2}} + 3\Big(\mathbb{E}\Big[C_{n,2}^2 (y_{n+1} - Y_n)^2 \Big| y(t_n) = y_n\Big]\Big)^{\frac{1}{2}} \\ & + 3\Big(\mathbb{E}\Big[C_{n,1}^2 \big(Y_n - y_n\big)^2 \Big| y(t_n) = y_n\Big]\Big)^{\frac{1}{2}} \\ := & 3\big(H_{4,1} + H_{4,2} + H_{4,3}\big). \end{aligned} \tag{49}$$

First estimate $H_{4,1}$,

$$\begin{aligned} H_{4,1} \leq & Kh(1+Ch)\Big(\mathbb{E}\Big[\Big(hf(t_n, y_n) + \Delta W_n g(t_n, y_n) + \frac{1}{2}\big[(\Delta W_n)^2 - h\big] g g'(y_n)\Big)^2 \Big| y(t_n) = y_n\Big]\Big)^{\frac{1}{2}} \\ \leq & Kh^{\frac{3}{2}}(1+Ch)\big(1 + |y_n|^2\big)^{\frac{1}{2}} \\ \leq & K\big(1 + |y_n|^2\big)^{\frac{1}{2}} h^{\frac{3}{2}}. \end{aligned} \tag{50}$$

In addition,

$$\begin{aligned} H_{4,2} \leq & Kh(1+Ch)\Big(\mathbb{E}\Big[\frac{1}{4}\big[(\Delta W_n)^2 - h\big]^2 \big(g g'(Y_n)\big)^2\Big) \Big| y(t_n) = y_n\Big]\Big)^{\frac{1}{2}} \\ \leq & Kh^2(1+Ch)\big(1 + |Y_n|^2\big)^{\frac{1}{2}} \\ \leq & K\big(1 + |y_n|^2\big)^{\frac{1}{2}} h^2, \end{aligned} \tag{51}$$

and $H_{4,3}$ can be obtained in the same way

$$H_{4,3} \leq K\big(1 + |y_n|^2\big)^{\frac{1}{2}} h^{\frac{3}{2}}. \tag{52}$$

Together with Lemma 3.3, we have

$$\begin{aligned} & \Big|\mathbb{E}\Big[y(t_{n+1}) - y_{n+1} \Big| y(t_n) = y_n\Big]\Big| \leq K\big(1 + |y_n|^2\big)^{\frac{1}{2}} h^2, \\ & \Big(\mathbb{E}\Big[\big(y(t_{n+1}) - y_{n+1}\big)^2 \Big| y(t_n) = y_n\Big]\Big)^{\frac{1}{2}} \leq K\big(1 + |y_n|^2\big)^{\frac{1}{2}} h^{\frac{3}{2}}. \end{aligned} \tag{53}$$

According to Theorem 3.2, estimation (39) holds, and this completes the proof. □

4. Stability analysis

In order to study the stability of the PSM method, we employ the test problem

$$dy(t) = ay(t)dt + by(t)dW(t), \quad y(t_0) = y_0, \tag{54}$$

with $a, b \in \mathbb{R}$. Suppose y_0 is independent of the wiener process $W(t)$, and the solution of Eq. (54) is

$$y(t) = y_0 \exp\big((a - b^2/2)t + bW(t)\big). \tag{55}$$

We see $\mathbb{E}\left[y(t)^2\right] = \mathbb{E}\left[y_0^2 \exp\left((2a - b^2)t\right)\exp\left(2bW(t)\right)\right]$, where $\exp\left(2bW(t)\right)$ is a log-normal random variable, and $\mathbb{E}[\exp\left(2bW(t)\right)] = \exp(2b^2 t)$. Then

$$\mathbb{E}\left[y(t)^2\right] = \exp\left[(2a + b^2)t\right]\mathbb{E}\left[y_0^2\right].$$

It means that the solution $y(t)$ of Eq. (1) is MS stable, if $2a + b^2 < 0$, which means the stable region of the solution is

$$S = \left\{(a, b^2)\middle| a + \frac{1}{2}b^2 < 0, a < 0\right\}. \tag{56}$$

Next, we will investigate the stability of the numerical solution by the PSM method.

Proposition 4.1. *Suppose the numerical solution of Eq. (54) by a one-step numerical method with time step size h can be represented as*

$$y_{n+1} = R(a, b, h, J)y_n, \quad J \sim N(0, 1), \tag{57}$$

then the one-step numerical method with time step size h is said to be MS stable, if the numerical method for Eq. (54) satisfies

$$\bar{R}(a, b, h) := \mathbb{E}\left[R^2(a, b, h, J)\right] < 1, \tag{58}$$

where $\bar{R}(a, b, h)$ is called MS stability function of the numerical method.

We apply the PSM method for Eq. (54) with $C_{n,1} = \beta ah, C_{n,2} = \frac{h}{2}\alpha b^2$, and we can obtain

$$Y_n = \frac{(1 + ah + b\Delta W_n + C_{n,1})y_n}{1 + C_{n,1}} = \frac{1 + (1 + \beta)ah + b\Delta W_n}{1 + \beta ah}y_n, \tag{59}$$

$$y_{n+1} = \frac{1 + 1 + \frac{1}{2}\left[(\Delta W_n)^2 - h\right]b^2 + C_{n,2}}{1 + C_{n,2}}Y_n = \frac{1 + \frac{1}{2}\Delta W_n^2 b^2 + \frac{h}{2}(\alpha - 1)b^2}{1 + \frac{h}{2}\alpha b^2}Y_n, \tag{60}$$

which leads to

$$R = \frac{1 + \frac{1}{2}\Delta W_n^2 b^2 + \frac{h}{2}(\alpha - 1)b^2}{1 + \frac{h}{2}\alpha b^2} \times \frac{1 + (1 + \beta)ah + b\Delta W_n}{1 + \beta ah}. \tag{61}$$

Since $\Delta W_n = \sqrt{h}J$, where $J \sim N(0, 1)$, with the notation $p = ah$ and $q = b^2 h$, the factor R can be rewritten as:

$$R = \frac{\left(1 + \frac{1}{2}qJ^2 + \frac{1}{2}(\alpha - 1)q\right)\left(1 + (1 + \beta)p + q^{\frac{1}{2}}J\right)}{\left(1 + \frac{1}{2}\alpha q\right)\left(1 + \beta p\right)}. \tag{62}$$

Therefore, the MS stability function of the PSM method is given by

$$\begin{aligned}\bar{R} =& \frac{1}{\left(2 + \alpha q\right)^2\left(1 + \beta p\right)^2}\Big[\left(q(\alpha - 1) + 2\right)^2\left((p + \beta p + 1)^2 + q\right) + (p\beta + p + 1)^2\left(2q^2(\alpha - 1) + 4q\right) \\ &+ 6q^3(\alpha - 1) + 12q^2 + 3(p + \beta p + 1)^2 q^2 + 15q^3\Big].\end{aligned} \tag{63}$$

In order to guarantee $\bar{R} < 1$, we need the denominator of $\bar{R}$ to be greater than its numerator. That is

$$\begin{aligned}&(\beta p+1)^2(\alpha q+2)^2-(p+\beta p+1)^2(\alpha q-q+2)^2-3q^2(p+\beta p+1)^2\\-&q(\alpha q-q+2)^2-12q^2-15q^3-6q^3(\alpha-1)-2q(p+\beta p+1)^2(\alpha q-q+2)>0.\end{aligned} \tag{64}$$

The pairs (p, q), which satisfy the inequality (64), constitute the MS stability region of the PSM method. According to the set (56), the stability region of SDE satisfies

$$p+\frac{1}{2}q<0. \tag{65}$$

Therefore, we will investigate the inequality (64) with such a restriction. We notice that, the region satisfying restriction (65) is a sector area, and the boundary is the straight line $q = -2p$.

Since $q = b^2h > 0$, we have $p < 0$ from the restriction (65). We aim for the PSM method to be A-stable, which means its stability region should fully encompass the stability region of the original SDE [33]. For simplicity, we focus only on the two boundaries of the stability region of SDE, i.e., $\{(p, q), p < 0, q = 0\}$ and $\{(p, q), p < 0, q = -2p\}$. Specifically, we recognize that all the points along these boundaries lie within the PSM's stability region, thereby satisfying the stability condition in inequality (64). For one thing, taking the $\{(p, q), p < 0, q = 0\}$ into inequality (64), we have $(\beta p + 1)^2 > (\beta p + p + 1)^2$, which can be done by choosing $p < 0$. It means that the boundary $\{(p, q), p < 0, q = 0\}$ is always included in the stability region of the PSM method. For another, substituting the expression $q = -2p$ into the left hand side of inequality (64), we can get

$$p^2\Big[(-2\alpha^2\beta-\alpha^2-2\beta^2-4\beta-2)p^2+(10\alpha-4\beta+4\alpha\beta+16)p-2\beta-11\Big]:=p^2\cdot G(p).$$

With $p < 0$ finding the sector stability region can be transformed into finding proper values for α and β, such that $G(p) > 0$ holds for any $p < 0$. Moreover, $G(p)$ can be regarded as a quadratic polynomial with respect to p. To ensure $G(p) > 0$ for all $p < 0$, the quadratic polynomial $G(p) := Ap^2 + Bp + C$ must satisfy specific criteria based on its coefficients A, B, C and discriminant

$$\Delta=(10\alpha-4\beta+4\alpha\beta+16)^2-4(2\alpha^2\beta+\alpha^2+2\beta^2+4\beta+2)(2\beta+11).$$

Four mutually scenarios guarantee positivity:

Case 1: When $A > 0$ and $\Delta < 0$ (no real roots), the quadratic polynomial remains entirely above the p-axis, thus $G(p) > 0$ holds globally for $p < 0$.

Case 2: When $A > 0$ and $\Delta \geq 0$ (one or two real roots), it is required that the axis of symmetry $-\dfrac{B}{2A} \geq 0$ and the intercept $G(p = 0) \geq 0$, which leads to $B \leq 0$ and $C \geq 0$. Then $G(p) > 0$ holds globally for all $p < 0$.

Case 3: When $A = 0$, $B \neq 0$, which implies $\alpha^2(2\beta + 1) = -2(\beta + 1)^2$, and we need to prove $B < 0$ and $G(p = 0) \geq 0$. The α and β that satisfy these three conditions form a curve.

Case 4: When $A = 0$ and $B = 0$, which implies $\alpha^2(2\beta + 1) = -2(\beta + 1)^2$ and $5\alpha - 2\beta + 2\alpha\beta + 8 = 0$, and we also need $C > 0$. Under these three constraints, the pair (α, β) is uniquely determined.

The values of the parameters α and β that satisfy Case 1 (green shaded area) or Case 2 (blue shaded area) above are shown in the shaded area of Figure 1. The curve in Case 3 (red dashed line) lies on the boundary of the shaded area, and the resulting pair (α, β) in Case 4 (red star) lies also on the boundary of the shaded area.

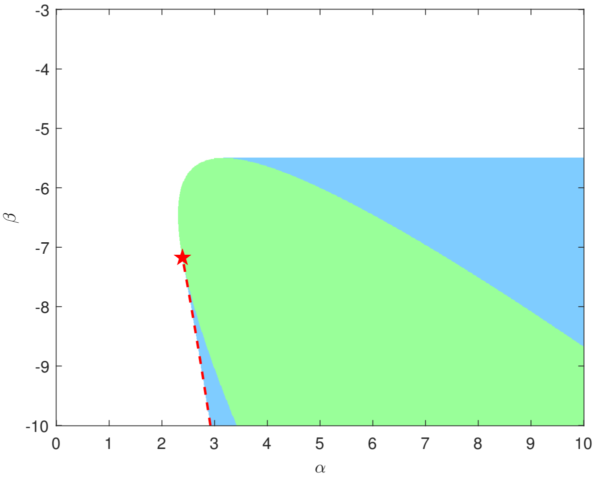

Figure 1: The range for (α, β) for the A-stability of the PSM method

Remark 4.2. *If the parameters α and β are chosen within the range shown in Figure 1, then the PSM method is A-stable. In fact, α primarily expands the stability region. Simultaneously, β refines stability near the origin, a critical area where numerical accuracy is highly sensitive.*

Remark 4.3. *For some values of the parameters α and β beyond the range in Figure 1, the corresponding PSM method can be nearly A-stable, with satisfied approximation effect. For example, when $\alpha = 10$ and $\beta = -1$, the stability region covers almost the entire stability regions except for a negligible vicinity around the origin, which is indicated in Figure 2.*

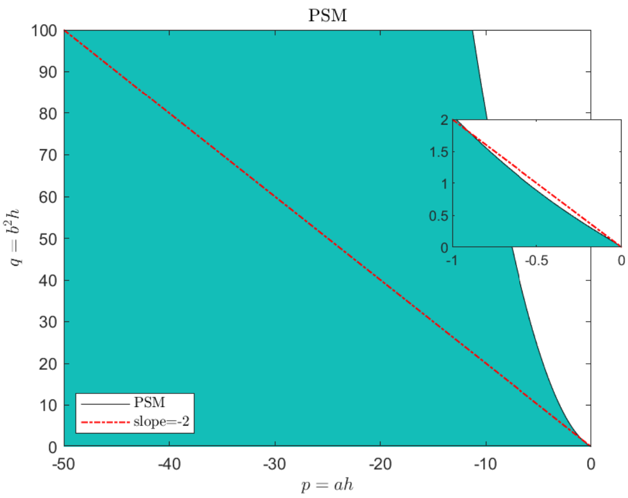

Figure 2: MS stability region of the PSM method with $(\alpha, \beta) = (10, -1)$.

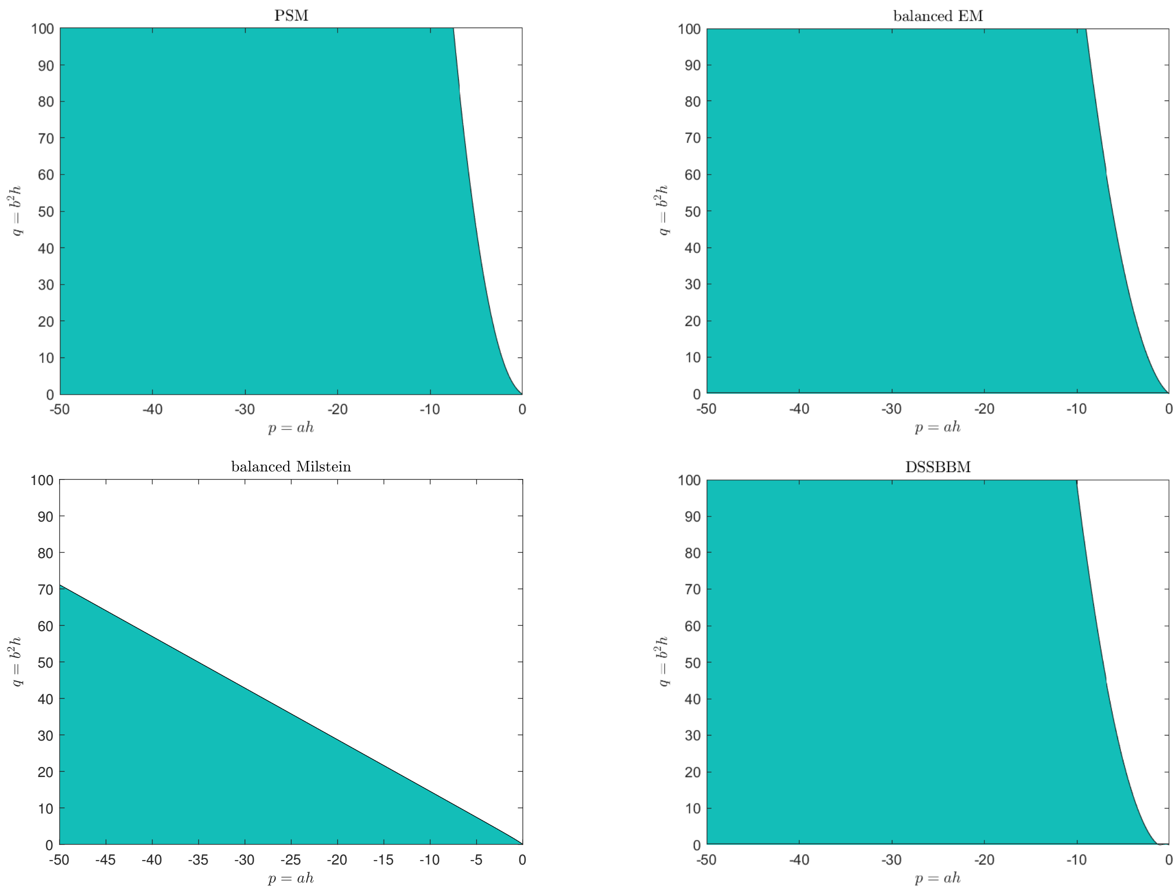

Figure 3: MS stability regions of the PSM method with $(\alpha, \beta) = (3.1, -5.5)$, balanced EM, balanced Milstein and DSSBBM methods.

Moreover, we also compare in Figure 3 the MS stability regions of the PSM method with those of balanced EM method, balanced Milstein method and the drifting split-step backward balanced Milstein (DSSBBM) method reported in [21]. The stability of the PSM method looks much better than that of the balanced Milstein method (the third subplot in Figure 3) and it is also a little bit better than the balanced EM method (the second subplot in Figure 3) and the DSSBBM method (the last subplot in Figure 3).

5. Numerical experiments

In this section, several numerical experiments are reported to confirm the convergence order and stability properties of the PSM method. All the tests are coded using Matlab R2017b on a HP desktop with Intel(R) Core(TM) i5-8265 CPU @1.60GHz. Denoting $y_N^{(i)}$ as the numerical approximation to $y^{(i)}(T)$ at time instant $t = T$ in the ith simulation within 5000 simulations, we use the means of absolute errors

$$Err = \Big(\frac{1}{5000}\sum_{i=1}^{5000}\big|y_N^{(i)} - y^{(i)}(T)\big|^2\Big)^{1/2} \tag{66}$$

to measure the convergence of the PSM method.

Example 5.1. Consider the following nonlinear SDE reported in [34]

$$dX(t) = \cos^2(X(t))\Big(3 - \frac{1}{2}\sin(2X(t))\Big)dt + \cos^2\Big(X(t)\Big)dW(t), \quad X_0 = -1, \quad t \in [0,2], \tag{67}$$

with the exact solution

$$X(t) = \arctan(3t + W(t) + \tan(X_0)). \tag{68}$$

In Table 1 we report the MS error of various methods for Eq. (67), including the balanced EM method, the balanced Milstein method, the first kind of three-stage Milstein (TSM1a) method in [22] and the DSSBBM method in [21]. For the PSM method, we select the parameters $\alpha = 10$, $\beta = -1$, we can find that the error of the PSM method is slightly smaller than other methods with the same step sizes. Moreover, in Figure 4 we plot the MS errors of the five methods with the step size $\Delta t = 2^{-i}, i = 12, 13, ..., 16$, we can conclude that the balanced Milstein method, the TSM1a method, the DSSBBM method and the PSM method have the first order of strong convergence, and all of the four methods look better than the balanced EM method. Among these method, our PSM method seems the most efficient. With parameter $\alpha = 3.1$, $\beta = -5.5$, the PSM method can still achieves first-order of strong convergence, but the resulting error is a little bit big, which is not shown here.

Table 1: Observing errors of various methods for Eq. (67) .

Δt	balanced EM	balanced Milstein	TSM1a	DSSBBM	PSM
2^{-12}	3.98e-06	3.59e-06	3.75e-06	4.20e-06	3.31e-06
2^{-13}	1.94e-06	1.80e-06	1.86e-06	2.13e-06	1.67e-06
2^{-14}	1.04e-06	9.06e-07	9.24e-07	1.05e-06	8.31e-07
2^{-15}	6.48e-07	4.45e-07	4.65e-07	5.21e-07	4.22e-07
2^{-16}	4.33e-07	2.26e-07	2.32e-07	2.60e-07	2.09e-07

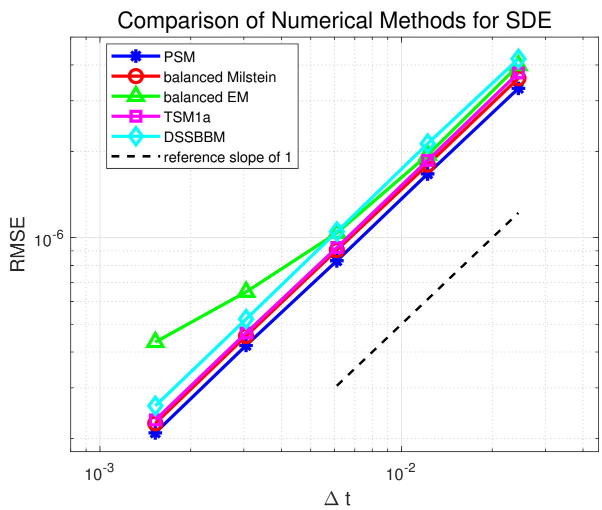

Figure 4: Numerical simulations of Eq. (67) by the PSM method, balanced EM, balanced Milstein methods, TSM1a and DSSBBM method.

Example 5.2. We consider a stochastic Brusselator system reported in [34]

$$\begin{aligned} dX_1(t) &= \Big((\theta - 1)X_1(t) + \theta X_1(t)^2 + (X_1(t) + 1)^2 X_2(t)\Big)dt + \gamma X_1(t)(1 + X_1(t)\Big)dW(t), \\ dX_2(t) &= \Big(- \theta X_1(t) - \theta X_1(t)^2 - (X_1(t) + 1)^2 X_2(t))dt - \gamma X_1(t)(1 + X_1(t)\Big)dW(t). \end{aligned} \tag{69}$$

The Brusselator system is a mathematical model to simulate the phenomenon of self-organization, such as cell cycles, skin spots and so on. The parameters are chosen as $\theta = 1.9$, $\gamma = 0.1$, $0 \le t \le 125$, and the initial value is $(X_1(0), X_2(0)) = (-0.1, 0)$. As time goes by, the mechanical energy of the system is gradually consumed, causing the amplitude of the oscillation to continuously decrease and eventually stop at the equilibrium position 0. We plot the performance of four different methods with time step size $\Delta t = 0.025$ in Figure 5. It can be seen that, the balanced EM method, TSM1a method and our PSM method with parameters $\alpha = 3.1, \beta = -5.5$ are able to asymptotically converge to the origin, which appear to be more consistent with the physical context compared to the balanced Milstein method. In addition, our PSM method exhibits the smallest amplitude, indicating better stability and simulation efficiency.

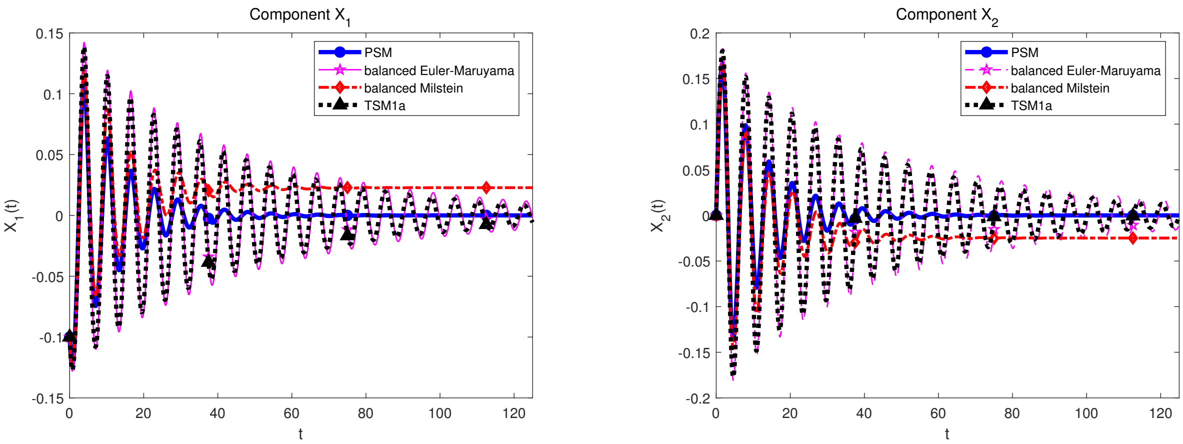

Figure 5: Numerical simulations of the PSM method, the TSM1a method, balanced EM and balanced Milstein methods for Eq. (69).

Example 5.3. We consider a stochastic model for HIV dynamics reported in [35]

$$\begin{aligned}
dX_1(t) &= (\lambda - \delta X_1(t) - (1-\gamma)\zeta X_1(t)X_3(t))dt + \sigma_1 X_1(t)dW_1(t),\\
dX_2(t) &= \Big((1-\gamma)\zeta X_1(t)X_3(t) - \xi X_2(t)\Big)dt + \sigma_2 X_2(t)dW_2(t),\\
dX_3(t) &= \Big((1-\eta)N\xi X_2(t) - uX_3(t) - \zeta X_1(t)X_3(t)\Big)dt + \sigma_3 X_3(t)dW_3(t).
\end{aligned} \tag{70}$$

The parameters are chosen as $\lambda = 10\text{day}^{-1}\text{dm}^{-3}$, $\zeta = 0.00024\text{day}^{-1}\text{dm}^{-3}$, $N = 500$ peer cell, $\gamma = 0.5$, $\eta = 0.5$, ξ=1day^{-1}, $\delta = 0.007\text{day}^{-1}$, $\sigma_1 = 0.008$, $\sigma_2 = 0.3$, $\sigma_3 = 1$, $u = 13\text{day}^{-1}$ and the initial value are $X_1(0) = 1000\text{dm}^{-3}, X_2(0) = 0.001\text{dm}^{-3}, X_3(0) = 0.001\text{dm}^{-3}$. The simulations of the PSM method with parameter $\alpha = 3.1, \beta = -5.5$, the DSSBBM method and the TSM1a method are shown in Figure 6. With the same step size, which is a little bit big, the approximation of the TSM1a method blows up, while the result of our method and the DSSBBM method indicate that the concentrations of healthy target cells, actively infected target cells and infections virus particles are in a dynamic equilibrium state, which is consistent with the biological context, though the simulations seem different.

In fact, we have carried out some more simulations with various step sizes, such as the classical Milstein method and the balanced Milstein method, which are not shown here any more for simplicity. Based on all the numerical observations, we can conclude that, for the long-term nonlinear simulation, explicit numerical solvers typically fail unless the time step is sufficiently small, such as the classical Milstein method and the TSM1a method. The DSSBBM method involves solving nonlinear equations, which are dealt with by the Newton's method and its simulation depends on the time steps. With small time step all the implicit methods behave almost the same, but the DSSBBM method exhibits oscillations when using a large time step (see 2nd line in Figure 6). The simulation of our method and the balanced Milstein method are independent of the time step, and show close agreement, then the balanced Milstein method is not shown here any more.

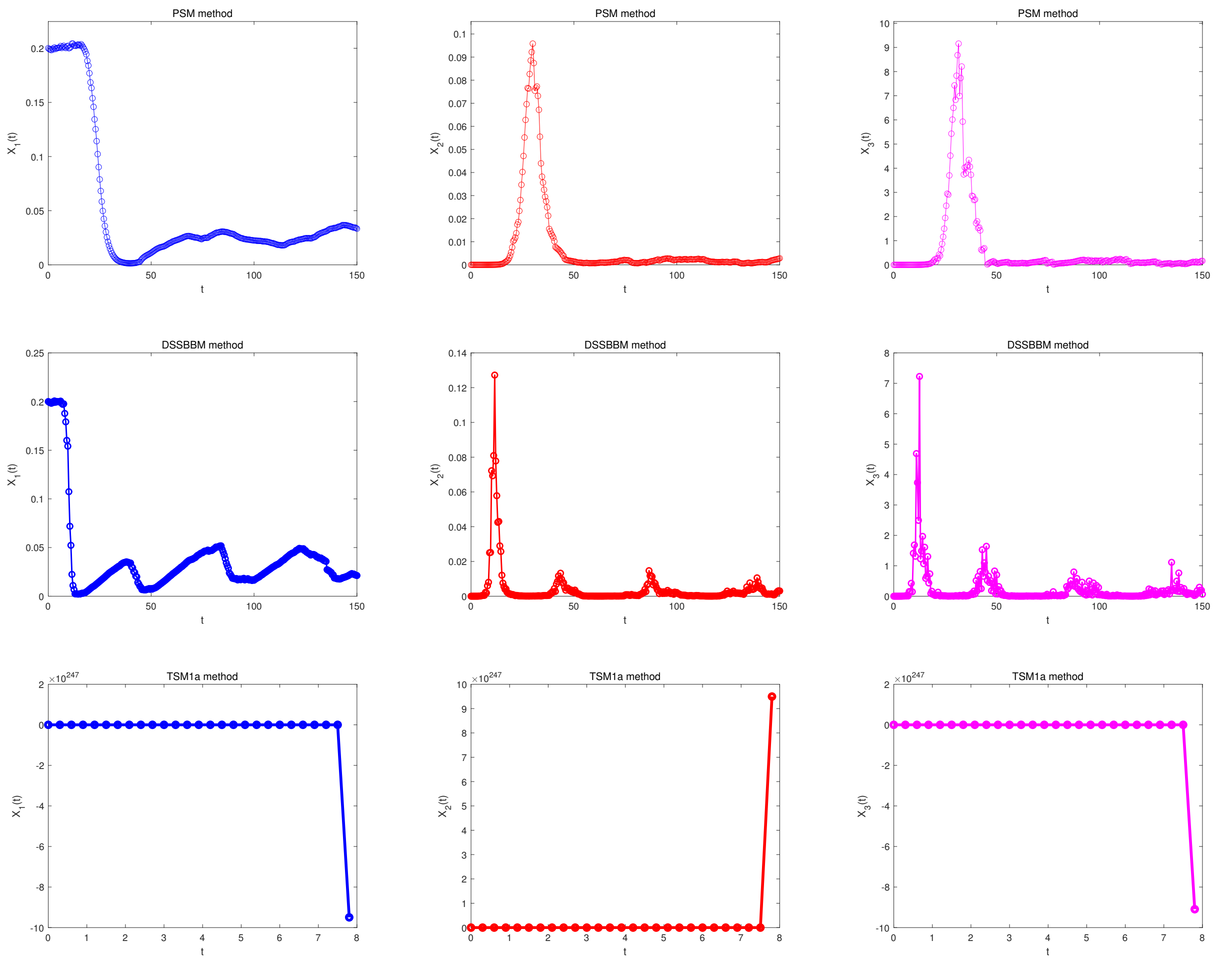

Figure 6: Numerical simulations for system (70) by the PSM method, the DSSBBM method and the TSM1a method with $\Delta t = 0.5$.

Example 5.4. We consider the following stochastic partial differential equation reported in [36, 37]

$$\begin{cases} dv = -\mu \dfrac{\partial v}{\partial x} dt + \dfrac{1}{2} \dfrac{\partial^2 v}{\partial x^2} dt - \sqrt{\rho} \dfrac{\partial v}{\partial x} dW(t), \quad 0 < x < 1, \quad 0 < t < 1, \\ v(0) = \sin(\pi x), \end{cases} \tag{71}$$

with homogeneous Dirichlet boundary conditions. The equation can be used to model the prices of basket assets driven by a common market factor. The variable x represents a normalized measure of a firm's log-asset value relative to its default barrier, and the function $v(t, x)$ is the probability density function of the firms' distance-to-default at time t. The risk-neutral drift parameter is given by $\mu = (r - \sigma^2/2)/\sigma$, where r is the risk-free interest rate and σ is the asset volatility, and ρ is the correlation parameter. In the experiment, we choose $\sigma = 0.22, r = 0.042$ and $\rho = 0.2$.

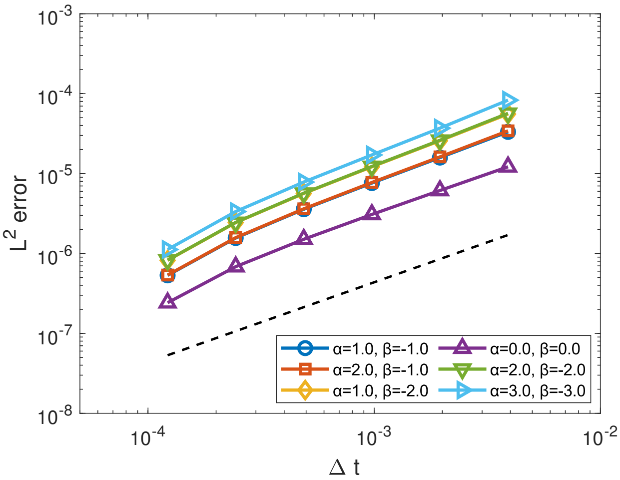

Figure 7: The observing errors of the PSM method of different parameters for Eq. (71) with $\mu = 0.081, \rho = 0.2$.

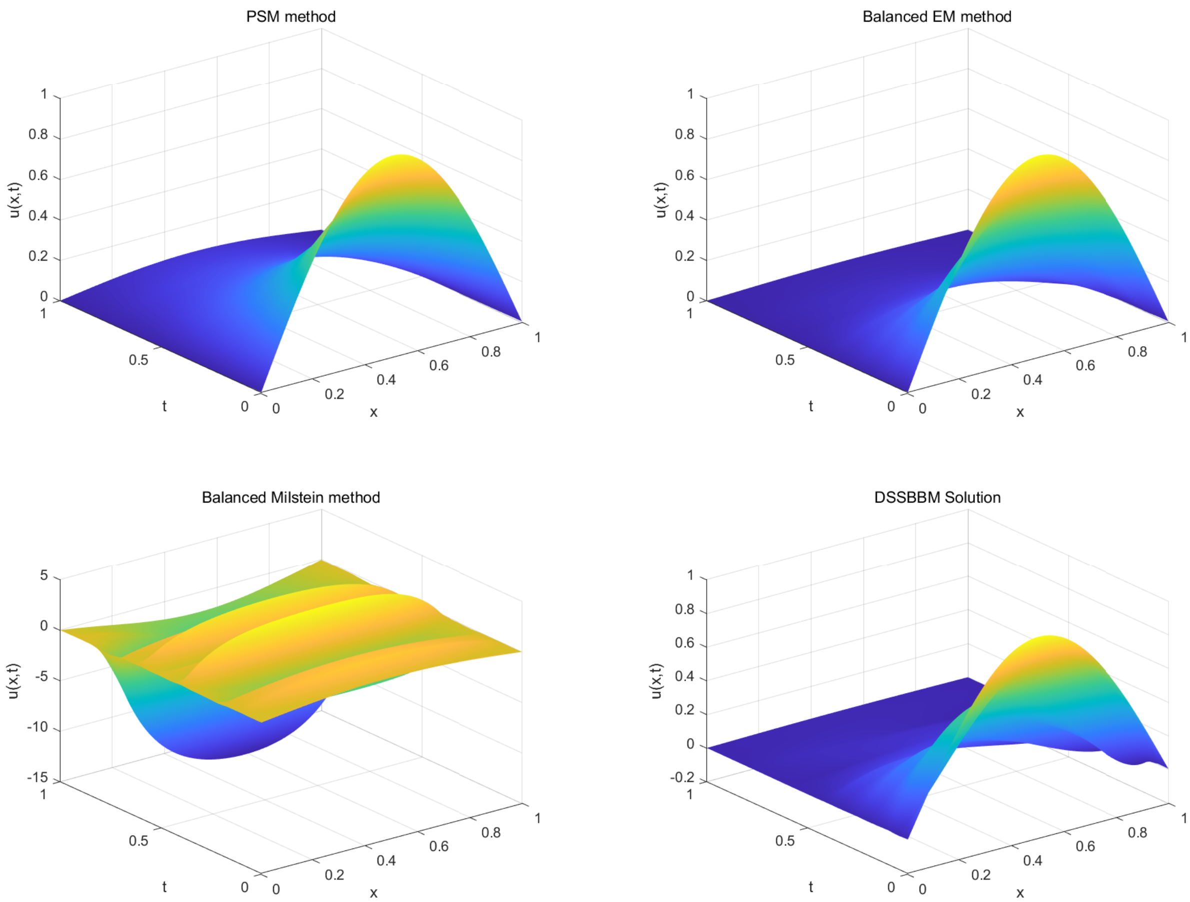

Figure 8: Numerical simulations of the PSM method, the balanced Milstein method, the balanced EM method and the DSSBBM method with $M = 500, \Delta t = 0.05$ for Eq. (71).

When numerically solving Eq. (71), we divide the interval $[0,1]$ for the variable x into $M = 500$ meshes, and partition the temporal interval $[0,1]$ uniformly, and denote by Δt the time step size. The PSM method for such a linear equation is

$$\begin{aligned}\Big(I - \frac{\beta\mu\Delta t}{2\Delta x}B + \frac{\beta\Delta t}{2\Delta x^2}A\Big)u^n &= \Big(I - (1+\beta)\frac{\beta\mu\Delta t}{2\Delta x}B - \frac{\sqrt{\rho}}{2\Delta x}B\Delta W_n + \frac{(1+\beta)\Delta t}{2\Delta x^2}A\Big)v^n,\\ \Big(I + \frac{\rho\alpha\Delta t}{2\Delta x^2}A\Big)v^{n+1} &= \Big(I - \frac{(1-\alpha)\beta\Delta t}{2\Delta x^2}A + \frac{\rho\Delta W_n^2}{2\Delta x^2}A\Big)u^n,\end{aligned} \tag{72}$$

where $A = \mathrm{diag}\{1,-2,1\}$, $B = \mathrm{diag}\{-1,0,1\}$ and u^n is an intermediate variable. For simplicity, we perform 50 numerical simulations, rather than 5000 simulations in the above examples. With the refined mesh technique, we can obtain a reference solution. The observing errors are plotted in Figure 7. We can see that, the strong convergence orders of the PSM methods of different parameters are approximately the first order.

We choose $\Delta t = 0.05$, and present an averaged simulation of 50 paths of the PSM method, the balanced Milstein method, balanced EM method and the DSSBBM method in Figure 8. By comparisons we can see that, the proposed PSM method and the balanced EM method work well, and the numerical simulation of the balanced Milstein method demonstrates a trend toward blow-up. The result of the DSSBBM method looks not smooth. It means that, the time step size $\Delta t = 0.05$ is too big for the balanced Milstein method, while such a value of Δt is acceptable for our PSM method, which fit well with the stability property in Section 4. In fact, we also compute Eq. (71) with the TSM1a method and the numerical result blows up even with a very small Δt, (e.g. $\Delta t = 2^{-10}$). It means that the TSM1a method is an explicit numerical method, and it is not suitable for stiff equation such as Eq. (71).

6. Concluding remarks

In this paper, we have developed the PSM method for nonlinear stochastic differential equations. The computation of the numerical solution is split into two steps, and two balancing terms are imposed to the two steps respectively, to improve the stability. We have proved the first order of strong convergence, and analyzed the stability of the PSM method, and also presented the available balancing parameters for the A-stability of the PSM method.

Several numerical experiments are implemented. It is shown that the results fit well with the theoretical first order of strong convergence, which is better than the balanced EM methods. It can also be observed that, the PSM method works well for some stiff SDEs and long time simulations, which confirms better stability than the balanced Milstein method and some existing numerical methods.

The convergence analysis presented in this work assumes the global Lipschitz condition. However, many SDEs arising in finance and biology only satisfy a local Lipschitz condition. Since our current theory does not cover such cases, extending the convergence proof of the PSM method to SDEs with local Lipschitz coefficients remains an important direction for our future research.

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