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The B_α-spectra of graph product

Xiaxia Zhang[a], Xiaoling Ma[a,*]

[a]*College of Mathematics and System Sciences, Xinjiang University, Urumqi Xinjiang 830017, P.R.China*

Abstract. For a graph G, let $A(G)$ and $L(G)$ denote its adjacency matrix and Laplacian matrix, respectively. Recently, Samanta et al. introduced the one-parameter family of matrices $B_\alpha(G)$, defined for any real $\alpha \in [0,1]$ as

$$B_\alpha(G) = \alpha A(G) + (1-\alpha)L(G).$$

The multiset of eigenvalues of $B_\alpha(G)$ is referred to as the B_α-spectrum of G. Let $G\Box H$, $G[H]$, $G\times H$, and $G\oplus H$ denote the Cartesian product, lexicographic product, direct (or tensor) product, and strong product of graphs G and H, respectively. In this paper, we determine the B_α-spectrum of $G\Box H$ for arbitrary graphs G and H, and that of $G[H]$ when G is arbitrary and H is regular. Furthermore, we characterize the B_α-spectrum of the generalized lexicographic product $G[H_1, H_2, \ldots, H_n]$, where G is an n-vertex graph and each H_i is regular. Finally, for any real $\alpha \in [\frac{1}{2}, 1]$, we derive explicit formulas for the spectral radii of $B_\alpha(G\times H)$ and $B_\alpha(G\oplus H)$, assuming G is arbitrary and H is regular.

1. Introduction

All graphs considered in this paper are undirected and simple. Let $G = \big(V(G), E(G)\big)$ be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. The *adjacency matrix* of G is a symmetric $n\times n$ matrix $A(G)$ whose (i,j)-th entry is 1 if v_i and v_j are adjacent, 0 otherwise. Let $d_i = d_G(v_i)$ be the *degree* of vertex v_i in G, and $D(G)$ be the diagonal matrix with diagonal entries $d_1, d_2, \ldots, d_n$. The *Laplacian matrix* and the *signless Laplacian matrix* of G are defined as $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$, respectively. Note that many of the spectral properties of adjacency, Laplacian and signless Laplacian matrices are quite different from each other.

In 2017, Nikiforov [7] introduced the convex combinations $A_\alpha(G)$ of $A(G)$ and $D(G)$ defined by

$$A_\alpha(G) := \alpha D(G) + (1-\alpha)A(G), \quad 0 \le \alpha \le 1.$$

Note that $A(G) = A_0(G)$, $D(G) = A_1(G)$ and $\frac{1}{2}Q(G) = A_{\frac{1}{2}}(G)$. It was claimed in [7] that the matrices $A_\alpha(G)$ can underpin a unified theory of $A(G)$ and $Q(G)$.

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* Corresponding author: Xiaoling Ma
Email addresses: 1123663044@qq.com (Xiaxia Zhang), mxling2018@163.com (Xiaoling Ma)
ORCID iD: https://orcid.org/0000-0001-8634-8566 (Xiaoling Ma)

In 2023, Samanta et. al [9] proposed to investigate the convex combination $B_\alpha(G)$ of $A(G)$ and $L(G)$ in order to understand how uniformly the spectral behavior transforms from one matrix to another. The $B_\alpha(G)$ of a graph G is defined as

$$B_\alpha(G) := \alpha A(G) + (1-\alpha)L(G), \quad 0 \le \alpha \le 1. \tag{1}$$

Notice that $A(G) = B_1(G)$, $D(G) = 2B_{\frac{1}{2}}(G)$, $L(G) = B_0(G)$ and $Q(G) = 3B_{\frac{2}{3}}(G)$. Therefore, the theory of B_α-matrices merges the theories of the adjacency matrix, Laplacian and signless Laplacian matrix of graphs. On the one hand, the spectral properties of $B_\alpha(G)$ may reveal the common connection among the spectral properties of all such well-known matrices. On the other hand, $B_\alpha(G)$ may analyze the structural and combinatorial properties of the graph G in a better way.

Let M be an $n \times n$ real symmetric matrix. Denote the eigenvalues of M by $\lambda_1(M) \ge \lambda_2(M) \ge \cdots \ge \lambda_n(M)$. The collection of eigenvalues of M together with multiplicities is called the *spectrum* of M, denoted by Spec(M). In particular, $\lambda_1(M)$ is called the *spectral radius* of M and $\lambda_n(M)$ is called the *least eigenvalues* of M. Particularly, if M is a nonnegative irreducible matrix, then by the Perron-Frobenius theorem, we know that $\lambda_1(M)$ is simple and positive, and there is a unique positive unit eigenvector corresponding to $\lambda_1(M)$, which we call the *Perron vector* of M.

Operations on graphs play an important role in graph theory. Several interesting graphs can be found by combining pairs of graphs. The *Cartesian product* $G\square H$ of two graphs G and H, is the graph with vertex set $V(G) \times V(H)$, in which two vertices (u, v) and (u', v') are adjacent if and only if $u = u'$ and $vv' \in E(H)$, or $v = v'$ and $uu' \in E(G)$ (see Fig. 1(a)).

The *direct product* $G \times H$ of two graphs G and H, is the graph with vertex set $V(G) \times V(H)$, in which two vertices (u, v) and (u', v') are adjacent if and only if $uu' \in E(G)$ and $vv' \in E(H)$ (see Fig. 1(b)).

The *strong product* $G \oplus H$ of two graphs G and H, is the graph with vertex set $V(G) \times V(H)$ and edge set $E(G\square H) \cup E(G \times H)$ (see Fig. 1(c)).

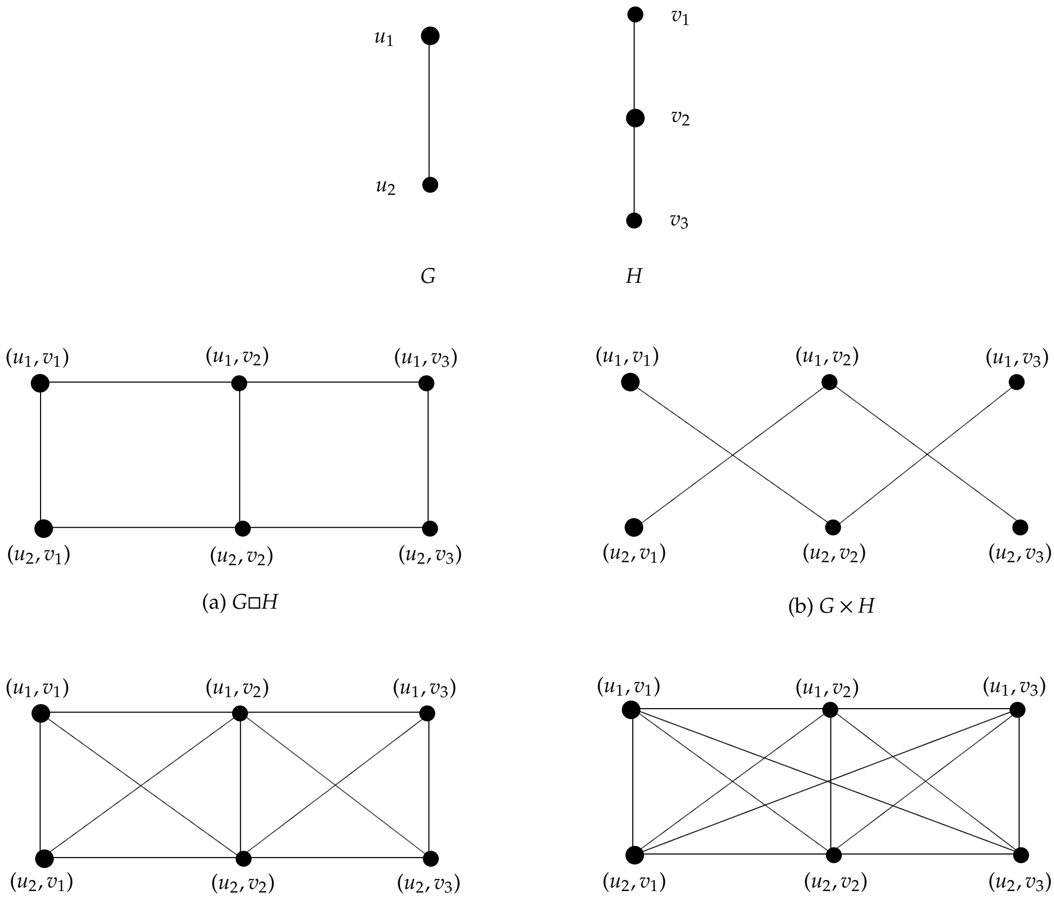
The *lexicographic product* $G[H]$ (also called the composition) of graphs G and H, is the graph with vertex set $V(G[H]) = V(G) \times V(H)$, in which two vertices $(u, v), (u', v')$ are adjacent if $uu' \in E(G)$, or if $u = u'$ and $vv' \in E(H)$ (see Fig. 1(d)).

The *generalized lexicographic product* (also called generalized composition or called generalized join operation) of graphs G, $H_1, H_2, \ldots, H_n$, where G is an arbitrary graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$, consider in [10] with the notation $G[H_1, H_2, \ldots, H_n]$, which is formed by taking the graphs $H_1, H_2, \ldots, H_n$ and joining every vertex of H_i to every vertex of H_j whenever v_i is adjacent to v_j in G.

For example, let G and H be the graphs illustrated in Fig. 1, then the graphs $G\square H$, $G \times H$, $G \oplus H$ and $G[H]$ can be illustrated in Fig. 1, respectively. Let G and H_1, H_2, H_3 be the graphs illustrated in Fig. 2, then the graph $G[H_1, H_2, H_3]$ can be illustrated in Fig. 2.

The adjacency spectra of the Cartesian product, the direct product and the strong product of two graphs were obtained early works [4]. In 2015, Barik et al. [1] investigated the Laplacian eigenvalues and eigenvectors of the product graphs for the four standard products, namely, the Cartesian product, the direct product, the strong product and the lexicographic product. In 1974, Fiedler [5] put forward the Fiedler's lemma. Applying Fiedler's lemma and its generalization, the adjacency, Laplacian and signless Laplacian spectra of the lexicographic product and generalized lexicographic product were investigated in [2, 3, 11]. In 2019, Li and Wang [6] introduced a complete characterization of the A_α-spectrum of $G\square H$ and $G[H]$ for two graphs G and H. Furthermore, they [6] also considered the A_α-spectrum of the generalized lexicographic product $G[H_1, H_2, \ldots, H_n]$ for n-vertex graph G and regular graphs H_i's.

Motivated by these researches, we consider the B_α-spectra of $G\square H$ and $G[H]$ for two graphs G and H. Meanwhile, we determine B_α-spectrum of the generalized lexicographic product $G\,[H_1, H_2, \ldots, H_n]$ for n-vertex graph G and regular graphs H_i's. Finally, we give the spectral radii of $B_\alpha(G \times H)$ and $B_\alpha(G \oplus H)$ for arbitrary graph G and regular graph H.



(a) $G \square H$ (b) $G \times H$ (c) $G \oplus H$ (d) $G[H]$

Figure 1: The graphs G, H, $G \square H$, $G \times H$, $G \oplus H$ and $G[H]$.

2. Preliminaries

In this section, we give some useful results which have an important role in the proof of the main results. Throughout the paper for any integers k, n_1 and n_2, I_k denotes the identity matrix of size k, $\mathbf{1}_k$ denotes the column vector of size k whose all entries are 1 and $J_{n_1 \times n_2}$ denotes $n_1 \times n_2$ matrix whose all entries are 1. We use $[n]$ to denote the set of $\{1, 2, \ldots, n\}$.

Lemma 2.1 ([8]). *The Kronecker product $A \otimes B$ of two matrices $A = (a_{ij})_{m \times n}$ and $B = (b_{ij})_{p \times q}$ is the $(mp) \times (nq)$ matrix obtained from A by replacing each element a_{ij} with the block $a_{ij}B$. The Kronecker product is an associative operation, and the following relations are well known.*

(*i*) $(M \otimes P)(N \otimes Q) = MN \otimes PQ$, *for matrices M, N, P, Q of suitable sizes;*

(*ii*) $\det(M \otimes N) = (\det M)^s (\det N)^k$, *where M is a square matrix of order k and N is a square matrix of order s;*

(*iii*) $(M \otimes N)^T = M^T \otimes N^T$, *for any two matrices M and N.*

(*iv*) $(M \otimes N)^{-1} = M^{-1} \otimes N^{-1}$, *where M and N are invertible matrices.*

(*v*) *If M and N are orthogonal matrices, then $M \otimes N$ is also an orthogonal matrix.*

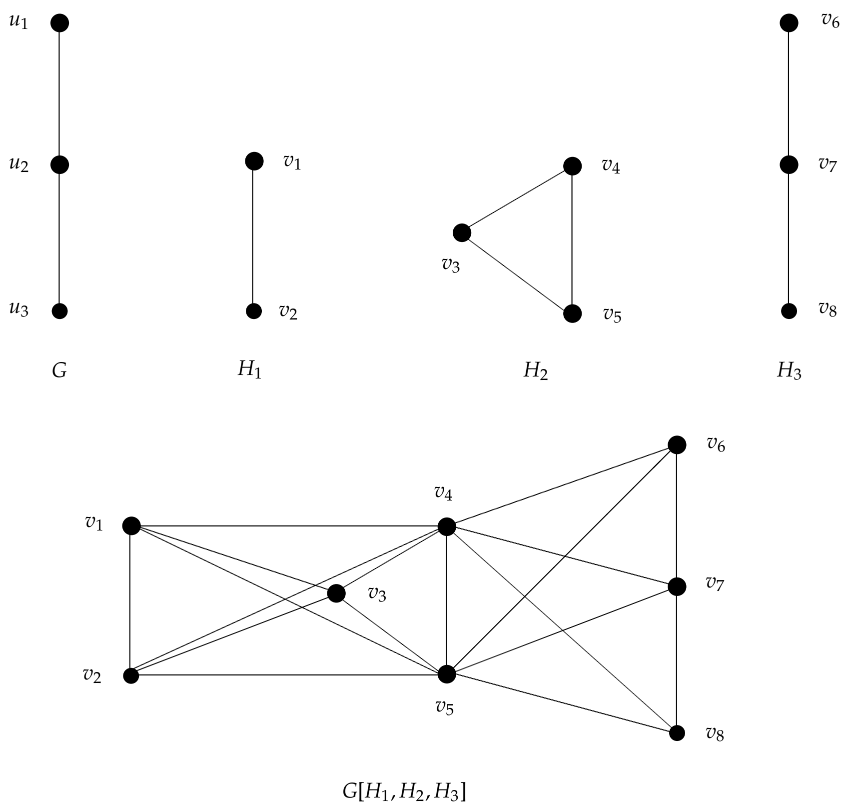


Figure 2: The graphs G, H_1, H_2, H_3 and $G[H_1, H_2, H_3]$.

3. The B_α-spectrum of the graphs $G\Box H$, $G[H]$ and $G[H_1, H_2, \dots, H_n]$

In this section, we determine the B_α-spectrum of $G\Box H$ for arbitrary graphs G and H and $G[H]$ for arbitrary graph G and regular graph H. Next, we characterize the B_α-spectrum of the generalized lexicographic product $G\,[H_1, H_2, \dots, H_n]$ for n-vertex graph G and regular graphs H_i's.

Theorem 3.1. *Suppose G and H are two graphs with order n and m, respectively. Let $Spec\Big(B_\alpha(G)\Big) = \{\lambda_1(B_\alpha(G)), \dots, \lambda_n(B_\alpha(G))\}$ and $Spec\Big(B_\alpha(H)\Big) = \{\lambda_1(B_\alpha(H)), \dots, \lambda_m(B_\alpha(H))\}$. Then*

$$Spec\Big(B_\alpha(G\Box H)\Big) = \bigcup_{i=1}^{n}\bigcup_{j=1}^{m}\{\lambda_i(B_\alpha(G)) + \lambda_j(B_\alpha(H))\}.$$

Proof. By the definition of Cartesian product of two graphs. The adjacency matrix and the degree diagonal matrix of $G\Box H$ can be expressed as follows:

$$A(G\Box H) = A(G) \otimes I_m + I_n \otimes A(H), \tag{2}$$

$$D(G\Box H) = D(G) \otimes I_m + I_n \otimes D(H). \tag{3}$$

Applying Lemmma 2.1, substituting (2) and (3) in (1), we have that

$$B_\alpha(G\Box H) = B_\alpha(G) \otimes I_m + I_n \otimes B_\alpha(H). \tag{4}$$

Suppose that

$$U = [u_1\ u_2\ \cdots\ u_n]$$

is an orthogonal matrix whose columns are eigenvectors corresponding to the eigenvalue $\lambda_1\Big(B_\alpha(G)\Big)$, $\lambda_2\Big(B_\alpha(G)\Big), \ldots, \lambda_n\Big(B_\alpha(G)\Big)$.

Let

$$W = [w_1\ w_2\ \cdots\ w_m]$$

be an orthogonal matrix whose columns are eigenvectors corresponding to the eigenvalue $\lambda_1\Big(B_\alpha(H)\Big)$, $\lambda_2\Big(B_\alpha(H)\Big), \ldots, \lambda_n\Big(B_\alpha(H)\Big)$. Then

$$U^T B_\alpha(G) U = \begin{pmatrix} \lambda_1\Big(B_\alpha(G)\Big) & & & \\ & \lambda_2\Big(B_\alpha(G)\Big) & & \\ & & \ddots & \\ & & & \lambda_n\Big(B_\alpha(G)\Big) \end{pmatrix}$$

and

$$W^T B_\alpha(H) W = \begin{pmatrix} \lambda_1\Big(B_\alpha(H)\Big) & & & \\ & \lambda_2\Big(B_\alpha(H)\Big) & & \\ & & \ddots & \\ & & & \lambda_m\Big(B_\alpha(H)\Big) \end{pmatrix}.$$

By Lemma 2.1(v), we know that $U \otimes W$ is an orthogonal matrix. Thus using (4) and Lemma 2.1(i), we get that

$$\begin{aligned}
&(U \otimes W)^T B_\alpha(G \square H)(U \otimes W) \\
=&(U \otimes W)^T \Big(B_\alpha(G) \otimes I_m + I_n \otimes B_\alpha(H)\Big)(U \otimes W) \\
=&\Big(U^T B_\alpha(G) U\Big) \otimes (W^T W) + (U^T U) \otimes \Big(W^T B_\alpha(H) W\Big) \\
=&\begin{pmatrix} \lambda_1\Big(B_\alpha(G)\Big) & & & \\ & \lambda_2\Big(B_\alpha(G)\Big) & & \\ & & \ddots & \\ & & & \lambda_n\Big(B_\alpha(G)\Big) \end{pmatrix} \otimes I_m \\
&+ I_n \otimes \begin{pmatrix} \lambda_1\Big(B_\alpha(H)\Big) & & & \\ & \lambda_2\Big(B_\alpha(H)\Big) & & \\ & & \ddots & \\ & & & \lambda_m\Big(B_\alpha(H)\Big) \end{pmatrix}.
\end{aligned}$$

Hence, it completes the proof of Theorem 3.1. □

In what follows, we characterize the spectrum of $B_\alpha\Big(G[H]\Big)$. By (1), we know that the matrix

$$B_\alpha(G) = (1-\alpha)D(G) + (2\alpha - 1)A(G). \tag{5}$$

Then, for r-regular graph G, we get that $B_\alpha(G) = (1-\alpha)rI + (2\alpha - 1)A(G)$. So we have the following lemma.

Lemma 3.2. *Assume H is a r-regular graph with m vertices. Let $p \geq \lambda_2(H) \geq \cdots \geq \lambda_m(H)$ be the spectrum of $A(H)$. Then*

$$Spec\Big(B_\alpha(H)\Big) = \{\alpha r, (1-\alpha)r + (2\alpha-1)\lambda_2(H), \ldots, (1-\alpha)r + (2\alpha-1)\lambda_m(H)\}.$$

Furthermore, let $W = [\mathbf{1}_m \ w_2 \ \cdots \ w_m]$ be an orthogonal matrix whose columns $\mathbf{1}_m, w_2, \cdots, w_m$ are eigenvectors corresponding to the eigenvalues $\alpha r, (1-\alpha)r + (2\alpha-1)\lambda_2(H), \ldots, (1-\alpha)r + (2\alpha-1)\lambda_m(H)$ of $B_\alpha(H)$, respectively.

Theorem 3.3. *Suppose G is a connected graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$, H is a r-regular graph with m vertices. Let $r \geq \lambda_2(H) \geq \cdots \geq \lambda_m(H)$ be the spectrum of $A(H)$, $\lambda_1(B_\alpha(G)) \geq \lambda_2(B_\alpha(G)) \geq \cdots \geq \lambda_n(B_\alpha(G))$ be the spectrum of $B_\alpha(G)$. Then*

$$Spec\Big(B_\alpha(G[H])\Big) = \bigcup_{i=1}^{n}\bigcup_{j=2}^{m}\Big\{(1-\alpha)r + (2\alpha-1)\lambda_j(H) + (1-\alpha)md_G(v_i)\Big\} \bigcup \bigcup_{i=1}^{n}\{\alpha r + m\lambda_i(B_\alpha(G))\}.$$

Proof. Let $A(G) = (a_{ij})_{n\times n}$ be the adjacency matrix of G and $d_G(v_i)$ be the degree of v_i of G for $i = 1, 2, \ldots, n$. By the definition of the lexicographic product of two graphs, we have

$$A(G[H]) = \begin{pmatrix} A(H) & a_{12}J_{m\times m} & \cdots & a_{1n}J_{m\times m} \\ a_{21}J_{m\times m} & A(H) & \cdots & a_{2n}J_{m\times m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}J_{m\times m} & a_{n2}J_{m\times m} & \cdots & A(H) \end{pmatrix} = I_n \otimes A(H) + A(G) \otimes J_{m\times m} \tag{6}$$

and

$$\begin{aligned} D(G[H]) &= \begin{pmatrix} \Big(r + d_G(v_1)m\Big)I_m & 0 & \cdots & 0 \\ 0 & \Big(r + d_G(v_2)m\Big)I_m & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Big(r + d_G(v_n)m\Big)I_m \end{pmatrix} \\ &= I_n \otimes D(H) + mD(G) \otimes I_m. \end{aligned} \tag{7}$$

From Lemma 2.1, we plug (6) and (7) into (5), which implies that

$$\begin{aligned} B_\alpha\Big(G[H]\Big) &= (1-\alpha)\Big(I_n \otimes D(H) + mD(G) \otimes I_m\Big) + (2\alpha-1)\Big(I_n \otimes A(H) + A(G) \otimes J_{m\times m}\Big) \\ &= I_n \otimes B_\alpha(H) + m(1-\alpha)D(G) \otimes I_m + (2\alpha-1)A(G) \otimes J_{m\times m}. \end{aligned}$$

Firstly, we show that $(1-\alpha)r + (2\alpha-1)\lambda_j(H) + m(1-\alpha)d_G(v_i)$ is an eigenvalue of $B_\alpha\Big(G[H]\Big)$ for $i = 1, 2, \ldots, n$ and $j = 2, 3, \ldots, m$.

Let $W = [\mathbf{1}_m \ w_2 \ \cdots \ w_m]$ be an orthogonal matrix whose columns $\mathbf{1}_m, w_2, \cdots, w_m$ are eigenvectors corresponding to the eigenvalues $r, \lambda_2(H), \ldots, \lambda_m(H)$, respectively. Thus for $j = 2, 3, \ldots, m$, $B_\alpha(H)w_j = \Big((1-\alpha)r + (1-\alpha)\lambda_j(H)\Big)w_j$ and $\mathbf{1}_m^T w_j = 0$ from Lemma 3.2. Let $e_i = (\underbrace{0, 0, \ldots, 1}_{i}, \ldots, 0)^T$ for $i = 1, 2, \ldots, n$. Then by Lemma 2.1, we obtain

$$\begin{aligned} B_\alpha\Big(G[H]\Big)(e_i \otimes w_j) &= \Big(I_n \otimes B_\alpha(H) + m(1-\alpha)D(G) \otimes I_m + (2\alpha-1)A(G) \otimes J_{m\times m}\Big)(e_i \otimes w_j) \\ &= e_i \otimes B_\alpha(H)w_j + m(1-\alpha)D(G)e_i \otimes w_j + (2\alpha-1)A(G)e_i \otimes (J_{m\times m}w_j) \\ &= \Big((1-\alpha)r + (2\alpha-1)\lambda_j(H)\Big)(e_i \otimes w_j) + m(1-\alpha)d_G(v_i)(e_i \otimes w_j) + 0 \\ &= \Big((1-\alpha)r + (2\alpha-1)\lambda_j(H) + m(1-\alpha)d_G(v_i)\Big)(e_i \otimes w_j). \end{aligned}$$

Hence, we deduce that $e_i \otimes Y_j$ is an eigenvector of $B_\alpha\Big(G[H]\Big)$ corresponding to $(1-\alpha)r + (2\alpha-1)\lambda_j(H) + m(1-\alpha)d_G(v_i)$.

Let u_i be the eigenvector of $B_\alpha(G)$ corresponding to $\lambda_i(B_\alpha(G))$ for $i = 1, 2, \ldots, n$. Then

$$\begin{aligned}
B_\alpha\Big(G[H]\Big)(u_i \otimes \mathbf{1}_m) &= \Big(I_n \otimes B_\alpha(H) + m(1-\alpha)D(G) \otimes I_m + (2\alpha-1)A(G) \otimes J_{m\times m}\Big)(u_i \otimes \mathbf{1}_m)\\
&= u_i \otimes \Big(B_\alpha(H)\mathbf{1}_m\Big) + m(1-\alpha)D(G)u_i \otimes \mathbf{1}_m + (2\alpha-1)A(G)u_i \otimes (J_{m\times m}\mathbf{1}_m)\\
&= \alpha r(u_i \otimes \mathbf{1}_m) + m(1-\alpha)D(G)u_i \otimes \mathbf{1}_m + m(2\alpha-1)A(G)u_i \otimes \mathbf{1}_m\\
&= \alpha r(u_i \otimes \mathbf{1}_m) + mB_\alpha(G)u_i \otimes \mathbf{1}_m\\
&= \Big(\alpha r + m\lambda_i(B_\alpha(G)\Big)(u_i \otimes \mathbf{1}_m).
\end{aligned}$$

It means that $u_i \otimes \mathbf{1}_m$ is an eigenvector of $B_\alpha\Big(G[H]\Big)$ corresponding to $\alpha r + m\lambda_i(B_\alpha(G))$.

Furthermore, it is easy to verify that $(e_{i_1} \otimes w_{j_1})^T(e_{i_2} \otimes w_{j_2}) = 0$ if $(i_1, j_1) \neq (i_2, j_2)$, and $(e_{i_1} \otimes w_j)^T(e_{i_2} \otimes \mathbf{1}_m) = 0$ for any $i_1, j_2 \in [n]$ and $j \in [m] \setminus \{1\}$, that is, all these eigenvectors of $B_\alpha\Big(G[H]\Big)$ are orthogonal. Therefore, it completes the proof of Theorem 3.3. □

Next, we determine the spectrum of $B_\alpha\Big(G[H_1, H_2, \ldots, H_n]\Big)$ as follows.

Theorem 3.4. *Let G be a connected graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$ and for $i = 1, 2, \ldots, n$, let H_i be a r_i-regular graph with order m_i. Suppose $A(G) = (a_{ij})_{n\times n}$ is the adjacency matrix of G and for $i = 1, 2, \ldots, n$, $s_i = \sum_{v_j \in N_G(v_i)} m_j$. If $r_i \geq \lambda_2(H_i) \geq \cdots \geq \lambda_{m_i}(H_i)$ are the spectrum of $A(H_i)$, then*

$$Spec\Big(B_\alpha(G[H_1, H_2, \ldots, H_n])\Big) = \bigcup_{i=1}^{n}\bigcup_{j=2}^{m_i}\{(1-\alpha)(r_i + s_i) + (2\alpha-1)\lambda_j(H_i)\} \cup Spec(C),$$

where

$$C = \begin{pmatrix} \alpha r_1 + (1-\alpha)s_1 & (2\alpha-1)a_{12}\sqrt{m_1m_2} & \cdots & (2\alpha-1)a_{1n}\sqrt{m_1m_n}\\ (2\alpha-1)a_{21}\sqrt{m_2m_1} & \alpha r_2 + (1-\alpha)s_2 & \cdots & (2\alpha-1)a_{2n}\sqrt{m_2m_n}\\ \vdots & \vdots & \ddots & \vdots\\ (2\alpha-1)a_{n1}\sqrt{m_nm_1} & (2\alpha-1)a_{n2}\sqrt{m_nm_2} & \cdots & \alpha r_n + (1-\alpha)s_n \end{pmatrix}_{n\times n}.$$

Proof. By the definition of the generalized lexicographic product of graphs. The adjacency matrix and the degree diagonal matrix of $G[H_1, H_2, \ldots, H_n]$ can be expressed below:

$$A\Big(G[H_1, H_2, \ldots, H_n]\Big) = \begin{pmatrix} A(H_1) & a_{12}J_{m_1\times m_2} & \cdots & a_{1n}J_{m_1\times m_n}\\ a_{21}J_{m_2\times m_1} & A(H_2) & \cdots & a_{2n}J_{m_2\times m_n}\\ \vdots & \vdots & \ddots & \vdots\\ a_{n1}J_{m_n\times m_1} & a_{n2}J_{m_n\times m_2} & \cdots & A(H_n) \end{pmatrix} \tag{8}$$

and

$$D\Big(G[H_1, H_2, \ldots, H_n]\Big) = \begin{pmatrix} (r_1+s_1)I_{m_1} & 0 & \cdots & 0\\ 0 & (r_2+s_2)I_{m_2} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & (r_n+s_n)I_{m_n} \end{pmatrix}. \tag{9}$$

Then submitting (8) and (9) in (5), we obtain

$$\begin{aligned}
&B_\alpha\Big(G[H_1, H_2, \ldots, H_n]\Big) = (1-\alpha)D\Big(G[H_1, H_2, \ldots, H_n]\Big) + (2\alpha-1)A\Big(G[H_1, H_2, \ldots, H_n]\Big)\\
&= \begin{pmatrix} B_\alpha(H_1) + (1-\alpha)s_1I_{m_1} & (2\alpha-1)a_{12}J_{m_1\times m_2} & \cdots & (2\alpha-1)a_{1n}J_{m_1\times m_n}\\ (2\alpha-1)a_{21}J_{m_2\times m_1} & B_\alpha(H_2) + (1-\alpha)s_2I_{m_2} & \cdots & (2\alpha-1)a_{2n}J_{m_2\times m_n}\\ \vdots & \vdots & \ddots & \vdots\\ (2\alpha-1)a_{n1}J_{m_n\times m_1} & (2\alpha-1)a_{n2}J_{m_n\times m_2} & \cdots & B_\alpha(H_n) + (1-\alpha)s_nI_{m_n} \end{pmatrix}
\end{aligned}$$

Firstly, we show that $(1-\alpha)(r_i+s_i)+(2\alpha-1)\lambda_j(H)$ is an eigenvalue of $B_\alpha\Big(G[H_1,H_2,\ldots,H_n]\Big)$ for $i=1,2,\ldots,n$ and $j=2,3,\ldots,m$.

Let $W_i=[\mathbf{1}_{m_i}\ w_{i2}\ \cdots\ w_{im_i}]$ be an orthogonal matrix whose columns $\mathbf{1}_{m_i},w_{i2},\ldots,w_{im_i}$ are eigenvectors corresponding to the eigenvalues $r_i,\lambda_2(H_i),\ldots,\lambda_{m_i}(H_i)$, respectively. By Lemma 3.2, for $j=2,3,\ldots,m_i$, $B_\alpha(H_i)w_{ij}=\Big((1-\alpha)r_i+(2\alpha-1)\lambda_j(H_i)\Big)w_{ij}$ and $\mathbf{1}_{m_i}^T w_{ij}=0$. Let $w'_{ij}=\Big(\mathbf{0}_{1\times m_1},\mathbf{0}_{1\times m_2},\ldots,w_{ij}^T,\ldots,\mathbf{0}_{1\times m_n}\Big)^T$. Note that

$$J_{m_j\times m_i}w_{ij}=\mathbf{0}_{m_j\times 1}$$

and

$$\Big(B_\alpha(H_i)+(1-\alpha)s_iI_{m_i}\Big)w_{ij}=\Big((1-\alpha)(r_i+s_i)+(2\alpha-1)\lambda_j(H_i)\Big)w_{ij}.$$

So, we have that

$$B_\alpha\Big(G[H_1,\ldots,H_n]\Big)w'_{ij}=\Big((1-\alpha)(r_i+s_i)+(2\alpha-1)\lambda_j(H_i)\Big)w'_{ij}.$$

Thus, w'_{ij} is an eigenvector of $B_\alpha\Big(G[H_1,\ldots,H_n]\Big)$ corresponding to $(1-\alpha)(r_i+s_i)+(2\alpha-1)\lambda_j(H_i)$.

Next, let $U=[u_1\ u_2\ \cdots\ u_n]$ be an orthogonal matrix whose column $u_i=(u_{i1},\ldots,u_{in})^T$ is an eigenvector corresponding to the eigenvalue $\lambda_i(C)$. Then $Cu_i=\lambda_i(C)u_i$ and $u_i^Tu_j=0$ for $i\neq j$. Let

$$\begin{aligned}u'_i&=(\underbrace{\frac{u_{i1}}{\sqrt{m_1}},\ldots,\frac{u_{i1}}{\sqrt{m_1}}}_{m_1},\underbrace{\frac{u_{i2}}{\sqrt{m_2}},\ldots,\frac{u_{i2}}{\sqrt{m_2}}}_{m_2}\ldots,\underbrace{\frac{u_{in}}{\sqrt{m_n}},\ldots,\frac{u_{in}}{\sqrt{m_n}}}_{m_n})^T\\&=(\frac{u_{i1}}{\sqrt{m_1}}\mathbf{1}_{m_1}^T,\frac{u_{i2}}{\sqrt{m_2}}\mathbf{1}_{m_2}^T,\ldots,\frac{u_{in}}{\sqrt{m_n}}\mathbf{1}_{m_n}^T)^T.\end{aligned}$$

It is not difficult to check that

$$\Big(B_\alpha(H_k)+(1-\alpha)s_kI_{m_k}\Big)\frac{u_{ik}}{\sqrt{m_k}}\mathbf{1}_{m_k}=\alpha r_k+(1-\alpha)s_k\frac{u_{ik}}{\sqrt{m_k}}\mathbf{1}_{m_k}$$

and for $t\in[n]\backslash\{k\}$,

$$J_{m_k\times m_t}\frac{u_{it}}{\sqrt{m_t}}\mathbf{1}_{m_t}=\sqrt{m_km_t}\frac{u_{it}}{\sqrt{m_k}}\mathbf{1}_{m_k}.$$

Hence, we have that

$$\begin{aligned}B_\alpha\Big(G[H_1,\ldots,H_n]\Big)u'_i&=\begin{pmatrix}\frac{1}{\sqrt{m_1}}\Big((\alpha r_1+(1-\alpha)s_1)u_{i1}+\sum_{t\in[n]\backslash\{1\}}(2\alpha-1)a_{1t}\sqrt{m_1m_t}u_{it}\Big)\mathbf{1}_{m_1}\\ \frac{1}{\sqrt{m_2}}\Big((\alpha r_2+(1-\alpha)s_2)u_{i2}+\sum_{t\in[n]\backslash\{2\}}(2\alpha-1)a_{2t}\sqrt{m_2m_t}u_{it}\Big)\mathbf{1}_{m_2}\\ \vdots\\ \frac{1}{\sqrt{m_n}}\Big((\alpha r_n+(1-\alpha)s_n)u_{in}+\sum_{t\in[n]\backslash\{n\}}(2\alpha-1)a_{nt}\sqrt{m_nm_t}u_{it}\Big)\mathbf{1}_{m_n}\end{pmatrix}\\&=\begin{pmatrix}\lambda_i\frac{u_{i1}}{\sqrt{m_1}}\mathbf{1}_{m_1}\\ \lambda_i\frac{u_{i2}}{\sqrt{m_2}}\mathbf{1}_{m_2}\\ \vdots\\ \lambda_i\frac{u_{in}}{\sqrt{m_n}}\mathbf{1}_{m_n}\end{pmatrix}=\lambda_i(C)u'_i.\end{aligned}$$

Therefore, u'_i is an eigenvector of $B_\alpha\Big(G[H_1,H_2,\ldots,H_n]\Big)$ corresponding to $\lambda_i(C)$.

Notice that $(w'_{i_1j_1})^Tw'_{i_2j_2}=0$ for $(i_1,j_1)\neq(i_2,j_2)$ and $(w'_{i_1j_1})^Tu'_i=0$ for any $i,i_1\in[n]$ and $j_1\in[m_{i_1}]\backslash\{1\}$, i.e., all these eigenvectors of $B_\alpha\Big(G[H_1,\ldots,H_n]\Big)$ are orthogonal. The proof is thus complete. □

4. The spectral radii of $B_\alpha(G \times H)$ and $B_\alpha(G \oplus H)$

In Section 3, we study the B_α-spectrum of the graphs $G\square H$, $G[H]$ and $G[H_1, H_2, \dots, H_n]$. Unfortunately, by a similar method, it is impossible to investigate the B_α-spectrum of the graphs $G \times H$ and $G \oplus H$. Hence, in this section, we determine the spectra radii of $B_\alpha(G \times H)$ and $B_\alpha(G \oplus H)$ for arbitrary graph G and regular graph H.

Here, by the definitions of the graphs $G \times H$ and $G \oplus H$, we mention some preliminary properties that will be needed for proving our main results.

Property 4.1. *For the graphs $G \times H$, we have the following properties:*

(i) $A(G \times H) = A(G) \otimes A(H)$;
(ii) $d_{G\times H}\big((v_i, u_j)\big) = d_G(v_i) \times d_H(u_j)$;
(iii) $D(G \times H) = D(G) \otimes D(H)$.

Property 4.2. *For the graphs $G \oplus H$, we have the following properties:*

(i) $A(G \oplus H) = A(G\square H) + A(G \times H) = A(G) \otimes I_m + I_n \otimes A(H) + A(G) \otimes A(H)$;
(ii) $d_{G\oplus H}\big((v_i, u_j)\big) = d_G(v_i) + d_H(u_j) + d_G(v_i) \times d_H(u_j)$;
(iii) $D(G \oplus H) = D(G) \otimes I_m + I_n \otimes D(H) + D(G) \otimes D(H)$.

By Property 4.1 and (5), we have

$$B_\alpha(G \times H) = (1-\alpha)D(G) \otimes D(H) + (2\alpha - 1)A(G) \otimes A(H).$$

Similarly, by Property 4.2 and and (5), we see

$$B_\alpha(G \oplus H) = B_\alpha(G) \otimes I_m + I_n \otimes B_\alpha(H) + B_\alpha(G \times H).$$

Note that for r regular graph of order m, $\mathbf{1}_m$ is an eigenvector of G corresponding to the spectral radius r.

Theorem 4.1. *Let G be a connected graph with n vertices, H be a r-regular graph with m vertices. Suppose $\lambda_1(B_\alpha(G))$ is the spectral radius of $B_\alpha(G)$. If $\alpha \in [\frac{1}{2}, 1]$, then*

$$\lambda_1\Big(B_\alpha(G \times H)\Big) = r\lambda_1\Big(B_\alpha(G)\Big),$$

$$\lambda_1\Big(B_\alpha(G \oplus H)\Big) = (r+1)\lambda_1\Big(B_\alpha(G)\Big) + \alpha r.$$

Proof. For $\alpha \in [\frac{1}{2}, 1]$, let $U_1 = (u_1, u_2, \dots, u_n)^T$ be the Perron vector of $B_\alpha(G)$ corresponding to $\lambda_1\Big(B_\alpha(G)\Big)$, that is, $B_\alpha(G)U_1 = \lambda_1\Big(B_\alpha(G)\Big)U_1$ and $u_i > 0, U_1^T U_1 = 1$. Then we have

$$\begin{aligned}
B_\alpha(G \times H)(U_1 \otimes \mathbf{1}_m) &= \Big((1-\alpha)D(G) \otimes D(H) + (2\alpha - 1)A(G) \otimes A(H)\Big)(U_1 \otimes \mathbf{1}_m) \\
&= \Big((1-\alpha)D(G)U_1\Big) \otimes (D(H)\mathbf{1}_m) + \Big((2\alpha-1)A(G)U_1\Big) \otimes (A(H)\mathbf{1}_m) \\
&= \Big((1-\alpha)rD(G)U_1\Big) \otimes (\mathbf{1}_m) + \Big(r(2\alpha-1)A(G)U_1\Big) \otimes (\mathbf{1}_m) \\
&= (rB_\alpha(G)U_1) \otimes \mathbf{1}_m \\
&= \Big(r\lambda_1(B_\alpha(G))U_1\Big) \otimes \mathbf{1}_m \\
&= r\lambda_1\Big(B_\alpha(G)\Big)(U_1 \otimes \mathbf{1}_m).
\end{aligned}$$

It implies that $U_1 \otimes \mathbf{1}_m$ is an eigenvetor of $B_\alpha(G \times H)$ corresponding to eigenvalue $r\lambda_1\Big(B_\alpha(G)\Big)$. Since every entries of $U_1 \otimes \mathbf{1}_m$ are positive, by the Perron-Frobenius theorem, we obtain

$$\lambda_1\Big(B_\alpha(G \times H)\Big) = r\lambda_1\Big(B_\alpha(G)\Big).$$

Analogously, we can see that $U_1 \otimes \mathbf{1}_m$ is also an eigenvector of $B_\alpha(G \oplus H)$ corresponding to $r\lambda_1\Big(B_\alpha(G)\Big) + \lambda_1\Big(B_\alpha(G)\Big) + \alpha r$. Therefore,

$$\lambda_1\Big(B_\alpha(G \oplus H)\Big) = (r+1)\lambda_1\Big(B_\alpha(G)\Big) + \alpha r.$$

It completes the proof of Theorem 4.1. □

Note that the graphs $G \times H \cong H \times G$ and $G \oplus H \cong H \oplus G$, the following corollary is obvious by Theorem 4.1.

Corollary 4.2. *Let H be a connected graph with m vertices, G be a r-regular graph with n vertices. Assume $\lambda_1\Big(B_\alpha(H)\Big)$ is the spectral radius of $B_\alpha(H)$. Then, $\alpha \in [\frac{1}{2}, 1]$, we get*

$$\lambda_1\Big(B_\alpha(G \times H)\Big) = r\lambda_1\Big(B_\alpha(H)\Big),$$

$$\lambda_1\Big(B_\alpha(G \oplus H)\Big) = (r+1)\lambda_1\Big(B_\alpha(H)\Big) + \alpha r.$$

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Conflicts of Interest

The authors declare no conflict of interest.

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