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A study on fractional Weddle's inequalities through twice differentiable functions

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Abstract. The present study develops fractional Weddle inequalities by employing middle point starting Riemann-Liouville operators in conjunction with h-convex functions. The paper presents a novel variant of the established fractional Weddle inequalities for twice differentiable functions, derived through basic computations involving the B-function. Furthermore, we discuss new results concerning Weddle inequalities related to the Riemann integral, s-convex functions, and P-functions.

1. Introduction

The analysis of integral inequalities significantly influences the accuracy of numerical quadrature methods.The efficiency of Simpson-type inequalities stands out among these methods. The third rule of Simpson, also known as Weddle's rule, represents a significant development in this field. The Weddle formula improves accuracy by using a more complex curve, typically a quadratic polynomial, to approximate the function in each segment.

Particularly for functions that exhibit complex behaviors over long intervals or rapid variations, Weddle's formula represents a more efficient mathematical approach that transcends Simpson's rule in terms of precision. The Simpson's rule is known for its simplicity and efficiency; however, it is limited to third-degree polynomial approximations. On the other hand, Weddle's formula is a more precise mathematical method, especially for functions that have rapid fluctuations [7].

The Weddle inequality is determined for functions with convex absolute values of their first derivatives

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[7, Remark.3].

$$\left|\frac{1}{20}\left[f(a)+5f\left(\frac{5a+b}{6}\right)+f\left(\frac{2a+b}{3}\right)+6f\left(\frac{a+b}{2}\right)+f\left(\frac{a+2b}{3}\right)+5f\left(\frac{a+5b}{6}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right|$$

$$\leq \frac{13(b-a)\left[\left|f'(a)\right|+\left|f'(b)\right|\right]}{450}.$$

Fractional calculus refers to the differential and integral calculus of positive real order. Due to its wide range of applications beyond mathematics, fractional calculus is currently a hot topic. The Riemann-Liouville fractional operator is the most famous and widely used in numerous disciplines. Assuming that $f \in L_1[a,b]$, the left and right-sided Riemann-Liouville fractional operators with order $\alpha > 0$ are defined as follows:

$$\Im^{\alpha}_{a^+} f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a,$$

$$\Im^{\alpha}_{b^-} f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b.$$

The left and right-sided middle point starting Riemann-Liouville fractional operators with order $\alpha > 0$ are defined as follows:

$$\Im^{\alpha}_{\left(\frac{a+b}{2}\right)^+} f(x) = \frac{1}{\Gamma(\alpha)}\int_{\frac{a+b}{2}}^x (x-t)^{\alpha-1} f(t)dt, \quad x > \frac{a+b}{2},$$

$$\Im^{\alpha}_{\left(\frac{a+b}{2}\right)^-} f(x) = \frac{1}{\Gamma(\alpha)}\int_x^{\frac{a+b}{2}} (t-x)^{\alpha-1} f(t)dt, \quad x < \frac{a+b}{2}.$$

In [10], the author develops a new class of functions known as h-convex functions.

Definition 1.1. *Let $h : [0,1] \to \mathbb{R}$ be a positive function. We define a function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ as h-convex if for any $x, y \in I$ and $\lambda \in [0,1]$, the following condition holds:*

$$f(\lambda x + (1-\lambda)y) \leq h(\lambda)f(x) + h(1-\lambda)f(y). \tag{1}$$

In the case when the inequality (1) is reversed, we say that f is h-concave.

- By making $h(\lambda) = \lambda$, Definition 1.1 simplifies to a convex function.
- By setting $h(\lambda) = 1$, Definition 1.1 simplifies to P-functions convex mapping [4, 9].
- By defining $h(\lambda) = \lambda^s$, Definition 1.1 simplifies to the concept of s-convex functions [3].

In recent work [1, 2], the authors introduce a novel class of functions termed B-functions, defined as follows:

Definition 1.2. *Let $g : [0,\infty) \to \mathbb{R}$ be a non-negative function. The function g is called a B-function if*

$$g(x-a) + g(b-x) \leq 2g\left(\frac{a+b}{2}\right) \tag{2}$$

where $a < x < b$ with $a, b \in [0,\infty)$.

If the inequality (2) is reversed, g is called A-function, or that g belongs to the class $A(a,b)$.
If we have equality in (2), g is called AB-function, or that g belongs to the class $AB(a,b)$.

Corollary 1.3. *Let $h : [0,1] \to \mathbb{R}$ be a non-negative function. The function h is a B-function, if for all $\lambda \in [0,1]$, we have*

$$h(\lambda) + h(1-\lambda) \le 2\,h\left(\frac{1}{2}\right). \tag{3}$$

- *The functions $h(\lambda) = \lambda$ and $h(\lambda) = 1$ are AB-function, B-function and A-function.*
- *The function $h(\lambda) = \lambda^s$, $s \in (0,1]$ is a B-function.*

Theorem 1.4 (Hölder inequality). *Let p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. If Ψ and Φ are real functions defined on $[\lambda_1, \lambda_2]$ and if $|\Psi|^p$, $|\Phi|^q$ are integrable functions on $[\lambda_1, \lambda_2]$, then*

$$\int_{\lambda_1}^{\lambda_2} |\Psi(t)\Phi(t)|\,dt \le \left(\int_{\lambda_1}^{\lambda_2} |\Psi(t)|^p\,dt\right)^{\frac{1}{p}} \left(\int_{\lambda_1}^{\lambda_2} |\Phi(t)|^q\,dt\right)^{\frac{1}{q}}.$$

The power-mean integral inequality, derived from the Hölder inequality, can be expressed as follows:

Theorem 1.5 (Power-mean integral inequality). *Let $p \ge 1$ and W, Φ be two real functions defined on $[\lambda_1, \lambda_2]$. If $|W|$, $|W||\Phi|^q$ are integrable functions on $[\lambda_1, \lambda_2]$, then*

$$\int_{\lambda_1}^{\lambda_2} |W(t)\Phi(t)|\,dt \le \left(\int_{\lambda_1}^{\lambda_2} |W(t)|\,dt\right)^{1-\frac{1}{p}} \left(\int_{\lambda_1}^{\lambda_2} |W(t)|\,|\Phi(t)|^p\,dt\right)^{\frac{1}{p}}.$$

This study uses an exploratory approach to find a new fractional Weddle inequality for h-convex functions, using Riemann-Liouville fractional operators on functions that are twice differentiable.

2. Basic Identity

Lemma 2.1. *Let $\alpha > 0$ and $f : I^\circ \to \mathbb{R}$ be twice differentiable mapping such that $f'' \in L_1([a,b])$ where $[a,b] \subset I^\circ \subseteq \mathbb{R}$, then the following equality holds.*

$$\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[\Im^\alpha_{\left(\frac{a+b}{2}\right)^-} f(a) + \Im^\alpha_{\left(\frac{a+b}{2}\right)^+} f(b)\right] - \frac{1}{20}\left\{f(a) + 5f\left(\frac{5a+b}{6}\right) + f\left(\frac{2a+b}{3}\right)\right.$$

$$\left. +6f\left(\frac{a+b}{2}\right) + f\left(\frac{a+2b}{3}\right) + 5f\left(\frac{a+5b}{6}\right) + f(b)\right\} \tag{4}$$

$$= \frac{(b-a)^2}{8(\alpha+1)} \int_0^1 H_\alpha(t)\left[f''\left(\left(1-\frac{t}{2}\right)a + \left(\frac{t}{2}\right)b\right) + f''\left(\left(\frac{t}{2}\right)a + \left(1-\frac{t}{2}\right)b\right)\right]dt,$$

where

$$H_\alpha(t) = \begin{cases} t^{\alpha+1} - \frac{(\alpha+1)}{10}t, & if\ 0 \le t \le \frac{1}{3}, \\ t^{\alpha+1} - \frac{3(\alpha+1)}{5}t + \frac{(\alpha+1)}{6}, & if\ \frac{1}{3} \le t \le \frac{2}{3}, \\ t^{\alpha+1} - \frac{7(\alpha+1)}{10}t + \frac{7(\alpha+1)}{30}, & if\ \frac{2}{3} \le t \le 1. \end{cases} \tag{5}$$

Proof. By using the integration by parts, we deduce

$$
\begin{aligned}
I_1 &= \int_0^{\frac{1}{3}} \left(t^{\alpha+1} - \frac{(\alpha+1)}{10}t\right)\left[f''\left(\left(1-\frac{t}{2}\right)a + \left(\frac{t}{2}\right)b\right) + f''\left(\left(\frac{t}{2}\right)a + \left(1-\frac{t}{2}\right)b\right)\right]dt \\
&= \left(\frac{2}{b-a}\right)\left(t^{\alpha+1} - \frac{(\alpha+1)}{10}t\right)\left[f'\left(\left(1-\frac{t}{2}\right)a + \left(\frac{t}{2}\right)b\right) - f'\left(\left(\frac{t}{2}\right)a + \left(1-\frac{t}{2}\right)b\right)\right]\Bigg|_0^{\frac{1}{3}} \\
&\quad - \left(\frac{2}{b-a}\right)\int_0^{\frac{1}{3}} \left((\alpha+1)\,t^{\alpha} - \frac{(\alpha+1)}{10}\right)\left[f'\left(\left(1-\frac{t}{2}\right)a + \left(\frac{t}{2}\right)b\right) - f'\left(\left(\frac{t}{2}\right)a + \left(1-\frac{t}{2}\right)b\right)\right]dt,
\end{aligned}
$$

thus

$$
\begin{aligned}
I_1 &= \frac{2}{b-a}\left\{\left(\left(\frac{1}{3}\right)^{\alpha+1} - \frac{(\alpha+1)}{30}\right)\left(f'\left(\frac{5a+b}{6}\right) - f'\left(\frac{a+5b}{6}\right)\right)\right\} \\
&\quad - \left(\frac{4}{(b-a)^2}\right)\left((\alpha+1)\,t^{\alpha} - \frac{(\alpha+1)}{10}\right)\left[f\left(\left(1-\frac{t}{2}\right)a + \left(\frac{t}{2}\right)b\right) + f\left(\left(\frac{t}{2}\right)a + \left(1-\frac{t}{2}\right)b\right)\right]\Bigg|_0^{\frac{1}{3}} \\
&\quad + \left(\frac{4\,\alpha(\alpha+1)}{(b-a)^2}\right)\int_0^{\frac{1}{3}} t^{\alpha-1}\left[f\left(\left(1-\frac{t}{2}\right)a + \left(\frac{t}{2}\right)b\right) + f\left(\left(\frac{t}{2}\right)a + \left(1-\frac{t}{2}\right)b\right)\right]dt \\
&= \frac{2}{b-a}\left\{\left(\left(\frac{1}{3}\right)^{\alpha+1} - \frac{(\alpha+1)}{30}\right)\left(f'\left(\frac{5a+b}{6}\right) - f'\left(\frac{a+5b}{6}\right)\right)\right\} \\
&\quad - \left(\frac{4}{(b-a)^2}\right)\left\{\left((\alpha+1)\left(\frac{1}{3}\right)^{\alpha} - \frac{(\alpha+1)}{10}\right)\left(f\left(\frac{5a+b}{6}\right) + f\left(\frac{a+5b}{6}\right)\right) + \frac{(\alpha+1)}{10}\left(f(a)+f(b)\right)\right\} \\
&\quad + \left(\frac{4\,\alpha(\alpha+1)}{(b-a)^2}\right)\int_0^{\frac{1}{3}} t^{\alpha-1}\left[f\left(\left(1-\frac{t}{2}\right)a + \left(\frac{t}{2}\right)b\right) + f\left(\left(\frac{t}{2}\right)a + \left(1-\frac{t}{2}\right)b\right)\right]dt.
\end{aligned}
$$

Similarly

$$
\begin{aligned}
I_2 &= \int_{\frac{1}{3}}^{\frac{2}{3}} \left(t^{\alpha+1} - \frac{3(\alpha+1)}{5}t + \frac{(\alpha+1)}{6}\right)\left[f''\left(\left(1-\frac{t}{2}\right)a + \left(\frac{t}{2}\right)b\right) + f''\left(\left(\frac{t}{2}\right)a + \left(1-\frac{t}{2}\right)b\right)\right]dt \\
&= \left(\frac{2}{b-a}\right)\left(t^{\alpha+1} - \frac{3(\alpha+1)}{5}t + \frac{(\alpha+1)}{6}\right)\left[f'\left(\left(1-\frac{t}{2}\right)a + \left(\frac{t}{2}\right)b\right) - f'\left(\left(\frac{t}{2}\right)a + \left(1-\frac{t}{2}\right)b\right)\right]\Bigg|_{\frac{1}{3}}^{\frac{2}{3}} \\
&\quad - \left(\frac{2}{b-a}\right)\int_{\frac{1}{3}}^{\frac{2}{3}} \left((\alpha+1)\,t^{\alpha} - \frac{3(\alpha+1)}{5}\right)\left[f'\left(\left(1-\frac{t}{2}\right)a + \left(\frac{t}{2}\right)b\right) - f'\left(\left(\frac{t}{2}\right)a + \left(1-\frac{t}{2}\right)b\right)\right]dt \\
&= \frac{2}{b-a}\left\{\left(\left(\frac{2}{3}\right)^{\alpha+1} - \frac{7(\alpha+1)}{30}\right)\left(f'\left(\frac{2a+b}{3}\right) - f'\left(\frac{a+2b}{3}\right)\right)\right. \\
&\quad \left. -\left(\left(\frac{1}{3}\right)^{\alpha+1} - \frac{(\alpha+1)}{30}\right)\left(f'\left(\frac{5a+b}{6}\right) - f'\left(\frac{a+5b}{6}\right)\right)\right\} \\
&\quad - \left(\frac{4}{(b-a)^2}\right)\left((\alpha+1)\,t^{\alpha} - \frac{3(\alpha+1)}{5}\right)\left[f\left(\left(1-\frac{t}{2}\right)a + \left(\frac{t}{2}\right)b\right) + f\left(\left(\frac{t}{2}\right)a + \left(1-\frac{t}{2}\right)b\right)\right]\Bigg|_{\frac{1}{3}}^{\frac{2}{3}} \\
&\quad + \left(\frac{4\,\alpha(\alpha+1)}{(b-a)^2}\right)\int_{\frac{1}{3}}^{\frac{2}{3}} t^{\alpha-1}\left[f\left(\left(1-\frac{t}{2}\right)a + \left(\frac{t}{2}\right)b\right) + f\left(\left(\frac{t}{2}\right)a + \left(1-\frac{t}{2}\right)b\right)\right]dt,
\end{aligned}
$$

then

$$
\begin{aligned}
I_2 &= \frac{2}{b-a}\left\{\left(\left(\frac{2}{3}\right)^{\alpha+1} - \frac{7(\alpha+1)}{30}\right)\left(f'\left(\frac{2a+b}{3}\right) - f'\left(\frac{a+2b}{3}\right)\right)\right. \\
&\quad \left. -\left(\left(\frac{1}{3}\right)^{\alpha+1} - \frac{(\alpha+1)}{30}\right)\left(f'\left(\frac{5a+b}{6}\right) - f'\left(\frac{a+5b}{6}\right)\right)\right\} \\
&\quad - \left(\frac{4}{(b-a)^2}\right)\left\{\left((\alpha+1)\left(\frac{2}{3}\right)^{\alpha} - \frac{3(\alpha+1)}{5}\right)\left(f\left(\frac{2a+b}{3}\right) + f\left(\frac{a+2b}{3}\right)\right)\right. \\
&\quad \left. - \left((\alpha+1)\left(\frac{1}{3}\right)^{\alpha} - \frac{3(\alpha+1)}{5}\right)\left(f\left(\frac{5a+b}{6}\right) + f\left(\frac{a+5b}{6}\right)\right)\right\} \\
&\quad + \left(\frac{4\,\alpha(\alpha+1)}{(b-a)^2}\right)\int_{\frac{1}{3}}^{\frac{2}{3}} t^{\alpha-1}\left[f\left(\left(1-\frac{t}{2}\right)a + \left(\frac{t}{2}\right)b\right) + f\left(\left(\frac{t}{2}\right)a + \left(1-\frac{t}{2}\right)b\right)\right]dt.
\end{aligned}
$$

And

$$I_3 = \int_{\frac{2}{3}}^{1} \left(t^{\alpha+1} - \frac{7(\alpha+1)}{10}t + \frac{7(\alpha+1)}{30}\right)\left[f''\left(\left(1-\frac{t}{2}\right)a + \left(\frac{t}{2}\right)b\right) + f''\left(\left(\frac{t}{2}\right)a + \left(1-\frac{t}{2}\right)b\right)\right]dt$$

$$= \left(\frac{2}{b-a}\right)\left(t^{\alpha+1} - \frac{7(\alpha+1)}{10}t + \frac{7(\alpha+1)}{30}\right)\left[f'\left(\left(1-\frac{t}{2}\right)a + \left(\frac{t}{2}\right)b\right) - f'\left(\left(\frac{t}{2}\right)a + \left(1-\frac{t}{2}\right)b\right)\right]\Bigg|_{\frac{2}{3}}^{1}$$

$$- \left(\frac{2}{b-a}\right)\int_{\frac{2}{3}}^{1} \left((\alpha+1)\,t^{\alpha} - \frac{7(\alpha+1)}{10}\right)\left[f'\left(\left(1-\frac{t}{2}\right)a + \left(\frac{t}{2}\right)b\right) - f'\left(\left(\frac{t}{2}\right)a + \left(1-\frac{t}{2}\right)b\right)\right]dt$$

$$= \frac{2}{b-a}\left\{-\left(\left(\frac{2}{3}\right)^{\alpha+1} - \frac{7(\alpha+1)}{30}\right)\left(f'\left(\frac{2a+b}{3}\right) - f'\left(\frac{a+2b}{3}\right)\right)\right\}$$

$$- \left(\frac{4}{(b-a)^2}\right)\left((\alpha+1)\,t^{\alpha} - \frac{7(\alpha+1)}{10}\right)\left[f\left(\left(1-\frac{t}{2}\right)a + \left(\frac{t}{2}\right)b\right) + f\left(\left(\frac{t}{2}\right)a + \left(1-\frac{t}{2}\right)b\right)\right]\Bigg|_{\frac{2}{3}}^{1}$$

$$+ \left(\frac{4\,\alpha(\alpha+1)}{(b-a)^2}\right)\int_{\frac{2}{3}}^{1} t^{\alpha-1}\left[f\left(\left(1-\frac{t}{2}\right)a + \left(\frac{t}{2}\right)b\right) + f\left(\left(\frac{t}{2}\right)a + \left(1-\frac{t}{2}\right)b\right)\right]dt,$$

hence

$$I_3 = \frac{2}{b-a}\left\{-\left(\left(\frac{2}{3}\right)^{\alpha+1} - \frac{7(\alpha+1)}{30}\right)\left(f'\left(\frac{2a+b}{3}\right) - f'\left(\frac{a+2b}{3}\right)\right)\right\}$$

$$- \left(\frac{4}{(b-a)^2}\right)\left\{\frac{3(\alpha+1)}{5}f\left(\frac{a+b}{2}\right) - \left((\alpha+1)\left(\frac{2}{3}\right)^{\alpha} - \frac{7(\alpha+1)}{10}\right)\left(f\left(\frac{2a+b}{3}\right) + f\left(\frac{a+2b}{3}\right)\right)\right\}$$

$$+ \left(\frac{4\,\alpha(\alpha+1)}{(b-a)^2}\right)\int_{\frac{2}{3}}^{1} t^{\alpha-1}\left[f\left(\left(1-\frac{t}{2}\right)a + \left(\frac{t}{2}\right)b\right) + f\left(\left(\frac{t}{2}\right)a + \left(1-\frac{t}{2}\right)b\right)\right]dt.$$

Consequently

$$I_1 + I_2 + I_3 = \left(\frac{4\,\alpha(\alpha+1)}{(b-a)^2}\right)\int_{0}^{1} t^{\alpha-1}\left[f\left(\left(1-\frac{t}{2}\right)a + \left(\frac{t}{2}\right)b\right) + f\left(\left(\frac{t}{2}\right)a + \left(1-\frac{t}{2}\right)b\right)\right]dt$$

$$- \left(\frac{4}{(b-a)^2}\right)\left\{\frac{(\alpha+1)}{10}\,(f(a)+f(b)) + \frac{(\alpha+1)}{2}\left(f\left(\frac{5a+b}{6}\right) + f\left(\frac{a+5b}{6}\right)\right)\right.$$

$$\left. + \frac{(\alpha+1)}{10}\left(f\left(\frac{2a+b}{3}\right) + f\left(\frac{a+2b}{3}\right)\right) + \frac{3(\alpha+1)}{5}f\left(\frac{a+b}{2}\right)\right\}.$$

Given that

$$\int_{0}^{1} t^{\alpha-1} f\left(\left(1-\frac{t}{2}\right)a + \left(\frac{t}{2}\right)b\right)dt = \frac{2^{\alpha}}{(b-a)^{\alpha}}\int_{a}^{\frac{a+b}{2}} (t-a)^{\alpha-1} f(t)\,dt,$$

and

$$\int_{0}^{1} t^{\alpha-1} f\left(\left(\frac{t}{2}\right)a + \left(1-\frac{t}{2}\right)b\right)dt = \frac{2^{\alpha}}{(b-a)^{\alpha}}\int_{\frac{a+b}{2}}^{b} (b-t)^{\alpha-1} f(t)\,dt,$$

as a result

$$I_1+I_2+I_3=\frac{2^{\alpha+2}(\alpha+1)\Gamma(\alpha+1)}{(b-a)^{\alpha+2}}\left[\mathfrak{I}^{\alpha}_{\left(\frac{a+b}{2}\right)^-}f(a)+\mathfrak{I}^{\alpha}_{\left(\frac{a+b}{2}\right)^+}f(b)\right]$$

$$-\left(\frac{4(\alpha+1)}{10(b-a)^2}\right)\left\{(f(a)+f(b))+5\left(f\left(\frac{5a+b}{6}\right)+f\left(\frac{a+5b}{6}\right)\right)\right.$$

$$\left.+\left(f\left(\frac{2a+b}{3}\right)+f\left(\frac{a+2b}{3}\right)\right)+6f\left(\frac{a+b}{2}\right)\right\}.$$

The desired result (4) is obtained by multiplying the previous equality by $\frac{(b-a)^2}{8(\alpha+1)}$. □

Putting $\alpha=1$ in Lemma 2.1, we obtain the following Corollary.

Corollary 2.2. *Let $f: I^{\circ}\to\mathbb{R}$ be twice differentiable mapping such that $f''\in L_1([a,b])$ where $[a,b]\subset I^{\circ}\subseteq\mathbb{R}$, then the following equality holds.*

$$\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{1}{20}\left[f(a)+5f\left(\frac{5a+b}{6}\right)+f\left(\frac{2a+b}{3}\right)+6f\left(\frac{a+b}{2}\right)+f\left(\frac{a+2b}{3}\right)+5f\left(\frac{a+5b}{6}\right)+f(b)\right]$$

$$=\frac{(b-a)^2}{16}\int_0^1 H(t)\left[f''\left(\left(1-\frac{t}{2}\right)a+\left(\frac{t}{2}\right)b\right)+f''\left(\left(\frac{t}{2}\right)a+\left(1-\frac{t}{2}\right)b\right)\right]dt,$$

where

$$H_1(t)=\begin{cases} t^2-\frac{1}{5}t, & if\ 0\le t\le\frac{1}{3},\\ t^2-\frac{6}{5}t+\frac{1}{3}, & if\ \frac{1}{3}\le t\le\frac{2}{3},\\ t^2-\frac{7}{5}t+\frac{7}{15}, & if\ \frac{2}{3}\le t\le 1.\end{cases}$$

3. Weddle inequality via power-mean integral inequality

The first estimation of the Weddle inequality is currently presented.

Theorem 3.1. *Let $p\ge 1$, h be a B-function on $[0,1]$ and assume that the assumptions of Lemma 2.1 are justified. If $|f''|^p$ is an h-convex mapping on $[a,b]$, then the next Weddle's inequality holds.*

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[\mathfrak{I}^{\alpha}_{\left(\frac{a+b}{2}\right)^-}f(a)+\mathfrak{I}^{\alpha}_{\left(\frac{a+b}{2}\right)^+}f(b)\right]-\frac{1}{20}\left\{f(a)+5f\left(\frac{5a+b}{6}\right)\right.\right.$$

$$\left.\left.+f\left(\frac{2a+b}{3}\right)+6f\left(\frac{a+b}{2}\right)+f\left(\frac{a+2b}{3}\right)+5f\left(\frac{a+5b}{6}\right)+f(b)\right\}\right| \tag{6}$$

$$\le\frac{(b-a)^2}{4(\alpha+1)}\left(\int_0^1|H_\alpha(t)|\,dt\right)\left(h\left(\frac{1}{2}\right)\right)^{\frac{1}{p}}\left(\left|f''(a)\right|^p+\left|f''(b)\right|^p\right)^{\frac{1}{p}},$$

where H_α is defined by (5).

Proof. The following inequality is required to establish the next results. Let $D, C \geq 0$ and $\delta > 0$:

$$D^{\delta} + C^{\delta} \leq \max(1, 2^{1-\delta})(D + C)^{\delta}. \tag{7}$$

By applying the absolute value of identity (4), we conclude

$$\begin{aligned}
&\left| \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}} \left[\Im^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}} f(a) + \Im^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}} f(b) \right] - \frac{1}{20} \left\{ f(a) + 5f\left(\frac{5a+b}{6}\right) \right.\right. \\
&\left.\left. + f\left(\frac{2a+b}{3}\right) + 6f\left(\frac{a+b}{2}\right) + f\left(\frac{a+2b}{3}\right) + 5f\left(\frac{a+5b}{6}\right) + f(b) \right\} \right| \\
&\leq \frac{(b-a)^2}{8(\alpha+1)} \int_0^1 |H_{\alpha}(t)| \left[\left| f''\left(\left(1-\frac{t}{2}\right)a + \frac{t}{2}b\right) \right| + \left| f''\left(\frac{t}{2}a + \left(1-\frac{t}{2}\right)b\right) \right| \right] dt.
\end{aligned} \tag{8}$$

For $p \geq 1$, the application of the power-mean integral inequality results in the following:

$$\begin{aligned}
&\int_0^1 |H_{\alpha}(t)| \left[\left| f''\left(\left(1-\frac{t}{2}\right)a + \frac{t}{2}b\right) \right| + \left| f''\left(\frac{t}{2}a + \left(1-\frac{t}{2}\right)b\right) \right| \right] dt \\
&\leq \left(\int_0^1 |H_{\alpha}(t)|\, dt \right)^{1-\frac{1}{p}} \left[\left(\int_0^1 |H_{\alpha}(t)| \left| f''\left(\left(1-\frac{t}{2}\right)a + \frac{t}{2}b\right) \right|^p dt \right)^{\frac{1}{p}} + \left(\int_0^1 |H_{\alpha}(t)| \left| f''\left(\frac{t}{2}a + \left(1-\frac{t}{2}\right)b\right) \right|^p dt \right)^{\frac{1}{p}} \right].
\end{aligned}$$

The inequality (7) yields $D^{\frac{1}{p}} + C^{\frac{1}{p}} \leq 2^{1-\frac{1}{p}}(D+C)^{\frac{1}{p}}$, hence

$$\begin{aligned}
&\int_0^1 |H_{\alpha}(t)| \left[\left| f''\left(\left(1-\frac{t}{2}\right)a + \frac{t}{2}b\right) \right| + \left| f''\left(\frac{t}{2}a + \left(1-\frac{t}{2}\right)b\right) \right| \right] dt \\
&\leq \left(\int_0^1 |H_{\alpha}(t)|\, dt \right)^{1-\frac{1}{p}} 2^{1-\frac{1}{p}} \left[\int_0^1 |H_{\alpha}(t)| \left(\left| f''\left(\left(1-\frac{t}{2}\right)a + \frac{t}{2}b\right) \right|^p + \left| f''\left(\frac{t}{2}a + \left(1-\frac{t}{2}\right)b\right) \right|^p \right) dt \right]^{\frac{1}{p}}.
\end{aligned}$$

Given that $|f''|^p$ is a h-convex function, the application of inequality (3) to $\lambda = \frac{t}{2}$ yields the following:

$$\left| f''\left(\left(1-\frac{t}{2}\right)a + \frac{t}{2}b\right) \right|^p \leq h\left(1-\frac{t}{2}\right) \left| f''(a) \right|^p + h\left(\frac{t}{2}\right) \left| f''(b) \right|^p,$$

then

$$\begin{aligned}
\left| f''\left(\left(1-\frac{t}{2}\right)a + \frac{t}{2}b\right) \right|^p + \left| f''\left(\frac{t}{2}a + \left(1-\frac{t}{2}\right)b\right) \right|^p &\leq \left[h\left(1-\frac{t}{2}\right) + h\left(\frac{t}{2}\right) \right] \left(\left| f''(a) \right|^p + \left| f''(b) \right|^p \right) \\
&\leq 2h\left(\frac{1}{2}\right) \left(\left| f''(a) \right|^p + \left| f''(b) \right|^p \right),
\end{aligned}$$

thus

$$\begin{aligned}
&\int_0^1 |H_{\alpha}(t)| \left[\left| f''\left(\left(1-\frac{t}{2}\right)a + \frac{t}{2}b\right) \right| + \left| f''\left(\frac{t}{2}a + \left(1-\frac{t}{2}\right)b\right) \right| \right] dt \\
&\leq \left(\int_0^1 |H_{\alpha}(t)|\, dt \right)^{1-\frac{1}{p}} 2^{1-\frac{1}{p}} \left[\int_0^1 |H_{\alpha}(t)|\, 2h\left(\frac{1}{2}\right) \left(\left| f''(a) \right|^p + \left| f''(b) \right|^p \right) dt \right]^{\frac{1}{p}}.
\end{aligned}$$

As a result

$$\begin{aligned}&\int_0^1 |H_\alpha(t)|\left[\left|f''\left(\left(1-\frac{t}{2}\right)a+\frac{t}{2}\,b\right)\right|+\left|f''\left(\frac{t}{2}\,a+\left(1-\frac{t}{2}\right)b\right)\right|\right]dt\\&\leq\left(\int_0^1 |H_\alpha(t)|\;dt\right)2\left(h\left(\frac{1}{2}\right)\right)^{\frac{1}{p}}\left(\left|f''(a)\right|^p+\left|f''(b)\right|^p\right)^{\frac{1}{p}}.\end{aligned}\tag{9}$$

Combine inequality (8) and inequality (9) to obtain

$$\begin{aligned}&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[\Im^\alpha_{\left(\frac{a+b}{2}\right)^-}f(a)+\Im^\alpha_{\left(\frac{a+b}{2}\right)^+}f(b)\right]-\frac{1}{20}\left\{f(a)+5f\left(\frac{5a+b}{6}\right)\right.\right.\\&\left.\left.+f\left(\frac{2a+b}{3}\right)+6f\left(\frac{a+b}{2}\right)+f\left(\frac{a+2b}{3}\right)+5f\left(\frac{a+5b}{6}\right)+f(b)\right\}\right|\\&\leq\frac{(b-a)^2}{4(\alpha+1)}\left(\int_0^1 |H_\alpha(t)|\;dt\right)\left(h\left(\frac{1}{2}\right)\right)^{\frac{1}{p}}\left(\left|f''(a)\right|^p+\left|f''(b)\right|^p\right)^{\frac{1}{p}}.\end{aligned}$$

This completes the proof of the inequality (6). □

By setting $p = 1$ in the above Theorem 3.1, we get the following result.

Corollary 3.2. *Let h be a B-function on $[0, 1]$ and assume that the assumptions of Lemma 2.1 hold. If $|f''|$ is an h-convex mapping on $[a, b]$, then the following Weddle inequality holds.*

$$\begin{aligned}&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[\Im^\alpha_{\left(\frac{a+b}{2}\right)^-}f(a)+\Im^\alpha_{\left(\frac{a+b}{2}\right)^+}f(b)\right]-\frac{1}{20}\left\{f(a)+5f\left(\frac{5a+b}{6}\right)\right.\right.\\&\left.\left.+f\left(\frac{2a+b}{3}\right)+6f\left(\frac{a+b}{2}\right)+f\left(\frac{a+2b}{3}\right)+5f\left(\frac{a+5b}{6}\right)+f(b)\right\}\right|\\&\leq\frac{(b-a)^2}{4(\alpha+1)}\left(\int_0^1 |H_\alpha(t)|\;dt\right)\left(h\left(\frac{1}{2}\right)\right)\left(\left|f''(a)\right|+\left|f''(b)\right|\right),\end{aligned}\tag{10}$$

where H_α is defined by (5).

By putting $\alpha = 1$ into Theorem 3.1, we obtain the next results pertaining to the Riemann integral.

Corollary 3.3. *Let $p \geq 1$, h be a B-function on $[0, 1]$ and assume that the assumptions of Lemma 2.1 hold. If $|f'|^p$ is an h-convex mapping on $[a, b]$, then the following Weddle inequality holds.*

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{1}{20}\left[f(a)+5f\left(\frac{5a+b}{6}\right)+f\left(\frac{2a+b}{3}\right)+6f\left(\frac{a+b}{2}\right)+f\left(\frac{a+2b}{3}\right)+5f\left(\frac{a+5b}{6}\right)+f(b)\right]\right|\\&\leq\frac{51+7\sqrt{21}+16\sqrt{6}}{54000}(b-a)^2\left(h\left(\frac{1}{2}\right)\right)^{\frac{1}{p}}\left[\left|f''(a)\right|^p+\left|f''(b)\right|^p\right]^{\frac{1}{p}}.\end{aligned}\tag{11}$$

Here

$$\int_0^{\frac{1}{3}}\left|t^2-\frac{1}{5}t\right|dt=\frac{79}{20250},\ \int_{\frac{1}{3}}^{\frac{2}{3}}\left|t^2-\frac{6}{5}t+\frac{1}{3}\right|dt=\frac{9+24\sqrt{6}}{10125}\text{ and }\int_{\frac{2}{3}}^{1}\left|t^2-\frac{7}{5}t+\frac{7}{15}\right|dt=\frac{56+21\sqrt{21}}{20250}.$$

Next, let's examine some particular cases of h-convexity.

1. Putting $h(t)=t^s$ with $s\in(0,1]$ in Theorem 3.1, Corollary 3.2 and Corollary 3.3, we get the following results.

Corollary 3.4. *Assume p, α and f are defined according to Theorem 3.1. If $|f''|^p$ is an s-convex function on $[a,b]$, then*

$$\begin{aligned}&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[\mathfrak{I}^\alpha_{\left(\frac{a+b}{2}\right)^-}f(a)+\mathfrak{I}^\alpha_{\left(\frac{a+b}{2}\right)^+}f(b)\right]-\frac{1}{20}\left\{f(a)+5f\left(\frac{5a+b}{6}\right)\right.\right.\\&\left.\left.+f\left(\frac{2a+b}{3}\right)+6f\left(\frac{a+b}{2}\right)+f\left(\frac{a+2b}{3}\right)+5f\left(\frac{a+5b}{6}\right)+f(b)\right\}\right|\\&\le\frac{(b-a)^2}{4(\alpha+1)}\left(\int_0^1|H_\alpha(t)|\,dt\right)\left(\frac{1}{2}\right)^{\frac{s}{p}}\left(\left|f''(a)\right|^p+\left|f''(b)\right|^p\right)^{\frac{1}{p}}.\end{aligned}\tag{12}$$

For $p=1$, we get

$$\begin{aligned}&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[\mathfrak{I}^\alpha_{\left(\frac{a+b}{2}\right)^-}f(a)+\mathfrak{I}^\alpha_{\left(\frac{a+b}{2}\right)^+}f(b)\right]-\frac{1}{20}\left\{f(a)+5f\left(\frac{5a+b}{6}\right)\right.\right.\\&\left.\left.+f\left(\frac{2a+b}{3}\right)+6f\left(\frac{a+b}{2}\right)+f\left(\frac{a+2b}{3}\right)+5f\left(\frac{a+5b}{6}\right)+f(b)\right\}\right|\\&\le\frac{(b-a)^2}{4(\alpha+1)}\left(\int_0^1|H_\alpha(t)|\,dt\right)\left(\frac{1}{2}\right)^{s}\left(\left|f''(a)\right|+\left|f''(b)\right|\right).\end{aligned}\tag{13}$$

For $\alpha=1$, we get

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{1}{20}\left[f(a)+5f\left(\frac{5a+b}{6}\right)+f\left(\frac{2a+b}{3}\right)+6f\left(\frac{a+b}{2}\right)+f\left(\frac{a+2b}{3}\right)\right.\right.\\&\left.\left.+5f\left(\frac{a+5b}{6}\right)+f(b)\right]\right|\le\frac{51+7\sqrt{21}+16\sqrt{6}}{54000}(b-a)^2\left(\frac{1}{2}\right)^{\frac{s}{p}}\left[\left|f''(a)\right|^p+\left|f''(b)\right|^p\right]^{\frac{1}{p}}.\end{aligned}\tag{14}$$

2. Setting $h(t)=t$ in Theorem 3.1, Corollary 3.2 and Corollary 3.3 gives the next Corollary.

Corollary 3.5. *Assume p α and f are defined according to Theorem 3.1. If $|f''|^p$ is a convex function on $[a,b]$,* *then*

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[\mathfrak{I}^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}}f(a)+\mathfrak{I}^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}}f(b)\right]-\frac{1}{20}\left\{f(a)+5f\left(\frac{5a+b}{6}\right)\right.\right.$$

$$\left.\left.+f\left(\frac{2a+b}{3}\right)+6f\left(\frac{a+b}{2}\right)+f\left(\frac{a+2b}{3}\right)+5f\left(\frac{a+5b}{6}\right)+f(b)\right\}\right| \tag{15}$$

$$\leq\frac{(b-a)^2}{4(\alpha+1)}\left(\int_0^1|H_\alpha(t)|\,dt\right)\left[\frac{\left|f''(a)\right|^p+\left|f''(b)\right|^p}{2}\right]^{\frac{1}{p}}.$$

For $p=1$, we get

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[\mathfrak{I}^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}}f(a)+\mathfrak{I}^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}}f(b)\right]-\frac{1}{20}\left\{f(a)+5f\left(\frac{5a+b}{6}\right)\right.\right.$$

$$\left.\left.+f\left(\frac{2a+b}{3}\right)+6f\left(\frac{a+b}{2}\right)+f\left(\frac{a+2b}{3}\right)+5f\left(\frac{a+5b}{6}\right)+f(b)\right\}\right| \tag{16}$$

$$\leq\frac{(b-a)^2}{8(\alpha+1)}\left(\int_0^1|H_\alpha(t)|\,dt\right)\left(\left|f''(a)\right|+\left|f''(b)\right|\right).$$

For $\alpha=1$, we get

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{1}{20}\left[f(a)+5f\left(\frac{5a+b}{6}\right)+f\left(\frac{2a+b}{3}\right)+6f\left(\frac{a+b}{2}\right)+f\left(\frac{a+2b}{3}\right)\right.\right.\\&\left.\left.+5f\left(\frac{a+5b}{6}\right)+f(b)\right]\right|\leq\frac{51+7\sqrt{21}+16\sqrt{6}}{54000}(b-a)^2\left[\frac{\left|f''(a)\right|^p+\left|f''(b)\right|^p}{2}\right]^{\frac{1}{p}}.\end{aligned}\tag{17}$$

Remark 3.6. *Considering that $\alpha=p=1$ in Theorem 3.1 and assuming that $|f''|$ is a convex function on $[a,b]$, then*

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{1}{20}\left[f(a)+5f\left(\frac{5a+b}{6}\right)+f\left(\frac{2a+b}{3}\right)+6f\left(\frac{a+b}{2}\right)+f\left(\frac{a+2b}{3}\right)\right.\right.\\&\left.\left.+5f\left(\frac{a+5b}{6}\right)+f(b)\right]\right|\leq\frac{51+7\sqrt{21}+16\sqrt{6}}{108000}(b-a)^2\left[\left|f''(a)\right|+\left|f''(b)\right|\right].\end{aligned}\tag{18}$$

3. The following new result for the class of P-functions convexity is obtained by supposing $h(t)=1$ in Theorem 3.1, Corollary 3.2 and Corollary 3.3. In the context of inequality (12), it is also analogous to the cases when $s\to 0^+$.

Corollary 3.7. *Assume p and f are defined according to Theorem 3.1. If $|f''|^p$ is a P-functions convex mapping* *on* $[a,b]$, *then*

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[\Im^{\alpha}_{\left(\frac{a+b}{2}\right)^-}f(a)+\Im^{\alpha}_{\left(\frac{a+b}{2}\right)^+}f(b)\right]-\frac{1}{20}\left\{f(a)+5f\left(\frac{5a+b}{6}\right)\right.\right.$$

$$\left.\left.+f\left(\frac{2a+b}{3}\right)+6f\left(\frac{a+b}{2}\right)+f\left(\frac{a+2b}{3}\right)+5f\left(\frac{a+5b}{6}\right)+f(b)\right\}\right| \tag{19}$$

$$\leq\frac{(b-a)^2}{4(\alpha+1)}\left(\int_0^1|H_\alpha(t)|\,dt\right)\left(\left|f''(a)\right|^p+\left|f''(b)\right|^p\right)^{\frac{1}{p}}.$$

For $p=1$, *we get*

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[\Im^{\alpha}_{\left(\frac{a+b}{2}\right)^-}f(a)+\Im^{\alpha}_{\left(\frac{a+b}{2}\right)^+}f(b)\right]-\frac{1}{20}\left\{f(a)+5f\left(\frac{5a+b}{6}\right)\right.\right.$$

$$\left.\left.+f\left(\frac{2a+b}{3}\right)+6f\left(\frac{a+b}{2}\right)+f\left(\frac{a+2b}{3}\right)+5f\left(\frac{a+5b}{6}\right)+f(b)\right\}\right| \tag{20}$$

$$\leq\frac{(b-a)^2}{4(\alpha+1)}\left(\int_0^1|H_\alpha(t)|\,dt\right)\left(\left|f''(a)\right|+\left|f''(b)\right|\right).$$

For $\alpha=1$, *we get*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{1}{20}\left[f(a)+5f\left(\frac{5a+b}{6}\right)+f\left(\frac{2a+b}{3}\right)+6f\left(\frac{a+b}{2}\right)+f\left(\frac{a+2b}{3}\right)\right.\right.$$
$$\left.\left.+5f\left(\frac{a+5b}{6}\right)+f(b)\right]\right|\leq\frac{51+7\sqrt{21}+16\sqrt{6}}{54000}(b-a)^2\left[\left|f''(a)\right|^p+\left|f''(b)\right|^p\right]^{\frac{1}{p}}. \tag{21}$$

4. Weddle's inequality through the use of Hölder inequality

Theorem 4.1. *Let h be a B-function on* $[0,1]$, $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$ *and assume that* f *is defined as in Lemma 2.1. If* $\left|f''\right|^p$ *is an h-convex mapping on* $[a,b]$,*then the ensuing Weddle inequality is valid.*

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[\Im^{\alpha}_{\left(\frac{a+b}{2}\right)^-}f(a)+\Im^{\alpha}_{\left(\frac{a+b}{2}\right)^+}f(b)\right]-\frac{1}{20}\left\{f(a)+5f\left(\frac{5a+b}{6}\right)+f\left(\frac{2a+b}{3}\right)+6f\left(\frac{a+b}{2}\right)+f\left(\frac{a+2b}{3}\right)\right.\right.$$
$$\left.\left.+5f\left(\frac{a+5b}{6}\right)+f(b)\right\}\right|\leq\frac{(b-a)^2}{4(\alpha+1)}\left(\int_0^1|H_\alpha(t)|^q\,dt\right)^{\frac{1}{q}}\left(h\left(\frac{1}{2}\right)\right)^{\frac{1}{p}}\left[\left|f''(a)\right|^p+\left|f''(b)\right|^p\right]^{\frac{1}{p}}, \tag{22}$$

where H_α *is defined by (5).*

Proof. Through the use of the absolute value of identity (4), we easily derive

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[\Im^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}}f(a)+\Im^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}}f(b)\right]-\frac{1}{20}\left\{f(a)+5f\left(\frac{5a+b}{6}\right)\right.\right.$$

$$\left.\left.+f\left(\frac{2a+b}{3}\right)+6f\left(\frac{a+b}{2}\right)+f\left(\frac{a+2b}{3}\right)+5f\left(\frac{a+5b}{6}\right)+f(b)\right\}\right|$$

$$\leq\frac{(b-a)^2}{8(\alpha+1)}\int_0^1|H_\alpha(t)|\left|f''\left(\left(1-\frac{t}{2}\right)a+\frac{t}{2}b\right)\right|dt+\frac{(b-a)^2}{8(\alpha+1)}\int_0^1|H_\alpha(t)|\left|f''\left(\frac{t}{2}a+\left(1-\frac{t}{2}\right)b\right)\right|dt,$$

applying Hölder inequality and $A^{\frac{1}{p}}+B^{\frac{1}{p}}\leq 2^{1-\frac{1}{p}}(A+B)^{\frac{1}{p}}$ yields

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[\Im^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}}f(a)+\Im^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}}f(b)\right]-\frac{1}{20}\left\{f(a)+5f\left(\frac{5a+b}{6}\right)\right.\right.$$

$$\left.\left.+f\left(\frac{2a+b}{3}\right)+6f\left(\frac{a+b}{2}\right)+f\left(\frac{a+2b}{3}\right)+5f\left(\frac{a+5b}{6}\right)+f(b)\right\}\right|$$

$$\leq\frac{(b-a)^2}{8(\alpha+1)}\left(\int_0^1|H_\alpha(t)|^q\,dt\right)^{\frac{1}{q}}\left(\int_0^1\left|f''\left(\left(1-\frac{t}{2}\right)a+\frac{t}{2}b\right)\right|^p dt\right)^{\frac{1}{p}}$$

$$+\frac{(b-a)^2}{8(\alpha+1)}\left(\int_0^1|H_\alpha(t)|^q\,dt\right)^{\frac{1}{q}}\left(\int_0^1\left|f''\left(\frac{t}{2}a+\left(1-\frac{t}{2}\right)b\right)\right|^p dt\right)^{\frac{1}{p}}$$

$$\leq\frac{(b-a)^2}{8(\alpha+1)}\left(\int_0^1|H_\alpha(t)|^q\,dt\right)^{\frac{1}{q}}2^{1-\frac{1}{p}}\left[\int_0^1\left|f''\left(\left(1-\frac{t}{2}\right)a+\frac{t}{2}b\right)\right|^p dt+\int_0^1\left|f''\left(\frac{t}{2}a+\left(1-\frac{t}{2}\right)b\right)\right|^p dt\right]^{\frac{1}{p}}.$$

Considering that $|f''|^p$ indicates an h-convex function,

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[\Im^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}}f(a)+\Im^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}}f(b)\right]-\frac{1}{20}\left\{f(a)+5f\left(\frac{5a+b}{6}\right)\right.\right.$$

$$\left.\left.+f\left(\frac{2a+b}{3}\right)+6f\left(\frac{a+b}{2}\right)+f\left(\frac{a+2b}{3}\right)+5f\left(\frac{a+5b}{6}\right)+f(b)\right\}\right|$$

$$\leq\frac{(b-a)^2}{8(\alpha+1)}\left(\int_0^1|H_\alpha(t)|^q\,dt\right)^{\frac{1}{q}}2^{\frac{1}{q}}\left[\int_0^1\left(h\left(1-\frac{t}{2}\right)\left|f''(a)\right|^p+h\left(\frac{t}{2}\right)\left|f''(b)\right|^p\right)dt\right.$$

$$\left.+\int_0^1\left(h\left(\frac{t}{2}\right)\left|f''(a)\right|^p+h\left(1-\frac{t}{2}\right)\left|f''(b)\right|^p\right)dt\right]^{\frac{1}{p}}$$

$$\leq\frac{(b-a)^2}{8(\alpha+1)}\left(\int_0^1|H_\alpha(t)|^q\,dt\right)^{\frac{1}{q}}2^{\frac{1}{q}}\left(\int_0^1\left[h\left(1-\frac{t}{2}\right)+h\left(\frac{t}{2}\right)\right]dt\right)^{\frac{1}{p}}\left[\left|f''(a)\right|^p+\left|f''(b)\right|^p\right]^{\frac{1}{p}}.$$

Assuming inequality (3) for $\lambda = \frac{t}{2}$, we get

$$\left| \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}} \left[\Im^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}} f(a) + \Im^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}} f(b) \right] - \frac{1}{20} \left\{ f(a) + 5f\left(\frac{5a+b}{6}\right) \right. \right.$$

$$\left. \left. + f\left(\frac{2a+b}{3}\right) + 6f\left(\frac{a+b}{2}\right) + f\left(\frac{a+2b}{3}\right) + 5f\left(\frac{a+5b}{6}\right) + f(b) \right\} \right|$$

$$\leq \frac{(b-a)^2}{4(\alpha+1)} \left(\int_0^1 |H_\alpha(t)|^q \, dt \right)^{\frac{1}{q}} \left(h\left(\frac{1}{2}\right) \right)^{\frac{1}{p}} \left[\left| f''(a) \right|^p + \left| f''(b) \right|^p \right]^{\frac{1}{p}}.$$

This completes the proof on inequality (22). □

The next step is to investigate some special cases involving h-convexity.

Corollary 4.2. *Assume p and f are defined according to Theorem 4.1.*

- *If $|f''|^p$ is an s-convex function on $[a,b]$, then*

$$\left| \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}} \left[\Im^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}} f(a) + \Im^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}} f(b) \right] - \frac{1}{20} \left\{ f(a) + 5f\left(\frac{5a+b}{6}\right) + f\left(\frac{2a+b}{3}\right) + 6f\left(\frac{a+b}{2}\right) \right. \right.$$

$$\left. \left. + f\left(\frac{a+2b}{3}\right) + 5f\left(\frac{a+5b}{6}\right) + f(b) \right\} \right| \leq \frac{(b-a)^2}{4(\alpha+1)} \left(\int_0^1 |H_\alpha(t)|^q \, dt \right)^{\frac{1}{q}} \left[\frac{\left| f''(a) \right|^p + \left| f''(b) \right|^p}{2^s} \right]^{\frac{1}{p}}.$$

- *If $|f''|^p$ is a convex function on $[a,b]$, then*

$$\left| \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}} \left[\Im^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}} f(a) + \Im^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}} f(b) \right] - \frac{1}{20} \left\{ f(a) + 5f\left(\frac{5a+b}{6}\right) + f\left(\frac{2a+b}{3}\right) + 6f\left(\frac{a+b}{2}\right) \right. \right.$$

$$\left. \left. + f\left(\frac{a+2b}{3}\right) + 5f\left(\frac{a+5b}{6}\right) + f(b) \right\} \right| \leq \frac{(b-a)^2}{4(\alpha+1)} \left(\int_0^1 |H_\alpha(t)|^q \, dt \right)^{\frac{1}{q}} \left[\frac{\left| f''(a) \right|^p + \left| f''(b) \right|^p}{2} \right]^{\frac{1}{p}}.$$

- *If $|f''|^p$ is a P-functions convex mapping on $[a,b]$, then*

$$\left| \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}} \left[\Im^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}} f(a) + \Im^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}} f(b) \right] - \frac{1}{20} \left\{ f(a) + 5f\left(\frac{5a+b}{6}\right) + f\left(\frac{2a+b}{3}\right) + 6f\left(\frac{a+b}{2}\right) \right. \right.$$

$$\left. \left. + f\left(\frac{a+2b}{3}\right) + 5f\left(\frac{a+5b}{6}\right) + f(b) \right\} \right| \leq \frac{(b-a)^2}{4(\alpha+1)} \left(\int_0^1 |H_\alpha(t)|^q \, dt \right)^{\frac{1}{q}} \left[\left| f''(a) \right|^p + \left| f''(b) \right|^p \right]^{\frac{1}{p}}.$$

5. Applications

for any positive values $a, b > 0$, we consider the following means:

- The weighted arithmetic mean:

$$W(\lambda_1, \lambda_2, a, b) = \frac{\lambda_1 a + \lambda_2 b}{\lambda_1 + \lambda_2}.$$

- The arithmetic mean:

$$A(a, b) = \frac{a+b}{2}.$$

- The m-logarithmic mean:

$$L_m(a, b) = \left(\frac{b^{m+1} - a^{m+1}}{(b-a)(m+1)} \right)^{\frac{1}{m}}, \quad m \in \mathbb{R} - \{-1, 0\}, \ b > a.$$

In [6], the following example is given: Let $s \in (0, 1)$ and $d, k, c \in \mathbb{R}$. We define a function $f : [0, +\infty) \to \mathbb{R}$, as

$$f(t) = \begin{cases} d & , t = 0; \\ k\, t^s + c & , t > 0. \end{cases}$$

If $k \geq 0$ and $0 \leq c \leq d$, then f is an s-convex function.

Example 5.1. *Let $t > 0$, $0 < s < 1$ and consider the function $f(t) = t^{s+2}$, then*

$$f''(t) = (s+2)\,(s+1)\ t^s.$$

Thus, for $d = c = 0$, $k = (s+2)\ (s+1)$, we have $|f''|$ is an s-convex function.

The following results are obtained by applying the preceding example to inequality (14) for $p = 1$.

Proposition 5.2. *Let $b > a > 0$, $0 < s < 1$ and $m = s + 2$. Then the following inequality holds:*

$$\left| \frac{1}{10} A(a^m, b^m) + \frac{1}{4} \left[W^m(1, 5, a, b) + W^m(5, 1, a, b) \right] + \frac{1}{4} \left[W^m(1, 2, a, b) + W^m(2, 1, a, b) \right] + \frac{3}{10} A^m(a, b) - L_m^m(a, b) \right|$$

$$\leq \frac{51 + 7\sqrt{21} + 16\sqrt{6}}{27000 \cdot 2^s} (b-a)^2\, (s+2)(s+3)\, A\,(a^s,\, b^s)\,.$$

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