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On the maximum atom-bond sum-connectivity index of cacti with given parameters

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Abstract. The atom-bond sum-connectivity (ABS) index of a graph G is defined as

$$ABS(G) = \sum_{uv \in E(G)} \sqrt{\frac{d_G(u) + d_G(v) - 2}{d_G(u) + d_G(v)}},$$

where $d_G(u)$ denotes the degree of vertex $u \in V(G)$. A connected graph G is called a cactus if any two of its cycles have no common edge. Let $\mathscr{C}(n,s)$ be the set of cacti of order n with s cycles, $\mathscr{P}(n,t)$ be the set of cacti of order n with t pendent vertices and $\mathscr{T}(2\beta,q)$ be the set of cacti of order 2β with a perfect matching M and q cycles, respectively. In this paper we determine the sharp upper bounds on the ABS index of cacti, respectively, among $\mathscr{C}(n,s)$, $\mathscr{P}(n,t)$ and $\mathscr{T}(2\beta,q)$. Moreover, the corresponding cacti are characterized at which the extremal values are attained.

1. Introduction

Let $G = (V(G), E(G))$ be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. For $u \in V(G)$, $d_G(u)$ and $N_G(u)$ denote the degree and the neighborhood of u, respectively. A pendent vertex is a vertex of degree 1. A support vertex is a vertex adjacent to a pendent vertex. Let $\mathcal{P}(G)$ be the set of all pendent vertices of G. A matching, denoted by M, in a graph is a set of edges such that no two edges are incident to a common vertex. The matching number of a graph G, denoted by β, is the maximum cardinality among all matchings in G. A perfect matching M is a matching such that each vertex of G is incident to exactly one edge of M. A graph G is called a cactus if any two cycles in G have no common edge.

In chemical graph theory, the structure of a molecule can usually be modeled by a graph, where each atom is represented by a vertex and each chemical bond between two atoms is represented by an edge. The

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topological indices of molecular graphs are invariants under graph isomorphism. One of the best known topological indices is the *connectivity index* or *Randić index* [14] defined as

$$R(G) = \sum_{uv\in E(G)} \frac{1}{\sqrt{d_G(u)d_G(v)}},$$

which is closely related to physical and chemical properties of various organic compounds [9]. For more on the Randić index, we refer the readers to the survey [11] and the references cited therein.

Since then, researchers have developed several modified versions of the *connectivity index*. In 1998, Estrada et al. [7] put forward *the atom-bond connectivity (ABC) index* as follows:

$$ABC(G) = \sum_{uv\in E(G)} \sqrt{\frac{d_G(u)+d_G(v)-2}{d_G(u)d_G(v)}}.$$

Zhou and Trinajstić [18] introduced the *sum-connectivity (SC) index* as

$$SC(G) = \sum_{uv\in E(G)} \frac{1}{\sqrt{d_G(u)+d_G(v)}}.$$

Combining the ABC index and SC index, Ali et al. [3] proposed a new topological index, named *the atom-bond sum-connectivity (ABS) index*, in the following:

$$ABS(G) = \sum_{uv\in E(G)} \sqrt{\frac{d_G(u)+d_G(v)-2}{d_G(u)+d_G(v)}} = \sum_{uv\in E(G)} \sqrt{1-\frac{2}{d_G(u)+d_G(v)}}.$$

Nithya et al. [8] revealed that the ABS index can be used to describe some properties of benzenoid hydrocarbons and some drug structures and to explain well the π-electron energy of benzenoid hydrocarbons. Zuo et al. [19] characterized the extremal chemical graphs with respect to the ABS index. Wang et al. [15] determined the maximum ABS index of molecular trees with a perfect matching. In [6, 12, 13], some lower and upper bounds were presented on the ABS index of trees with given number of pendent vertices. Zhang et al. [17] determined the extremal values of the ABS index of trees in terms of matching number or domination number. Extremal problems on the ABS index of unicyclic graphs were derived in [5, 8, 16]. Aarthi et al. [1] identified the extremal values of the ABS index in bicyclic graphs. A review with respect to the ABS index can be found in [4]. For more results on the ABS index, we refer to [2, 3, 10].

Motivated by the above results, we aim to investigate the maximum ABS index of cacti with given parameters. Let $\mathscr{C}(n,s)$ be the set of cacti of order n with s cycles, $\mathscr{P}(n,t)$ be the set of cacti of order n with t pendent vertices, $\mathscr{T}(2\beta,q)$ be the set of cacti of order 2β with a perfect matching M and q cycles. In Section 2, we introduce some notations and preliminary results. In Sections 3, 4 and 5, we determine the sharp upper bounds for the ABS index of cacti among $\mathscr{C}(n,s)$, $\mathscr{P}(n,t)$ and $\mathscr{T}(2\beta,q)$, respectively, with the corresponding cacti at which the extremal values are attained.

2. Preliminaries

Throughout this paper, for $X \subseteq V(G)$, we denote by $G-X$ the graph obtained from G by deleting vertices in X and all edges incident to them. In particular, if $X=\{v\}$, we write $G-v$ instead of $G-\{v\}$. For $S \subseteq E(G)$, let $G-S$ be the graph obtained from G by deleting edges in S. If S is a subset of the edge set of the complement of G, we denote by $G+S$ the graph obtained from G by adding the edges in S. If $S=\{e\}$, we write $G-e$ and $G+e$ for $G-\{e\}$ and $G+\{e\}$, respectively. As usual, the path, cycle and star with n vertices are denoted by P_n, C_n and S_n, respectively.

The following result can be easily proved and its proof is omitted here.

Lemma 2.1. *Let a and b be two integers with $a \geq b+1 \geq 0$, the function $f(x) = \sqrt{1-\frac{2}{x+a}} - \sqrt{1-\frac{2}{x+b}}$ with $x \geq 2$. Then $f(x)$ is monotonically decreasing for $x \geq 2$.*

Lemma 2.2. *Let integer $k \geq 1$ and the function $g(x) = k\sqrt{1-\frac{2}{x+1}} + (x-k)\left(\sqrt{1-\frac{2}{x+2}} - \sqrt{1-\frac{2}{x+2-k}}\right)$ with $x \geq k+1 \geq 2$. Then $g(x)$ is monotonically increasing for $x \geq 2$.*

Proof. With some direct computation, we have

$$\begin{aligned} g'(x) &= \frac{k}{(x-1)^{\frac{1}{2}}(x+1)^{\frac{3}{2}}} + \frac{x-k}{x^{\frac{1}{2}}(x+2)^{\frac{3}{2}}} - \frac{(x-k)^{\frac{1}{2}}}{(x-k+2)^{\frac{3}{2}}} - \frac{(x-k)^{\frac{1}{2}}}{(x-k+2)^{\frac{1}{2}}} + \frac{x^{\frac{1}{2}}}{(x+2)^{\frac{1}{2}}} \\ &= \frac{k}{(x-1)^{\frac{1}{2}}(x+1)^{\frac{3}{2}}} + \frac{x^2+3x-k}{x^{\frac{1}{2}}(x+2)^{\frac{3}{2}}} - \frac{(x-k)^{\frac{1}{2}}(x-k+3)}{(x-k+2)^{\frac{3}{2}}} \\ &> \frac{k}{x^{\frac{1}{2}}(x+2)^{\frac{3}{2}}} + \frac{x^2+3x-k}{x^{\frac{1}{2}}(x+2)^{\frac{3}{2}}} - \frac{(x-k)^{\frac{1}{2}}(x-k+3)}{(x-k+2)^{\frac{3}{2}}} \\ &= \frac{x^{\frac{1}{2}}(x+3)}{(x+2)^{\frac{3}{2}}} - \frac{(x-k)^{\frac{1}{2}}(x-k+3)}{(x-k+2)^{\frac{3}{2}}} \end{aligned}$$

Let $\varphi(x) = \frac{x^{\frac{1}{2}}(x+3)}{(x+2)^{\frac{3}{2}}}$ for $x \geq 2$, then $\varphi'(x) = \frac{3}{x^{\frac{1}{2}}(x+2)^{\frac{5}{2}}} > 0$. Therefore, $\varphi(x)$ is monotonically increasing. Then

$$g'(x) > \frac{x^{\frac{1}{2}}(x+3)}{(x+2)^{\frac{3}{2}}} - \frac{(x-k)^{\frac{1}{2}}(x-k+3)}{(x-k+2)^{\frac{3}{2}}} = \varphi(x) - \varphi(x-k) > 0.$$

We finish the proof of this result. □

Lemma 2.3. *Let $h_1(x) = x\sqrt{1-\frac{2}{x+2}} - (x-2)\sqrt{1-\frac{2}{x}}$, $h_2(x) = (x-1)\sqrt{1-\frac{2}{x+2}} - (x-2)\sqrt{1-\frac{2}{x+1}}$ with $x > 2$. Then $h_1(x)$ and $h_2(x)$ are both monotonically increasing for $x > 2$.*

Proof. Let $\phi_1(x) = (x-2)\sqrt{1-\frac{2}{x}}$ for $x \geq 2$, then $h_1(x) = \phi_1(x+2) - \phi_1(x)$. For $x > 2$, we have $\phi_1''(x) = \frac{3}{x^{\frac{5}{2}}(x-2)^{\frac{1}{2}}} > 0$. Then $h_1'(x) = \phi_1'(x+2) - \phi_1'(x) > 0$. The conclusion holds for $h_1(x)$. Similarly, the result holds for $h_2(x)$. □

Lemma 2.4. *Let the function $\Theta(x) = x(\sqrt{1-\frac{2}{x+3}} + \sqrt{\frac{1}{3}} - \sqrt{\frac{1}{2}} - \sqrt{\frac{2}{3}}) + \sqrt{1-\frac{2}{x+2}}$, where $x \geq 5$. Then $\Theta(x)$ is monotonically increasing for $x \geq 5$.*

Proof. Let $\theta_1(x) = x(\sqrt{1-\frac{2}{x+3}} + \sqrt{\frac{1}{3}} - \sqrt{\frac{1}{2}} - \sqrt{\frac{2}{3}})$ and $\theta_2(x) = \sqrt{1-\frac{2}{x+2}}$ with $x \geq 5$. Then $\Theta(x) = \theta_1(x) + \theta_2(x)$. Clearly, $\theta_2(x)$ is monotonically increasing for $x \geq 5$. It suffices to prove that $\theta_1(x)$ is monotonically increasing for $x \geq 5$. Note that $\sqrt{\frac{1}{3}} - \sqrt{\frac{1}{2}} - \sqrt{\frac{2}{3}} \approx -0.9463 > -0.95$. Then

$$\theta_1{}'(x) = \frac{x^2+5x+3}{(x+1)^{\frac{1}{2}}(x+3)^{\frac{3}{2}}} + \sqrt{\frac{1}{3}} - \sqrt{\frac{1}{2}} - \sqrt{\frac{2}{3}} > \frac{x^2+5x+3}{(x+1)^{\frac{1}{2}}(x+3)^{\frac{3}{2}}} - 0.95.$$

We now prove $\frac{x^2+5x+3}{(x+1)^{\frac{1}{2}}(x+3)^{\frac{3}{2}}} > 0.95$, i.e., $(x^2+5x+3)^2 > 0.95^2(x+1)(x+3)^3$ for $x \geq 5$. Expanding gives $(x^2+5x+3)^2 - 0.95^2(x+1)(x+3)^3 = 0.0975x^4 + 0.975x^3 - 1.49x^2 - 18.735x - 15.3675$. Using Matlab, the largest real root of this quartic polynomial is less than 5. Thus $(x^2+5x+3)^2 > 0.95^2(x+1)(x+3)^3$ follows for $x \geq 5$. The conclusion holds. □

Lemma 2.5. *Let* $\Psi_1(x) = (x-1)\sqrt{1-\frac{2}{x+2}} - (x-3)\sqrt{1-\frac{2}{x+1}} - \sqrt{1-\frac{2}{x}}$, $\Psi_2(x) = (x-1)\sqrt{1-\frac{2}{x+2}} + \sqrt{1-\frac{2}{x+1}} - (x-3)\sqrt{1-\frac{2}{x}} - \sqrt{1-\frac{2}{x-1}}$ *with* $x \geq 5$. *Then* $\Psi_1(x)$, $\Psi_2(x)$ *are both monotonically increasing for* $x \geq 5$.

Proof. Let $\psi_1(x) = (x-2)\sqrt{1-\frac{2}{x+1}} + \sqrt{1-\frac{2}{x}}$, then $\Psi_1(x) = \psi_1(x+1) - \psi_1(x)$. Moreover,

$$\begin{aligned}\psi_1''(x) &= \frac{(5-2x)x-2}{(x-1)^{\frac{3}{2}}(x+1)^{\frac{5}{2}}} + \frac{2}{(x-1)^{\frac{1}{2}}(x+1)^{\frac{3}{2}}} - \frac{2}{x^{\frac{5}{2}}(x-2)^{\frac{1}{2}}} - \frac{1}{x^{\frac{5}{2}}(x-2)^{\frac{3}{2}}}\\ &= \frac{5x-4}{(x-1)^{\frac{3}{2}}(x+1)^{\frac{5}{2}}} - \frac{2x-3}{x^{\frac{5}{2}}(x-2)^{\frac{3}{2}}}\\ &= A - B, \end{aligned} \tag{1}$$

where $A = \frac{5x-4}{(x-1)^{\frac{3}{2}}(x+1)^{\frac{5}{2}}} > 0$, $B = \frac{2x-3}{x^{\frac{5}{2}}(x-2)^{\frac{3}{2}}} > 0$ for $x \geq 5$. Then

$$A^2 - B^2 = \frac{(5x-4)^2}{(x-1)^3(x+1)^5} - \frac{(2x-3)^2}{x^5(x-2)^3} = \frac{C}{x^5(x-2)^3(x-1)^3(x+1)^5},$$

where $C = 21x^{10} - 186x^9 + 579x^8 - 794x^7 + 458x^6 - 98x^5 + 64x^4 - 22x^3 - 38x^2 + 6x + 9$. We can write $C = x^8(21x^2 - 186x + 420) + x^6(159x^2 - 794x + 400) + x^5(58x - 98) + 22x^3(x-1) + x^2(42x^2 - 38) + 6x + 9 > 0$ for $x \geq 5$. By (1), we have $\psi_1''(x) = A - B > 0$. Hence $\Psi_1'(x) = \psi_1'(x+1) - \psi_1'(x) > 0$ for $x \geq 5$. The conclusion holds for $\Psi_1(x)$.

Let $\psi_2(x) = (x-3)\sqrt{1-\frac{2}{x}} + \sqrt{1-\frac{2}{x-1}}$, then $\Psi_2(x) = \psi_2(x+2) - \psi_2(x)$. Similarly, the conclusion holds for $\Psi_2(x)$. We complete the proof of this result. □

Note that $\sqrt{1-x} = \frac{1}{\sqrt{1+\frac{x}{1-x}}} \geq \frac{1}{1+\frac{x}{2(1-x)}} = 1 - \frac{x}{2-x}$ with $x < 1$ from Bernoulli's inequality. Then the following result holds immediately.

Lemma 2.6. *Let* $x < 1$, *then* $1 - \frac{x}{2-x} \leq \sqrt{1-x} \leq 1 - \frac{x}{2}$.

Lemma 2.7. *Let* $\Gamma_1(k) = 2\sqrt{1-\frac{2}{k+3}} + (k-3)\sqrt{1-\frac{2}{k+2}} - (k-3)\sqrt{1-\frac{2}{k+1}} - \sqrt{1-\frac{2}{k}}$ *with integer* $k \geq 4$, $\Gamma_2(k) = (k-1)\sqrt{1-\frac{2}{k+2}} - (k-2)\sqrt{1-\frac{2}{k+1}} + \sqrt{1-\frac{2}{k+3}} - \sqrt{1-\frac{2}{k}}$ *with integer* $k \geq 3$. *Then* $\Gamma_1(k) \leq \Gamma_1(4)$, $\Gamma_2(k) \leq \Gamma_2(3)$.

Proof. By using Matlab software, $\Gamma_1(k) \leq \Gamma_1(4)$ holds for $4 \leq k \leq 239$. For $k \geq 240$, by Lemma 2.6, we obtain

$$\begin{aligned}\Gamma_1(k) &\leq 2(1 - \frac{1}{k+3}) + (k-3)(1 - \frac{1}{k+2}) - (k-3)(1 - \frac{\frac{2}{k+1}}{2 - \frac{2}{k+1}}) - (1 - \frac{\frac{2}{k}}{2 - \frac{2}{k}})\\ &= 2 - \frac{2}{k+3} - \frac{k-3}{k+2} + \frac{k-3}{k} - 1 + \frac{1}{k-1}\\ &= 1 - \frac{2}{k+3} + \frac{5}{k+2} - \frac{3}{k} + \frac{1}{k-1}\\ &< 1 + \frac{5}{k+2} + \frac{1}{k-1}\\ &\leq 1 + \frac{5}{240+2} + \frac{1}{240-1}\\ &\approx 1.0248 < \Gamma_1(4) \approx 1.0251.\end{aligned}$$

Thus $\Gamma_1(k) \leq \Gamma_1(4)$ holds for $k \geq 4$. By using a similar method, $\Gamma_2(k) \leq \Gamma_2(3)$ holds for $k \geq 3$. □

Lemma 2.8 ([16]). *Let* uv *be a non-pendent edge of graph* G *with* $N_G(u) \bigcap N_G(v) = \emptyset$. *Let* $G^* = G - \{vw : w \in N_G(v) \setminus \{u\}\} + \{uw : w \in N_G(v) \setminus \{u\}\}$, *shown in Figure 1. Then* $ABS(G) < ABS(G^*)$.

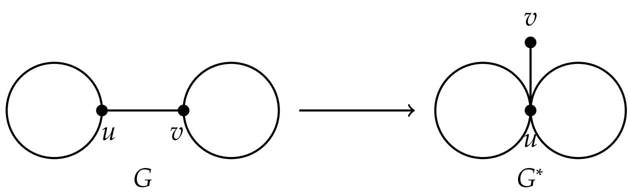


Figure 1: Graphs G and G^*.

3. The maximum ABS index of cacti among $\mathscr{C}(n,s)$

In this section, we determine the maximum ABS index of cacti of order n with s cycles and characterize the corresponding cacti. Let $G^+[n,s]$ be the cactus obtained from the star S_n by inserting s mutually independent edges, as shown in Figure 2.

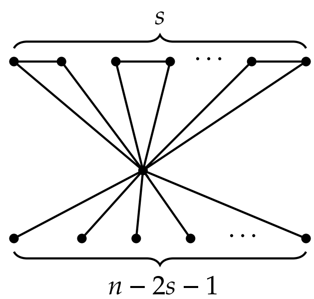


Figure 2: $G^+[n,s]$.

Lemma 3.1 ([3]). *Let $G \in \mathscr{C}(n,0)$ with $n \geq 2$. Then $ABS(G) \leq (n-1)\sqrt{1-\frac{2}{n}}$ with equality if and only if $G \cong S_n$.*

Lemma 3.2 ([5]). *Let $G \in \mathscr{C}(n,1)$ with $n \geq 4$. Then $ABS(G) \leq \frac{1}{\sqrt{2}} + 2\sqrt{1-\frac{2}{n+1}} + (n-3)\sqrt{1-\frac{2}{n}}$ with equality if and only if $G \cong G^+[n,1]$.*

Theorem 3.3. *Let $G \in \mathscr{C}(n,s)$. Then $ABS(G) \leq \frac{s}{\sqrt{2}} + 2s\sqrt{1-\frac{2}{n+1}} + (n-2s-1)\sqrt{1-\frac{2}{n}}$ with equality if and only if $G \cong G^+[n,s]$.*

Proof. We will prove the result by induction on $n+s$. The results hold for $s=0$ and $s=1$ by Lemmas 3.1 and 3.2, respectively. We consider $s \geq 2$ and $n \geq 5$. If $n=5$, then $\mathscr{C}(5,2) = \{G^+[5,2]\}$ and the result holds immediately. Let $F(n,s) = \frac{s}{\sqrt{2}} + 2s\sqrt{1-\frac{2}{n+1}} + (n-2s-1)\sqrt{1-\frac{2}{n}}$. In what follows, we assume $G \in \mathscr{C}(n,s)$ with $n \geq 6$ and $s \geq 2$. We will analyze the following two cases.

Case 1. $\mathcal{P}(G) \neq \emptyset$.

Let $v_1 \in \mathcal{P}(G)$ with $wv_1 \in E(G)$. Let $d_G(w) = r$ and $N_G(w) = \{v_1, v_2, \ldots, v_r\}$. Without loss of generality, we assume $N_G(w) \cap \mathcal{P}(G) = \{v_1, v_2, \ldots, v_k\}$ and $d_G(v_i) \geq 2$ for $k+1 \leq i \leq r$, where $k \geq 1$ and $r \leq n-1$. Let $G_1 = G - \{v_1, v_2, \ldots, v_k\}$. Then $G_1 \in \mathscr{C}(n-k,s)$. By Lemma 2.1 and inductive hypothesis, we have

$$\begin{aligned} ABS(G) &= ABS(G_1) + k\sqrt{1-\frac{2}{r+1}} + \sum_{i=k+1}^{r}\left[\sqrt{1-\frac{2}{d_G(v_i)+r}} - \sqrt{1-\frac{2}{d_G(v_i)+r-k}}\right] \\ &\leq ABS(G_1) + k\sqrt{1-\frac{2}{r+1}} + \sum_{i=k+1}^{r}\left(\sqrt{1-\frac{2}{2+r}} - \sqrt{1-\frac{2}{2+r-k}}\right) \end{aligned}$$

$$\leq F(n-k,s)+k\sqrt{1-\frac{2}{r+1}}+(r-k)\left(\sqrt{1-\frac{2}{r+2}}-\sqrt{1-\frac{2}{r+2-k}}\right)$$
$$=F(n-k,s)+g(r),$$

where $g(x)$ is defined in Lemma 2.2. By Lemma 2.2, we get $g(r)\leq g(n-1)$. Then

$$\begin{aligned}
ABS(G)&\leq F(n-k,s)+k\sqrt{1-\frac{2}{n}}+(n-1-k)\left(\sqrt{1-\frac{2}{n+1}}-\sqrt{1-\frac{2}{n-k+1}}\right)\\
&=F(n,s)-2s\sqrt{1-\frac{2}{n+1}}-(n-2s-1)\sqrt{1-\frac{2}{n}}+(n-2s-1-k)\sqrt{1-\frac{2}{n-k}}\\
&\quad+2s\sqrt{1-\frac{2}{n-k+1}}+k\sqrt{1-\frac{2}{n}}+(n-1-k)\left(\sqrt{1-\frac{2}{n+1}}-\sqrt{1-\frac{2}{n-k+1}}\right)\\
&=F(n,s)+(n-2s-1-k)\left(\sqrt{1-\frac{2}{n+1}}-\sqrt{1-\frac{2}{n}}\right)\\
&\quad-(n-2s-1-k)\left(\sqrt{1-\frac{2}{n-k+1}}-\sqrt{1-\frac{2}{n-k}}\right)\\
&=F(n,s)+(n-2s-1-k)[f(n)-f(n-k)] \qquad (2)
\end{aligned}$$

where $f(x)$ is defined in Lemma 2.1 with $a=1,b=0$. By Lemma 2.1 and (2), we deduce that $ABS(G)\leq F(n,s)$ with equality if and only if $d_G(v_i)=2$ for $i=k+1,\dots,r$ and $G_1\cong G^+[n-k,s]$ with $r=n-1$, $n-2s-1-k=0$, i.e., $G\cong G^+[n,s]$.

Case 2. $\mathcal{P}(G)=\emptyset$.

Let C be a cycle of G with $\{v_1,v_2,v_3\}\subseteq V(C)$ and $v_1v_2,v_2v_3\in E(C)$, $d_G(v_1)=r\geq 3$, $d_G(v_2)=d_G(v_3)=2$. We will classify the following two subcases by $|E(C)|$.

Subcase 2.1. $|E(C)|\geq 4$.

Let $G_1=G-v_2v_3+v_1v_3$. Then $G_1\in\mathscr{C}(n,s)$ and $\mathcal{P}(G_1)\neq\emptyset$. By Lemma 2.8 and the argument in Case 1, we have $ABS(G)<ABS(G_1)\leq F(n,s)$.

Subcase 2.2. $|E(C)|=3$.

Let $G_1=G-\{v_2,v_3\}$ and $N_G(v_1)\setminus\{v_2,v_3\}=\{u_1,u_2,\dots,u_{r-2}\}$, $d_G(u_i)\geq 2$ for $i=1,2,\dots,r-2$. Then $G_1\in\mathscr{C}(n-2,s-1)$. By Lemma 2.1 and inductive hypothesis, we have

$$\begin{aligned}
ABS(G)&=ABS(G_1)+\frac{1}{\sqrt{2}}+2\sqrt{1-\frac{2}{r+2}}+\sum_{i=1}^{r-2}\left(\sqrt{1-\frac{2}{d_G(u_i)+r}}-\sqrt{1-\frac{2}{d_G(u_i)+r-2}}\right)\\
&\leq ABS(G_1)+\frac{1}{\sqrt{2}}+2\sqrt{1-\frac{2}{r+2}}+\sum_{i=1}^{r-2}\left(\sqrt{1-\frac{2}{2+r}}-\sqrt{1-\frac{2}{2+r-2}}\right)\\
&\leq F(n-2,s-1)+\frac{1}{\sqrt{2}}+2\sqrt{1-\frac{2}{r+2}}+\sum_{i=1}^{r-2}\left(\sqrt{1-\frac{2}{2+r}}-\sqrt{1-\frac{2}{2+r-2}}\right)\\
&=F(n-2,s-1)+\frac{1}{\sqrt{2}}+r\sqrt{1-\frac{2}{r+2}}-(r-2)\sqrt{1-\frac{2}{r}}\\
&=F(n-2,s-1)+\frac{1}{\sqrt{2}}+h_1(r)
\end{aligned}$$

where $h_1(x)$ is defined in Lemma 2.3. By Lemma 2.3, we get $h_1(r)\leq h_1(n-1)$. Then

$$ABS(G)\leq F(n-2,s-1)+\frac{1}{\sqrt{2}}+(n-1)\sqrt{1-\frac{2}{n+1}}-(n-3)\sqrt{1-\frac{2}{n-1}}$$

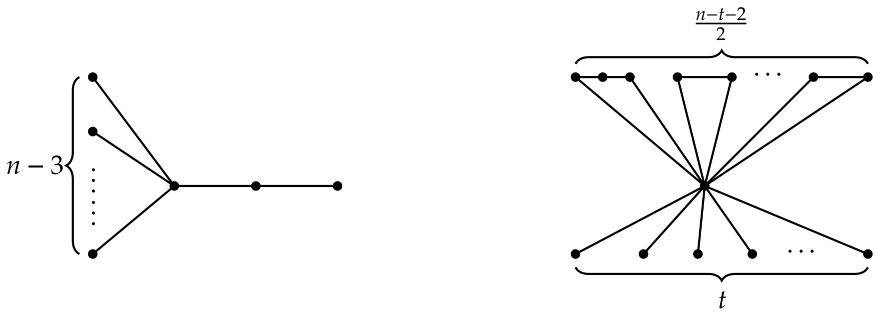
$$= F(n,s) - \frac{s}{\sqrt{2}} - 2s\sqrt{1-\frac{2}{n+1}} - (n-2s-1)\sqrt{1-\frac{2}{n}} + \frac{s-1}{\sqrt{2}} + 2(s-1)\sqrt{1-\frac{2}{n-1}}$$
$$+ (n-2s-1)\sqrt{1-\frac{2}{n-2}} + \frac{1}{\sqrt{2}} + (n-1)\sqrt{1-\frac{2}{n+1}} - (n-3)\sqrt{1-\frac{2}{n-1}}$$
$$= F(n,s) + (n-2s-1)\left[\left(\sqrt{1-\frac{2}{n+1}} - \sqrt{1-\frac{2}{n}}\right) - \left(\sqrt{1-\frac{2}{n-1}} - \sqrt{1-\frac{2}{n-2}}\right)\right]$$
$$\leq F(n,s) \text{ (by Lemma 2.1 with } a = 1, b = 0)$$

with equalities if and only if $d_G(u_1) = d_G(u_2) = \cdots = d_G(u_{r-2}) = 2$ and $G_1 \cong G^+[n-2, s-1]$ with $n-2s-1 = 0$, that is, $G \cong G^+[n,s]$. Thus we complete the proof. □

4. The maximum ABS index of cacti among $\mathscr{P}(n,t)$

In this section, we determine the maximum ABS index of cacti of order n with t pendent vertices and characterize the corresponding cacti. Denote by $S_{n,n-1}$ the graph obtained by attaching $n-3$ pendent edges to a leaf of P_3, shown in Figure 3 (a). Denote by $G^*[n,t]$ the cactus obtained from the star S_n by inserting $\frac{n-t-1}{2}$ mutually independent edges if $n-t$ is odd with $t \leq n-3$. It is clear that $G^*[n,t] \cong G^+[n, \frac{n-t-1}{2}]$, shown in Figure 2. Denote by $G^{**}[n,t]$ the cactus obtained from the star S_{n-1} by adding $\frac{n-t-2}{2}$ mutually independent edges if $n-t$ is even with $t \leq n-4$ such that one of the independent edges is subdivided, as shown in Figure 3 (b).

Lemma 4.1 ([13]). *Let $G \in \mathscr{P}(n, n-2)$ where $n \geq 4$. Then $ABS(G) \leq \frac{1}{\sqrt{3}} + \sqrt{1-\frac{2}{n}} + (n-3)\sqrt{1-\frac{2}{n}}$ with equality if and only if $G \cong S_{n,n-1}$.*



(a) $S_{n,n-1}$ (b) $G^{**}[n,t]$

Figure 3: Graphs $S_{n,n-1}$ and $G^{**}[n,t]$.

Theorem 4.2. *Let $G \in \mathscr{P}(n,t)$ where $n \geq 4$.*

(i) If $t = n-1$, then $ABS(G) = (n-1)\sqrt{1-\frac{2}{n}}$ and $G \cong S_n$.

(ii) If $t = n-2$, then $ABS(G) \leq \frac{1}{\sqrt{3}} + \sqrt{1-\frac{2}{n}} + (n-3)\sqrt{1-\frac{2}{n}}$ with equality if and only if $G \cong S_{n,n-1}$.

(iii) If $t \leq n-3$, then $ABS(G) \leq \frac{n-t-1}{2\sqrt{2}} + (n-t-1)\sqrt{1-\frac{2}{n+1}} + t\sqrt{1-\frac{2}{n}}$ where $n-t$ is odd with $t \leq n-3$, with equality if and only if $G \cong G^*[n,t]$; $ABS(G) \leq \frac{n-t}{2\sqrt{2}} + (n-t-2)\sqrt{1-\frac{2}{n}} + t\sqrt{1-\frac{2}{n-1}}$ where $n-t$ is even with $t \leq n-4$, with equality if and only if $G \cong G^{**}[n,t]$.

Proof. Since $\mathscr{P}(n, n-1) = \{S_n\}$, the result (i) holds. We can get the result (ii) by Lemma 4.1.

Next we prove the result (iii) by induction on $n + t$. If $n + t = 4$ for $n \geq 4$, then $n = 4, t = 0$. Since $\mathscr{P}(4, 0) = \{C_4\}$ and $C_4 \cong G^{**}[4, 0]$, the result holds for $n + t = 4$. We consider the case when $n + t \geq 5$ with $n - t$ being even. Let $H(n, t) = \frac{n-t}{2\sqrt{2}} + (n - t - 2)\sqrt{1 - \frac{2}{n}} + t\sqrt{1 - \frac{2}{n-1}}$ where $n - t$ is even with $t \leq n - 4$. We divide the proof into the following two cases.

Case 1. $\mathcal{P}(G) \neq \emptyset$.

Let $v_1 \in \mathcal{P}(G)$ with $wv_1 \in E(G)$. Denote $d_G(w) = r$ and $N_G(w) = \{v_1, v_2, \ldots, v_r\}$. Without loss of generality, we assume $N_G(w) \cap \mathcal{P}(G) = \{v_1, v_2, \ldots, v_k\}$ and $d_G(v_i) \geq 2$ for $k + 1 \leq i \leq r$, where $k \geq 1$. Let $G_1 = G - \{v_1, v_2, \ldots, v_k\}$, then $G_1 \in \mathscr{P}(n - k, t - k)$.

By Lemma 2.1 and inductive hypothesis, we have

$$\begin{aligned}
ABS(G) &= ABS(G_1) + k\sqrt{1 - \frac{2}{r+1}} + \sum_{i=k+1}^{r}\left[\sqrt{1 - \frac{2}{d_G(v_i) + r}} - \sqrt{1 - \frac{2}{d_G(v_i) + r - k}}\right] \\
&\leq ABS(G_1) + k\sqrt{1 - \frac{2}{r+1}} + (r - k)\left(\sqrt{1 - \frac{2}{2+r}} - \sqrt{1 - \frac{2}{2+r-k}}\right) \\
&\leq H(n-k, t-k) + k\sqrt{1 - \frac{2}{r+1}} + (r - k)\left(\sqrt{1 - \frac{2}{r+2}} - \sqrt{1 - \frac{2}{r+2-k}}\right). \qquad (3)
\end{aligned}$$

Since $n - t$ is even, then $n - k - (t - k)$ is even. For $r \leq n - 2$, applying Lemma 2.2 yields $g(r) = k\sqrt{1 - \frac{2}{r+1}} + (r - k)\left(\sqrt{1 - \frac{2}{r+2}} - \sqrt{1 - \frac{2}{r+2-k}}\right) \leq g(n - 2)$. By (3), we have

$$\begin{aligned}
ABS(G) &\leq H(n-k, t-k) + k\sqrt{1 - \frac{2}{n-1}} + (n - 2 - k)\left(\sqrt{1 - \frac{2}{n}} - \sqrt{1 - \frac{2}{n-k}}\right) \\
&= H(n, t) - (n - t - 2)\sqrt{1 - \frac{2}{n}} - t\sqrt{1 - \frac{2}{n-1}} + (n - t - 2)\sqrt{1 - \frac{2}{n-k}} \\
&\quad + (t - k)\sqrt{1 - \frac{2}{n-k-1}} + k\sqrt{1 - \frac{2}{n-1}} + (n - 2 - k)\left(\sqrt{1 - \frac{2}{n}} - \sqrt{1 - \frac{2}{n-k}}\right) \\
&= H(n, t) + (t - k)\left[\left(\sqrt{1 - \frac{2}{n}} - \sqrt{1 - \frac{2}{n-1}}\right) - \left(\sqrt{1 - \frac{2}{n-k}} - \sqrt{1 - \frac{2}{n-k-1}}\right)\right] \\
&\leq H(n, t) \text{ (by Lemma 2.1, with } a = 1, b = 0),
\end{aligned}$$

with equalities if and only if $d_G(v_{k+1}) = d_G(v_{k+2}) = \cdots = d_G(v_r) = 2$ and $G_1 \cong G^{**}[n - k, t - k]$ with $r = n - 2$, $t - k = 0$, that is, $G \cong G^{**}[n, t]$.

Case 2. $\mathcal{P}(G) = \emptyset$.

Let C be a cycle of G with $\{v_1, v_2, v_3\} \subseteq V(C)$ and $v_1v_2, v_2v_3 \in E(C)$, $d_G(v_1) = r \geq 3, d_G(v_2) = d_G(v_3) = 2$. We will classify the following two subcases by $|E(C)|$.

Subcase 2.1. $|E(C)| \geq 4$.

Let $G_1 = G - v_2 + v_1v_3$. Then $G_1 \in \mathscr{P}(n - 1, 0)$. Since $n - t = n - 0$ is even, then $n - 1$ is odd. By inductive hypothesis, we have $ABS(G_1) \leq \frac{n-2}{2\sqrt{2}} + (n - 2)\sqrt{1 - \frac{2}{n}}$. Thus

$$ABS(G) = ABS(G_1) + \frac{1}{\sqrt{2}} \leq \frac{n-2}{2\sqrt{2}} + (n - 2)\sqrt{1 - \frac{2}{n}} + \frac{1}{\sqrt{2}} = H(n, 0)$$

with equality if and only if $G_1 \cong G^{*}[n - 1, 0]$, that is, $G \cong G^{**}[n, 0]$.

Subcase 2.2. $|E(C)| = 3$.

Let $G_1 = G - \{v_2, v_3\}$ and $N_G(v_1) \setminus \{v_2, v_3\} = \{u_1, u_2, \ldots, u_{r-2}\}$, $d_G(u_i) \geq 2$ for $i = 1, 2, \ldots, r-2$. Then $G_1 \in \mathscr{P}(n-2, 0)$. Similarly as Subcase 2.2 in the proof of Theorem 3.3, the conclusion holds, which completes the proof when $n - t$ is even. Analogously, the conclusion holds when $n - t$ is odd. □

5. The maximum ABS index of cacti among $\mathscr{T}(2\beta, q)$

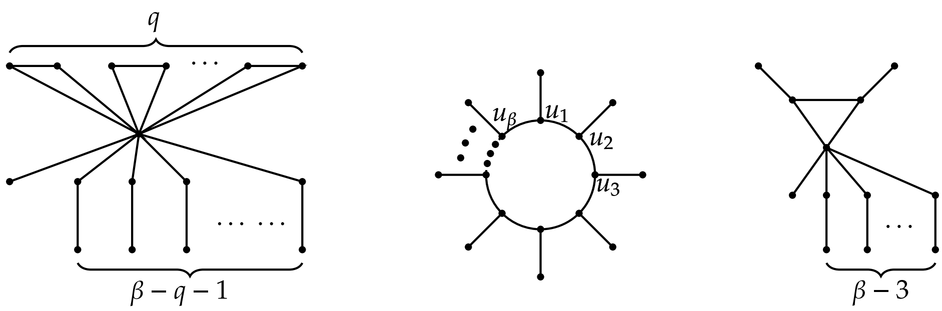
In this section, we determine the maximum ABS index of cacti of order 2β with a perfect matching M and q cycles, and characterize the corresponding cacti. Denote by $G'[2\beta, q]$ the cactus obtained by attaching q triangles and $(\beta - q - 1)$ pendent paths of length 2 to one vertex of P_2, as shown in Figure 4 (a). Denote by $Q_{2\beta}$ the cactus obtained by attaching a pendent vertex to each vertex of C_β, shown in Figure 4 (b). Denote by $R_{2\beta}$ the cactus obtained by attaching a pendent vertex to each vertex of C_3, and attaching $\beta - 3$ pendent paths of length 2 to one vertex of C_3, shown in Figure 4 (c). For convenience, we denote

$$\Phi(\beta, q) = ABS(G'[2\beta, q]) = (\beta + q - 1)\sqrt{1 - \frac{2}{\beta + q + 2}} + \sqrt{1 - \frac{2}{\beta + q + 1}} + (\beta - q - 1)\sqrt{\frac{1}{3}} + q\sqrt{\frac{1}{2}}. \tag{4}$$

Then, we get

$$\begin{aligned}
\Phi(\beta - 1, q - 1) &= (\beta + q - 3)\sqrt{1 - \frac{2}{\beta + q}} + \sqrt{1 - \frac{2}{\beta + q - 1}} + (\beta - q - 1)\sqrt{\frac{1}{3}} + (q - 1)\sqrt{\frac{1}{2}} \\
&= \Phi(\beta, q) - \Bigg[(\beta + q - 1)\sqrt{1 - \frac{2}{\beta + q + 2}} + \sqrt{1 - \frac{2}{\beta + q + 1}} \\
&\quad - (\beta + q - 3)\sqrt{1 - \frac{2}{\beta + q}} - \sqrt{1 - \frac{2}{\beta + q - 1}}\Bigg] - \sqrt{\frac{1}{2}} \\
&= \Phi(\beta, q) - \Psi_2(\beta + q) - \sqrt{\frac{1}{2}},
\end{aligned} \tag{5}$$

where $\Psi_2(x)$ is defined in Lemma 2.5.



(a) $G'[2\beta, q]$ (b) $Q_{2\beta}$ (c) $R_{2\beta}$

Figure 4: Graphs $G'[2\beta, q]$, $Q_{2\beta}$ and $R_{2\beta}$.

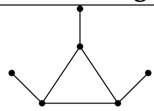
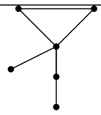
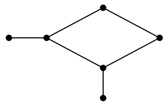
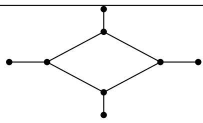
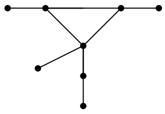
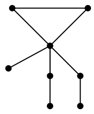
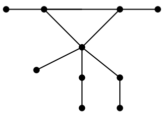
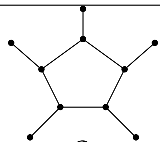
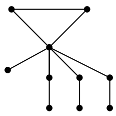
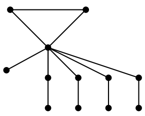
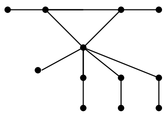
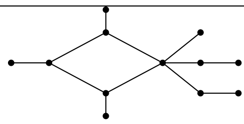
Since G contains a perfect matching, the following result holds immediately.

Lemma 5.1. *Let $G \in \mathscr{T}(2\beta, q)$ where $\beta \geq 1, q \geq 0$. For any $v \in V(G)$, we have $|N_G(v) \cap \mathcal{P}(G)| \leq 1$.*

Lemma 5.2 ([17]). *Let $G \in \mathscr{T}(2\beta, 0)$ where $\beta \geq 1$. Then $ABS(G) \leq \Phi(\beta, 0)$ with equality if and only if $G \cong G'[2\beta, 0]$.*

Using Python, we obtained the top three ABS indices of graphs $G \in \mathscr{T}(2\beta, 1)$ for $\beta \in \{3, 4, 5, 6\}$. The results are presented in Table 1. The lower right corner of each graph in Table 1 shows its approximate ABS index value. From Table 1, for each graph $G \in \mathscr{T}(2\beta, 1) \setminus \{Q_6, Q_8, R_8, R_{10}, Q_{10}\}$ with $\beta \in \{3, 4, 5, 6\}$, we have $ABS(G) \leq \Phi(\beta, 1)$ with equality if and only if $G \cong G'[2\beta, 1]$.

Table 1: The top three ABS indices of graphs among $\mathscr{T}(2\beta, 1)$ for $\beta \in \{3, 4, 5, 6\}$.

β	**The First Largest**	**The Second Largest**	**The Third Largest**
3	 Q_6 4.5708	 $G'[6, 1]$ 4.5085	 W_1 4.4870
4	 Q_8 6.0944	 R_8 6.0895	 $G'[8, 1]$ 6.0589
5	 R_{10} 7.6243	 Q_{10} 7.6180	 $G'[10, 1]$ 7.6144
6	 $G'[12, 1]$ 9.1740	 R_{12} 9.1698	 W_2 9.1479

Lemma 5.3. *Let $G \in \mathscr{T}(2\beta, 1)$ for $\beta \geq 2$. If $G \in \mathscr{T}(2\beta, 1) \setminus \{Q_6, Q_8, R_8, R_{10}, Q_{10}\}$, then $ABS(G) \leq \Phi(\beta, 1)$ with equality if and only if $G \cong G'[2\beta, 1]$.*

Proof. If $\beta = 2$, we have $\mathscr{T}(4, 1) = \{G'[4, 1], C_4\}$. Since $ABS(G'[4, 1]) > ABS(C_4)$, the conclusion holds for $\beta = 2$. From Table 1, the results hold for $\beta \in \{3, 4, 5, 6\}$. Now we consider $\beta \geq 7$. We assume that G has the maximum ABS index of cacti among $\mathscr{T}(2\beta, 1)$. Let $C_l = u_1u_2u_3 \ldots u_lu_1$ be the unique cycle of G. Then at least one of edges u_1u_2 and u_1u_l is not in M. Without loss of generality, we assume $u_1u_2 \notin M$. If $\mathcal{P}(G) = \emptyset$, then $G = C_{2\beta}$. Let $C' = C_{2\beta} - u_1u_2 + u_1u_3$, then $C' \in \mathscr{T}(2\beta, 1)$. By Lemma 2.8, we have $ABS(C_{2\beta}) < ABS(C')$. Thus $\mathcal{P}(G) \neq \emptyset$. We will divide into the following two cases.

Case 1. There is no support vertex of degree 2 in G.

By Lemma 5.1, G consists of a cycle C_l ($\beta \leq l < 2\beta$) together with some pendent vertices, each of which is attached to exactly one vertex of C_l. Consequently, every vertex of G has degree at most 3. If $\beta < l$, there exists at least one edge in $E(C_l) \bigcap M$. Without loss of generality, we assume $u_1u_l \in E(C_l) \bigcap M$. Let $G_1 = G - u_{l-1}u_l + u_1u_{l-1}$, then $G_1 \in \mathscr{T}(2\beta, 1)$ and $ABS(G_1) > ABS(G)$ by Lemma 2.8. Hence $\beta = l$. It indicates that each vertex of C_β is adjacent to one pendent vertex, i.e., $G \cong Q_{2\beta}$. Then $ABS(G) = \beta(\sqrt{\frac{1}{2}} + \sqrt{\frac{2}{3}})$. For $\beta \geq 7$, we obtain

$$\begin{aligned}\Phi(\beta, 1) - ABS(G) &= \beta\sqrt{1 - \frac{2}{\beta + 3}} + (\beta - 2)\sqrt{\frac{1}{3}} + \sqrt{1 - \frac{2}{\beta + 2}} + \sqrt{\frac{1}{2}} - \beta\left(\sqrt{\frac{1}{2}} + \sqrt{\frac{2}{3}}\right) \\ &= \beta\left(\sqrt{1 - \frac{2}{\beta + 3}} + \sqrt{\frac{1}{3}} - \sqrt{\frac{1}{2}} - \sqrt{\frac{2}{3}}\right) + \sqrt{1 - \frac{2}{\beta + 2}} - 2\sqrt{\frac{1}{3}} + \sqrt{\frac{1}{2}}\end{aligned}$$

$$= \Theta(\beta) - 2\sqrt{\frac{1}{3}} + \sqrt{\frac{1}{2}}, \tag{6}$$

where $\Theta(x)$ is defined in Lemma 2.4. By (6) and Lemma 2.4, we have $\Phi(\beta,1) - ABS(G) = \Theta(\beta) - 2\sqrt{\frac{1}{3}} + \sqrt{\frac{1}{2}} \geq \Theta(7) - 2\sqrt{\frac{1}{3}} + \sqrt{\frac{1}{2}} = 7\sqrt{\frac{4}{5}} + 5\sqrt{\frac{1}{3}} + \sqrt{\frac{7}{9}} - 6\sqrt{\frac{1}{2}} - 7\sqrt{\frac{2}{3}} \approx 0.0715 > 0$, a contradiction. The conclusion holds.

Case 2. There exists at least one support vertex of degree 2 in G.

We prove the conclusion by induction on $\beta \geq 6$. The conclusion holds for $\beta = 6$ by Table 1. We assume that v is a support vertex of degree 2 in G. Let $N_G(v) = \{u, w\}$ where u is a pendent vertex and w is a vertex of degree k. Since G contains a perfect matching, then $2 \leq k \leq \beta + 1$. Let $N_G(w) = \{v, w_1, w_2, \dots, w_{k-1}\}$ where $d_G(w_1) \geq 1$, $d_G(w_i) \geq 2$, $2 \leq i \leq k-1$. Let $G_2 = G - \{u, v\}$. Then $G_2 \in \mathscr{T}(2\beta - 2, 1)$. By Lemma 2.1 and $d_G(w_1) \geq 1$, $d_G(w_i) \geq 2$, $2 \leq i \leq k-1$, we obtain

$$\begin{aligned}
&\sum_{i=1}^{k-1}\left(\sqrt{1 - \frac{2}{k + d_G(w_i)}} - \sqrt{1 - \frac{2}{k + d_G(w_i) - 1}}\right) \\
&\leq \left(\sqrt{1 - \frac{2}{k+1}} - \sqrt{1 - \frac{2}{k}}\right) + (k-2)\left(\sqrt{1 - \frac{2}{k+2}} - \sqrt{1 - \frac{2}{k+1}}\right) \\
&= (k-2)\sqrt{1 - \frac{2}{k+2}} - (k-3)\sqrt{1 - \frac{2}{k+1}} - \sqrt{1 - \frac{2}{k}}
\end{aligned} \tag{7}$$

with equality if and only $d_G(w_1) = 1, d_G(w_i) = 2, 2 \leq i \leq k-1$. By (7) and inductive hypothesis, we have

$$\begin{aligned}
ABS(G) &= ABS(G_2) + \sqrt{\frac{1}{3}} + \sqrt{1 - \frac{2}{k+2}} + \sum_{i=1}^{k-1}\left(\sqrt{1 - \frac{2}{k + d_G(w_i)}} - \sqrt{1 - \frac{2}{k + d_G(w_i) - 1}}\right) \\
&\leq \Phi(\beta - 1, 1) + \sqrt{\frac{1}{3}} + \sqrt{1 - \frac{2}{k+2}} + (k-2)\sqrt{1 - \frac{2}{k+2}} - (k-3)\sqrt{1 - \frac{2}{k+1}} - \sqrt{1 - \frac{2}{k}} \\
&= \Phi(\beta - 1, 1) + \sqrt{\frac{1}{3}} + (k-1)\sqrt{1 - \frac{2}{k+2}} - (k-3)\sqrt{1 - \frac{2}{k+1}} - \sqrt{1 - \frac{2}{k}} \\
&= \Phi(\beta - 1, 1) + \sqrt{\frac{1}{3}} + \Psi_1(k),
\end{aligned} \tag{8}$$

where $\Psi_1(x)$ is defined in Lemma 2.5. Applying Lemma 2.5 yields $\Psi_1(k) \leq \Psi_1(\beta + 1)$ for $k \in \{5, 6, \dots, \beta + 1\}$. Since $\Psi_1(4) < \Psi_1(3) < \Psi_1(5)$, it follows that $\Psi_1(k) \leq \Psi_1(\beta + 1)$ for $k \in \{3, 4, \dots, \beta + 1\}$. Hence, by (8), $ABS(G) \leq \Phi(\beta - 1, 1) + \sqrt{\frac{1}{3}} + \Psi_1(\beta + 1) = \Phi(\beta, 1)$ with equality if and only if $G_2 \cong G'[2\beta - 2, 1], k = \beta + 1, d_G(w_1) = 1, d_G(w_2) = \cdots = d_G(w_\beta) = 2$, i.e., $G \cong G'[2\beta, 1]$. We complete the proof. □

Lemma 5.4. *Let $G \in \mathscr{T}(2\beta, q) \setminus \{W_8, Z_{10}\}$ for $\beta \geq 3, q \geq 2$. The graphs W_8 and Z_{10} are depicted in Figure 5. Then $ABS(G) \leq \Phi(\beta, q)$ with equality if and only if $G \cong G'[2\beta, q]$.*

Figure 5: The graphs W_8 and Z_{10}.

Proof. We prove the conclusion by induction on $\beta+q$. We know that $\mathscr{T}(6,2)=\{G'[6,2],G_{1a},G_{1b},G_{1c}\}$, shown in Figure 6. It is straightforward to check that $ABS(G)\le\Phi(3,2)$ with equality if and only if $G\cong G'[6,2]$ for $G\in\mathscr{T}(6,2)$. The conclusion holds for $\beta=3,q=2$. Note that $ABS(W_8)\approx 7.1767>\Phi(4,2)\approx 7.1688$ and $ABS(Z_{10})\approx 8.7420>\Phi(5,2)\approx 8.7264$. We assume $G\in\mathscr{T}(2\beta,q)\setminus\{W_8,Z_{10}\}$ for $\beta\ge 4$, $q\ge 2$. We divide into the following two cases.

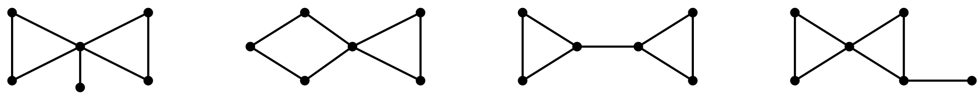


Figure 6: $\mathscr{T}(6,2)$.

Case 1. $\mathcal{P}(G)=\emptyset$.

We assume that the cycle $C_l=v_1v_2\dots v_lv_1$ of G such that $3\le d_G(v_1)=k\le\beta+q-1, d_G(v_2)=\cdots=d_G(v_l)=2$. Let $N_G(v_1)=\{v_2,v_l,w_1,w_2,\dots,w_{k-2}\}$, $d_G(w_i)\ge 2$ for $1\le i\le k-2$. Then at least one of v_1v_2 and v_1v_l is not in M. Without loss of generality, assume that $v_1v_2\notin M$. Let $G_1=G-v_1v_2$, then $G_1\in\mathscr{T}(2\beta,q-1)\setminus\{Q_6,Q_8,R_8,R_{10},Q_{10},W_8,Z_{10}\}$. By inductive hypothesis and Lemma 2.1, we have

$$\begin{aligned}
ABS(G)&=ABS(G_1)+\sqrt{\frac12}-\sqrt{\frac13}+\sqrt{1-\frac{2}{k+2}}+\sum_{i=1}^{k-2}\left(\sqrt{1-\frac{2}{k+d_G(w_i)}}-\sqrt{1-\frac{2}{k+d_G(w_i)-1}}\right)\\
&\le\Phi(\beta,q-1)+\sqrt{\frac12}-\sqrt{\frac13}+\sqrt{1-\frac{2}{k+2}}+(k-2)\left(\sqrt{1-\frac{2}{k+2}}-\sqrt{1-\frac{2}{k+1}}\right)\\
&=\Phi(\beta,q-1)+\sqrt{\frac12}-\sqrt{\frac13}+(k-1)\sqrt{1-\frac{2}{k+2}}-(k-2)\sqrt{1-\frac{2}{k+1}}\\
&=\Phi(\beta,q-1)+\sqrt{\frac12}-\sqrt{\frac13}+h_2(k),
\end{aligned}\tag{9}$$

where $h_2(x)$ is defined in Lemma 2.3. Applying Lemma 2.3 yields $h_2(k)\le h_2(\beta+q-1)<h_2(\beta+q)$. Then

$$\begin{aligned}
ABS(G)&<\Phi(\beta,q-1)+\sqrt{\frac12}-\sqrt{\frac13}+h_2(\beta+q)\\
&=\Phi(\beta,q-1)+\sqrt{\frac12}-\sqrt{\frac13}+(\beta+q-1)\sqrt{1-\frac{2}{\beta+q+2}}-(\beta+q-2)\sqrt{1-\frac{2}{\beta+q+1}}\\
&=\Phi(\beta,q)+\sqrt{1-\frac{2}{\beta+q}}-\sqrt{1-\frac{2}{\beta+q+1}}\\
&<\Phi(\beta,q)
\end{aligned}$$

Case 2. $\mathcal{P}(G)\neq\emptyset$.

We divide into the following two subcases.

Subcase 2.1. There exists at least one support vertex of degree 2 in G.

We assume that v is a support vertex of degree 2 in G. Let $N_G(v)=\{u,w\}$ where u is a pendent vertex and w is a vertex of degree k, $k\ge 2$. Let $G_2=G-\{u,v\}$, then $G_2\in\mathscr{T}(2\beta-2,q)$. Let $G_{2a},G_{2b},G_{2c},G_{2d},G_{2e}$ be the graphs obtained from W_8 by attaching a pendent path of length 2 to the vertices v_1,v_2,v_4,v_5,v_7 of W_8, respectively. The graph G_{2a} is depicted in the Figure 7. If $G_2\cong W_8$, then $G\cong G_{2i}, i=a,b,c,d,e$. It is straightforward to check that $ABS(G_{2i})<\Phi(5,2)$, $i=a,b,c,d,e$. Hence the conclusion holds for $G_2\cong W_8$. Similarly, the conclusion holds for $G_2\cong Z_{10}$. If $G_2\in\mathscr{T}(2\beta-2,q)\setminus\{W_8,Z_{10}\}$, similarly as Case 2 in the proof of Lemma 5.3, the conclusion holds.

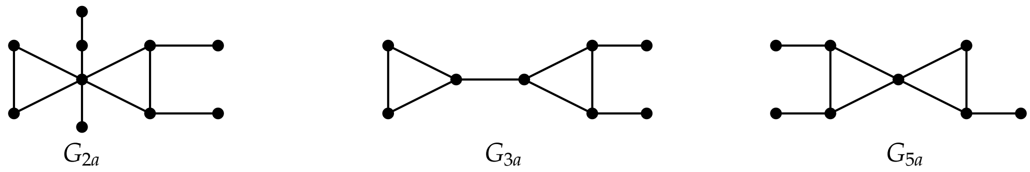


Figure 7: The graphs G_{2a}, G_{3a} and G_{5a}.

Subcase 2.2. There is no support vertex of degree 2 in G.

We can assume that G contains a cycle $C_l = v_1v_2 \ldots v_lv_1$ such that $3 \le d_G(v_1) = k \le \beta + q$ and $d_G(v_i) \in \{2,3\}, i = 2,3,\ldots,l$. If $d_G(v_i) = 3$ for $i = 2,3,\ldots,l$, then v_i is adjacent to a pendent vertex. Let $N_G(v_1) = \{v_2, v_l, w_1, w_2, \ldots, w_{k-2}\}$ where $d_G(w_1) \ge 1$ and $d_G(w_i) \ge 2$ for $i = 2,\ldots,k-2$. We divide into the following two subcases.

Subcase 2.2.1. $l = 3$.

(I) If $d_G(v_2) = d_G(v_3) = 2$, then $v_2v_3 \in M$. Let $G_3 = G - \{v_2, v_3\}$, then $G_3 \in \mathscr{T}(2\beta - 2, q - 1)$. If $G_3 \cong Q_6$, we have $\beta = 4, q = 2$ and $G \cong G_{3a}$ or $G \cong W_8$ where G_{3a} and W_8 are depicted in Figure 7 and Figure 5, respectively. Note that $G \in \mathscr{T}(2\beta, q) \setminus \{W_8, Z_{10}\}$. Since $ABS(G_{3a}) \approx 6.9365 < \Phi(4,2) \approx 7.1688$, the conclusion holds for $G_3 \cong Q_6$. Similarly, we can obtain that the conclusions hold for $G_3 \cong Q_8$, $G_3 \cong R_8$, $G_3 \cong Q_{10}$, $G_3 \cong W_8$ and $G_3 \cong Z_{10}$. Note that $G_3 \not\cong R_{10}$. Next we consider $G_3 \in \mathscr{T}(2\beta - 2, q - 1) \setminus \{Q_6, Q_8, R_8, Q_{10}, R_{10}, W_8, Z_{10}\}$. By Lemma 2.1 and $d_G(w_1) \ge 1, d_G(w_i) \ge 2, 2 \le i \le k-2$, we obtain

$$\begin{aligned}
&\sum_{i=1}^{k-2}\left(\sqrt{1-\frac{2}{k+d_G(w_i)}}-\sqrt{1-\frac{2}{k+d_G(w_i)-2}}\right)\\
&\le\left(\sqrt{1-\frac{2}{k+1}}-\sqrt{1-\frac{2}{k-1}}\right)+(k-3)\left(\sqrt{1-\frac{2}{k+2}}-\sqrt{1-\frac{2}{k}}\right),
\end{aligned}\tag{10}$$

with equality if and only $d_G(w_1) = 1, d_G(w_i) = 2, 2 \le i \le k-2$. By (10) and inductive hypothesis, we have

$$\begin{aligned}
ABS(G) &= ABS(G_3) + \sqrt{\frac{1}{2}} + 2\sqrt{1-\frac{2}{k+2}} + \sum_{i=1}^{k-2}\left(\sqrt{1-\frac{2}{k+d_G(w_i)}}-\sqrt{1-\frac{2}{k+d_G(w_i)-2}}\right)\\
&\le \Phi(\beta-1, q-1) + \sqrt{\frac{1}{2}} + 2\sqrt{1-\frac{2}{k+2}} + \left(\sqrt{1-\frac{2}{k+1}}-\sqrt{1-\frac{2}{k-1}}\right)\\
&\quad + (k-3)\left(\sqrt{1-\frac{2}{k+2}}-\sqrt{1-\frac{2}{k}}\right)\\
&= \Phi(\beta-1, q-1) + \sqrt{\frac{1}{2}} + (k-1)\sqrt{1-\frac{2}{k+2}} + \sqrt{1-\frac{2}{k+1}}\\
&\quad - (k-3)\sqrt{1-\frac{2}{k}} - \sqrt{1-\frac{2}{k-1}}\\
&= \Phi(\beta-1, q-1) + \sqrt{\frac{1}{2}} + \Psi_2(k),
\end{aligned}\tag{11}$$

where $\Psi_2(x)$ is defined in Lemma 2.5. By Lemma 2.5, (11) and (5), we get $ABS(G) \le \Phi(\beta-1, q-1) + \sqrt{\frac{1}{2}} + \Psi_2(\beta + q) = \Phi(\beta, q)$ with equality if and only if $G_3 \cong G'[2\beta-2, q-1]$, $k = \beta + q$, $d_G(w_1) = 1, d_G(w_2) = \ldots = d_G(w_l) = 2$, i.e., $G \cong G'[2\beta, q]$. The conclusion holds.

(II) If $d_G(v_2) = d_G(v_3) = 3$, we denote by $\mathscr{T}^*(2\beta, q)$ the subset of $\mathscr{T}(2\beta, q)$ consisting of cacti satisfying this condition. If $\beta + q = 6$, then $\beta = 4, q = 2$. It is easy to check that W_8 and G_{5a}, shown in Figure 5 and Figure 7, have the maximum and second maximum ABS indices of cacti among $\mathscr{T}^*(8,2)$, respectively. If $\beta + q = 7$, then $\beta = 5, q = 2$. We derive that Z_{10} and G_{6b}, shown in Figure 5 and Figure 9, have the maximum and second maximum ABS indices of cacti among $\mathscr{T}^*(10,2)$, respectively. It is straightforward to check $ABS(G_{5a}) < \Phi(4,2) < ABS(W_8)$ and $ABS(G_{6b}) < \Phi(5,2) < ABS(Z_{10})$. Then $ABS(G) < \Phi(\beta,q)$ for $G \in \mathscr{T}^*(2\beta,q) \setminus \{W_8, Z_{10}\}$ with $\beta + q \in \{6,7\}$. If $\beta + q = 8$, then $\beta = 6, q = 2$ or $\beta = 5, q = 3$. If $\beta = 6, q = 2$, then G_{4a}, shown in Figure 8, has the maximum ABS index of cacti among $\mathscr{T}^*(12,2)$. If $\beta = 5, q = 3$, it is easy to check that G_{4b}, shown in Figure 8, has the the maximum ABS index of cacti among $\mathscr{T}^*(10,3)$. Since $ABS(G_{4a}) \approx 10.2656 < \Phi(6,2) \approx 10.2892$, $ABS(G_{4b}) \approx 9.8275 < \Phi(5,3) \approx 9.8416$, we have $ABS(G) < \Phi(\beta,q)$ for $G \in \mathscr{T}^*(2\beta,q)$ with $\beta + q = 8$. Thus the conclusion holds for $\beta + q \in \{6,7,8\}$.

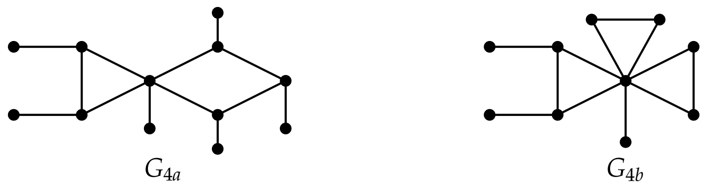


Figure 8: The graphs G_{4a} and G_{4b}.

Next we consider $\beta + q \geq 9$. Since $d_G(v_2) = d_G(v_3) = 3$, there exist two pendent vertices, v_2' and v_3', such that $v_2v_2' \in M$ and $v_3v_3' \in M$. If $d_G(v_1) = k = 3$, there exists a vertex $u \in N_G(v_1) \setminus \{v_2, v_3\}$ such that $v_1u \in M$. Let $G' = G - \{uu_i : u_i \in N_G(u) \setminus \{v_1\}\} + \{v_1u_i : u_i \in N_G(u) \setminus \{v_1\}\}$, then $d_{G'}(v_1) \geq 4$ and $ABS(G') > ABS(G)$ by Lemma 2.8. Therefore, to maximize the ABS index of G, we may assume $d_G(v_1) = k \geq 4$. Let $G_4 = G - \{v_3, v_3'\}$, then $G_4 \in \mathscr{T}(2\beta - 2, q-1) \setminus \{Q_6, Q_8, R_8, R_{10}, Q_{10}, W_8, Z_{10}\}$. Recall that $N_G(v_1) = \{v_2, v_l, w_1, w_2, \ldots, w_{k-2}\}$, $d_G(w_1) \geq 1$, $d_G(w_i) \geq 2$ for $2 \leq i \leq k-2$. By Lemma 2.1, we obtain

$$\begin{aligned}
&\sum_{i=1}^{k-2}\left(\sqrt{1-\frac{2}{k+d_G(w_i)}}-\sqrt{1-\frac{2}{k+d_G(w_i)-1}}\right)\\
&\leq \left(\sqrt{1-\frac{2}{k+1}}-\sqrt{1-\frac{2}{k}}\right)+(k-3)\left(\sqrt{1-\frac{2}{k+2}}-\sqrt{1-\frac{2}{k+1}}\right) \qquad (12)\\
&= (k-3)\sqrt{1-\frac{2}{k+2}}-(k-4)\sqrt{1-\frac{2}{k+1}}-\sqrt{1-\frac{2}{k}},
\end{aligned}$$

with equality holding if and only $d_G(w_1) = 1$, $d_G(w_i) = 2, 2 \leq i \leq k-2$. By (12) and inductive hypothesis, we obtain

$$\begin{aligned}
ABS(G) &= ABS(G_4)+\sum_{i=1}^{k-2}\left(\sqrt{1-\frac{2}{k+d_G(w_i)}}-\sqrt{1-\frac{2}{k+d_G(w_i)-1}}\right)\\
&\quad +2\sqrt{1-\frac{2}{k+3}}-\sqrt{1-\frac{2}{k+1}}+\sqrt{\frac{2}{3}}+2\sqrt{\frac{1}{2}}-\sqrt{\frac{1}{3}}\\
&\leq \Phi(\beta-1,q-1)+(k-3)\sqrt{1-\frac{2}{k+2}}-(k-4)\sqrt{1-\frac{2}{k+1}}-\sqrt{1-\frac{2}{k}}\\
&\quad +2\sqrt{1-\frac{2}{k+3}}-\sqrt{1-\frac{2}{k+1}}+\sqrt{\frac{2}{3}}+2\sqrt{\frac{1}{2}}-\sqrt{\frac{1}{3}}\\
&= \Phi(\beta-1,q-1)+2\sqrt{1-\frac{2}{k+3}}+(k-3)\sqrt{1-\frac{2}{k+2}}-(k-3)\sqrt{1-\frac{2}{k+1}}\\
&\quad -\sqrt{1-\frac{2}{k}}+\sqrt{\frac{2}{3}}+2\sqrt{\frac{1}{2}}-\sqrt{\frac{1}{3}}
\end{aligned}$$

$$= \Phi(\beta-1,q-1)+\Gamma_1(k)+\sqrt{\frac{2}{3}}+2\sqrt{\frac{1}{2}}-\sqrt{\frac{1}{3}}, \tag{13}$$

where $\Gamma_1(k)$ is defined in Lemma 2.7. By (13) and Lemma 2.7, we have

$$\begin{aligned}
ABS(G) &\le \Phi(\beta-1,q-1)+\Gamma_1(4)+\sqrt{\frac{2}{3}}+2\sqrt{\frac{1}{2}}-\sqrt{\frac{1}{3}}\\
&= \Phi(\beta,q)-\Psi_2(\beta+q)+\Gamma_1(4)+\sqrt{\frac{2}{3}}+\sqrt{\frac{1}{2}}-\sqrt{\frac{1}{3}} \quad \text{(by (5))}\\
&\le \Phi(\beta,q)-\Psi_2(9)+\Gamma_1(4)+\sqrt{\frac{2}{3}}+\sqrt{\frac{1}{2}}-\sqrt{\frac{1}{3}} \quad \text{(by Lemma 2.5)}\\
&\le \Phi(\beta,q)+6\sqrt{\frac{7}{9}}+2\sqrt{\frac{5}{7}}+\sqrt{\frac{3}{4}}+2\sqrt{\frac{2}{3}}-8\sqrt{\frac{9}{11}}-\sqrt{\frac{4}{5}}-\sqrt{\frac{3}{5}}-\sqrt{\frac{1}{3}}\\
&\approx \Phi(\beta,q)-0.0018<\Phi(\beta,q).
\end{aligned}$$

The conclusion holds.

(III) If $d_G(v_2)=2, d_G(v_3)=3$ or $d_G(v_2)=3, d_G(v_3)=2$, without loss of generality, we assume that $d_G(v_2)=2, d_G(v_3)=3$. Then there exists a pendent vertex $v_3^{'}$ such that $v_3v_3^{'}\in M$. Let $G_5=G-\{v_3,v_3^{'}\}$, then $G_5\in\mathscr{T}(2\beta-2,q-1)$. If $G_5\cong Q_6$, we have $G\cong G_{5a}$, shown in Figure 7. It is easy to check that $ABS(G_{5a})<\Phi(4,2)$. The conclusion holds for $G_5\cong Q_6$. Similarly, we can obtain that the conclusions hold for $G_5\cong Q_8$, $G_5\cong R_8$, $G_5\cong Q_{10}$, $G_5\cong W_8$ and $G_5\cong Z_{10}$. Note that $G_5\ncong R_{10}$. Next we consider $G_5\in\mathscr{T}(2\beta-2,q-1)\setminus\{Q_6,Q_8,R_8,Q_{10},R_{10},W_8,Z_{10}\}$. By inductive hypothesis, we obtain

$$\begin{aligned}
ABS(G) &= ABS(G_5)+\sum_{i=1}^{k-2}\left(\sqrt{1-\frac{2}{k+d_G(w_i)}}-\sqrt{1-\frac{2}{k+d_G(w_i)-1}}\right)\\
&\quad+\sqrt{1-\frac{2}{k+3}}+\sqrt{1-\frac{2}{k+2}}-\sqrt{1-\frac{2}{k}}+\sqrt{\frac{1}{2}}+\sqrt{\frac{3}{5}}\\
&\le ABS(G_5)+(k-2)\left(\sqrt{1-\frac{2}{k+2}}-\sqrt{1-\frac{2}{k+1}}\right) \quad \text{(by Lemma 2.1)}\\
&\quad+\sqrt{1-\frac{2}{k+3}}+\sqrt{1-\frac{2}{k+2}}-\sqrt{1-\frac{2}{k}}+\sqrt{\frac{1}{2}}+\sqrt{\frac{3}{5}}\\
&\le \Phi(\beta-1,q-1)+\left[(k-1)\sqrt{1-\frac{2}{k+2}}-(k-2)\sqrt{1-\frac{2}{k+1}}\right.\\
&\quad\left.+\sqrt{1-\frac{2}{k+3}}-\sqrt{1-\frac{2}{k}}\right]+\sqrt{\frac{1}{2}}+\sqrt{\frac{3}{5}}\\
&= \Phi(\beta-1,q-1)+\Gamma_2(k)+\sqrt{\frac{1}{2}}+\sqrt{\frac{3}{5}},
\end{aligned} \tag{14}$$

where $\Gamma_2(k)$ is defined in Lemma 2.7. By (14) and Lemma 2.7, we deduce

$$\begin{aligned}
ABS(G) &\le \Phi(\beta-1,q-1)+\Gamma_2(3)+\sqrt{\frac{1}{2}}+\sqrt{\frac{3}{5}}\\
&= \Phi(\beta,q)-\Psi_2(\beta+q)+\Gamma_2(3)+\sqrt{\frac{3}{5}} \quad \text{(by (5))}\\
&\le \Phi(\beta,q)-\Psi_2(6)+\Gamma_2(3)+\sqrt{\frac{3}{5}} \quad \text{(by Lemma 2.5)}
\end{aligned}$$

$$= \Phi(\beta,q) + 4\sqrt{\frac{3}{5}} + 4\sqrt{\frac{2}{3}} - 5\sqrt{\frac{3}{4}} - \sqrt{\frac{5}{7}} - \sqrt{\frac{1}{2}} - \sqrt{\frac{1}{3}}$$
$$\approx \Phi(\beta,q) - 0.0954 < \Phi(\beta,q).$$

The conclusion holds.

Subcase 2.2.2. $l \geq 4$.

(I) If $d_G(v_2) = 3$, there exists a pendent vertex v_2' such that $v_2v_2' \in M$. Let $G_6 = G - \{v_2, v_2'\} + \{v_1v_3\}$, then $G_6 \in \mathscr{T}(2\beta-2,q)$. If $G_6 \cong W_8$, then $\beta = 5, q = 2$. Thus $G \cong G_{6a}$ or $G \cong G_{6b}$, shown in Figure 9. Since $ABS(G_{6b}) \approx 8.6792 < ABS(G_{6a}) \approx 8.7003 < \Phi(5,2) \approx 8.7264$, the conclusion holds for $G_6 \cong W_8$. Similarly, the conclusion holds for $G_6 \cong Z_{10}$. Next we consider $G_6 \in \mathscr{T}(2\beta-2,q) \setminus \{W_8, Z_{10}\}$. By inductive hypothesis, we

Figure 9: The graphs G_{6a} and G_{6b}.

obtain

$$\begin{aligned}
ABS(G) &= ABS(G_6) + \sqrt{1-\frac{2}{k+3}} - \sqrt{1-\frac{2}{k+d_G(v_3)}} + \sqrt{1-\frac{2}{d_G(v_3)+3}} + \sqrt{\frac{1}{2}}\\
&\leq \Phi(\beta-1,q) + \sqrt{1-\frac{2}{k+3}} - \sqrt{1-\frac{2}{k+d_G(v_3)}} + \sqrt{1-\frac{2}{d_G(v_3)+3}} + \sqrt{\frac{1}{2}}\\
&= \Phi(\beta,q) + \sqrt{1-\frac{2}{k+3}} - \sqrt{1-\frac{2}{k+d_G(v_3)}} + \sqrt{1-\frac{2}{d_G(v_3)+3}} + \sqrt{\frac{1}{2}} - \sqrt{\frac{1}{3}}\\
&\quad - \left[(\beta+q-1)\sqrt{1-\frac{2}{\beta+q+2}} - (\beta+q-3)\sqrt{1-\frac{2}{\beta+q+1}} - \sqrt{1-\frac{2}{\beta+q}}\right]\\
&= \Phi(\beta,q) + \sqrt{1-\frac{2}{k+3}} - \sqrt{1-\frac{2}{k+d_G(v_3)}} \qquad (15)\\
&\quad + \sqrt{1-\frac{2}{d_G(v_3)+3}} + \sqrt{\frac{1}{2}} - \sqrt{\frac{1}{3}} - \Psi_1(\beta+q)
\end{aligned}$$

where $\Psi_1(x)$ is defined in Lemma 2.5. By Lemma 2.5 and (15), we have

$$ABS(G) \leq \Phi(\beta,q) + \sqrt{1-\frac{2}{k+3}} - \sqrt{1-\frac{2}{k+d_G(v_3)}} + \sqrt{1-\frac{2}{d_G(v_3)+3}} + \sqrt{\frac{1}{2}} - \sqrt{\frac{1}{3}} - \Psi_1(6). \qquad (16)$$

If $d_G(v_3) = 3$, by (16), then $ABS(G) \leq \Phi(\beta,q) + \sqrt{\frac{2}{3}} + \sqrt{\frac{1}{2}} - \sqrt{\frac{1}{3}} - \Psi_1(6) = \Phi(\beta,q) + 3\sqrt{\frac{5}{7}} + 2\sqrt{\frac{2}{3}} + \sqrt{\frac{1}{2}} - 5\sqrt{\frac{3}{4}} - \sqrt{\frac{1}{3}} \approx \Phi(\beta,q) - 0.0319 < \Phi(\beta,q)$. By Lemma 2.1 and $k \geq 3$, taking $a = 3, b = 2$, we have $\sqrt{1-\frac{2}{k+3}} - \sqrt{1-\frac{2}{k+2}} \leq \sqrt{1-\frac{2}{6}} - \sqrt{1-\frac{2}{5}} = \sqrt{\frac{2}{3}} - \sqrt{\frac{3}{5}}$. If $d_G(v_3) = 2$, by (16), $ABS(G) \leq \Phi(\beta,q) + \sqrt{1-\frac{2}{k+3}} - \sqrt{1-\frac{2}{k+2}} + \sqrt{\frac{3}{5}} + \sqrt{\frac{1}{2}} - \sqrt{\frac{1}{3}} - \Psi_1(6) \leq \Phi(\beta,q) + \sqrt{\frac{2}{3}} - \sqrt{\frac{3}{5}} + \sqrt{\frac{3}{5}} + \sqrt{\frac{1}{2}} - \sqrt{\frac{1}{3}} - \Psi_1(6) = \Phi(\beta,q) + 3\sqrt{\frac{5}{7}} + 2\sqrt{\frac{2}{3}} + \sqrt{\frac{1}{2}} - 5\sqrt{\frac{3}{4}} - \sqrt{\frac{1}{3}} \approx \Phi(\beta,q) - 0.0319 < \Phi(\beta,q)$.

(II) If $d_G(v_2) = 2$, then at least one of v_1v_2 and v_1v_l is not in M. Suppose that $v_1v_2 \notin M$. Thus $v_2v_3 \in M$ and $d_G(v_3) = 2$. Let $G_7 = G - v_3v_4 + v_2v_4$, then $G_7 \in \mathscr{T}(2\beta, q)$ and $d_{G_7}(v_2) = 3$. By Lemma 2.8, we obtain $ABS(G) < ABS(G_7)$. If $l - 1 = 3$, according to the argument in Subcase 2.2.1, we have $ABS(G) < ABS(G_7) \leq \Phi(\beta, q)$. If $l - 1 \geq 4$, according to the argument in Subcase 2.2.2 (I), we have $ABS(G) < ABS(G_7) < \Phi(\beta, q)$. We complete the proof. □

By Lemmas 5.2, 5.3 and 5.4, together with Table 1, we obtain our conclusion as follows:

Theorem 5.5. *Let $G \in \mathscr{T}(2\beta, q)$ where $\beta \geq 1$, $q \geq 0$.*

(i) If $\beta = 3$, $q = 1$, then $ABS(G) \leq 3\left(\sqrt{\frac{2}{3}} + \sqrt{\frac{1}{2}}\right)$ with equality if and only if $G \cong Q_6$.

(ii) If $\beta = 4$, $q = 1$, then $ABS(G) \leq 4\left(\sqrt{\frac{2}{3}} + \sqrt{\frac{1}{2}}\right)$ with equality if and only if $G \cong Q_8$.

(iii) If $\beta = 5$, $q = 1$, then $ABS(G) \leq 2\left(\sqrt{\frac{3}{4}} + \sqrt{\frac{5}{7}} + \sqrt{\frac{2}{3}} + \sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}}\right)$ with equality if and only if $G \cong R_{10}$.

(iv) If $\beta = 4$, $q = 2$, then $ABS(G) \leq 3\sqrt{\frac{1}{2}} + 2\sqrt{\frac{3}{4}} + 2\sqrt{\frac{5}{7}} + 2\sqrt{\frac{2}{3}}$ with equality if and only if $G \cong W_8$.

(v) If $\beta = 5$, $q = 2$, then $ABS(G) \leq 2\sqrt{3} + 2\sqrt{2} + \sqrt{6}$ with equality if and only if $G \cong Z_{10}$.

(vi) If $(\beta, q) \notin \{(3,1), (4,1), (5,1), (4,2), (5,2)\}$, then $ABS(G) \leq \Phi(\beta, q)$ with equality if and only if $G \cong G'[2\beta, q]$.

Declarations

Conflict of interest The authors declare no conflict of interest.

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