Filomat 40:10 (2026), 3841–3853
https://doi.org/10.2298/FIL2610841H

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Weighted extropy for residual life based on order statistics

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Abstract. In this paper, we discuss novel discoveries regarding the weighted residual extropy of random events. We specifically investigate the weighted extropy of the i-th order statistic and its connection to the weighted residual extropy of the i-th order statistic derived from a random sample following a uniform distribution. Also, we explore some characterization results, monotone properties and bounds of weighted residual extropy of order statistics. Furthermore, we examine the monotonic tendencies of the weighted residual extropy in order statistics. Non-parametric estimations of the proposed measures based on the kernel and empirical estimators are discovered and by using some simulation studies, the efficiency of them are compared based on their bias and mean square error. Finally, we give a real example to show the usefulness of the proposed estimators for the problem of goodness-of-fit test.

1. Introduction

The idea of entropy was initially introduced by physicist Boltzmann [7] to represent the level of disorder in a physical system. Shannon [20] later built upon this concept, which has become fundamental and widely used in fields like communication theory, information theory, and statistics. Entropy serves as a measure of the uncertainty associated with a random variable in information theory. For more details, refer to Cover and Thomas [9].

To measure the uncertainty contained in X, the entropy was defined by Shannon [20] as follows,

$$H(X) = -E\left[\log f(x)\right] = -\int_0^{+\infty} f(x)\log f(x)dx, \tag{1}$$

where " log " means the natural logarithm and the convention that $0\log 0 = 0$, which is easily justified by continuity since $x\log x \to 0$ as $x \to 0$.

As highlighted by Asadi and Ebrahimi [2], when X is viewed as the lifetime of a new unit, the traditional entropy measure $H(X)$ does not effectively capture the uncertainty related to the unit's remaining lifetime. In these contexts, it is more appropriate to consider the concept of residual entropy for X. This concept, introduced by Ebrahimi [12], focuses on the entropy of the conditional random variable $X_t = [X - t|X \geq t]$,

2020 *Mathematics Subject Classification*. Primary 62F10; Secondary 62N05.
Keywords. Extropy, Hazard rate function, Order statistics, Residual extropy, Stochastic order.
Received: 12 May 2025; Revised: 30 December 2025; Accepted: 27 January 2026
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which represents the remaining lifetime of the unit given that it has already survived for a time t. This approach provides a more relevant measure of uncertainty for assessing the future performance or reliability of a unit beyond a certain point in time, i.e.,

$$H\left(X_t\right) = -\int_t^{+\infty} \frac{f(x)}{\bar{F}(t)} \log \frac{f(x)}{\bar{F}(t)} \mathrm{d}x, \tag{2}$$

where $\bar{F}(.)$ is the survival function (SF) of X. Analogous to Eq. (2), the residual entropy of X is defined as the entropy of X_t in this paper.

The weighted entropy, referred by Di Crescenzo and Longobardi [11] in agreement with Belis and Guiasu [6], is defined as

$$H^w(X) = -E[W(X) \log f(X)] = -\int_0^{+\infty} w(X) f(x) \log f(x) dx. \tag{3}$$

The residual measure of entropy has been extended to the length-biased weighted residual entropy.

Di Crescenzo and Longobardi [11] discussed weighted versions of the past and residual entropies. The weighted residual entropy is defined as

$$H^w\left(X_t\right) = -\int_t^{+\infty} x \frac{f(x)}{\bar{F}(t)} \log \frac{f(x)}{\bar{F}(t)} \mathrm{d}x,$$

while the weighted past entropy is defined as

$$H^w\left({}_tX\right) = -\int_0^t x \frac{f(x)}{F(t)} \log \frac{f(x)}{F(t)} \mathrm{d}x.$$

The factor x in the integral on right hand side yields a length-biased shift dependent information measure assigning greater importance to larger values of the random variable X.

1.1. Extropy

In their study, Lad et al. [17] introduced a distinct approach to quantify the uncertainty associated with a random variable, which they termed extropy. Extropy serves as a measure of information that can be viewed as the opposite or inverse of entropy. More precisely, X's extropy can be expressed as follows:

$$\mathcal{J}\left(X\right) = -\frac{1}{2}\int_0^{\infty} f^2(x) dx = -\frac{1}{2}\int_0^1 f\left(F^{-1}(u)\right) du = -\frac{1}{2}\int_0^{\infty} f(x) dF(x). \tag{4}$$

The extropy is a measure of uncertainty in the distribution of a random variable, similar to entropy but shift-independent. While the entropy and extropy of a binary distribution are the same, they differ in other scenarios. Both measures reach their maximum value with a uniform distribution and are unaffected by permutations of their mass functions. However, they assess distribution refinement differently. The relationship between entropy and extropy can be compared to positive and negative photographic images. Extropy is commonly used to evaluate forecasting distributions and in speech recognition. It can also be used to compare uncertainties between two random variables. If the extropy of one variable is less than another, it indicates the former has more uncertainty. Extropy is easy to calculate, making it valuable for creating goodness-of-fit tests and inferential methods. Its potential applications are of great interest in various fields.

Analogous to the weighted entropy, Balakrishnan et al. [3] introduce the concept of weighted extropy as

$$\mathcal{J}^w\left(X\right) = -\frac{1}{2} E[X f(X)] = -\frac{1}{2}\int_0^{+\infty} x f^2(x) dx, \tag{5}$$

which can also be alternatively expressed as

$$\mathcal{J}^{w}(X) = -\frac{1}{2}\int_{0}^{+\infty} f^{2}(x)\int_{0}^{x} \mathrm{d}y\,\mathrm{d}x = -\frac{1}{2}\int_{0}^{+\infty}\mathrm{d}y\int_{y}^{+\infty} f^{2}(x)\mathrm{d}x.$$

Also, they introduce and study weighted past extropy of X at time t as

$$\mathcal{J}^{w}({}_{t}X) = -\frac{1}{2F^{2}(t)}\int_{0}^{t} x f^{2}(x)\mathrm{d}x.$$

This paper is organized as follows. In Section 2, we explore some bounds of weighted residual extropy. Also, we show that the weighted residual extropy of a random variable X is uniquely determined by its failure rate function (FRF). Based on this point, several distributions are characterized in terms of its weighted residual extropy. In Section 3, some monotone properties of weighted residual extropy of the first order statistic are built. We also show that the underlying distributions can be characterized by the weighted residual extropy of order statistics. In Section 4, two Non-parametric estimations based on the kernel and empirical estimators are provided. Using simulation studies, we compare these estimations based on their bias and mean square error. Section 5 gives a real example to show the usefulness of the proposed estimators for the problem of goodness-of-fit test.

2. Weighted residual extropy

In this section, we introduce and study a concept referred to as the weighted residual extropy (WREX) for a random variable X, which is defined as

$$\mathcal{RJ}^{w}(X_{t}) = -\frac{1}{2}\int_{0}^{\infty} x\left[\frac{f(x+t)}{\bar{F}(t)}\right]^{2} dx = -\frac{1}{2\bar{F}^{2}(t)}\int_{t}^{\infty} x f^{2}(x)dx,\ t \geq 0. \tag{6}$$

It is evident that the weighted residual extropy of a continuous distribution is consistently negative, in contrast, the weighted residual entropy of a continuous distribution can assume any value across the entire real line, including negative infinity and positive infinity. Notably, setting $t = 0$ in Eq. (6) results in $\mathcal{RJ}^{w}(X_{0}) = \mathcal{J}^{w}(X)$, aligning with what is presented in Eq. (5).

Example 2.1. *Let X be a Power random variable with cumulative distribution function (CDF) $F(x) = x^{c}$, $0 < x < 1$. We get*

$$\mathcal{RJ}^{w}(X) = -\frac{ct^{c} + c}{4t^{c} - 4}, \tag{7}$$

it was assumed that $2c - 1 > 0$. It shows that $\mathcal{RJ}^{w}(X)$ is increasing in t.

Example 2.2. *Suppose that X be a exponential random variable with probability density function (PDF) $f(x) = \theta e^{-\theta x}$ and CDF $F(x) = 1 - e^{-\theta x}$. we get*

$$\mathcal{RJ}^{w}(X) = -\frac{2t\theta + 1}{8}. \tag{8}$$

This example shows that $\mathcal{RJ}^{w}(X)$ is decreasing in t.

In the following, relationships equivalent to the concept of weighted residual extropy in relation (6) are presented.

Proposition 2.3. *Suppose that X is a non-negative and continuous random variable with PDF $f(x)$ and CDF $F(x)$. Then*

$$\mathcal{RJ}^{w}(X_{t}) = -\frac{1}{2}\int_{t}^{\infty} c(x,t)\lambda_{X}^{2}(x)dx, \tag{9}$$

where $c(x,t) = x\left[\frac{\bar{F}(x)}{\bar{F}(t)}\right]^{2}$ and $\lambda_{X}(x) = f(x)/\bar{F}(x),\ x \geq 0$.

Remark 2.4. *Assume that X is a non-negative and continuous random variable with CDF $F(x)$ and PDF $f(x)$. Then we have*

$$\mathcal{RJ}^w(X_t) = [\bar{F}(t)]^{-2}\mathcal{J}^w(X) + A(t), \tag{10}$$

where $A(t) = \frac{1}{2}\int_0^t x\left(\frac{f(x)}{\bar{F}(x)}\right)^2 dx$.

The upcoming theorem indicates that the weighted residual extropy of a random variable is uniquely defined by its FRF.

Theorem 2.5. *The unique determination of the weighted residual extropy $\mathcal{RJ}^w(X_t)$ of X is based on the failure rate function $\lambda_X(t)$.*

Proof. It is obvious from Eq. (6) that

$$\begin{aligned}\frac{\partial\mathcal{RJ}^w(X_t)}{\partial t} &= -\frac{1}{2\bar{F}^4(t)}\left[-tf^2(t)\bar{F}^2(t) + 2\bar{F}(t)f(t)\int_t^\infty xf^2(x)\mathrm{d}x\right]\\ &= \frac{t}{2}\lambda_X^2(t) + 2\lambda_X(t)\mathcal{RJ}^w(X_t)\,.\end{aligned}$$

Thus, we have

$$\frac{\partial\mathcal{RJ}^w(X_t)}{\partial t} - 2\lambda_X(t)\mathcal{RJ}^w(X_t) = \frac{t}{2}\lambda_X^2(t). \tag{11}$$

Solving the aforementioned differential equation results in

$$\mathcal{RJ}^w(X_t) = \mathrm{e}^{2\int\lambda_X(t)\mathrm{d}t}\left[\frac{1}{2}\int t\lambda_X^2(t)\mathrm{e}^{-2\int\lambda_X(t)dt}\,\mathrm{d}t + B\right],$$

where B is a constant and is determined by $\mathcal{RJ}^w(X_t)|_{t=0} = \mathcal{J}^w(X)$. The proof is now complete. □

2.1. Bounds for the weighted residual extropy

In this section, we give several bounds for the weighted residual extropy by using assertions established in Section 2.

In the sequel, we provide upper bound for the relation weighted residual extropy.

Proposition 2.6. *Let $M = f(m) < \infty$ where $m = sup\{x; f(x) \leq M\}$ is the mode of the distribution. We get*

$$\mathcal{RJ}^w(X_t) \leq \bar{F}^2(t)\left[\mathcal{J}^w(X) + \left(\frac{Mt}{2}\right)^2\right].$$

Theorem 2.7. *Let X is a non-negative and continuous random variable with decreasing PDF $f(x)$ such that $f(0) \leq 1$. Then*

$$-\frac{E(X)}{2\bar{F}^2(t)} \leq \mathcal{RJ}^w(X_t) \leq 0.$$

Remark 2.8. *Based on Eq. (11), the function $\mathcal{RJ}^w(X_t)$ will increase (decrease) as t increases if and only if $\mathcal{RJ}^w(X_t) \geq (\leq) -\frac{t}{4}\lambda_X(t)$.*

In the following, We provide lower bound for the relation weighted residual extropy.

Remark 2.9. *Suppose that X is a non-negative random variable with PDF $f(x)$ and CDF $F(x)$. Then, we get*

$$\mathcal{RJ}^w(X_t) \geq [\bar{F}(t)]^{-2}\mathcal{J}^w(X)\,. \tag{12}$$

Next, we express some lower bounds for $\mathcal{RJ}^w(X_t)$ in terms of the HRF.

Remark 2.10. *Suppose that X is a non-negative and continuous random variable with PDF $f(x)$ and CDF $F(x)$. Then*

$$\mathcal{RJ}^w(X_t) \geq -\frac{1}{2}\int_t^\infty x\lambda_X^2(x)dx.$$

Remark 2.11. *Assume that X is a continuous and non-negative random variable with CDF $F(x)$ and PDF $f(x)$. Then*

$$\mathcal{RJ}^w(X_t) \geq -\int_t^\infty x\lambda_X^2(x)dx.$$

Next, we will investigate the relationship between $\mathcal{RJ}^w(X_t)$ and $E(X^2)$.

Proposition 2.12. *Suppose that X is a continuous and non-negative random variable with PDF $f(x)$ and CDF $F(x)$. Then*

$$\mathcal{RJ}^w(X_t) \geq -\frac{E(X^2)}{4\bar{F}^2(t)}.$$

Proposition 2.13. *Assume that X is a continuous and non-negative random variable with CDF $F(x)$ and PDF $f(x)$. Then*

$$\mathcal{RJ}^w(X_t) \geq -\frac{1}{2}\int_t^\infty x\lambda_X^2(x)b(x,t)dx,$$

where $b(x,t) = x\bar{F}(x)\bar{F}^{-1}(t)$.

2.2. Stochastic order

In the following steps, we will explore different scenarios where $\mathcal{RJ}^w(X_t)$ decreases as t changes. To start off, we will revisit the definitions of three stochastic orders.

We want to prove a property of weighted residual extropy measure using some properties of stochastic ordering. We present the following definition.

Definition 2.14. *Suppose that X and Y be two random variables with CDF's $F(x)$ and $G(x)$, SFs $\bar{F}(x)$ and $\bar{G}(x)$, PDF's $f(x)$ and $g(x)$, respectively.*

i) a random variable X is said to be less than Y in the stochastic ordering (denoted by $X \leq_{st} Y$) if $\bar{F}(x) \leq \bar{G}(x)$ for all x;

ii) a random variable X is said to be less than Y in the likelihood ratio order (denoted by $X \leq_{\text{lr}} Y$), if $f(x)/g(x)$ is decreasing in x;

iii) a random variable X is said to be less than Y in the hazard rate order, (denoted by $X \leq_{\text{hr}} Y$), if $r_X(x) \geq \lambda_Y(x)$ for all x, where $\lambda_X(x) = f(x)/\bar{F}(x)$ ($\lambda_Y = g(x)/\bar{G}(x)$) is the hazard rate function of $X(Y)$;

Remark 2.15. *It is well known that $X \leq_{\text{lr}} Y \Longrightarrow X \leq_{\text{st}} Y$, and $X \leq_{\text{st}} Y$ if and only if $\mathrm{E}\phi(X) \leq \mathrm{E}\phi(Y)$ for all increasing functions ϕ.*

Remark 2.16. *The function $f\left[F^{-1}(x)\right]$ is referred to as the density-quantile function (DQF) in academic researches and is utilized for estimating the moments of order statistics(OS).*

Theorem 2.17. *Suppose that X be a non-negative and continuous random variable with CDF $F(x)$ and PDF $f(x)$. If $f\left(F^{-1}(x)\right)$ is increasing in $x \geq 0$, then $\mathcal{RJ}^w(X_t)$ is decreasing in $t \geq 0$.*

Proof. Assume W_t is a random variable uniformly distributed on $(F(t), 1)$ with PDF $h_t(x) = 1/\bar{F}(t)$, $F(t) < x < 1$. Then, it follows from (6) that

$$\mathcal{RJ}^{w}(X_t) = -\frac{1}{2\bar{F}(t)}\mathrm{E}\left[F^{-1}(W_t)f\left(F^{-1}(W_t)\right)\right]$$

Suppose that $0 \leq t_1 < t_2$. Then $h_{t_1}(x)/h_{t_2}(x) = \infty$ if $F(t_1) < x \leq F(t_2)$, and $h_{t_1}(x)/h_{t_2}(x) = \bar{F}(t_2)/\bar{F}(t_1)$, a constant, if $F(t_2) < x < 1$. Thus, $h_{t_1}(x)/h_{t_2}(x)$ is decreasing in $x \in (F(t_1), 1)$, which implies $W_{t_1} \leq_{\mathrm{lr}} W_{t_2}$. Hence, $W_{t_1} \leq_{\mathrm{st}} W_{t_2}$ and $0 \leq \mathrm{E}\left[F^{-1}(W_{t_1}) f\left(F^{-1}(W_{t_1})\right)\right] \leq \mathrm{E}\left[F^{-1}(W_{t_2}) f\left(F^{-1}(W_{t_2})\right)\right]$ by the assumption that $f\left(F^{-1}(x)\right)$ is an increasing function. Since $0 < 1/\bar{F}(t_1) \leq 1/\bar{F}(t_2)$, we conclude the desired result by noting that

$$\mathcal{RJ}^{w}(X_{t_2}) = \frac{-1}{2\bar{F}(t_2)}\mathrm{E}\left[F^{-1}(W_{t_2}) f\left(F^{-1}(W_{t_2})\right)\right] \leq \frac{-1}{2\bar{F}(t_1)}\mathrm{E}\left[F^{-1}(W_{t_1}) f\left(F^{-1}(W_{t_1})\right)\right] = \mathcal{RJ}^{w}(X_{t_1}).$$

□

Example 2.18. *Assuming X is a Beta$(2,1)$ random variable with a CDF of $F(x) = x^2$ for x in the interval $(0,1)$, and $f(F^{-1}(x)) = 2x^{\frac{1}{2}}$ for x in $(0,1)$, the condition specified in Theorem 2.17 is met. Consequently, the weighted residual extropy of X decreases as t varies.*

3. Weighted residual extropy of OS

Suppose that $X_1, X_2, \ldots, X_n$ represent independent random samples of size n from the population X with CDF $F(x)$ and PDF $f(x)$. The OS of $X_1, X_2, \ldots, X_n$ are denoted by $X_{1:n} \leq \ldots \leq X_{n:n}$. The SF and PDF of $X_{1:n}$ are given by $\bar{F}_{1:n}(x) = \bar{F}^n(x)$ and $f_{1:n}(x) = n\bar{F}^{n-1}(x)f(x), x \geq 0$, respectively. If $f(x)$ is decreasing, then $f_{1:n}\left(F_{1:n}^{-1}(x)\right) = nx^{1-1/n}f\left[F^{-1}\left(1 - x^{1/n}\right)\right]$ is increasing in $x \in (0,1)$.

These OS have applications in various fields including statistical inference, reliability theory, strength of materials, goodness-of-fit tests, quality control, outlier detection, and characterization of probability distributions. In reliability theory, OS are used to model the lifetimes of systems. Each OS in a sample represents the lifetime of a specific subset of the system. For example, in reliability theory, OS is used for statistical modelling. The i-th OS in a sample of size n represents the lifetime of an $(n-i+1)$-out-of-n system.

Researchers have utilized order statistics to address a wide range of issues and gain valuable insights. we refer to David and Nagaraja [10] and Arnold et al. [1] for an extended set of references. In a study conducted by Qiu [19], the characterization and symmetric properties of the extropy of order statistics and record values were investigated. Studies by Baratpour et al. [4] demonstrated that Shannon entropy and Rényi entropy of specific order statistics can uniquely identify the distribution. Similar findings have been reported by Baratpour [5] for cumulative residual entropy and by Gupta et al. [14] for dynamic entropy of OS.

3.1. Monotone properties

This section explores the monotonic properties of weighted residual extropy and highlights how the residual extropy of OS can be used to uniquely determine the underlying distribution.

Theorem 3.1. *If the PDF of X decreases on the interval $[0,\infty)$, then weight residual extropy $\mathcal{RJ}^{w}(X_{1:n})$ is decreasing in $t \geq 0$.*

The following counterexample demonstrates that the outcome described in Theorem 3.1 cannot be extended to $X_{i:n}$ for $i > 1$.

Example 3.2. *Assume that X be a random variable with decreasing PDF $f(x) = 1/(x+1)^2, x \geq 0$. The CDF of $X_{2:2}$ is given by $F_{2:2}(x) = x^2/(x+1)^2, x \geq 0$. Thus,*

$$\mathcal{RJ}^{w}(X_{2:2}) = -\frac{10t^3 + 10t^2 + 5t + 1}{10(t+1)(2t+1)^2}.$$

After some algebraic manipulations, we have

$$\mathcal{RJ}^w_{0.3}(X_{2:2}) = -0.183, \mathcal{RJ}^w_{0.7}(X_{2:2}) = -0.131.$$

Thus, $\mathcal{RJ}^w(X_{2:2})$ *is increasing in* t.

In the following, we show that the weighted residual extropy is also decreasing in n if $f(x)$ is decreasing.

Theorem 3.3. *If the PDF of* X *is decreasing on the interval* $[0,\infty)$, *then* $\mathcal{RJ}^w(X_{1:n})$ *decreases as* n *increases.*

Proof. We note that the PDF of $[X_{1:n} \mid X_{1:n} > t]$ is given by

$$h^t_{1:n}(x) = \frac{n\bar{F}^{n-1}(x)f(x)}{\bar{F}^n(t)}, \quad x \geq t.$$

Also,

$$\frac{h^t_{1:(2n-1)}(x)}{h^t_{1:(2n+1)}(x)} = \frac{(2n-1)\bar{F}^2(t)}{(2n+1)\bar{F}^2(x)}$$

is increasing in $x \in [t,\infty)$. Therefore,

$$\left[X_{1:(2n+1)} \mid X_{1:(2n+1)} > t\right] \leq_{\text{lr}} \left[X_{1:(2n-1)} \mid X_{1:(2n-1)} > t\right],$$

which implies

$$\left[X_{1:(2n+1)} \mid X_{1:(2n+1)} > t\right] \leq_{\text{st}} \left[X_{1:(2n-1)} \mid X_{1:(2n-1)} > t\right].$$

If $f(.)$ is decreasing, then,

$$\mathrm{E}\left[f\left(X_{1:(2n+1)}\right) \mid X_{1:(2n+1)} > t\right] \geq \mathrm{E}\left[f\left(X_{1:(2n-1)}\right) \mid X_{1:(2n-1)} > t\right]$$

Now, it can be deduced from Eq. (6) that

$$\mathcal{RJ}^w(X_{1:n}) = -\frac{n^2}{2(2n-1)\bar{F}(t)}\mathrm{E}\left[Xf\left(X_{1:(2n-1)}\right) \mid X_{1:(2n-1)} > t\right].$$

Therefore

$$\frac{\mathcal{RJ}^w(X_{1:n})}{\mathcal{RJ}^w\left(X_{1:(n+1)}\right)} = k\frac{\mathrm{E}\left[Xf\left(X_{1:(2n-1)}\right) \mid X_{1:(2n-1)} > t\right]}{\mathrm{E}\left[Xf\left(X_{1:(2n+1)}\right) \mid X_{1:(2n+1)} > t\right]} \leq 1$$

where $k = \frac{n^2(2n+1)}{(n+1)^2(2n-1)}$. The proof is completed. □

3.2. Characterizations

The general characterization problem is to determine when the proposed weighted residual extropy characterizes the distribution function uniquely. Numerous recent research studies have delved into exploring methods to describe a sample's fundamental distribution by utilizing metrics like extropy or enhanced variations of order statistics.

In this section, we further establish that the weighted residual extropy of the i-th order statistic also provides a unique characterization of the underlying distribution.

Theorem 3.4. *Assume that* $\mathcal{RJ}^w(X_{1:n})$ *and* $\mathcal{RJ}^w(Y_{1:n})$ *represent the weighted residual extropy of the first order statistic from* X *and* Y, *respectively. The statement* $X \stackrel{d}{=} Y$ *holds true if and only if* $\mathcal{RJ}^w(X_{1:n}) = \mathcal{RJ}^w(Y_{1:n})$, $\forall\, t \geq 0$ *and* $n \geq 1$.

Proof. To establish the sufficiency, we need to demonstrate that the residual lifetime of $X_{1:n}$ at age $t \geq 0$, is denoted by $X_{1:n;t} = [X_{1:n} - t \mid X_{1:n} \geq t]$. The SF of $X_{1:n;t}$ is calculated as $\bar{F}_{1:n;t}(x) = \bar{F}_{1:n}(t+x)/\bar{F}_{1:n}(t) = [\bar{F}(t+x)/\bar{F}(t)]^n$, $x \geq 0$, $t \geq 0$.

Therefore,

$$X_{1:n;t} \stackrel{d}{=} \min\{X_{1;t}, X_{2;t}, \ldots, X_{n;t}\}, \tag{13}$$

where $X_{i;t} = [X_i - t \mid X_i \geq t]$, $i = 1, 2, \ldots, n$. If $\forall t \geq 0$ and $n \geq 1$, $\mathcal{RJ}^w(X_{1:n}) = \mathcal{RJ}^w(Y_{1:n})$, then by Eq. (13) and [19] that $\forall t \geq 0$, $X_t \stackrel{d}{=} Y_t$. i.e. $\forall\, x \geq 0$ and $t \geq 0$, $F_t(x) = G_t(x)$, where $F_t(x)$ and $G_t(x)$ are CDFs of X_t and Y_t, respectively. Therefore, $\forall x \geq 0$ and $t \geq 0$ $F(x+t) = \bar{F}(t)G(x+t)/\bar{G}(t)$. As x approaches infinity, it can be deduced that $\bar{F}(t) = \bar{G}(t)$ $\forall t \geq 0$, indicating that X and Y share the same distribution function. □

In the sequel, to extend Theorem 3.4 from $X_{1:n}$ to $X_{i:n}$, where $i \geq 1$, we explore the task of determining a sufficient condition for the unique solution of the initial value problem (IVP) given by the differential equation:

$$\frac{\partial y}{\partial x} = g(x, y), \quad y(x_0) = y_0. \tag{14}$$

Here, $g(.)$ represents a two-variable function defined on a region $M \subset \mathbb{R}^2$, (x_0, y_0) denotes a point within M, and y is the function to be determined. Through the solution of Eq. (14), we obtain a function $\varphi(.)$ that meets the following criteria:

I- $\varphi(.)$ is differentiable on interval I,

II- the growth of $\varphi(.)$ is within region M, and

III- $\varphi(x_0) = y_0$,

IV- $\forall x \in I$, $\varphi'(x) = g(x, \varphi(x))$.

This theorem will be utilized in demonstrating our characterization outcomes.

Theorem 3.5. *Assume that $g(.)$ is a continuous function defined in a domain $M \subset \mathbb{R}^2$ and satisfies the Lipschitz condition (with respect to y) in M, meaning $\left|g(x, y_1) - g(x, y_2)\right| \leq k\left|y_1 - y_2\right|, k > 0$ for all points (x, y_1) and (x, y_2) in M. Then, the function $y = \varphi(x)$ that solves the initial value problem $y' = g(x, y)$ with $y(x_0) = y_0$ for $x \in I$ is uniquely determined within M.*

We utilize the lemma to introduce a condition that is sufficient to ensure the satisfaction of the Lipschitz condition.

Lemma 3.6. *Let the function $g(.)$ is continuous in a convex region $M \subset \mathbb{R}^2$, $\frac{\partial g}{\partial y}$ exists and is continuous in M. Then $g(.)$ satisfies the Lipschitz condition in M.*

Theorem 3.7. *Suppose that $\mathcal{RJ}^w(Y_{i:n})$ and $\mathcal{RJ}^w(X_{i:n})$ be the weighted residual extropy of the i-th order statistic from Y and X, respectively. Then $Y \stackrel{d}{=} X$ if and only if $\mathcal{RJ}^w(Y_{i:n}) = \mathcal{RJ}^w(X_{i:n})$, $\forall t \geq 0$ and $n \geq i$.*

Proof. From Eq. (11), we have

$$\frac{\partial}{\partial t}\mathcal{RJ}^w(X_{i:n}) - 2\lambda_{X_{i:n}}(t)\mathcal{RJ}^w(X_{1:n}) = \frac{t}{2}\lambda^2_{x_{i:n}}(t), \quad t \geq 0.$$

Taking the derivative of the above equation concerning t, we have

$$\frac{\partial}{\partial t}\lambda_{X_{i:n}}(t) = \frac{\frac{d^2}{dt^2}\mathcal{RJ}^w(X_{i:n}) - 2\lambda_{X_{i:n}}(t)\frac{\partial}{\partial t}\mathcal{RJ}^w(X_{i:n}) - \frac{1}{2}\lambda^2_{X_{i:n}}(t)}{2\mathcal{RJ}^w(X_{i:n}) + t}.$$

Assume that $\forall t \geq 0, n \geq i$, $\mathcal{RJ}^{w}(Y_{i:n}) = \mathcal{RJ}^{w}(X_{i:n}) = \eta(t)$. Then for all $t \geq 0$,

$$\frac{\partial}{\partial t}\lambda_{X_{i:n}}(t) = \psi\left(t, \lambda_{X_{i:n}}(t)\right), \quad \frac{\partial}{\partial t}\lambda_{Y_{i:n}}(t) = \psi\left(t, \lambda_{Y_{i:n}}(t)\right),$$

where

$$\psi(t,y) = \frac{\eta''(t) - 2y\eta'(t) - \frac{1}{2}y^2}{2\eta(t) + t}.$$

In simpler terms, based on Theorem 3.5 and Lemma 3.6, it can be concluded that the functions $\lambda_{X_{i:n}}(t)$ and $\lambda_{Y_{i:n}}(t)$ are equal for all $t \geq 0$. This equality further implies that the CDFs $F_{i:n}(t)$ and $G_{i:n}(t)$ for $X_{i:n}$ and $Y_{i:n}$ are also equal for all $t \geq 0$. Given that $F(t)$ is defined as the ratio of $F_{i:n}(t)$ to $B_{i,n-i+1}$ and $G(t)$ is defined as the ratio of $G_{i:n}(t)$ to $B_{i,n-i+1}$, where $B_{i,n-i+1}(t)$ represents the CDF of the beta distribution with parameters i and $(n-i+1)$, it follows that for all $t \geq 0$, $F(t)$ is equal to $G(t)$. The desired result is proved. □

In the sequel, we characterize some distributions in terms of weighted residual extropy.

Theorem 3.8. *Assume that X be a non-negative continuous random variable with FRF $\lambda_X(t)$. If $\forall t \geq 0$, $\mathcal{RJ}^{w}(X) = -c\,\lambda_X(t)$, where c is a non-negative constant. Then X has*

I. *a power distribution if and only if* $c = \dfrac{a\left(t^{2a+1} - 1\right)}{4\,(2a+1)}$,

II. *an exponential distribution if and only if* $c = \dfrac{2t\theta + 1}{8\theta}$,

III. *a uniform distribution if and only if* $c = \dfrac{t+b}{4}$,

IV. *a power distribution if and only if* $c = \frac{a(t^{2a-1}-1)(1-t^{a})}{2(2a-1)(t^{a}-1)^{2}t^{a-1}}$.

We showed that the weighted residual extropy of X is determined uniquely by its FRF.

4. Non-parametric estimation

Assume that $X_1, \ldots, X_n$ is a random sample obtained from a population where $f(\cdot)$ and $F(\cdot)$ are its PDF and CDF, respectively. In this section, we propose a non-parametric estimation of $\mathcal{RJ}^{w}(X_{k:n})$, where $X_{k:n}$ is the k-th order statistic of the random sample $X_1, \ldots, X_n$. By referring to [1], the PDF and CDF of the k-th order statistic $X_{k:n}$ can be given as follows. Its PDF is

$$f_{k:n}(x) = \frac{F^{k-1}(x)\bar{F}^{n-k}(x)f(x)}{B(k, n-k+1)}, \tag{15}$$

where $B(\alpha, \beta)$ is the beta function with parameters α and β, and the SF of $X_{k:n}$ is

$$\bar{F}_{k:n}(x) = \frac{\bar{B}_{F(x)}(k, n-k+1)}{B(k, n-k+1)}, \tag{16}$$

where

$$\bar{B}_{F(x)}(\alpha, \beta) = \int_{F(x)}^{1} u^{\alpha-1}(1-u)^{\beta-1}.$$

The $\mathcal{RJ}^{w}(X_{k:n})$ then can be written as

$$\mathcal{RJ}^{w}(X_{k:n}) = -\frac{1}{2\bar{F}_{k:n}(t)}\int_{t}^{\infty} x f_{k:n}^{2}(x)dx. \tag{17}$$

One can use kernel density estimation method for estimating of $\mathcal{RJ}^w(X_{k:n})$. So, the estimator $\widehat{\mathcal{RJ}}^w(X_{k:n})$ can be obtained by replacing the kernel density estimation of $f(x)$ and $F(x)$ in (17). Let $\hat{f}(x)$ be a kernel density estimation of $f(x)$. Then one can see that

$$\hat{f}(x) = \frac{1}{nh_n}\sum_{i=1}^{n} K(\frac{x - X_i}{h_n}),$$

where h_n is a bandwidth and $K(x)$ is a kernel function. Two choices are examined for estimating $F(x)$ in this part. One is based on the kernel method and the second can be obtained by using empirical cumulative distribution function (ECDF). By [18], the kernel estimation of $F(x)$ can be provided as

$$\hat{F}(x) = \frac{1}{n}\sum_{i=1}^{n} W(\frac{x - X_i}{h_n}),$$

where $W(x)$ can be obtained by

$$W(x) = \int_{-\infty}^{x} K(s)ds.$$

Also, the ECDF of $F(x)$ is

$$F_n(x) = \frac{1}{n}\sum_{i=1}^{n} I(X_i \leq x),$$

where $I(.)$ stands for the indicator function. In the following section, we compare two estimators $\widehat{\mathcal{RJ}}_1^w(X_{k:n})$ and $\widehat{\mathcal{RJ}}_2^w(X_{k:n})$ based on kernel method and ECDF, respectively, using some simulation studies.

4.1. Simulation study

In this section, we compare two estimators $\widehat{\mathcal{RJ}}_1^w(X_{k:n})$ and $\widehat{\mathcal{RJ}}_2^w(X_{k:n})$ given in the former section in terms of their bias and root mean square error (RMSE). Our comparison is based on the simulation. We consider a case that the random sample follows the exponential (Exp(λ)) distribution. The sample size n is 50 and 100. Also, the order parameter k in (15) is 1, 5 and 10. For Exp(λ), parameter λ is 0.1, 0.2 and 0.3. The simulation results are based on 5000 runs. Based on the results of Table 1, we observe that the estimator $\widehat{\mathcal{RJ}}_1^w$ is better than the $\widehat{\mathcal{RJ}}_2^w$ in terms of Bias and RMSE. This result shows that the estimator $\widehat{\mathcal{RJ}}_1^w$ based on the kernel method is superior to ECDF estimation of CDF. It is seen that the RMSE decreases as the sample size n increases. The value of RMSE increases when the order parameter k increases. Beside these results, one can see that RMSE of these proposed estimators is an decreasing function of time t. In all settings, we take a Gaussian kernel with bandwidth $h_n = n^{-1/5}$. Totally, we can say that estimator $\widehat{\mathcal{RJ}}_1^w$ can be applied for estimating the weighted residual extropy $\mathcal{RJ}^w$.

5. Real data

In this section, a real example is given to support the obtained results in the former sections of the paper. The following data are the active repair times for an airborne communications transceiver.

0.2, 0.3, 0.5, 0.5, 0.5, 0.5, 0.6, 0.6, 0.7, 0.7, 0.7, 0.8, 0.8, 1.0, 1.0, 1.0, 1.0, 1.1, 1.3, 1.5, 1.5, 1.5, 1.5, 2.0, 2.0, 2.2, 2.5, 2.7, 3.0, 3.0, 3.3, 3.3, 4.0, 4.0, 4.5, 4.7, 5.0, 5.4, 5.4, 7.0, 7.5, 8.8, 9.0, 10.3, 22.0, 24.5

We consider two distributions for these data. The first one is the inverse Gaussian model ($IG(\mu, \lambda)$) from Chhikara and Folks [8] and the second one is the log-normal ($LN(\mu, \sigma)$) from Hashempour and Mohammadi [15]. The maximum likelihood estimation (MLE) of unknown parameters of IG and LN distributions as well as the Kolmogorov-Smirnov statistic and its p-value are given in Table 2.

			n=50				n=100			
			$\widehat{\mathcal{RJ}}_1^w$		$\widehat{\mathcal{RJ}}_2^w$		$\widehat{\mathcal{RJ}}_1^w$		$\widehat{\mathcal{RJ}}_2^w$	
Parameter (λ)	order (k)	Time (t)	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
0.1	1	0.5	0.0215	0.0800	-0.1035	0.5849	0.0072	0.0120	-0.0119	0.1733
		1	-0.0050	0.0413	-0.0440	0.2681	0.0000	0.0007	-0.0005	0.0066
		1.5	-0.0033	0.0213	-0.0103	0.0497	0.0000	0.0000	0.0000	0.0002
	5	0.5	0.1005	0.1587	0.1279	0.1990	0.1601	0.2420	0.0883	0.3191
		1	0.0585	0.1739	0.0415	0.2521	0.0200	0.0912	-0.0228	0.2122
		1.5	-0.0019	0.1766	-0.0537	0.2719	-0.0009	0.0175	-0.0083	0.0600
	10	0.5	0.0340	0.3123	0.0786	0.3288	0.0627	0.2199	0.1183	0.2637
		1	0.0273	0.2838	0.0411	0.3194	0.0783	0.2856	0.0501	0.3703
		1.5	0.0261	0.2329	0.0406	0.2849	0.0150	0.2273	-0.0378	0.3458
0.2	1	0.5	0.0074	0.0106	-0.0058	0.1179	0.0000	0.0001	0.0000	0.0012
		1	0.0000	0.0007	-0.0022	0.0491	0.0000	0.0000	0.0000	0.0000
		1.5	0.0000	0.0000	-0.0001	0.0010	0.0000	0.0000	0.0000	0.0000
	5	0.5	0.1538	0.2299	0.0987	0.2975	0.0557	0.0573	0.0410	0.1209
		1	0.0185	0.1007	-0.0195	0.1968	0.0000	0.0001	0.0000	0.0063
		1.5	-0.0039	0.0469	-0.0224	0.1424	0.0000	0.0000	0.0000	0.0000
	10	0.5	0.1119	0.2240	0.1527	0.2678	0.3491	0.4030	0.2874	0.4284
		1	0.0580	0.2602	0.0566	0.3355	0.0189	0.0354	0.0123	0.0770
		1.5	0.0198	0.2371	-0.0397	0.3583	0.0000	0.0013	0.0000	0.0058
0.3	1	0.5	0.0010	0.0014	0.0005	0.0006	0.0000	0.0000	0.0000	0.0000
		1	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
		1.5	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	5	0.5	0.1496	0.1700	0.1019	0.2306	0.0031	0.0031	0.0031	0.0035
		1	0.0015	0.0158	-0.0033	0.0448	0.0000	0.0000	0.0000	0.0000
		1.5	0.0000	0.0001	-0.0003	0.0036	0.0000	0.0000	0.0000	0.0000
	10	0.5	0.0617	0.2441	0.0983	0.3105	0.1710	0.1720	0.1682	0.1745
		1	0.0840	0.2239	0.0407	0.3163	0.0000	0.0000	0.0000	0.0000
		1.5	-0.0010	0.0506	-0.0128	0.0922	0.0000	0.0000	0.0000	0.0000

Table 1: Bias and RMSE for $\widehat{\mathcal{RJ}}_1^w$ and $\widehat{\mathcal{RJ}}_2^w$ for the exponential distribution.

statistic	MLE of Parametrs	K-S	p-value
IG	$\hat{\mu} = 3.6065, \hat{\lambda} = 0.6028$	0.0682	0.9830
LN	$\hat{\mu} = 0.6583, \hat{\sigma} = 1.1017$	0.0945	0.8059

Table 2: MLE of parameters, K-S statistics and p-values of the proposed models.

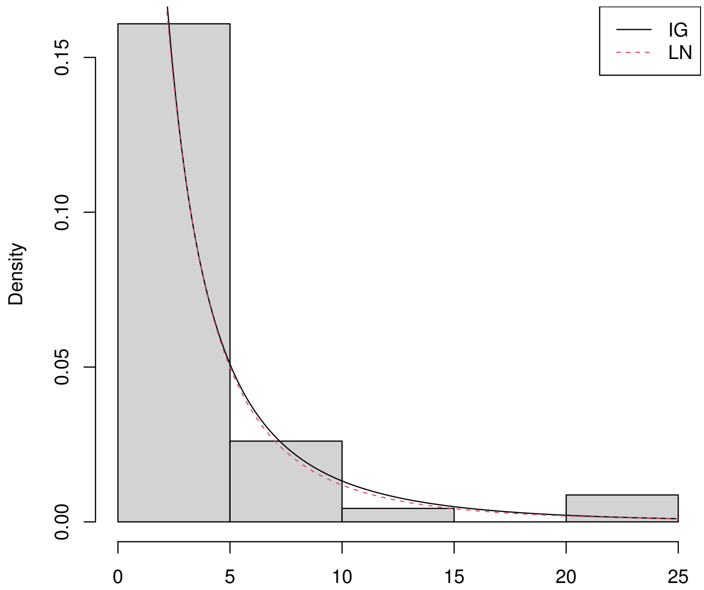

Figure 1: Histogram of the data with the IG and LN distributions curves.

Also, Figure 1 shows the histogram of the data as well as the PDF of the IG and LN distribution curves. The theoretic formula for $\mathcal{RJ}^w(X_{1:n})$ is computed based on IG and LN distributions for the first order statistic $X_{1:n}$. Due to the results of simulation section, that the efficiency of the estimator $\widehat{\mathcal{RJ}}_1^w(X_{1:n})$ is better than $\widehat{\mathcal{RJ}}_2^w(X_{1:n})$, so, we only obtain the value of the estimator $\widehat{\mathcal{RJ}}_1^w(X_{1:n})$ based on the above data. The results are depicted in Figure 2 for different values of time t. This Figure 2 indicates that again the IG is the best distribution to model these data as it is seen in Table 2 that the inverse Gaussian model is better than the log-normal distribution.

6. Conclusion

In this paper, we delved into the idea of weighted residual extropy to measure uncertainty in order statistics. We focused on calculating the weighted residual extropy for the i-th order statistic and establishing its connection with the weighted residual extropy of the i-th order statistic from a random sample drawn from a uniform distribution. Furthermore, we examined the symmetrical properties of weighted residual extropy in order statistics and investigate the monotonicity characteristics of weighted residual extropy in order statistics. Also, we proposed two estimators for weighted residual extropy based on the order statistics. Using simulations, we found that the estimator based on the kernel method is superior to that of based on the ECDF. Finally, we give a real data to support our findings. In this real data example, we showed that how our estimation can be applied for the problem of goodness-of-fit test.

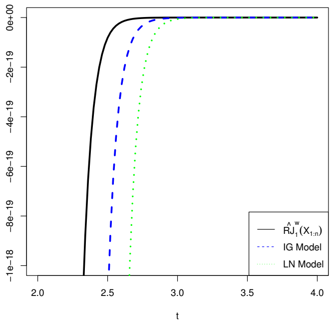

Figure 2: The plot of $\widehat{\mathcal{RJ}}_1^w(X_{1:n})$ and theoretical $\mathcal{RJ}^w(X_{1:n})$ for IG and LN models, respectively, as a function of time t.

Acknowledgements

The authors thank the editor-in-chief, the associate editor, and the anonymous reviewer for their useful comments on the earlier version of this paper.

Statements and Declarations

The authors certify that they have NO affiliations with or involvement in any organization or entity with any financial interest or non-financial interest in the subject matter or materials discussed in this manuscript. The authors did not receive support from any organization for the submitted work. No funding was received to assist with the preparation of this manuscript.

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