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On the ℓ_p and $\ell_{p,q}$ norms of Cauchy–Toeplitz matrices

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This paper is dedicated to the memory of my father and teacher, Vandansanj Tserendorj.

Abstract. New upper and lower bounds for the $\ell_p (1 < p < \infty)$ norms of Cauchy–Toeplitz matrices in the form $T_n = [2/(1 + 2(i - j))]_{i,j=1}^n$ are derived. Moreover, we give a complete answer to a conjecture proposed by D. Bozkurt.

1. Introduction

Let $C = \left[1/\left(x_i - y_j\right)\right]_{i,j=1}^n \left(x_i \neq y_j\right)$ be a Cauchy matrix, and $T_n = \left[t_{j-i}\right]_{i,j=0}^n$ be a Toeplitz matrix. In generally Cauchy–Toeplitz matrix is being defined as

$$T_n = \left[\frac{1}{g + (i-j)h}\right]_{i,j=1}^n,$$

where $h \neq 0$, g and h are any numbers and $\frac{g}{h}$ is not integer. If we substitute $g = \frac{1}{2}$ and $h = 1$, then we have

$$T_n = \left[\frac{2}{1 + 2(i-j)}\right]_{i,j=1}^n. \tag{1}$$

In [7], Tyrtyshnikov obtained lower bounds for the spectral norm of the Cauchy–Toeplitz matrices as in (1). The following upper and lower bounds for the ℓ_p norm of Cauchy–Toeplitz matrices as in (1) was published in 1998 by Bozkurt [1]: Let $p > 1$, then

$$\left[(2^p - 1)\,\zeta(p)\right]^{\frac{1}{p}} \leq n^{-\frac{1}{p}} \|T_n\|_p \leq 2^{\frac{1}{p}} \left[(2^p - 1)\,\zeta(p)\right]^{\frac{1}{p}}, \tag{2}$$

where $\zeta(p)$ is Riemann zeta function.

Unfortunately, the first inequality is not true for $1 < p < \varepsilon_n$ because of $\lim_{\varepsilon \to 0} \zeta(1 + \varepsilon) = \infty$. In the same paper, he conjectured the following.

Conjecture. *Let the matrix T_n be as in (1). Then the inequalities*

$$n^{-\frac{1}{q}} \|T_n\|_{p,q} < 4\left(\frac{1}{2} + \frac{1}{2^p - 1}\right)^{\frac{1}{p}}, \quad (p \geq q) \tag{3}$$

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and

$$n^{-\frac{1}{q}}\|T_n\|_{p,q} \geq 4\left(\frac{1}{2^p-1}\right)^{\frac{1}{p}}, \quad (p<q) \tag{4}$$

are valid for the $\ell_{p,q}$ $(1 \leq p,q \leq \infty)$ norm of the matrix T_n.
For some related results of the norm of Cauchy–Toeplitz matrices, the reader is referred to papers [3–6, 8].

The main objective of this paper is to derive new upper and lower bounds for the $\ell_p(1<p<\infty)$ norms of Cauchy–Toeplitz matrices as in (1). Moreover, we give a complete answer to the conjecture.

2. Preliminaries

In order to prove our main result, we need the following notations and lemmas.

Let A be any n-square matrix. The ℓ_p and $\ell_{p,q}(1 \leq p,q \leq \infty)$ norms of the matrix A are

$$\|A\|_p = \left[\sum_{i,j=1}^{n} \left|a_{ij}\right|^p\right]^{1/p}$$

and

$$\|A\|_{p,q} = \left[\sum_{j=1}^{n}\left(\sum_{i=1}^{n}\left|a_{ij}\right|^p\right)^{\frac{q}{p}}\right]^{\frac{1}{q}}.$$

For $r \in \mathbb{R}$, the power mean of the order r of positive numbers $x_1, x_2, \ldots, x_n$ is defined by

$$M_r(\underline{x}) = \left(\frac{1}{n}\sum_{k=1}^{n} x_k^r\right)^{\frac{1}{r}},$$

where $\underline{x} = (x_1, x_2, \ldots, x_n)$. The power mean has the following nice properties (see [2]).

Lemma 2.1. *1. (Monotonicity) If $r<s$, then*

$$M_r(\underline{x}) \leq M_s(\underline{x}).,\tag{5}$$

which is the well–known power mean inequality.
2. (Continuity) The power mean is a continuous function for r, that is

$$\lim_{\varepsilon\to 0} M_{r+\varepsilon}(\underline{x}) = M_r(\underline{x}).$$

Lemma 2.2. *Let $p \geq 1$. Then the following inequality*

$$3 - 2^p + \frac{1}{3^p} - \left(\frac{2}{3}\right)^p > 0$$

holds for $1 \leq p < \delta$. The opposite inequality holds $p > \delta$. Here $\delta = 1.40485...$ is the only solution of the equation $2^\delta + 6^\delta = 3^{\delta+1} + 1$.

Proof. Considering the function g on $[1,+\infty)$ defined by

$$g(p) = 3 - 2^p + \frac{1}{3^p} - \left(\frac{2}{3}\right)^p$$

and differentiation yields

$$g'(p) = -2^p \ln 2 - \frac{\ln 3}{3^p} - \left(\frac{2}{3}\right)^p \ln\frac{2}{3} = -2^p\left(\ln 2 - \frac{\ln 3}{3^p}\right) - \frac{\ln 3}{3^p} - \left(\frac{2}{3}\right)^p \ln 2 < 0,$$

which implies g is decreasing on $[1,+\infty[$. From $g(1) = \frac{2}{3}$, $g(2) = -\frac{4}{3}$ and g is continuous on $[1,+\infty[$, there is a unique $\delta \in (1,2)$ to satisfy equation $g(\delta)=0$. Therefore, $g(p)>0$ for $1 \leq p < \delta$ and $g(p)<0$ for $p>\delta$, which proves the desired result. □

Lemma 2.3. *Let $p > 1$. Then the following inequality*

$$\left(1-\frac{1}{2^p}\right)\zeta(p) > 2^{p-1}\left(\frac{1}{2}+\frac{1}{2^p-1}\right)$$

holds for $1 < p < \mu$. The opposite inequality holds for $p > \mu$. Here $\mu = 1.6181...$ is the only solution of the equation

$$\left(1-\frac{1}{2^\mu}\right)\zeta(\mu) = 2^{\mu-1}\left(\frac{1}{2}+\frac{1}{2^\mu-1}\right).$$

Proof. Let

$$f(p) = \left(1-\frac{1}{2^p}\right)\zeta(p) - 2^{p-1}\left(\frac{1}{2}+\frac{1}{2^p-1}\right).$$

We show that the function $f(p)$ is strictly decreasing on $(1, +\infty)$. Differentiation yields

$$f'(p) = -\sum_{n=2}^{\infty}\frac{\ln(2n-1)}{(2n-1)^p} - 2^{p-2}\ln 2\cdot\frac{2^{2p}-2^{p+1}-1}{(2^p-1)^2}.$$

To prove $f'(p) < 0$ for $p > 1$, we consider two cases.

Case (i): $1 < p \le \mu$. Since a function $\frac{\ln(2x-1)}{(2x-1)^p}$ is strictly decreasing on $[2, \infty)$, we have

$$\begin{aligned}
f'(p) &< -\int_2^\infty \frac{\ln(2x-1)}{(2x-1)^p}\,dx - 2^{p-2}\ln 2\cdot\frac{2^{2p}-2^{p+1}-1}{(2^p-1)^2}\\
&= -\frac{\ln 3}{2(p-1)\cdot 3^{p-1}} - \frac{1}{2(p-1)^2\cdot 3^{p-1}} - 2^{p-2}\ln 2 + \frac{2^{p-1}\ln 2}{(2^p-1)^2}.
\end{aligned}$$

Using an obvious inequalities $1+\ln 3 > 2\ln 2$, $2^{p-1} < 2^p - 1$ and $\frac{1}{2^p-1} - 2^{p-2} < \frac{1}{2}$ lead to

$$\begin{aligned}
f'(p) &< -\frac{1+\ln 3}{2\cdot 3^{p-1}} - 2^{p-2}\ln 2 + \frac{2^{p-1}\ln 2}{(2^p-1)^2}\\
&< -\frac{\ln 2}{3^{p-1}} - 2^{p-2}\ln 2 + \frac{2^{p-1}\ln 2}{(2^p-1)^2}\\
&< -\frac{\ln 2}{3^{p-1}} - 2^{p-2}\ln 2 + \frac{\ln 2}{2^p-1}\\
&= \ln 2\cdot\left(\frac{1}{2^p-1} - 2^{p-2} - \frac{1}{3^{p-1}}\right)\\
&< \ln 2\cdot\left(\frac{1}{2} - \frac{1}{3^{p-1}}\right) \le \ln 2\cdot\left(\frac{1}{2} - \frac{1}{3^{\mu-1}}\right) \approx -0.005 < 0.
\end{aligned}$$

Case (ii): $p > \mu$. Since $f_1(p) = 2^{2p} - 2^{p+1} - 1$ is increasing on (μ, ∞), we obtain

$$2^{2p} - 2^{p+1} - 1 > 0.$$

This implies that $f'(p) < 0$ for $p \in (\mu, \infty)$.

Clearly, the function $f(p)$ is continuous in $(1, +\infty)$. So, there is a unique $\mu \in (1, \infty)$ to satisfy equation $f(\mu) = 0$. Hence $f(p) > 0$ for $1 < p < \mu$ and $f(p) < 0$ for $p > \mu$. The proof of Lemma 2.3 is complete. □

Lemma 2.4. *Let $p \ge 1$. Then*

$$n^{-1}\|T_n\|_{p,1} \le (n+1)^{-1}\|T_{n+1}\|_{p,1} \tag{6}$$

and

$$\lim_{n\to\infty} n^{-1}\|T_n\|_{p,1} = 2^{\frac{1}{p}}\left((2^p-1)\,\zeta(p)\right)^{\frac{1}{p}}, \quad p > 1. \tag{7}$$

Proof. Let $n_0 = \left[\frac{n}{2}\right] + 1$. Since $\sum_{i=1}^{n} |a_{im}|^p = \sum_{i=2}^{n+1} |a_{i(m+1)}|^p$, we have

$$\sum_{i=1}^{n+1} |a_{i(m+1)}|^p > \sum_{i=1}^{n} |a_{im}|^p$$

and

$$\sum_{i=1}^{n+1} |a_{im}|^p > \sum_{i=1}^{n} |a_{im}|^p.$$

Moreover, it is easy to see that

$$\max_{1\le m\le n} \sum_{i=1}^{n} |a_{im}|^p = \sum_{i=1}^{n} |a_{in_0}|^p.$$

Thus, we have

$$\begin{aligned}
&n\|T_{n+1}\|_{p,1} - (n+1)\|T_n\|_{p,1} \\
&\quad = n\sum_{j=1}^{n+1}\left(\sum_{i=1}^{n+1}|a_{ij}|^p\right)^{\frac{1}{p}} - (n+1)\sum_{j=1}^{n}\left(\sum_{i=1}^{n}|a_{ij}|^p\right)^{\frac{1}{p}} \\
&\quad = n\left(\sum_{\substack{1\le j\le n+1\\ j\ne n_0}}\left(\sum_{i=1}^{n+1}|a_{ij}|^p\right)^{\frac{1}{p}} - \sum_{j=1}^{n}\left(\sum_{i=1}^{n}|a_{ij}|^p\right)^{\frac{1}{p}}\right) + n\left(\sum_{i=1}^{n+1}|a_{in_0}|^p\right)^{\frac{1}{p}} - \sum_{j=1}^{n}\left(\sum_{i=1}^{n}|a_{ij}|^p\right)^{\frac{1}{p}} \\
&\quad > n\left(\sum_{\substack{1\le j\le n+1\\ j\ne n_0}}\left(\sum_{i=1}^{n+1}|a_{ij}|^p\right)^{\frac{1}{p}} - \sum_{j=1}^{n}\left(\sum_{i=1}^{n}|a_{ij}|^p\right)^{\frac{1}{p}}\right) + n\left(\sum_{i=1}^{n+1}|a_{in_0}|^p\right)^{\frac{1}{p}} - n\cdot\max_{1\le j\le n}\left(\sum_{i=1}^{n}|a_{ij}|^p\right)^{\frac{1}{p}} \\
&\quad = n\left(\sum_{\substack{1\le j\le n+1\\ j\ne n_0}}\left(\sum_{i=1}^{n+1}|a_{ij}|^p\right)^{\frac{1}{p}} - \sum_{j=1}^{n}\left(\sum_{i=1}^{n}|a_{ij}|^p\right)^{\frac{1}{p}}\right) + n\left(\sum_{i=1}^{n+1}|a_{in_0}|^p\right)^{\frac{1}{p}} - n\left(\sum_{i=1}^{n}|a_{in_0}|^p\right)^{\frac{1}{p}} > 0.
\end{aligned}$$

By the Stolz–Cesàro theorem, we get

$$\begin{aligned}
\lim_{n\to\infty} n^{-1}\|T_n\|_{p,1} &= \lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\left(\sum_{i=1}^{n}|a_{ij}|^p\right)^{\frac{1}{p}} \\
&= \lim_{n\to\infty}\left[\sum_{j=1}^{n}\left(\sum_{i=1}^{n}|a_{ij}|^p\right)^{\frac{1}{p}} - \sum_{j=1}^{n-1}\left(\sum_{i=1}^{n-1}|a_{ij}|^p\right)^{\frac{1}{p}}\right] \\
&= \lim_{n\to\infty}\left[\left(\sum_{i=1}^{n}|a_{in_0}|^p\right)^{\frac{1}{p}} + \sum_{j=1}^{n-1}\left(b_j^{\frac{1}{p}} - \left(\sum_{i=1}^{n-1}|a_{ij}|^p\right)^{\frac{1}{p}}\right)\right],
\end{aligned}$$

where $b_j = \begin{cases} \sum_{i=1}^{n} |a_{i(j+1)}|^p, & j \ge n_0 \\ \sum_{i=1}^{n} |a_{ij}|^p, & j < n_0 \end{cases}$.

Using Lagrange's mean value theorem, we have

$$b_j^{\frac{1}{p}} - \left(\sum_{i=1}^{n-1}|a_{ij}|^p\right)^{\frac{1}{p}} = \begin{cases} \frac{1}{p}|a_{1(j+1)}|^p\cdot C_{nj}^{\frac{1}{p}-1}, & j \ge n_0 \\ \frac{1}{p}|a_{nj}|^p\cdot C_{nj}^{\frac{1}{p}-1}, & j \ge n_0 \end{cases}$$

where, $\sum_{i=1}^{n-1}|a_{ij}|^p < C_{nj} < b_j$. Thus, we get that

$$
\begin{aligned}
0 < \sum_{j=1}^{n-1}\left(b_j^{\frac{1}{p}} - \left(\sum_{i=1}^{n-1}|a_{ij}|^p\right)^{\frac{1}{p}}\right) &= \frac{1}{p}\left(\sum_{j=1}^{n_0-1}|a_{nj}|^p\cdot C_{nj}^{\frac{1}{p}-1} + \sum_{j=n_0}^{n-1}|a_{1(j+1)}|^p\cdot C_{nj}^{\frac{1}{p}-1}\right)\\
&< \frac{1}{p}\left(\sum_{j=1}^{n_0-1}|a_{nj}|^p\left(\sum_{i=1}^{n-1}|a_{ij}|^p\right)^{\frac{1}{p}-1} + \sum_{j=n_0}^{n-1}|a_{1(j+1)}|^p\left(\sum_{i=1}^{n-1}|a_{ij}|^p\right)^{\frac{1}{p}-1}\right)\\
&< \frac{1}{p\cdot(n-1)^{1-\frac{1}{p}}}\left(\sum_{j=1}^{n_0-1}|a_{nj}| + \sum_{j=n_0}^{n-1}|a_{1(j+1)}|\right).
\end{aligned}
$$

Applying the Stolz–Cesàro theorem, we obtain

$$
\begin{aligned}
&\lim_{n\to\infty}\frac{1}{p\cdot(n-1)^{1-\frac{1}{p}}}\left(\sum_{j=1}^{n_0-1}|a_{nj}| + \sum_{j=n_0}^{n-1}|a_{1(j+1)}|\right)\\
&\quad= \frac{1}{p}\lim_{n\to\infty}\frac{1}{n^{1-\frac{1}{p}}-(n-1)^{1-\frac{1}{p}}}\left(\sum_{j=1}^{(n+1)_0-1}|a_{(n+1)j}| + \sum_{j=(n+1)_0}^{n}|a_{1(j+1)}| - \sum_{j=1}^{n_0-1}|a_{nj}| - \sum_{j=n_0}^{n-1}|a_{1(j+1)}|\right)\\
&\quad= \frac{1}{p}\lim_{n\to\infty}\frac{1}{n^{1-\frac{1}{p}}-(n-1)^{1-\frac{1}{p}}}S_n = 0,
\end{aligned}
$$

since $S_n = \begin{cases}\frac{2}{2n-1}, & \text{if } n \text{ is even}\\ \frac{4n^2+1}{4n^3-n}, & \text{if } n \text{ is odd}\end{cases}$. Hence

$$
\lim_{n\to\infty}\sum_{j=1}^{n-1}\left(b_j^{\frac{1}{p}} - \left(\sum_{i=1}^{n-1}|a_{ij}|^p\right)^{\frac{1}{p}}\right) = 0.
$$

In view of above, we have that

$$
\lim_{n\to\infty} n^{-1}\|T_n\|_{p,1} = \lim_{n\to\infty}\left(\sum_{i=1}^{n}|a_{in_0}|^p\right)^{\frac{1}{p}} = \lim_{n\to\infty} 2\left(\sum_{i=1}^{n_0-1}\frac{2}{(2k-1)^p} + \frac{(-1)^{n+1}+1}{2(2n_0-1)^p}\right)^{\frac{1}{p}} = 2^{\frac{1}{p}}\left((2^p-1)\,\zeta(p)\right)^{\frac{1}{p}}.
$$

This completes the proof of Lemma 2.4. □

3. Main Results

From the definition of ℓ_p norm, we have

$$
\|T_n\|_p = \left[\sum_{i,j=1}^{n}\left|a_{ij}\right|^p\right]^{\frac{1}{p}} = \left[2^p\sum_{k=1}^{n}\frac{2n-2k+1}{(2k-1)^p}\right]^{\frac{1}{p}}. \tag{8}
$$

We begin with the following theorem that gives a lower bound for $n^{-\frac{1}{p}}\|T_n\|_p$, which is independent of the size.

Theorem 3.1. *Let T_n be as in (1), and let $p \geq 1$. If $m \geq n$, then*

$$
2 \leq n^{-\frac{1}{p}}\|T_n\|_p \leq m^{-\frac{1}{p}}\|T_m\|_p. \tag{9}
$$

Proof. From the equality (8), we have

$$\begin{aligned}\frac{1}{n+1}\|T_{n+1}\|_p^p-\frac{1}{n}\|T_n\|_p^p&=\frac{2^p}{n+1}\sum_{k=1}^{n+1}\frac{2n-2k+3}{(2k-1)^p}-\frac{2^p}{n}\sum_{k=1}^{n}\frac{2n-2k+1}{(2k-1)^p}\\&=2^p\left(\frac{1}{n+1}\sum_{k=1}^{n+1}\frac{2n-2k+3}{(2k-1)^p}-\frac{1}{n}\sum_{k=1}^{n}\frac{2n-2k+1}{(2k-1)^p}\right)\\&=2^p\left(\frac{1}{(n+1)(2n+1)^p}+\sum_{k=1}^{n}\left(\frac{2n-2k+3}{(n+1)(2k-1)^p}-\frac{2n-2k+1}{n(2k-1)^p}\right)\right)\\&=2^p\left(\frac{1}{(2n+1)^p}+\frac{1}{n(n+1)}\sum_{k=1}^{n}\frac{1}{(2k-1)^{p-1}}\right)>0.\end{aligned}$$

Thus, we have

$$2=\|T_1\|_p\le n^{-\frac{1}{p}}\|T_n\|_p\le m^{-\frac{1}{p}}\|T_m\|_p.$$

The theorem is proved. □

Remark 3.2. *By the calculation in [1], we have*

$$\lim_{m\to\infty}m^{-\frac{1}{p}}\|T_m\|_p=2^{\frac{1}{p}}\left[(2^p-1)\zeta(p)\right]^{\frac{1}{p}},$$

hence

$$2\le n^{-\frac{1}{p}}\|T_n\|_p<2^{\frac{1}{p}}\left[(2^p-1)\zeta(p)\right]^{\frac{1}{p}}.$$

We assume that $1<p<\mu$. Then, by the Lemma 2.3, there is a positive integer $N_{1,p}$ such that

$$N_{1,p}^{-\frac{1}{p}}\|T_{N_{1,p}}\|_p\le 4\left(\frac{1}{2}+\frac{1}{2^p-1}\right)^{\frac{1}{p}}\le (N_{1,p}+1)^{-\frac{1}{p}}\|T_{N_{1,p}+1}\|_p,$$

for fixed p.

Remark 3.3. *Similarly to the above remark, if $1<p<\mu$, then by Lemma 2.3 and Lemma 2.4, there is a positive integer $N_{2,p}$ such that*

$$N_{2,p}^{-1}\|T_{N_{2,p}}\|_{p,1}\le 4\left(\frac{1}{2}+\frac{1}{2^p-1}\right)^{\frac{1}{p}}\le (N_{2,p}+1)^{-1}\|T_{N_{2,p}+1}\|_{p,1},$$

for fixed p. In Table 1, we give some values of $N_{1,p}$ and $N_{2,p}$.

p	1.1	1.2	1.3	1.4	1.5	1.51	1.52	1.53	1.54	1.55	1.56
$N_{1,p}$	8	10	13	20	44	49	56	65	76	92	115
$N_{2,p}$	8	10	13	20	44	50	57	66	78	94	118

Table 1:

The next theorem gives new bounds for $n^{-\frac{1}{p}}\|T_n\|_p$ that depend on n and p.

Theorem 3.4. *Let T_n be as in (1), and let $p>1, n\ge 2$. Then*

$$\left[\left(1+\frac{2}{n^2}\right)(2^p-1)\zeta(p)-C'(n,p)\right]^{\frac{1}{p}}<n^{-\frac{1}{p}}\|T_n\|_p<\left(2-\frac{1}{n}\right)^{\frac{1}{p}}\left[(2^p-1)\zeta(p)-C''(n,p)\right]^{\frac{1}{p}},\tag{10}$$

where

$$C'(n,p)=\frac{n^2+2}{(p-1)n^{p+1}}+\frac{2^{p+1}}{n(2n-1)^p},\quad C''(n,p)=\frac{2^{p-1}}{(p-1)(2n+1)^{p-1}}.$$

Proof. By the equality (8) and using the Binet–Cauchy identity with $a_k = \frac{1}{n}, b_k = 2n-2k+1, c_k = \frac{1}{n}, d_k = \frac{1}{(2k-1)^p}$, we have

$$\begin{aligned}
\|T\|_p &= 2\left[\sum_{k=1}^{n} \frac{2n-2k+1}{(2k-1)^p}\right]^{\frac{1}{p}} \\
&= 2\left[\frac{1}{n}\left(\sum_{k=1}^{n} b_k\right)\left(\sum_{k=1}^{n} d_k\right) + \frac{1}{n}\sum_{1\le k<m\le n}(b_m-b_k)(d_m-d_k)\right]^{\frac{1}{p}} \\
&= 2\left[n\sum_{k=1}^{n}\frac{1}{(2k-1)^p} + \frac{2}{n}\sum_{k=1}^{n-1}\sum_{m=k+1}^{n}(m-k)\left(\frac{1}{(2k-1)^p} - \frac{1}{(2m-1)^p}\right)\right]^{\frac{1}{p}}. \qquad (11)
\end{aligned}$$

Since $d_k \ge d_{m-1}$ and $m-k \ge 1$ for $1 \le k < m \le n$, we have

$$\begin{aligned}
\sum_{k=1}^{n-1}\sum_{m=k+1}^{n}(m-k)\left(\frac{1}{(2k-1)^p} - \frac{1}{(2m-1)^p}\right) &> \sum_{k=1}^{n-1}\sum_{m=k+1}^{n}\left(\frac{1}{(2m-3)^p} - \frac{1}{(2m-1)^p}\right) \\
&= \sum_{k=1}^{n-1}\left(\frac{1}{(2k-1)^p} - \frac{1}{(2n-1)^p}\right) \\
&= \sum_{k=1}^{n}\frac{1}{(2k-1)^p} - \frac{n}{(2n-1)^p},
\end{aligned}$$

hence

$$\|T\|_p > 2n^{\frac{1}{p}}\left[\left(1+\frac{2}{n^2}\right)\sum_{k=1}^{n}\frac{1}{(2k-1)^p} - \frac{2}{n(2n-1)^p}\right]^{\frac{1}{p}}.$$

On the other hand, the function $\frac{1}{(2x-1)^p}$ is a convex on interval $[k-\frac{1}{2}, k+\frac{1}{2}]$, application of the well-known Hermite–Hadamard inequality on intervals $[k-\frac{1}{2}, k+\frac{1}{2}],\ k \ge n+1$ yields

$$\begin{aligned}
\sum_{k=1}^{n}\frac{1}{(2k-1)^p} &= \sum_{k=1}^{\infty}\frac{1}{(2k-1)^p} - \sum_{k=n+1}^{\infty}\frac{1}{(2k-1)^p} \\
&\ge \left(1-\frac{1}{2^p}\right)\zeta(p) - \sum_{k=n+1}^{\infty}\int_{k-\frac{1}{2}}^{k+\frac{1}{2}}\frac{1}{(2x-1)^p}dx \\
&= \left(1-\frac{1}{2^p}\right)\zeta(p) - \int_{n+\frac{1}{2}}^{\infty}\frac{1}{(2x-1)^p}dx \\
&= \left(1-\frac{1}{2^p}\right)\zeta(p) - \frac{1}{2^p(p-1)n^{p-1}}.
\end{aligned}$$

Then it follows

$$n^{-\frac{1}{p}}\|T\|_p > \left[\left(1+\frac{2}{n^2}\right)(2^p-1)\zeta(p) - \frac{n^2+2}{(p-1)n^{p+1}} - \frac{2^{p+1}}{n(2n-1)^p}\right]^{\frac{1}{p}}.$$

Now we shall prove the last inequality in (10). From (11), we have

$$\begin{aligned}
\|T_n\|_p &= 2\left[n\sum_{k=1}^{n}\frac{1}{(2k-1)^p}+\frac{2}{n}\sum_{k=1}^{n-1}\sum_{m=k+1}^{n}(m-k)\left(\frac{1}{(2k-1)^p}-\frac{1}{(2m-1)^p}\right)\right]^{\frac{1}{p}}\\
&< 2\left[n\sum_{k=1}^{n}\frac{1}{(2k-1)^p}+\frac{2}{n}\sum_{k=1}^{n-1}\frac{1}{(2k-1)^p}\sum_{m=k+1}^{n}(m-k)\right]^{\frac{1}{p}}\\
&= 2\left[n\sum_{k=1}^{n}\frac{1}{(2k-1)^p}+\frac{1}{n}\sum_{k=1}^{n-1}\frac{(n-k)(n-k+1)}{(2k-1)^p}\right]^{\frac{1}{p}}\\
&\le 2\left[n\sum_{k=1}^{n}\frac{1}{(2k-1)^p}+\frac{1}{n}\sum_{k=1}^{n}\frac{(n-1)n}{(2k-1)^p}\right]^{\frac{1}{p}}\\
&= 2n^{\frac{1}{p}}\left[\left(2-\frac{1}{n}\right)\sum_{k=1}^{n}\frac{1}{(2k-1)^p}\right]^{\frac{1}{p}}.
\end{aligned}$$

By the decreasing property of the function $\frac{1}{(2x-1)^p}$, it follows that

$$\begin{aligned}
\sum_{k=1}^{n}\frac{1}{(2k-1)^p} &= \sum_{k=1}^{\infty}\frac{1}{(2k-1)^p}-\sum_{k=n+1}^{\infty}\frac{1}{(2k-1)^p}\\
&< \left(1-\frac{1}{2^p}\right)\zeta(p)-\int_{n+1}^{\infty}\frac{1}{(2x-1)^p}dx\\
&= \left(1-\frac{1}{2^p}\right)\zeta(p)-\frac{1}{2(p-1)(2n+1)^{p-1}}.
\end{aligned}$$

Thus, we get

$$n^{-\frac{1}{p}}\|T_n\|_p < \left(2-\frac{1}{n}\right)^{\frac{1}{p}}\left[(2^p-1)\zeta(p)-\frac{2^{p-1}}{(p-1)(2n+1)^{p-1}}\right]^{\frac{1}{p}}.$$

The proof is complete. □

Remark 3.5. *It should be noticed here that the right-hand side of (10) improves the right-hand side of (2).*

The following two theorems give an answer to the conjecture.

Theorem 3.6. *Let T_n be as in (1), and let $1 \le p < q \le \infty$. Then the following inequality*

$$n^{-\frac{1}{q}}\|T_n\|_{p,q} \ge 4\left(\frac{1}{2^p-1}\right)^{\frac{1}{p}}, \tag{12}$$

holds for $n \ge 3$ and for $n = 2, (p \ge \delta$ or $1 < p < \delta, q \ge \delta_p)$, where δ is as in the Lemma 2.2. The opposite inequality holds for $n = 1$ and for $n = 2, (p = 1$ or $1 < p < \delta, q < \delta_p)$. Here $q = \delta_p$ is the only solution of the following equation

$$2^{-p}\left(\left(1+\frac{1}{3^p}\right)^{\frac{q}{p}}+2^{\frac{q}{p}}\right)^p = 2^{pq}\left(\frac{1}{2^p-1}\right)^q, \tag{13}$$

for fixed p.

Proof. Case (i): $n \geq 3$. By the power mean inequality, we have

$$\begin{aligned}
n^{-\frac{1}{q}} \|T_n\|_{p,q} &= n^{-\frac{1}{q}} \left[\sum_{j=1}^{n} \left(\sum_{i=1}^{n} |a_{ij}|^p \right)^{\frac{q}{p}} \right]^{\frac{1}{q}} = \left[\frac{1}{n} \sum_{j=1}^{n} \left(\sum_{i=1}^{n} |a_{ij}|^p \right)^{\frac{q}{p}} \right]^{\frac{1}{q}} \\
&= M_q \left(\underline{\left(\sum_{i=1}^{n} |a_{ij}|^p \right)^{\frac{1}{p}}} \right) \\
&> M_p \left(\underline{\left(\sum_{i=1}^{n} |a_{ij}|^p \right)^{\frac{1}{p}}} \right) = \left[\frac{1}{n} \sum_{j=1}^{n} \left(\sum_{i=1}^{n} |a_{ij}|^p \right)^{\frac{p}{p}} \right]^{\frac{1}{p}} \\
&= n^{-\frac{1}{p}} \left[\sum_{j=1}^{n} \sum_{i=1}^{n} |a_{ij}|^p \right]^{\frac{1}{p}} = n^{-\frac{1}{p}} \|T_n\|_p.
\end{aligned} \tag{14}$$

Applying (9), gives

$$n^{-\frac{1}{p}} \|T_n\|_p > 3^{-\frac{1}{p}} \|T_3\|_p = 3^{-\frac{1}{p}} \left(2^p \left(5 + \frac{1}{3^{p-1}} + \frac{1}{5^p} \right) \right)^{\frac{1}{p}}. \tag{15}$$

Combining (14) and (15) leads to

$$n^{-\frac{1}{q}} \|T_n\|_{p,q} > 3^{-\frac{1}{p}} \left(2^p \left(5 + \frac{1}{3^{p-1}} + \frac{1}{5^p} \right) \right)^{\frac{1}{p}}.$$

To prove the inequality (12) for $n \geq 3$, it suffices to show that the following inequality

$$3^{-\frac{1}{p}} \left(2^p \left(5 + \frac{1}{3^{p-1}} + \frac{1}{5^p} \right) \right)^{\frac{1}{p}} > 4 \left(\frac{1}{2^p - 1} \right)^{\frac{1}{p}},$$

i.e.,

$$2^{p+1} - 5 + 3 \left(\frac{2}{3} \right)^p + \left(\frac{2}{5} \right)^p - \frac{1}{3^{p-1}} - \frac{1}{5^p} > 0.$$

Considering the function h on $[1, +\infty)$ defined by

$$h(p) = 2^{p+1} - 5 + 3 \left(\frac{2}{3} \right)^p + \left(\frac{2}{5} \right)^p - \frac{1}{3^{p-1}} - \frac{1}{5^p}.$$

Differentiation yields

$$\begin{aligned}
h'(p) &= 2^{p+1} \ln 2 + 3 \left(\frac{2}{3} \right)^p \ln \frac{2}{3} + \left(\frac{2}{5} \right)^p \ln \frac{2}{5} + \frac{1}{3^{p-1}} \ln 3 + \frac{1}{5^p} \ln 5 \\
&> 2^{p+1} \ln 2 + 3 \left(\frac{2}{3} \right)^p \ln \frac{2}{3} + \left(\frac{2}{5} \right)^p \ln \frac{2}{5} := h_1(p)
\end{aligned}$$

and

$$h_1'(p) = 2^{p+1} (\ln 2)^2 + 3 \left(\frac{2}{3} \right)^p \left(\ln \frac{2}{3} \right)^2 + \left(\frac{2}{5} \right)^p \left(\ln \frac{2}{5} \right)^2 > 0.$$

Thus, we deduce that for $p \geq 1$,

$$h_1(p) \geq h_1(1) = 4 \ln 2 + 2 \ln \frac{2}{3} + \frac{2}{5} \ln \frac{2}{5} \approx 1.5951 \ldots > 0.$$

Therefore, the function $h(p)$ is increasing on $[1, +\infty[$, which implies that

$$h(p) > h(1) = \frac{1}{5} > 0$$

for $p \geq 1$.

Case (ii): $n = 2, p \geq \delta$. From the inequality (14), we obtain

$$2^{-\frac{1}{q}} \|T_2\|_{p,q} > 2^{-\frac{1}{p}} \|T_2\|_p .$$

Clearly, it suffices to show that

$$2^{-\frac{1}{p}} \|T_2\|_p \geq 4\left(\frac{1}{2^p - 1}\right)^{\frac{1}{p}}, \tag{16}$$

This inequality is equivalent to

$$3 - 2^p + \frac{1}{3^p} - \left(\frac{2}{3}\right)^p \leq 0,$$

which is true by the Lemma 2.2.

Case (iii): $n = 2, 1 \leq p < \delta$. If $p = 1$, then by the power mean inequality, we have

$$2^{-\frac{1}{q}} \|T_2\|_{1,q} = 2^{1-\frac{1}{q}} \left(\left(\frac{4}{3}\right)^q + 2^q\right)^{\frac{1}{q}} < \lim_{q\to\infty} 2^{1-\frac{1}{q}} \left(\left(\frac{4}{3}\right)^q + 2^q\right)^{\frac{1}{q}} = 4.$$

In other words, the inequality (12) does not hold for $p = 1$.

Similarly to the previous case, one has for $1 \leq p < \delta$,

$$2^{-\frac{1}{p}} \|T_2\|_p < 4\left(\frac{1}{2^p - 1}\right)^{\frac{1}{p}} . \tag{17}$$

If $1 < p < \delta$, then by the power mean inequality, we have

$$2^{-\frac{1}{q}} \|T_2\|_{p,q} = 2^{1-\frac{1}{q}} \left(\left(1 + \frac{1}{3^p}\right)^{\frac{q}{p}} + 2^{\frac{q}{p}}\right)^{\frac{1}{q}} < \lim_{q\to+\infty} 2^{1-\frac{1}{q}} \left(\left(1 + \frac{1}{3^p}\right)^{\frac{q}{p}} + 2^{\frac{q}{p}}\right)^{\frac{1}{q}} = 2^{1+\frac{1}{p}}.$$

It is easy to see that

$$4\left(\frac{1}{2^p - 1}\right)^{\frac{1}{p}} < 2^{1+\frac{1}{p}}.$$

By the continuity and monotonicity of power mean, there are q' and q'' such that $p < q' \leq q''$ and

$$2^{-\frac{1}{p}} \|T_2\|_p < 2^{-\frac{1}{q}} \|T_2\|_{p,q'} \leq 4\left(\frac{1}{2^p - 1}\right)^{\frac{1}{p}} \leq 2^{-\frac{1}{q}} \|T_2\|_{p,q''} < 2^{1+\frac{1}{p}}.$$

Hence, there is exist a unique $q = \delta_p$ for fixed p, such that

$$2^{-\frac{1}{q}} \|T_2\|_{p,q} = 4\left(\frac{1}{2^p - 1}\right)^{\frac{1}{p}},$$

that is,

$$2^{-p} \left(\left(1 + \frac{1}{3^p}\right)^{\frac{\delta_p}{p}} + 2^{\frac{\delta_p}{p}}\right)^p = 2^{p\delta_p} \left(\frac{1}{2^p - 1}\right)^{\delta_p}.$$

Therefore, by the power mean inequality, we have

$$2^{-\frac{1}{q}} \|T_2\|_{p,q} < 4\left(\frac{1}{2^p - 1}\right)^{\frac{1}{p}},$$

for $1 < q < \delta_p$, and

$$2^{-\frac{1}{q}} \|T_2\|_{p,q} \geq 4\left(\frac{1}{2^p - 1}\right)^{\frac{1}{p}},$$

for $q \geq \delta_p$. We thus complete the proof. □

Remark 3.7. *In Table 2, we give some values of δ_p.*

p	1.1	1.15	1.2	1.25	1.3	1.35	1.4
δ_p	11.583...	7.900...	5.801...	4.347...	3.222...	2.289...	1.478...

Table 2:

Theorem 3.8. *Let T_n be as in (1), and let $1 \leq q \leq p \leq \infty$. Let μ be as stated in Lemma 2.3, and let N_1 and N_2 be as defined in Remarks 3.2 and 3.3, respectively. Then the following inequality*

$$n^{-\frac{1}{q}} \|T_n\|_{p,q} < 4\left(\frac{1}{2} + \frac{1}{2^p - 1}\right)^{\frac{1}{p}}, \tag{18}$$

holds for

- $p = 1, 1 \leq n \leq 7$
- $\mu \leq p$
- $1 < p < \mu, n \leq N_{1,p}, (N_{1,p} \geq 7)$
- $1 < p < \mu, N_{1,p} < n \leq N_{2,p}, 1 \leq q < \eta_{p,n}$.

The opposite inequality holds for other values of n, p, q. Here $q = \eta_{p,n}$ is the only solution of the following

$$n^{-\frac{1}{q}} \|T_n\|_{p,q} = 4\left(\frac{1}{2} + \frac{1}{2^p - 1}\right)^{\frac{1}{p}},$$

for fixed p $(1 < p < \mu)$ and n $(N_{1,p} < n \leq N_{2,p})$.

Proof. By the power mean inequality, we have

$$n^{-1} \|T_n\|_{p,1} \leq n^{-\frac{1}{q}} \|T_n\|_{p,q} \leq n^{-\frac{1}{p}} \|T_n\|_p. \tag{19}$$

To prove our result, we consider four cases.

Case (i): $p = 1$. We have

$$\lim_{n\to\infty} n^{-1} \|T_n\|_{1,1} = \lim_{n\to\infty} n^{-1} \|T_n\|_1 = \lim_{n\to\infty} \left[2\left(\sum_{k=1}^{n} \frac{1}{2k-1}\right) - 1\right] = \infty.$$

Hence, by the inequality (6) and simple computation, we get

$$n^{-1} \|T_n\|_{1,1} < 6$$

for $n \leq 7$, and

$$n^{-1}\|T_n\|_{1,1} > 6$$

for $n \geq 8$.

Case (ii): $\mu \leq p$. By inequality (19) and Theorem 3.1, we have

$$n^{-\frac{1}{q}}\|T_n\|_{p,q} \leq n^{-\frac{1}{p}}\|T_n\|_p < 2^{\frac{1}{p}}\left[(2^p-1)\zeta(p)\right]^{\frac{1}{p}}.$$

If we show that

$$2^{\frac{1}{p}}\left[(2^p-1)\zeta(p)\right]^{\frac{1}{p}} \leq 4\left(\frac{1}{2}+\frac{1}{2^p-1}\right)^{\frac{1}{p}},$$

then the desired result follows. This inequality is equivalent to

$$\left(1-\frac{1}{2^p}\right)\zeta(p) \leq 2^{p-1}\left(\frac{1}{2}+\frac{1}{2^p-1}\right),$$

which is true by the Lemma 2.3.

Case (iii): $1 < p < \mu, n \leq N_{1,p}$. By inequality (19), Theorem 3.1 and Remark 3.2, we have

$$n^{-\frac{1}{q}}\|T_n\|_{p,q} \leq n^{-\frac{1}{p}}\|T_n\|_p \leq N_{1,p}^{-\frac{1}{p}}\|T_{N_{1,p}}\|_p \leq 4\left(\frac{1}{2}+\frac{1}{2^p-1}\right)^{\frac{1}{p}}.$$

Now, we will prove that $N_{1,p} \geq 7$. By inequality (19) and Theorem 3.1, we have

$$n^{-\frac{1}{q}}\|T_n\|_{p,q} \leq n^{-\frac{1}{p}}\|T_n\|_p \leq 7^{-\frac{1}{p}}\|T_7\|_p.$$

Hence it is enough to prove that

$$7^{-\frac{1}{p}}\|T_7\|_p < 4\left(\frac{1}{2}+\frac{1}{2^p-1}\right)^{\frac{1}{p}}.$$

This inequality equivalent to

$$7\cdot 2^{p-1}+\frac{7\cdot 2^p}{2^p-1}-13-\frac{11}{3^p}-\frac{9}{5^p}-\frac{1}{7^{p-1}}-\frac{5}{9^p}-\frac{3}{11^p}-\frac{1}{13^p} > 0.$$

Let

$$\psi(p) = 7\cdot 2^{p-1}+\frac{7\cdot 2^p}{2^p-1}-13-\frac{11}{3^p}-\frac{9}{5^p}-\frac{1}{7^{p-1}}-\frac{5}{9^p}-\frac{3}{11^p}-\frac{1}{13^p}.$$

Differentiation yields

$$\psi'(p) = 7\cdot 2^{p-1}\ln 2-\frac{7\cdot 2^p\ln 2}{(2^p-1)^2}+\frac{11\ln 3}{3^p}+\frac{9\ln 5}{5^p}+\frac{\ln 7}{7^{p-1}}+\frac{5\ln 9}{9^p}+\frac{3\ln 11}{11^p}+\frac{\ln 13}{13^p}.$$

Using an obvious inequalities $\ln 3 > \ln 2$, $\ln 5 > 2\ln 2$, $\ln 7 > 2\ln 2$ and $2^{p-1} < 2^p - 1$ lead to

$$\begin{aligned}
\psi'(p) &> 7\cdot 2^{p-1}\ln 2-\frac{7\cdot 2^p\ln 2}{(2^p-1)^2}+\frac{11\ln 3}{3^p}+\frac{9\ln 5}{5^p}+\frac{\ln 7}{7^{p-1}}\\
&> 7\cdot 2^{p-1}\ln 2-\frac{7\cdot 2^p\ln 2}{(2^p-1)^2}+\frac{11\ln 2}{3^p}+\frac{18\ln 2}{5^p}+\frac{2\ln 2}{7^{p-1}}\\
&> \left(7\cdot 2^{p-1}-\frac{7}{2^{p-2}}+\frac{11}{3^p}+\frac{18}{5^p}+\frac{2}{7^{p-1}}\right)\ln 2\\
&> \left(7\cdot 2^{p-1}-14+\frac{11}{3^p}+\frac{18}{5^p}+\frac{2}{7^{p-1}}\right)\ln 2 := \psi_1(p)\ln 2.
\end{aligned}$$

Next, we show that $\psi_1(p) > 0$ for $1 \le p \le \mu$. Differentiation yields

$$\psi_1'(p) = 7 \cdot 2^{p-1} \ln 2 - \frac{11 \ln 3}{3^p} - \frac{18 \ln 5}{5^p} - \frac{2 \ln 7}{7^{p-1}},$$

$$\psi_1''(p) = 7 \cdot 2^{p-1} (\ln 2)^2 + \frac{11 (\ln 3)^2}{3^p} + \frac{18 (\ln 5)^2}{5^p} + \frac{2 (\ln 7)^2}{7^{p-1}} > 0.$$

This reveals that for $1 \le p \le \mu$,

$$\psi_1(p) \ge \psi_1(1.5) + \psi_1'(1.5)(p - 1.5) \approx 0.38 + 0.47(p - 1.5) > 0.$$

Thus, $\psi(p)$ is strictly increasing on $[1, \mu]$. Therefore, we get

$$\psi(p) > \psi(1) = \frac{4042}{6435} > 0,$$

for $1 < p < \mu$.

Case (iv): $1 < p \le \nu, n > N_{1,p}$: By inequality (19), we get $N_{2,p} \ge N_{1,p}$. We consider two sub-cases.

Case (iv-a): $N_{1,p} < n \le N_{2,p}$. By the Remark 3.2 and Remark 3.3, we have

$$n^{-1} \|T_n\|_{p,1} \le 4 \left(\frac{1}{2} + \frac{1}{2^p - 1} \right)^{\frac{1}{p}} \le n^{-\frac{1}{p}} \|T_n\|_p.$$

From the inequality (19) and continuity of the power means, there is unique number $\eta_{p,n}$ such that $1 < \eta_{p,n} < p$ and

$$n^{-\frac{1}{\eta_{p,n}}} \|T_n\|_{p,\eta_{p,n}} = 4 \left(\frac{1}{2} + \frac{1}{2^p - 1} \right)^{\frac{1}{p}},$$

for fixed p. Thus, by monotonicity of power means

$$n^{-\frac{1}{q}} \|T_n\|_{p,q} < 4 \left(\frac{1}{2} + \frac{1}{2^p - 1} \right)^{\frac{1}{p}},$$

for $q < \eta_{p,n}$ and

$$n^{-\frac{1}{q}} \|T_n\|_{p,q} \ge 4 \left(\frac{1}{2} + \frac{1}{2^p - 1} \right)^{\frac{1}{p}},$$

for $\eta_{p,n} \le q \le p$.

Case (iv-b): $N_{2,p} < n$. By the inequality (19), Lemma 2.4 and Remark 3.3, we have

$$n^{-\frac{1}{q}} \|T_n\|_{p,q} \ge n^{-1} \|T_n\|_{p,1} \ge (N_{2,p} + 1)^{-1} \|T_{N_{2,p}+1}\|_{p,1} \ge 4 \left(\frac{1}{2} + \frac{1}{2^p - 1} \right)^{\frac{1}{p}}.$$

The proof is complete. □

Remark 3.9. *In Table 3, we give some values of* $\eta_{p,n}$.

p	1.51	1.52	1.53	1.54	1.54	1.55	1.55	1.56	1.56	1.56
n	50	57	66	77	78	93	94	116	117	118
$\eta_{p,n}$	1.236...	1.190...	1.144...	1.436...	1.073...	1.470..	1.155...	1.545...	1.276...	1.021...

Table 3:

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