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Quasi-isometric embedding maps between $*$-algebras and their relation with an amenability question

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Abstract. In this paper, we investigate the following question: under which conditions the amenability of Banach algebras can be transferred back, i.e., if there exists a continuous homomorphism from a Banach algebra A into an amenable Banach algebra B, does it follow that A is also amenable? To answer this question, first, we define a new version of quasi-isometric embedding map between $*$-algebras and obtain some results related to this notion. Moreover, as a main result, we show that if φ is an injective quasi-isometry between $*$-Banach algebras A and B, then the amenability of B implies the amenability of A.

1. Introduction

Let (X, d_X) and (Y, d_Y) be two metric spaces. A map $f : X \longrightarrow Y$ is called quasi-isometric embedding if there exist $\alpha \geq 1$ and $\beta \geq 0$ such that

$$\frac{1}{\alpha} d_X(x, y) - \beta \leq d_Y(f(x), f(y)) \leq \alpha d_X(x, y) + \beta,$$

for all $x, y \in X$. In the above inequalities if $\beta = 0$ then the quasi-isometric embedding f is called a bi-Lipschitz map, for some results we refer to [1, 4, 11, 14, 17]. The concept of quasi-isometric embedding is a useful tool for investigating Cayley and hyperbolic graphs, for more details and applications, we refer to [2, 8, 9, 16]. Ceccherini-Silberstein and Samet-Vaillant [6] introduced the notion of quasi-isometric embeddings for finitely generated algebras and derived many interesting geometric properties. For more details about geometric properties and other works related to these properties, we refer to [6, Introduction] and the references therein.

Throughout this paper, by a $*$-algebra we mean a Banach algebra with the involution $*$. Let A and B be two $*$-algebras. For simplicity, we show the involutions on A and B by $*$. A linear map $\varphi : A \longrightarrow B$ is called a $*$-map, if $\varphi(a^*) = \varphi(a)^*$, for all $a \in A$. Moreover, a linear map $\varphi : A \longrightarrow B$ is called a homomorphism if

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$\varphi(ab) = \varphi(a)\varphi(b)$, for all $a, b \in A$. A linear map between two $*$-algebras is called a $*$-homomorphism if it is a $*$-map and a homomorphism.

The amenability of Banach algebras was introduced by Johnson in [12] and its relation to the homological properties of Banach algebras was introduced by Helemskii in [10]. Let A be a Banach algebra and X be a Banach A-bimodule. A derivation from A into X is a bounded linear map D such that $D(ab) = a\cdot D(b)+D(a)\cdot b$, for all $a, b \in A$. Moreover, if there exists $x \in X$ such that $D(a) = a \cdot x - x \cdot a$, for every $a \in A$, then D is called an inner derivation. A Banach algebra A is called amenable if every derivation from A into the first dual of any Banach A-bimodule X is inner. For a locally compact group G, Johnson proved that the group algebra $L^1(G)$ is amenable if and only if G is amenable [12]. Let A, B be two Banach algebras and φ be a continuous dense range (onto) homomorphism. If A is amenable, then B is too [12, Proposition 5.3]. This result shows that the amenability of Banach algebras can be transferred under a continuous dense range homomorphism.

In this paper, we consider $*$-algebras and quasi-isometric embedding maps on these algebras, where our definition is different from the quasi-isometric embedding maps on metric spaces and algebras that were defined before. By this new definition, we will consider the inverse of [12, Proposition 5.3], i.e., we investigate under which condition, if the right hand of a map is amenable, the left-hand side (domain) can be amenable.

2. Quasi-isometric embedding

The notion of quasi-isometric (uniform) embedding on groups was introduced by Ceccherini-Silberstein and Coornaert in their celebrated monograph [5]. Let G and H be two groups. A map $\varphi : G \longrightarrow H$ is called a quasi-isometric (uniform) embedding if (i) for every finite subset K of G, there is a finite subset F of H such that $g_1^{-1}g_2 \in K$ implies that $\varphi(g_1)^{-1}\varphi(g_2) \in F$ and (ii) for every finite subset F of H, there is a finite subset K of G such that $\varphi(g_1)^{-1}\varphi(g_2) \in F$ implies that $g_1^{-1}g_2 \in K$. Moreover, the quasi-isometric (uniform) embedding $\varphi : G \longrightarrow H$ is called a quasi-isometry if there is a finite subset C of H such that $\varphi(G)C = H$. In this section, inspired by the above definition on groups, we generalise the above two notions, i.e., quasi-isometric (uniform) embedding and quasi-isometry to Banach $*$-algebras. We commence with the following definitions:

Definition 2.1. *Let A and B be two $*$-algebras. We say a map $\varphi : A \longrightarrow B$ is a quasi-isometric embedding of A into B if the following two conditions hold:*

(i) for every finite subset $F \subset A$ there is a finite subset $F' \subset B$ such that if $a_1^*a_2 \in F$, then $\varphi(a_1)^*\varphi(a_2) \in F'$, for all $a_1, a_2 \in A$,

(ii) for every finite subset $F' \subset B$ there is a finite subset $F \subset A$ such that if $\varphi(a_1)^*\varphi(a_2) \in F'$, then $a_1^*a_2 \in F$, for all $a_1, a_2 \in A$.

Definition 2.2. *Let A and B be two $*$-algebras. A quasi-isometry from A into B is a quasi-isometric embedding $\varphi : A \longrightarrow B$ for which there is a finite-dimensional two-sided ideal $K \subset B$ such that $\varphi(A) + K = B$, i.e., for every $b \in B$ there exist $k \in K$ and $a \in A$ such that $\varphi(a) + k = b$. If there is a quasi-isometry from A into B, then we say that A is quasi-isometric to B.*

A physical example can be found in quantum systems with finite-dimensional noise. Consider two quantum systems described by their observable algebras (which are C*-algebras):

System A: A purely infinite-dimensional quantum system, such as a free quantum field on a non-compact spacetime (e.g., Minkowski space). Its algebra of observables A is typically a C*-algebra or von Neumann algebra (e.g., the algebra of canonical commutation relations or a Weyl algebra).

System B: The same quantum system but now coupled to a finite-dimensional "noise" or "environment", such as a finite-dimensional quantum system (e.g., a spin chain or a qubit register). The algebra B is then a direct sum or an extension of A by a finite-dimensional ideal K (representing the noise/environment).

We now continue with the following example that says there is an example of quasi-isometric embedding between $*$-algebras that is not an isometry in the classical case.

Example 2.3. *(i) Let A,B be two $*$-algebras and $\varphi : A \longrightarrow B$ be an injective $*$-homomorphism. We claim that φ is a quasi-isometric embedding of A into B. To verify this, we show that φ satisfies the conditions of Definition 2.1. Assume that $F \subset A$ is a finite subset and set $\varphi(F) = F'$. Clearly, $F' \subset B$ is finite. Then for all $a_1, a_2 \in A$ such that $a_1^* a_2 \in F$, we have*

$$\varphi(a_1)^* \varphi(a_2) = \varphi(a_1^* a_2) \in F'.$$

This implies that the condition (i) holds. For (ii), suppose that F' is a finite subset of B and set $F = \varphi^{-1}(F')$. Then F is a finite subset of A, because φ is injective. For all $a_1, a_2 \in A$ satisfying in $\varphi(a_1)^* \varphi(a_2) \in F'$, we have $\varphi(a_1^* a_2) \in F'$. This implies that $a_1^* a_2 \in F$.

(ii) Let G be a locally compact group and H be an open subgroup of G. Consider the group algebras $L^1(G)$ and $L^1(H)$ with the convolution product and the involution $f^*(x) = \Delta_G(x^{-1}) f(x^{-1})$, where Δ_G is the modular function of G, similarly, the involution on H can be defined by the modular function Δ_H. Now, consider the inclusion function $\imath : L^1(H) \longrightarrow L^1(G)$ by $\imath(f) = f$, for all $f \in L^1(H)$. Then, by (i), $\imath$ is a quasi-isometric embedding.

(iii) Let A be a $*$-Banach algebra and I be a closed ideal of A. Then similar to (ii), $\imath : I \longrightarrow A$ is a quasi-isometric embedding.

(iv) Let $A = M_n(\mathbb{C})$ and $B = M_{n+k}(\mathbb{C})$ for $k \geq 0$. The natural embedding $\varphi : A \hookrightarrow B$ defined by:

$$\varphi(a) = \begin{pmatrix} a & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$$

is a quasi-isometric embedding. The finite-dimensional ideal K can be taken as the subspace of matrices with non-zero entries only in the last k rows and columns.

Let A be a $*$-algebra and $F_1, F_2 \subset A$ be arbitrary subsets, by $\bigoplus_{f \in F_2}(F_1 + f)$ we means disjoint union of $F_1 + f$'s. In the following, we give necessary and sufficient conditions on the defined map in Example 2.3 that becomes a quasi-isometry.

Proposition 2.4. *Let A,B be two $*$-algebras and $\varphi : A \longrightarrow B$ be a quasi-isometric embedding with the closed range. Then φ is a quasi-isometry if and only if $\varphi(A)$ is a subspace of B with finite codimension.*

Proof. Assume that φ is a quasi-isometry. Then, there is a finite-dimensional ideal K of B such that $\varphi(A) + K = B$. Hence

$$\dim \frac{B}{\varphi(A)} \leq \dim K < \infty.$$

Conversely, suppose that $\varphi(A)$ has a finite codimension in B. Let K be the set of representatives for the right cosets of $\varphi(A)$ in B. Thus, K is a finite-dimensional ideal of B and

$$B = \bigoplus_{k \in K} (\varphi(A) + k) = \varphi(A) + K.$$

This implies that φ is a quasi-isometry. □

Example 2.5. *Let $A = C_0(\mathbb{R}^2)$, the $*$-algebra of complex-valued continuous functions vanishing at infinity on $\mathbb{R}^2$ (with pointwise multiplication and conjugation) and $B = C_0(\mathbb{R}^2) \oplus \mathbb{C}^n$, where $n \in \mathbb{N}$ and $\mathbb{C}^n$ is a finite-dimensional $*$-algebra (in the physical sense it representing "noise" or "local corrections"). Since $\varphi(A)$ is a subspace of B with finite codimension, Proposition 2.4 implies that φ is a quasi-isometry.*

Corollary 2.6. *Let A be a commutative $*$-Banach algebra and I be a maximal ideal of A. Then $\varphi : I \longrightarrow A$ is a quasi-isometry.*

Note that in the above result, commutativity of A is essential, because if we set $A = \mathcal{B}(E)$, the Banach algebra contains all linear bounded operators on the infinite-dimensional Banach space E. Then $M = \{T \in \mathcal{B}(E) : Tx = 0, \text{ for all } x \in E\}$ is a maximal ideal of $A = \mathcal{B}(E)$ which has infinite codimension.

Let A,B be two $*$-algebras and $\varphi : A \longrightarrow B$ be a $*$-map, i.e., $\varphi(a^*) = \varphi(a)^*$, for all $a \in A$. Then φ is called unitary preserving $*$-map if for every unitary element $a \in A$, $\varphi(a)$ is a unitary element in B. Assume that e_A and e_B are the unit elements of A and B, respectively. We denote the set of all unitary elements of a $*$-algebra A by $U(A)$ and by $|U(A)|$ we mean the cardinal number of $U(A)$. If φ is a quasi-isometric embedding, then we could not say that $\varphi(e_A) = e_B$, in general. By the following, we are seeking a quasi-isometric embedding that has this property. A Banach algebra A is called without order if, $ab = 0$ implies that $a = 0$ or $b = 0$.

Lemma 2.7. *Let A and B be two unital $*$-algebras with units e_A and e_B, respectively. If there is a quasi-isometric embedding from A into B that is a unitary preserving $*$-map, then there is a quasi-isometric embedding ϕ from A into B such that $\phi(e_A) = e_B$. Moreover, if the given map is a quasi-isometry, then ϕ is also a quasi-isometry.*

Proof. Suppose that $\varphi : A \longrightarrow B$ is a quasi-isometric embedding that satisfies the stated conditions. Define $\phi : A \longrightarrow B$ by $\phi(a) = \varphi(e_A)^*\varphi(a)$ for all $a \in A$. Then

$$\phi(a_1)^*\phi(a_2) = \varphi(a_1)^*\varphi(e_A)\varphi(e_A)^*\varphi(a_2) = \varphi(a_1)^*\varphi(a_2),$$

for all $a_1, a_2 \in A$. Thus, ϕ is a quasi-isometric embedding. Also, according to definition of ϕ, we have $\phi(e_A) = e_B$.

Now, let φ be a quasi-isometry. Thus, there is a finite-dimensional ideal K of B such that $\varphi(A) + K = B$. Set $K' = \varphi(e_A)^*K$. Clearly, K' is a finite-dimensional ideal of B. Then

$$\phi(A) + K' = \varphi(e_A)^*\varphi(A) + \varphi(e_A)^*K = \varphi(e_A)^*B. \tag{1}$$

Define $\psi : B \longrightarrow \varphi(e_A)^*B$ by $\psi(b) = \varphi(e_A)^*b$, for all $b \in B$. It is easy to verify that ψ is an isomorphism. Hence, from this fact and (1), we have $\varphi(e_A)^*B \cong B$. This shows that ϕ is a quasi-isometry. □

Proposition 2.8. *Let A and B be two Banach $*$-algebras such that A is finite-dimensional and B is unital. If $\varphi : A \longrightarrow B$ is a continuous quasi-isometry and linear map, then B is finite-dimensional.*

Proof. Since $\varphi : A \longrightarrow B$ is a quasi-isometry, there is a finite-dimensional ideal K_B of B such that $B = \varphi(A)+K_B$. Let e_A, e_B be the units of A and B, respectively, $\dim A = n$ and let $F = \{e_A, a_1, ..., a_m\} \subset A$ such that generates A and $m \geq n$. Since φ is a quasi-isometric embedding, there is a finite subset F' of B such that $\varphi(x)^*\varphi(y) \in F'$, whenever $x^*y \in F$. So, if we take $x = e_A$, then by Lemma 2.7, $\varphi(y) \in F'$, whenever, $y \in F$. This means that $\varphi(a_i) \in F'$, for all $a_i \in F$, $1 \leq i \leq m$. We show that $\overline{F} = F' \cup K_B$ generates B. Let $b \in B$, then there exist $b' \in \varphi(A)$ and $k \in K_B$ such that $b = b' + k$. Then there exists $a \in A$ such that $\varphi(a) = b'$, so we have $b = \varphi(a) + k$. Moreover, there exist $\alpha_1, \ldots, \alpha_m \in \mathbb{C}$ such that $a = \sum_{i=1}^m \alpha_i a_i$. Then,

$$b = b' + k = \varphi\left(\sum_{i=1}^m \alpha_i a_i\right) + k = \sum_{i=1}^m \alpha_i \varphi(a_i) + k.$$

This shows that F generates B and hence B is finite-dimensional. □

By the following result, we show that the composition of the quasi-isometric embedding maps is a quasi-isometric embedding map.

Proposition 2.9. *Let A, B and C be three $*$-algebras. Assume that $\varphi : A \longrightarrow B$ and $\psi : B \longrightarrow C$ are quasi-isometric embedding maps, then $\psi \circ \varphi : A \longrightarrow C$ is a quasi-isometric embedding map.*

Proof. Let F be a finite subset of A. Then one can find a finite subset F' of B such that if $a_1^*a_2 \in F$, then $\varphi(a_1)^*\varphi(a_2) \in F'$. Moreover, since ψ is a quasi-isometric embedding, there is finite subset F'' of C such that $\psi(\varphi(a_1))^*\psi(\varphi(a_2)) \in F''$. Similarly, for any finite subset $\overline{F}$ of C one can find a finite subset F of A such that $\psi(\varphi(a_1))^*\psi(\varphi(a_2)) \in \overline{F}$ implies that $a_1^*a_2 \in F$. □

Corollary 2.10. *Let A and B be Banach $*$-algebras, I be a $*$-ideal of A and $\varphi : A \longrightarrow B$ be a quasi-isometric embedding. Then the restriction map $\varphi|_I : I \longrightarrow B$ is a quasi-isometric embedding.*

Proof. Consider the inclusion map $\imath : I \longrightarrow A$. Since $\imath$ is an injective $*$-homomorphism, by Example 2.3, $\imath$ is a quasi-isometric embedding. Then by Proposition 2.9, $\varphi|_I = \varphi \circ \imath$ is a quasi-isometric embedding. □

Theorem 2.11. *Let A, B and C be three $*$-algebras. Assume that $\varphi : A \longrightarrow B$ and $\psi : B \longrightarrow C$ are quasi-isometries. Then the composition $\psi \circ \varphi$ is a quasi-isometry when the following conditions hold:*

(i) C is a unital $*$-algebra.

(ii) ψ is a $*$-homomorphism.

Proof. We first verify that $\psi \circ \varphi$ is a quasi-isometric embedding. Since φ and ψ are quasi-isometric embeddings, their composition $\psi \circ \varphi$ is a quasi-isometric embedding by Proposition 2.9.

Next, we show the existence of a finite-dimensional two-sided ideal K in C such that $(\psi \circ \varphi)(A) + K = C$. Since φ is a quasi-isometry, there exists a finite-dimensional two-sided ideal K_1 in B such that

$$\varphi(A) + K_1 = B.$$

Since ψ is a quasi-isometry, there exists a finite-dimensional two-sided ideal K_2 in C such that

$$\psi(B) + K_2 = C.$$

Now, define $L = \psi(K_1) + K_2$. Since K_1 is finite-dimensional and ψ is linear, $\psi(K_1)$ is finite-dimensional. Also, K_2 is finite-dimensional, so L is a finite-dimensional subspace of C.

We claim that L is a two-sided ideal in C. Let $x \in L$ and $c \in C$. Then $x = \psi(k_1) + k_2$ for some $k_1 \in K_1$, $k_2 \in K_2$. Since $C = \psi(B) + K_2$, write $c = \psi(b) + k_2'$ for some $b \in B$, $k_2' \in K_2$. Then

$$cx = (\psi(b) + k_2')(\psi(k_1) + k_2) = \psi(b)\psi(k_1) + \psi(b)k_2 + k_2'\psi(k_1) + k_2'k_2.$$

Now $\psi(b)\psi(k_1) = \psi(bk_1) \in \psi(K_1)$ because K_1 is an ideal in B and $bk_1 \in K_1$. Since K_2 is an ideal in C, $\psi(b)k_2 \in K_2$, $k_2'\psi(k_1) \in K_2$, and $k_2'k_2 \in K_2$.

Thus, $cx \in \psi(K_1) + K_2 = L$. Similarly, $xc \in L$. Therefore, L is a two-sided ideal in C. Finally, we have

$$C = \psi(B) + K_2 = \psi(\varphi(A) + K_1) + K_2 = \psi(\varphi(A)) + \psi(K_1) + K_2 = (\psi \circ \varphi)(A) + L.$$

Hence, $\psi \circ \varphi$ is a quasi-isometry. □

Note that in the above result, the unitality of C ensures that the finite-dimensional subspace $\psi(K_1) + K_2$ can account for all elements of C when combined with $\psi(\varphi(A))$. Moreover, linearity of ψ is crucial to guarantee that $\psi(K_1)$ remains finite-dimensional. Thus, without these conditions, the sum $\psi(K_1) + K_2$ might not be finite-dimensional, breaking the quasi-isometry property.

Example 2.12. *Consider $A = M_n(\mathbb{C})$, $B = M_{n+1}(\mathbb{C})$, $C = M_{n+2}(\mathbb{C})$, $\varphi : A \hookrightarrow B$ the upper-left corner embedding, and $\psi : B \hookrightarrow C$ the upper-left corner embedding. Then $\psi \circ \varphi : A \to C$ is a quasi-isometry with*

$$K = \left\{ \begin{pmatrix} 0 & \cdots & 0 & 0 \\ \vdots & \cdots & \vdots & \vdots \\ 0 & \cdots & * & * \\ 0 & \cdots & * & * \end{pmatrix} \right\} \subset C$$

which is a finite-dimensional ideal satisfying $(\psi \circ \varphi)(A) + K = C$.

Proposition 2.13. *Let A, B, C and D be Banach $*$-algebras. Assume that $\varphi : A \longrightarrow B$ and $\psi : C \longrightarrow D$ are quasi-isometries, then* *(i)* $\varphi \times \psi : A \times C \longrightarrow B \times D$ *is a quasi-isometry;*

(ii) $\varphi \otimes \psi : A\widehat{\otimes}C \longrightarrow B\widehat{\otimes}D$ *is not a quasi-isometry.*

Proof. Case (i) is straightforward. For case (ii), there are finite-dimensional ideals K_B and K_D such that $\varphi(A) + K_B = B$ and $\psi(C) + K_D = D$. Then

$$B\widehat{\otimes}D = \Big(\varphi(A)\widehat{\otimes}\psi(C)\Big) + \Big(\varphi(A)\widehat{\otimes}K_D\Big) + \Big(K_B\widehat{\otimes}\psi(C)\Big) + \Big(K_B\widehat{\otimes}K_D\Big).$$

Thus, $\varphi \otimes \psi$ is not a quasi-isometry. □

3. Connection with amenability

A Banach algebra A is said to have property (**G**) if there exists an amenable locally compact group G and a continuous homomorphism $\varphi : L^1(G) \longrightarrow A$ with dense range, this concept was introduced by Kepert [13] and he used the above-mentioned result in [12]. The following result is a special case of the inverse of [12, Proposition 5.3].

Proposition 3.1. *Let A,B be two $*$-algebras and $\varphi : A \longrightarrow B$ be a continuous homomorphism. If φ is an injective quasi-isometry and B is amenable, then A is amenable.*

Proof. Let K be a finite-dimensional ideal of B such that $\varphi(A) + K = B$. Since φ is a homomorphism, $\varphi(A)$ becomes a subalgebra of B. It is well-known that every finite-dimensional ideal of amenable Banach algebra is unital and it is equivalent to the amenability of the finite-dimensional ideal, see [12, Proposition 5.1]. Thus, the amenability of B and K implies the amenability of $\varphi(A)$. Now, define $\psi : A \longrightarrow \varphi(A)$, by $\psi(a) = \varphi(a)$, for all $a \in A$. Clearly, ψ is a bijective homomorphism. Hence, $\varphi(A) \cong A$. Then by [12, Proposition 5.3], A is amenable. □

Let A be an amenable Banach algebra and I be a closed two-sided ideal of A. If I possess a bounded approximate identity, then I is amenable [12, Proposition 5.1]. By the following result, for commutative $*$-Banach algebras, we omit the existence of a bounded approximate identity condition for maximal ideals of these algebras. For a Banach algebra A, by $\sigma(A)$, we mean the character space of A.

Corollary 3.2. *Let A be an amenable commutative Banach $*$-algebra and $\varphi \in \sigma(A)$. Then* $\ker\varphi$ *is amenable*

Proof. The space $\ker\varphi$ is a maximal ideal of A; indeed, all maximal ideals of A are kernels of members of $\sigma(A)$. Now, consider the inclusion map $\imath : \ker\varphi \longrightarrow A$. Then by Corollary 2.6, $\imath$ is an injective quasi-isometry and Proposition 3.1 implies that $\ker\varphi$ is amenable. □

Example 3.3. *Let G be an Abelian locally compact group and $\varphi \in \sigma(L^1(G))$. Then by Corollary 3.2,* $\ker\varphi$ *is amenable.*

The result in Proposition 3.1 is a special case of the inverse of Property (**G**). So, similar to property (**G**), we say a Banach $*$-algebra A has property (**G´**), if there exist an amenable locally compact group G and an injective quasi-isometry $\varphi : A \longrightarrow L^1(G)$. Clearly, for every amenable locally compact group G, $L^1(G)$ has property (**G´**). Furthermore, Example 3.3 demonstrates that the kernel of φ satisfies property (**G´**).

Now, this question arises: is there any amenable Banach $*$-algebra without property (**G´**)? We answer this question with the following example.

Example 3.4. *Let A and B be two Banach $*$-algebras that have property (**G´**). Thus, there are amenable locally compact groups G, H and injective quasi-isometries $\varphi : A \longrightarrow L^1(G)$ and $\psi : B \longrightarrow L^1(H)$. By Proposition 3.1, A and B are amenable. Hence, $A\widehat{\otimes}B$ is amenable [7, Corollary 2.9.62]. Now, consider $\varphi \otimes \psi : A\widehat{\otimes}B \longrightarrow L^1(G)\widehat{\otimes}L^1(H)$ defined by $\varphi \otimes \psi(a \otimes b) = \varphi(a) \otimes \psi(b)$, for all $a \otimes b \in A\widehat{\otimes}B$. Then by Proposition 2.13, $\varphi \otimes \psi$ is not a quasi-isometry. Hence, $A\widehat{\otimes}B$ has no property (**G´**).*

Theorem 3.5. *Restriction of quasi-isometries on ideals is not necessarily quasi-isometry.*

Proof. Let G be an Abelian locally compact group and consider $L^1(G)$ as a Banach $*$-algebra. Let $I = S^1(G)$ be an abstract Segal algebra, see [15], for definition and details and $\varphi : L^1(G) \longrightarrow L^1(G)$ be the identity map. Then $\varphi|_I$ is not a quasi-isometry, because if it is a quasi-isometry, then by Proposition 3.1, $S^1(G)$ becomes amenable, a contradiction because $S^1(G)$ is amenable whenever $S^1(G) = L^1(G)$ [3]. $\square$

The above result shows if a Banach $*$-algebra has property (**G**´), then its ideals maybe has not property (**G**´).

4. Problems

In this section we ask some questions related to quasi-isometries:

1. Under which conditions on I, does the restriction $\varphi|_I : I \longrightarrow B$ of a quasi-isometry $\varphi : A \longrightarrow B$ remain a quasi-isometry? Moreover, if a Banach $*$-algebra has property (**G**´), when its ideals has property (**G**´)?
2. Which homological and cohomological properties Banach $*$-algebras and C^*-algebras can be investigated by quasi-isometries?
3. If $\varphi : A \longrightarrow B$ is a quasi-isometry between Banach $*$-algebras, and B is Arens regular, is A Arens regular?

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