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Weighted Hölder and numerical radius inequalities

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Abstract. The main aim of the current work is a discussion of weighted bounds for singular values and numerical radii of Hilbert space operators.

1. Introduction

In the sequel, $\mathcal{H}$ is a separable complex Hilbert space, and $\mathcal{B}(\mathcal{H})$ is the C^*-algebra of all bounded linear operators from $\mathcal{H}$ to itself, with zero O and identity I. The subset of $\mathcal{B}(\mathcal{H})$ of all compact operators will be denoted by $\mathcal{K}(\mathcal{H})$. It is known that $\mathcal{K}(\mathcal{H})$ is a two-sided ideal in the sense that $TK, KT \in \mathcal{K}(\mathcal{H})$, whenever $T \in \mathcal{B}(\mathcal{H})$ and $K \in \mathcal{K}(\mathcal{H})$.

If $K \in \mathcal{K}(\mathcal{H})$, the singular values of K will be denoted by $s_1(K), s_2(K), \ldots,$ with the convention that $s_j(K) \geq s_{j+1}(K)$, for all j.

It is of a renowned interest in the literature to find bounds for the singular values of certain operator forms in terms of their individual operators. For example, for any $T_1, T_2 \in \mathcal{B}(\mathcal{H})$ and $K \in \mathcal{K}(\mathcal{H})$, we have [17, p. 27]

$$s_j(T_1KT_2) \leq \|T_1\| s_j(K) \|T_2\|, \; j = 1, 2, \ldots, \tag{1}$$

where $\|\cdot\|$ denotes the usual operator norm, defined on $\mathcal{B}(\mathcal{H})$ by $\|T\| = \sup_{\|x\|=1} \|Tx\|$. In the sequel, whenever we write $s_j(X) \leq s_j(Y)$, we implicitly mean that this is valid for $j = 1, 2, \ldots,$ unless otherwise stated. The significance of (1) partially refers to the fact that $s_j(\cdot)$ is not sub-multiplicative. That is, if $X, Y \in \mathcal{K}(\mathcal{H})$, then $s_j(XY) \leq s_j(X)s_j(Y)$ is not necessarily true, although $s_1(XY) \leq s_1(X)s_1(Y)$ holds true, because $s_1(\cdot) = \|\cdot\|$.

Generalizations of (1) have been seen frequently in the literature. For example, it was shown in [26] that if $A, B, X, Y \in \mathcal{B}(\mathcal{H})$ are such that $X, Y \in \mathcal{K}(\mathcal{H})$, then

$$s_j(AX - YB) \leq 2 \max(\|A\|, \|B\|) s_j(X \oplus Y). \tag{2}$$

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This is considered as a generalization of other known bounds, such as [25]

$$s_j(AX - XB) \leq \max(\|A\|, \|B\|)s_j(X \oplus X),$$

which is valid for positive $A, B, X \in \mathcal{B}(\mathcal{H})$ and compact X, and

$$s_j(AX - XA) \leq \frac{\|A\|}{2} s_j(X \oplus X)$$

if both A, X are positive, and X is compact [24]. A better bound than (2) is [19]

$$s_j(AX - YB) \leq (\|A\| + \|B\|)s_j(X \oplus Y), \tag{3}$$

when $X, Y \in \mathcal{K}(\mathcal{H})$ and $A, B \in \mathcal{B}(\mathcal{H})$.

In fact, earlier than all of this, a simpler bound that is valid for positive $A, B \in \mathcal{K}(\mathcal{H})$ was shown in [35] as follows:

$$s_j(A - B) \leq s_j(A \oplus B).$$

This was extended a little later in [24] to the form

$$s_j(AX - XB) \leq \|X\| s_j(A \oplus B) \tag{4}$$

for positive $A, B \in \mathcal{K}(\mathcal{H})$ and arbitrary $X \in \mathcal{B}(\mathcal{H})$.

When $\mathcal{H}$ is finite dimensional, say of dimension n, $\mathcal{B}(\mathcal{H})$ is just the algebra of all $n \times n$ complex matrices $\mathcal{M}_n$. An extension of (4) was shown in [19] at the cost of an additional assumption that $A, B \in \mathcal{M}_n$ are positive semidefinite, while $X, Y \in \mathcal{M}_n$ are arbitrary, as follows:

$$s_j(AX - YB) \leq \frac{\|X\| + \|Y\| + \|X - Y\|}{2} s_j(A \oplus B). \tag{5}$$

Another generalization that appeared in [19] employed the polar decompositions $A = U|A|, B = V|B|$ of A and B to show that

$$s_j(AX - XB) \leq \left(\|X\| + \frac{\|UX - XV\|}{2}\right) s_j(A \oplus B),$$

with the special case

$$s_j(AX - XA) \leq \left(\|X\| + \frac{\|UX - XU\|}{2}\right) s_j(A \oplus A).$$

We refer the reader to [2.2-2.2], [19, 24, 28, 33, 35–37], for further readings on singular value bounds and to see the type of forms we are usually interested in

The first part of this work will present some bounds related to the above forms.

After that, some numerical radius bounds of quantities similar to those above will be discussed. In this context, we recall that the numerical radius of $X \in \mathcal{B}(\mathcal{H})$ is defined by $\omega(X) = \sup_{\|x\|=1} |\langle Xx, x\rangle|$. This defines a norm on $\mathcal{B}(\mathcal{H})$ that is equivalent to the operator norm, as we have [18, Theorem 1.3-1]

$$\frac{1}{2}\|X\| \leq \omega(X) \leq \|X\|, X \in \mathcal{B}(\mathcal{H}). \tag{6}$$

Research involving the numerical radius focus on finding sharper bounds and further relations is a renowned topic, as one can see in [1, 2, 27, 29–31]. Among that basic relation for the numerical radius, we have [1]

$$\omega(AXB) \leq \frac{\|X\|}{2} \left\| |A^*|^2 + |B|^2 \right\|, \tag{7}$$

and, as a direct consequence of the definition of $\omega(\cdot)$,

$$\omega(AXA^*) \leq \|A\|^2 \omega(X), \tag{8}$$

where A^* is the adjoint of A, and $|A^*|^2 = AA^*$. Although the literature about the numerical radius is huge, we mention only these as they will be needed in our proofs.

2. Results

This section presents our results where singular value and numerical radius bounds are shown. For organizational purposes, we discuss these two concerns in two different subsections.

2.1. Singular value bounds

As mentioned earlier, we will discuss generalized forms of the singular value bounds we saw in the introduction.

It follows from [13] that if $A, B, C \in \mathcal{B}(\mathcal{H}), X_1, X_2, X_3 \in \mathcal{K}(\mathcal{H})$, then

$$s_j(AX_1 - X_2B - X_3C) \le (\|A\| + \sqrt{2}L)s_j(X_1 \oplus X_2 \oplus X_3), \tag{9}$$

where

$$L = \sqrt{\|B\|^2 + \|C\|^2}.$$

Remark 2.1.

(i) In (9)*, if we replace B, C by $\frac{1}{2}B$ and X_2, X_3 by Y, we obtain (3).*

(ii) We may restate (9) *as*

$$s_j(AX_1 + X_2B + X_3C) \le (\|A\| + \sqrt{2}L)s_j(X_1 \oplus X_2 \oplus X_3),$$

by replacing B, C with $-B, -C$.

The first result treats the relation between the singular values of the sum of two operators and those of their direct sums.

Theorem 2.2. *Let $S, T \in \mathscr{K}(\mathscr{H})$. Then for any $0 \le t \le 1$*

$$s_j(S+T) \le \sqrt{2\left(\|S\|^{2(1-t)} + \|T\|^{2(1-t)}\right)}s_j\left(|S|^t \oplus |T|^t\right).$$

Proof. It follows from Remark 2.1 that

$$s_j(X_2B + X_3C) \le \sqrt{2\left(\|B\|^2 + \|C\|^2\right)}s_j(X_2 \oplus X_3). \tag{10}$$

Let $S = U|S|$ and $T = V|T|$ be the polar decompositions of S and T, respectively. If we substitute $X_2 = |S|^t$, $B = |S|^{1-t}U^*$, $X_3 = |T|^t$, and $C = |T|^{1-t}V^*$, in (10), we infer that

$$\begin{aligned}
& s_j(S+T) \\
& = s_j((S+T)^*) \\
& = s_j(S^* + T^*) \\
& \le \sqrt{2\left(\left\||S|^{1-t}U^*\right\|^2 + \left\||T|^{1-t}V^*\right\|^2\right)}s_j\left(|S|^t \oplus |T|^t\right) \\
& = \sqrt{2\left(\left\||S|^{1-t}U^*U|S|^{1-t}\right\| + \left\||T|^{1-t}V^*V|T|^{1-t}\right\|\right)}s_j\left(|S|^t \oplus |T|^t\right) \\
& = \sqrt{2\left(\left\||S|^{2(1-t)}\right\| + \left\||T|^{2(1-t)}\right\|\right)}s_j\left(|S|^t \oplus |T|^t\right) \\
& = \sqrt{2\left(\|\,|S|\,\|^{2(1-t)} + \|\,|T|\,\|^{2(1-t)}\right)}s_j\left(|S|^t \oplus |T|^t\right) \\
& = \sqrt{2\left(\|S\|^{2(1-t)} + \|T\|^{2(1-t)}\right)}s_j\left(|S|^t \oplus |T|^t\right),
\end{aligned}$$

as required. □

For the next result, we state the following lemma first.

Lemma 2.3. *[37] Let $A, B \in \mathcal{B}(\mathcal{H})$ and $X \in \mathcal{K}(\mathcal{H})$ be positive. Then*

$$s_j(A^*XB) \le \frac{1}{2} s_j\left(\left(|A^*|^2 + |B^*|^2\right)^{\frac{1}{2}} X \left(|A^*|^2 + |B^*|^2\right)^{\frac{1}{2}}\right).$$

Now, we prove a new auxiliary result that we will use to present a new proof of (5).

Corollary 2.4. *Let $A, B \in \mathcal{B}(\mathcal{H})$ and $X, Y \in \mathcal{K}(\mathcal{H})$ be positive. Then*

$$s_j(AX - YB) \le \frac{\sqrt{\left\||A^*|^2 + I\right\| \left\||B|^2 + I\right\|} + \|A - B\|}{2} s_j(X \oplus Y).$$

Proof. It follows from Lemma 2.3 that, when X is positive,

$$\begin{aligned}
s_j(A^*XB) &\le \frac{\left\|\frac{\|B\|}{\|A\|}|A^*|^2 + \frac{\|A\|}{\|B\|}|B^*|^2\right\|}{2} s_j(X) \\
&\le \frac{\|A\|\|B\| + \||A^*||B^*|\|}{2} s_j(X) \\
&= \frac{\|A\|\|B\| + \|A^*B\|}{2} s_j(X).
\end{aligned}$$

So, when X, Y are positive, we have

$$\begin{aligned}
&s_j(AX - YB) \\
&= s_j\left(\begin{bmatrix} A & -I \\ O & O \end{bmatrix}\begin{bmatrix} X & O \\ O & Y \end{bmatrix}\begin{bmatrix} I & O \\ B & O \end{bmatrix}\right) \\
&\le \frac{\left\|\begin{bmatrix} A & -I \\ O & O \end{bmatrix}\right\|\left\|\begin{bmatrix} I & O \\ B & O \end{bmatrix}\right\| + \left\|\begin{bmatrix} A & -I \\ O & O \end{bmatrix}\begin{bmatrix} I & O \\ B & O \end{bmatrix}\right\|}{2} s_j(X \oplus Y) \\
&= \frac{\left\|\begin{bmatrix} A & -I \\ O & O \end{bmatrix}\right\|\left\|\begin{bmatrix} I & O \\ B & O \end{bmatrix}\right\| + \left\|\begin{bmatrix} A - B & O \\ O & O \end{bmatrix}\right\|}{2} s_j(X \oplus Y) \\
&= \frac{\sqrt{\left\||A^*|^2 + I\right\| \left\||B|^2 + I\right\|} + \|A - B\|}{2} s_j(X \oplus Y).
\end{aligned}$$

This completes the proof.
□

Now Corollary 2.4 may be used to present a new proof of (5).

Proposition 2.5. *Let $A, B, X, Y \in \mathscr{M}_n$ such that $X, Y \ge O$. Then*

$$s_j(AX - YB) \le \frac{\|A\| + \|B\| + \|A - B\|}{2} s_j(X \oplus Y).$$

Proof. It follows from Corollary 2.4 that

$$s_j(AX - YB) \le \frac{\sqrt{\left(\|A\|^2 + 1\right)\left(\|B\|^2 + 1\right)} + \|A - B\|}{2} s_j(X \oplus Y).$$

If we replace A and B by tA and tB and then take the infimum over $t > 0$, we deduce that

$$\begin{aligned} s_j(AX - YB) &\leq \frac{\min_{t>0} \sqrt{\left(\|A\|^2 + \frac{1}{t^2}\right)\left(\|B\|^2 + \frac{1}{t^2}\right)} + \|A - B\|}{2} s_j(X \oplus Y) \\ &= \frac{\|A\| + \|B\| + \|A - B\|}{2} s_j(X \oplus Y), \end{aligned}$$

as required. □

Proposition 2.6. *Let $A, B, C, D \in \mathcal{K}(\mathcal{H})$. Then*

$$s_j(C^*A - D^*B) \leq \sqrt{\|C^*C + D^*D\| \, s_j(A^*A + B^*B)}.$$

In particular,

$$s_j(A - B) \leq \sqrt{2 s_j(A^*A + B^*B)}. \tag{11}$$

Proof. Noting (1), we have

$$\begin{aligned} s_j(C^*A - D^*B \oplus O) &= s_j\left(\begin{bmatrix} C^*A - D^*B & O \\ O & O \end{bmatrix}\right) \\ &= s_j\left(\begin{bmatrix} C^* & \mathrm{i}D^* \\ O & O \end{bmatrix}\begin{bmatrix} A & O \\ \mathrm{i}B & O \end{bmatrix}\right) \\ &\leq \left\|\begin{bmatrix} C^* & \mathrm{i}D^* \\ O & O \end{bmatrix}\right\| s_j\left(\begin{bmatrix} A & O \\ \mathrm{i}B & O \end{bmatrix}\right) \\ &= \left\|\begin{bmatrix} C^* & \mathrm{i}D^* \\ O & O \end{bmatrix}\right\| s_j\left(\left|\begin{bmatrix} A & O \\ \mathrm{i}B & O \end{bmatrix}\right|\right) \\ &= \left\|\begin{bmatrix} C^* & \mathrm{i}D^* \\ O & O \end{bmatrix}\right\| s_j\left(\begin{bmatrix} A^*A + B^*B & O \\ O & O \end{bmatrix}^{\frac{1}{2}}\right) \\ &= \left\|\begin{bmatrix} C^* & iD^* \\ O & O \end{bmatrix}\right\| s_j^{\frac{1}{2}}\left(\begin{bmatrix} A^*A + B^*B & O \\ O & O \end{bmatrix}\right) \\ &= \sqrt{\|C^*C + D^*D\| \, s_j\left(\begin{bmatrix} A^*A + B^*B & O \\ O & O \end{bmatrix}\right)} \\ &= \sqrt{\|C^*C + D^*D\| \, s_j(A^*A + B^*B \oplus O)}, \end{aligned}$$

as required. □

Remark 2.7. *Notice that* (11) *provides a new upper bound for the singular values of the difference $A - B$. We give a numerical example where this new bound is better than* (5)*. If we let*

$$A = \begin{bmatrix} 0 & -1 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ -1 & 1 & 1 \end{bmatrix},$$

numerical calculations show that

$$s(A - B) \approx \{3.40268, 1.71871, 0.683969\}, \sqrt{2 s_j(A^*A + B^*B)} \approx \{3.4641, 2.82843, 1.41421\},$$

while

$$\frac{\|A\| + \|B\| + \|A - B\|}{2} \approx 3.52619.$$

So, in this example (11) *is better than* (5) *for* $j = 1, 2, 3$. *We point out that in other examples,* (5) *can be better for some values of* j.

Corollary 2.8. *Let* $S, T \in \mathcal{K}(\mathcal{H})$. *Then*

$$\begin{aligned} & s_j(S - T) \\ & \leq \min\left\{ \sqrt{\left\| |S|^{2(1-t)} + |T|^{2(1-t)} \right\| s_j\left(|S^*|^{2t} + |T^*|^{2t}\right)}, \sqrt{\left\| |S^*|^{2(1-t)} + |T^*|^{2(1-t)} \right\| s_j\left(|S|^{2t} + |T|^{2t}\right)} \right\}. \end{aligned}$$

Proof. Put $C^* = U|S|^{1-t}$, $A = |S|^t$, $D^* = V|T|^{1-t}$, $B = |T|^t$, in (10), we get

$$s_j(S - T) \leq \sqrt{\left\| |S^*|^{2(1-t)} + |T^*|^{2(1-t)} \right\| s_j\left(|S|^{2t} + |T|^{2t}\right)}. \tag{12}$$

If we replace S and T by S^* and T^*, in (12), we obtain

$$\begin{aligned} s_j(S - T) &= s_j\left((S - T)^*\right) \\ &= s_j(S^* - T^*) \\ &\leq \sqrt{\left\| |S|^{2(1-t)} + |T|^{2(1-t)} \right\| s_j\left(|S^*|^{2t} + |T^*|^{2t}\right)}. \end{aligned}$$

We get the desired result by combining the last two inequalities. □

Remark 2.9. *Since [21]*

$$s_j(X + Y) \leq 2 s_j(X \oplus Y),$$

we have

$$\begin{aligned} & s_j(S - T) \\ & \leq \sqrt{2} \min\left\{ \sqrt{\left\| |S|^{2(1-t)} + |T|^{2(1-t)} \right\| s_j\left(|S^*|^{2t} \oplus |T^*|^{2t}\right)}, \sqrt{\left\| |S^*|^{2(1-t)} + |T^*|^{2(1-t)} \right\| s_j\left(|S|^{2t} \oplus |T|^{2t}\right)} \right\} \\ & = \sqrt{2} \min\left\{ \sqrt{\left\| |S|^{2(1-t)} + |T|^{2(1-t)} \right\|} s_j\left(|S^*|^{t} \oplus |T^*|^{t}\right), \sqrt{\left\| |S^*|^{2(1-t)} + |T^*|^{2(1-t)} \right\|} s_j\left(|S|^{t} \oplus |T|^{t}\right) \right\}. \end{aligned}$$

In particular,

$$s_j(S - T) \leq 2 \min\left\{ s_j(|S^*| \oplus |T^*|), s_j(|S| \oplus |T|) \right\}. \tag{13}$$

Corollary 2.10. *Let* $A, X \in \mathcal{K}(\mathcal{H})$. *Then*

$$s_j(AX^* - AX) \leq 2 \min\left\{ \|A\| s_j(|X^*| \oplus |X|), \|X\| s_j(|A^*| \oplus |A^*|) \right\}.$$

Proof. We have by (13),

$$\begin{aligned} s_j(AX^* - AX) &\leq 2 s_j(|AX^*| \oplus |AX|) \\ &= 2 s_j\left((XA^*AX^*)^{\frac{1}{2}} \oplus (X^*A^*AX)^{\frac{1}{2}}\right) \\ &\leq 2 s_j\left(\|A\| (XX^*)^{\frac{1}{2}} \oplus \|A\| (X^*X)^{\frac{1}{2}}\right) \\ &= 2 \|A\| s_j(|X^*| \oplus |X|). \end{aligned}$$

That is,

$$s_j\left(AX^* - AX\right) \le 2\left\|A\right\| s_j\left(|X^*| \oplus |X|\right).$$

Similarly, using (13), we can prove that

$$\begin{aligned} s_j\left(AX^* - AX\right) &\le 2s_j\left(|XA^*| \oplus |X^*A^*|\right) \\ &= 2\left\|X\right\| s_j\left(|A^*| \oplus |A^*|\right). \end{aligned}$$

This completes the proof. □

2.2. Numerical radius bounds

For the rest of the paper, we present some numerical radius bounds consistent with the singular value bounds we found earlier. To better understand the proof, we recall the following Davidson-Power-type inequality for positive $A, B \in \mathcal{B}(\mathcal{H})$ (see [23]):

$$\|A + B\| \le \max\{\|A\|, \|B\|\} + \left\|A^{\frac{1}{2}}B^{\frac{1}{2}}\right\|. \tag{14}$$

Theorem 2.11. *Let $A, B, X \in \mathcal{B}(\mathcal{H})$. Then*

$$\omega\left(AXB\right) \le \frac{\|X\|}{2}\left(\sqrt{\|A\|\,\|B\|}\max\left\{\|A\|, \|B\|\right\} + \|BA\|\right).$$

Proof. If we replace A and B by $\sqrt{\frac{\|X\|}{\|A\|}}A$ and $\sqrt{\frac{\|X\|}{\|B\|}}B$, in (7), we infer that

$$\begin{aligned} \omega\left(AXB\right) &\le \frac{\sqrt{\|A\|\,\|B\|}}{2}\left\|\frac{\|X\|}{\|A\|}|A^*|^2 + \frac{\|X\|}{\|B\|}|B|^2\right\| \\ &\le \frac{\|X\|}{2}\left(\sqrt{\|A\|\,\|B\|}\max\left\{\|A\|, \|B\|\right\} + \||A^*|\,|B|\|\right) \quad (by\ (14)) \\ &= \frac{\|X\|}{2}\left(\sqrt{\|A\|\,\|B\|}\max\left\{\|A\|, \|B\|\right\} + \|A^*B^*\|\right) \\ &= \frac{\|X\|}{2}\left(\sqrt{\|A\|\,\|B\|}\max\left\{\|A\|, \|B\|\right\} + \left\|(BA)^*\right\|\right) \\ &= \frac{\|X\|}{2}\left(\sqrt{\|A\|\,\|B\|}\max\left\{\|A\|, \|B\|\right\} + \|BA\|\right), \end{aligned}$$

as required. □

Remark 2.12. *Notice that Theorem 2.11 and* (7) *provide upper bounds for $\omega(AXB)$. In this remark, we give an example where Theorem 2.11 can be better than* (7). *Indeed, if we let*

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 2 & -2 \\ -2 & -2 \end{bmatrix}, X = \begin{bmatrix} -2 & -1 \\ 2 & -3 \end{bmatrix},$$

we find that

$$\omega(AXB) = 10, \frac{\|X\|}{2}\left\||A^*|^2 + |B|^2\right\| \approx 16.3016,$$

and

$$\frac{\|X\|}{2}\left(\sqrt{\|A\|\,\|B\|}\max\left\{\|A\|, \|B\|\right\} + \|BA\|\right) \approx 13.7391.$$

This shows that Theorem 2.11 provides a sharper bound than (7) *for these matrices. We emphasize that other examples show the converse, which indicates that the two bounds are independent.*

We state the following numerical radius bound.

Theorem 2.13. *Let $A, B, C, X_1, X_2, X_3 \in \mathcal{B}(\mathcal{H})$. Then*

$$\omega\left(AX_1 - X_2B - X_3C\right) \le \frac{\max\left\{\|X_1\|, \|X_2\|, \|X_3\|\right\}}{2}\left\||A^*|^2 + |B|^2 + |C|^2 + 3I\right\|.$$

Proof. We have

$$\begin{aligned}
&\omega\left(AX_1 - X_2B - X_3C\right)\\
&= \omega\left(\begin{bmatrix} AX_1 - X_2B - X_3C & O & O \\ O & O & O \\ O & O & O \end{bmatrix}\right)\\
&= \omega\left(\begin{bmatrix} A & I & -I \\ O & O & O \\ O & O & O \end{bmatrix}\begin{bmatrix} X_1 & O & O \\ O & X_2 & O \\ O & O & X_3 \end{bmatrix}\begin{bmatrix} I & O & O \\ -B & O & O \\ C & O & O \end{bmatrix}\right)\\
&\le \frac{\left\|\begin{bmatrix} X_1 & O & O \\ O & X_2 & O \\ O & O & X_3 \end{bmatrix}\right\|}{2}\left\|\left|\begin{bmatrix} A & I & -I \\ O & O & O \\ O & O & O \end{bmatrix}^*\right|^2 + \left|\begin{bmatrix} I & O & O \\ -B & O & O \\ C & O & O \end{bmatrix}\right|^2\right\| \quad (by\ (7))\\
&= \frac{\left\|\begin{bmatrix} X_1 & O & O \\ O & X_2 & O \\ O & O & X_3 \end{bmatrix}\right\|}{2}\left\|\begin{bmatrix} |A^*|^2 + 2I & O & O \\ O & O & O \\ O & O & O \end{bmatrix} + \begin{bmatrix} I + |B|^2 + |C|^2 & O & O \\ O & O & O \\ O & O & O \end{bmatrix}\right\|\\
&= \frac{\left\|\begin{bmatrix} X_1 & O & O \\ O & X_2 & O \\ O & O & X_3 \end{bmatrix}\right\|}{2}\left\|\begin{bmatrix} |A^*|^2 + |B|^2 + |C|^2 + 3I & O & O \\ O & O & O \\ O & O & O \end{bmatrix}\right\|\\
&= \frac{\max\left\{\|X_1\|, \|X_2\|, \|X_3\|\right\}}{2}\left\||A^*|^2 + |B|^2 + |C|^2 + 3I\right\|,
\end{aligned}$$

as required. □

We also have the following numerical radius bound.

Proposition 2.14. *Let $A, X_1, X_2, X_3 \in \mathcal{B}(\mathcal{H})$. Then*

$$\omega\left(AX_1A^* + X_2 + X_3\right) \le \left\||A^*|^2 + 2I\right\| \max\left\{\omega\left(X_1\right), \omega\left(X_2\right), \omega\left(X_3\right)\right\}.$$

Proof. By (8), we can argue that

$$\begin{aligned}
\omega\left(AX_1A^* + X_2 + X_3\right) &= \omega\left(\begin{bmatrix} AX_1A^* + X_2 + X_3 & O & O \\ O & O & O \\ O & O & O \end{bmatrix}\right) \\
&= \omega\left(\begin{bmatrix} A & I & -I \\ O & O & O \\ O & O & O \end{bmatrix}\begin{bmatrix} X_1 & O & O \\ O & X_2 & O \\ O & O & X_3 \end{bmatrix}\begin{bmatrix} A^* & O & O \\ I & O & O \\ -I & O & O \end{bmatrix}\right) \\
&\le \left\|\begin{bmatrix} A & I & -I \\ O & O & O \\ O & O & O \end{bmatrix}\right\|^2 \omega\left(\begin{bmatrix} X_1 & O & O \\ O & X_2 & O \\ O & O & X_3 \end{bmatrix}\right) \\
&= \left\|\begin{bmatrix} |A^*|^2 + 2I & O & O \\ O & O & O \\ O & O & O \end{bmatrix}\right\| \omega\left(\begin{bmatrix} X_1 & O & O \\ O & X_2 & O \\ O & O & X_3 \end{bmatrix}\right) \\
&= \left\||A^*|^2 + 2I\right\| \max\left\{\omega\left(X_1\right), \omega\left(X_2\right), \omega\left(X_3\right)\right\},
\end{aligned}$$

as required. □

Finally, we present the following consequences of (12).

Corollary 2.15. *Let $A \in \mathscr{K}(\mathscr{H})$. Then for any $0 \le t \le 1$*

$$s_j(A) \le 2\omega^{1-t}(A)\, s_j\left(\left|\Re A\right|^t \oplus \left|\Im A\right|^t\right).$$

In particular,

$$\frac{1}{2}\|A\| \le \omega^{1-t}(A)\max\left\{\left\|\Re A\right\|^t, \left\|\Im A\right\|^t\right\},$$

where $\Re A$ and $\Im A$ are the real and imaginary parts of A, respectively.

Proof. From (12), we infer that

$$\begin{aligned}
s_j(A) &\le \sqrt{2\left(\left\|\Re A\right\|^{2(1-t)} + \left\|\Im A\right\|^{2(1-t)}\right)}\, s_j\left(\left|\Re A\right|^t \oplus \left|\Im A\right|^t\right) \\
&\le 2\omega^{1-t}(A)\, s_j\left(\left|\Re A\right|^t \oplus \left|\Im A\right|^t\right),
\end{aligned}$$

where we have used the fact that $\|\Re A\|, \|\Im A\| \le \omega(A)$ to obtain the last inequality. □

Notice that Corollary 2.15 provides a refinement of the first inequality in (6), because $\|\Re A\|, \|\Im A\| \le \omega(A)$.

Our next result is an upper bound for the numerical radius of the commutator $AB - BA$. Although this form is consistent with the forms we have discussed so far, the proof of this bound will differ. We recall that if $A, B \in \mathcal{B}(\mathcal{H})$, then [22]

$$\omega(AB - BA) \le 2\|A\|\omega(B). \tag{15}$$

Using this inequality, we can present a refinement of (15), as follows.

Proposition 2.16. *Let $A, B \in \mathcal{B}(\mathcal{H})$. Then*

$$\omega(AB - BA) \le 2\inf_{\lambda \in \mathbb{C}}\|A - \lambda I\|\omega(B).$$

Proof. Since, for any $\lambda \in \mathbb{C}$,

$$AB - BA = (A - \lambda) B - B (A - \lambda),$$

we infer from (15) that

$$\omega (AB - BA) \leq 2 \|A - \lambda\| \omega (B),$$

which implies the desired conclusion. □

Theorem 2.17. *Let $A, B, X, Y \in \mathcal{B}(\mathcal{H})$, and $\alpha, \beta \in \mathbb{C}$. Then*

$$\begin{aligned}&\omega (AX - YB)\\ &\leq 2\sqrt{\left(\omega^2\left(\begin{bmatrix} O & B-\beta \\ A-\alpha I & O\end{bmatrix}\right) - \frac{1}{2} m\left((A-\alpha I)(B-\beta)\right)\right)\left\| |X|^2 + |Y^*|^2 \right\|} + \omega (\alpha X - \beta Y),\end{aligned}$$

where $m(X) = \inf_{\|x\|=1} |\langle Xx, x\rangle|$.

Proof. Applying the triangle inequality, and [1, Eq. (32)], one can easily see that

$$\begin{aligned}&\omega (AX - YB)\\ &= \omega ((A-\alpha I) X - Y (B-\beta) + \alpha X - \beta Y)\\ &\leq \omega ((A-\alpha I) X - Y (B-\beta)) + \omega (\alpha X - \beta Y)\\ &\leq \sqrt{\left\| |A^* - \overline{\alpha} I|^2 + \left|B-\beta\right|^2 \right\| \left\| |X|^2 + |Y^*|^2 \right\|} + \omega (\alpha X - \beta Y).\end{aligned}$$

Furthermore, it follows from [1, Eq. (33)] that

$$\sqrt{\left\| |S|^2 + |T^*|^2 \right\|} \leq 2\sqrt{\omega^2\left(\begin{bmatrix} O & S \\ T & O\end{bmatrix}\right) - \frac{1}{2} m (TS)}.$$

So,

$$\begin{aligned}&\omega (AX - YB)\\ &\leq 2\sqrt{\left(\omega^2\left(\begin{bmatrix} O & B-\beta \\ A-\alpha I & O\end{bmatrix}\right) - \frac{1}{2} m\left((A-\alpha I)(B-\beta)\right)\right)\left\| |X|^2 + |Y^*|^2 \right\|} + \omega (\alpha X - \beta Y),\end{aligned}$$

as required. □

Remark 2.18. *If we put $\beta = 0$ and $\alpha = \frac{\|A\|}{2}$, in Theorem 2.17, we infer that*

$$\begin{aligned}&\omega (AX - YB)\\ &\leq 2\sqrt{\left(\omega^2\left(\begin{bmatrix} O & B \\ A-\frac{\|A\|}{2} I & O\end{bmatrix}\right) - \frac{1}{2} m\left(\left(A - \frac{\|A\|}{2} I\right) B\right)\right)\left\| |X|^2 + |Y^*|^2 \right\|} + \frac{\|A\| \omega (X)}{2}.\end{aligned}$$

In particular, by letting $Y = B = O$, we obtain

$$\omega (AX) \leq \left\| A - \frac{\|A\|}{2} I \right\| \|X\| + \frac{\|A\| \omega (X)}{2}.$$

Using the fact that [19, Lemma 3.1]:

$$\left\| S-\frac{\|S\|}{2}I \right\| \le \frac{\|S\|}{2},$$

which holds for $S \ge O$, we get

$$\begin{aligned}\omega(AX) &\le \left\| A-\frac{\|A\|}{2}I \right\| \|X\| + \frac{\|A\|\,\omega(X)}{2} \\ &\le \frac{\|A\|}{2}\left(\|X\|+\omega(X)\right),\end{aligned} \tag{16}$$

provided that A is positive. Notice that

$$\omega(AX) \le \frac{\|A\|}{2}\left(\|X\|+\omega(X)\right),$$

has been shown in [32], so, (16) *provides a refinement of this inequality.*

Remark 2.19. *We can write from* (16) *that*

$$\begin{aligned}\omega(T) &= \omega(T^*) \\ &= \omega\left(|T|^{1-t}|T|^t U^*\right) \\ &\le \left\| |T|^{1-t} - \frac{\|T\|^{1-t}}{2}I \right\| \|T\|^t + \frac{\|T\|^{1-t}\omega\left(U|T|^t\right)}{2} \\ &\le \frac{\|T\|^{1-t}}{2}\left(\|T\|^t + \omega\left(U|T|^t\right)\right),\end{aligned}$$

i.e.,

$$\omega(T) \le \left\| |T|^{1-t} - \frac{\|T\|^{1-t}}{2}I \right\| \|T\|^t + \frac{\|T\|^{1-t}\omega\left(U|T|^t\right)}{2} \le \frac{\|T\|^{1-t}}{2}\left(\|T\|^t + \omega\left(U|T|^t\right)\right).$$

In particular,

$$\omega(T) \le \left\| |T| - \frac{\|T\|}{2}I \right\| + \frac{\|T\|\,\omega(U)}{2} \le \frac{\|T\|}{2}\left(1+\omega(U)\right),$$

which is a refinement of [16, Proposition 4.1].

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References

[1] A. Abu-Omar, F. Kittaneh, *Numerical radius inequalities for products and commutators of operators*, Houston J. Math. **41** (2015), 1163–1173.
[2] A. Abu-Omar, F. Kittaneh, *Upper and lower bounds for the numerical radius with an application to involution operators*, Rocky Mountain J. Math. **45** (2015), 1055–1065.
[3] W. Audeh, H. Moradi, M. Sababheh, Davidson-Power type and singular value inequalities, Hokkaido Math. J., new.
[4] W. Audeh, H. Moradi, M. Sababheh, Singular values inequalities via matrix monotone functions, Anal.Math.Phys. 13 (2023). https://doi.org/10.1007/s13324-023-00832-8.
[5] W. Audeh, Singular value inequalities for operators and matrices, Ann. Funct. Anal. 13 (2022).
[6] W. Audeh, Singular value inequalities for accretive-dissipative normal operators, J. Math. Inequal. 16 (2022), 729-737.
[7] W. Audeh, Singular value and norm inequalities of Davidson-Power type, J. Math. Inequal. 15 (2021), 1311-1320.
[8] M. Al-Labadi, R. Al-Naimi and W. Audeh, Singular value inequalities for generalized anticommutators, J. Inequal. Appl. 15(2025).
[9] W. Audeh, A. Al-Boustanji, M. Al-Labadi and R. Al-Naimi, Singular value inequalities of matrices via increasing functions, J. Inequal. Appl. 114(2024).
[10] W. Audeh, H. R. Moradi and M. Sababheh, *Commutator bounds via singular values with applications to the numerical radius*, Mediterr. J. Math. **22** (2025), Article 8.
[11] W. Audeh, H. R. Moradi and M. Sababheh, *Generalizations of recent singular value inequalities for sums of products of matrices*, Filomat **38** (2024), 9905–9919.
[12] W.Audeh, Singular value inequalities with applications, J. Math. Computer Sci., 24(2022), 323-329.
[13] W. Audeh, *More commutator inequalities for Hilbert space operators*, Internat. J. Funct. Anal. Oper. Theory Appl. **6** (2014), 85–95.
[14] R. Bhatia, *Matrix Analysis*, GTM 169, Springer-Verlag, New York, 1997.
[15] R. Bhatia, F. Kittaneh, *Commutators, pinchings and spectral variation*, Oper. Matrices **2** (2008), 143–151.
[16] T. Bottazzi, C. Conde, *Generalized Buzano inequality*, Filomat **37** (2023), 9377–9390.
[17] I. C. Gohberg, M. G. Krein, *Introduction to the Theory of Linear Nonselfadjoint Operators*, Amer. Math. Soc., Providence, RI, 1969.
[18] K. E. Gustafson, D. K. M. Rao, *Numerical Range*, Springer, New York, 1997.
[19] O. Hirzallah, *Commutator inequalities for Hilbert space operators*, Linear Algebra Appl. **431** (2009), 1571–1578.
[20] O. Hirzallah, F. Kittaneh, K. Shebrawi, *Numerical radius inequalities for commutators of Hilbert space operators*, Numer. Funct. Anal. Optim. **32** (2011), 739–749.
[21] O. Hirzallah, F. Kittaneh, *Singular values, norms, and commutators*, Linear Algebra Appl. **432** (2010), 1322–1336.
[22] O. Hirzallah, F. Kittaneh, K. Shebrawi, *Numerical radius inequalities for commutators of Hilbert space operators*, Numer. Funct. Anal. Optim. **32** (2011), 739–749.
[23] F. Kittaneh, *Norm inequalities for certain operator sums*, J. Funct. Anal. **143** (1997), 337–348.
[24] F. Kittaneh, *Inequalities for commutators of positive operators*, J. Funct. Anal. **250** (2007), 132–143.
[25] F. Kittaneh, *Norm inequalities for commutators of self-adjoint operators*, Integral Equ. Oper. Theory **62** (2008), 129–135.
[26] F. Kittaneh, *Singular value inequalities for commutators of Hilbert space operators*, Linear Algebra Appl. **430** (2009), 2362–2367.
[27] F. Kittaneh, H. R. Moradi, M. Sababheh, *On the numerical radius of the product of Hilbert space operators*, Hokkaido Math. J. **53** (2024), 1–26.
[28] F. Kittaneh, H. R. Moradi, M. Sababheh, *Singular value inequalities with applications to norms and means of matrices*, Acta Sci. Math. (Szeged) (2024).
[29] H. R. Moradi, M. Sababheh, *New estimates for the numerical radius*, Filomat **35** (2021), 4957–4962.
[30] H. R. Moradi, M. Sababheh, *More accurate numerical radius inequalities II*, Linear Multilinear Algebra **69** (2021), 921–933.
[31] M. Sababheh, H. R. Moradi, *More accurate numerical radius inequalities (I)*, Linear Multilinear Algebra **69** (2021), 1964–1973.
[32] M. Sababheh, H. R. Moradi, Z. Heydarbeygi, *Buzano, Kreĭn and Cauchy–Schwarz inequalities*, Oper. Matrices **16** (2022), 239–259.
[33] M. Sababheh, S. Furuichi, S. Sheybani, H. R. Moradi, *Singular values inequalities for matrix means*, J. Math. Inequal. **16** (2022), 169–179.
[34] Y.-Q. Wang, H.-K. Du, *Norms for commutators of self-adjoint operators*, J. Math. Anal. Appl. **342** (2008), 747–751.
[35] X. Zhan, *Singular values of differences of positive semidefinite matrices*, SIAM J. Matrix Anal. Appl. **22** (2000), 819–823.
[36] J. Zhao, *Inequalities of singular values and unitarily invariant norms for sums and products of matrices*, Filomat **36** (2022), 31–38.
[37] L. Zou, *An arithmetic-geometric mean inequality for singular values and its applications*, Linear Algebra Appl. **528** (2017), 25–32.