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Dynamical properties of multiplication operators on the Zygmund spaces

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Abstract. In this paper, firstly we establish bounds on the norm of multiplication operators on the Zygmund spaces of the unit disk via weighted composition operators. Also the power boundedness and (uniformly) mean ergodicity of multiplication operators induced by analytic functions ψ are studied on the Zygmund spaces $\mathscr{Z}$ and $\mathscr{Z}_0$. We characterize power bounded and (uniformly) mean ergodic multiplication operators in terms of ψ. Lastly we describe the spectrum of the multiplication operators in terms of the range of the symbol.

1. Introduction

We introduce some basic notation that will be used throughout the paper. Denote by $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ the open unit disk in the complex plane $\mathbb{C}$, with $\partial\mathbb{D}$ as its boundary, by $H(\mathbb{D})$ the space of all analytic functions on $\mathbb{D}$, and by $H^\infty(\mathbb{D})$ the space of all bounded analytic functions on $\mathbb{D}$. Let φ be an analytic self-map on $\mathbb{D}$ and let $\psi \in H(\mathbb{D})$. *The weighted composition operator* $W_{\psi,\varphi}$ induced by φ and ψ on $H(\mathbb{D})$ is defined by

$$W_{\psi,\varphi} f = \psi\,(f \circ \varphi)$$

for all $f \in H(\mathbb{D})$. Setting $M_\psi f = \psi f$ and $C_\varphi f = f \circ \varphi$, we may write $W_{\psi,\varphi} = M_\psi C_\varphi$. Then C_φ is called *composition operator with symbol* φ and, in particular, for an analytic function ψ on $\mathbb{D}$ the linear operator M_ψ defined by

$$M_\psi f = \psi f$$

for all $f \in H(\mathbb{D})$ is called *multiplication operator with symbol* ψ.

Denote by $\mathscr{Z}$ the class of all $f \in H(\mathbb{D}) \cap C(\overline{\mathbb{D}})$ such that

$$\sup \frac{|f(e^{i(\theta+h)}) + f(e^{i(\theta-h)}) - 2f(e^{i\theta})|}{h} < \infty$$

where the supremum is taken over all $e^{i\theta} \in \partial\mathbb{D}$ and $h > 0$. From a theorem of Zygmund ([11, Theorem 5.3]) and the Closed Graph Theorem we see that $f \in \mathscr{Z}$ if and only if

$$\sup_{z\in\mathbb{D}} (1 - |z|^2)|f''(z)| < \infty.$$

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It is easy to see that $\mathscr{Z}$ is a Banach space under the norm $\|\cdot\|_{\mathscr{Z}}$ where

$$\|f\|_{\mathscr{Z}} = |f(0)| + |f'(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)|f''(z)|.$$

We call $\mathscr{Z}$ *the Zygmund space*. The little Zygmund space $\mathscr{Z}_0$ is the closed subspace of $\mathscr{Z}$ consisting of the functions $f \in \mathscr{Z}$ satisfying

$$\lim_{|z|\to 1}(1-|z|^2)|f''(z)| = 0.$$

The norm and terminology of the Zygmund space were introduced in [16] and for more information on the Zygmund space on the unit disk, see, for example, [11]. Recently, there have been numerous papers on various aspects of classes of operators on Zygmund spaces. Composition operators, weighted composition operators, and related operators acting between the Zygmund space and certain analytic function spaces have been studied in [2, 13, 17, 18, 22, 29]. In addition to the cited papers, the works [19–21, 23, 24] have also been influential and have inspired numerous subsequent studies on linear operators between analytic function spaces on the unit disk, the upper half-plane and the unit ball. Together with these important contributions, Colonna and Li [4] provided several characterizations of the bounded and the compact weighted composition operators from the Bloch space and the analytic Besov spaces $\mathcal{B}_p$ with $1 < p < \infty$ into the Zygmund space $\mathscr{Z}$. Moreover, Li [15] investigated the boundedness and compactness of weighted composition operators between minimal Möbius invariant spaces and Zygmund spaces on the unit disc and particularly, in [26], which characterized the boundedness and compactness of the weighted composition operator on the Zygmund space and will be referenced whenever necessary in the present work, the authors proved that $W_{\psi,\varphi}$ is a bounded operator on the Zygmund space if and only if $\psi \in \mathscr{Z}$ and the following two properties are satisfied:

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)|\psi(z)||\varphi'(z)|^2}{1-|\varphi(z)|^2} < \infty, \tag{1}$$

$$\sup_{z\in\mathbb{D}}(1-|z|^2)\left|2\psi'(z)\varphi'(z) + \psi(z)\varphi''(z)\right| \log\frac{1}{1-|\varphi(z)|^2} < \infty. \tag{2}$$

Note that if $W_{\psi,\varphi}$ is bounded on the Zygmund space, by taking $f \equiv 1$ in $\mathscr{Z}$, then we can easily obtain ψ is a Zygmund function. For more information on the Zygmund space on the unit disk, see [11].

There exists an extensive literature concerning the dynamics properties of linear continuous operators on Banach, Fréchet, and more general locally convex spaces, see [3, 5, 27]. The study of mean ergodicity in the space of linear operators defined on a Banach space goes back to J. von Neumann (1931), who proved the mean ergodicity of unitary operators on a Hilbert space. Since von Neumann's work, the topic of mean ergodic operators and their applications has remained a focal point of intensive research. In this context, Bonet and Ricker considering the multiplication operators defined on the weighted Banach spaces H_v^∞ consisting of holomorphic functions on the open unit disc investigated the relationship between power boundedness, mean ergodicity and uniform mean ergodicity of such operators [6]. Further, in [1], the authors focused on the spectral analysis and ergodic behavior of multiplication operators defined on the Schwartz space $S(\mathbb{R})$ of rapidly decreasing functions. Moreover, in [8], the mean ergodicity and power boundedness of the weighted composition operator $W_{\psi,\varphi}$ with symbol φ and multiplier ψ, are investigated on spaces of holomorphic functions $H(\mathbb{D})$ in the unit disc. Allen and Colonna [14] studied the norm behavior of multiplication operators on the Bloch space of the unit disk using weighted composition operators. They also derive norm estimates for these weighted composition operators.

Multiplication operators naturally arise in various spaces of analytic functions. In this paper, we investigate the power boundedness and (uniform) mean ergodicity of such operators on the Zygmund space. But before focusing on this subject, we first determine estimates on the norm of the weighted composition operator $W_{\psi,\varphi}$ the Zygmund space. Then we deduce estimates on the norm of the multiplication operator M_ψ, and finaly we show that the spectrum of the multiplication operator is the closure of the range of the symbol.

2. Norm estimates on weighted composition operators

To determine an upper and a lower bound on $\|W_{\psi,\varphi}\|$, we use the following lemma which collects some useful facts about functions in the Zygmund space.

Lemma 2.1. *For every $f \in \mathscr{Z}$, we have*

$$|f'(z)| \leq C \log \frac{2}{1-|z|^2} \|f\|_{\mathscr{Z}} \quad \textit{and} \quad |f(z)| \leq \|f\|_{\mathscr{Z}}$$

for every $z \in \mathbb{D}$, where $C > 0$.

The estimates given in Lemma 2.1 are well-known in the literature (see [16, 25]), and more general versions were established in [25, Lemma 2.4], which implies that the functions in the Zygmund space satisfy the growth condition $\|f\|_\infty \leq \|f\|_{\mathscr{Z}}$ for any $f \in \mathscr{Z}$ and so $\mathscr{Z} \subset H^\infty(\mathbb{D})$.

In this section we establish estimates on $\|W_{\psi,\varphi}\|$. In order to write these estimates in a succinct way, we introduce the following notation. Let $\psi \in \mathscr{Z}$ and $\varphi : \mathbb{D} \to \mathbb{D}$ be analytic functions satisfying conditions (1) and (2). Let

$$\tau_{\psi,\varphi} := \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)|\psi(z)||\varphi'(z)|^2}{1-|\varphi(z)|^2},$$

$$\sigma_{\psi,\varphi} := \sup_{z\in\mathbb{D}} (1-|z|^2)\left|2\psi'(z)\varphi'(z) + \psi(z)\varphi''(z)\right| C \log \frac{2}{1-|\varphi(z)|^2},$$

where are both finite. Note that if φ is the identity map, then $\tau_\psi = \|\psi\|_\infty$ and

$$\sigma_\psi := \sup_{z\in\mathbb{D}} (1-|z|^2)|\psi'(z)| C \log \frac{4}{1-|z|^2}. \tag{3}$$

Theorem 2.2. *Suppose that ψ is an analytic function on $\mathbb{D}$ and φ is an analytic self-map of $\mathbb{D}$ inducing a bounded weighted composition operator $W_{\psi,\varphi}$ on $\mathscr{Z}$. Then*

$$\max\left\{C_1, \|\psi\|_{\mathscr{Z}}\right\} \leq \|W_{\psi,\varphi}\| \leq \left\{C_1 + \|\psi\|_{\mathscr{Z}} + \sigma_{\psi,\varphi} + \tau_{\psi,\varphi}\right\}$$

where $C_1 := |\psi(0)\varphi'(0)| C \log \frac{2}{1-|\varphi(0)|^2}$.

Proof. Let $f \in \mathscr{Z}$ such that $\|f\|_{\mathscr{Z}} = 1$. Then

$$\begin{aligned} \|W_{\psi,\varphi} f\|_{\mathscr{Z}} \leq\ & \left|\psi(0)f(\varphi(0))\right| + \left|\psi'(0)f(\varphi(0)) + \psi(0)\varphi'(0)f'(\varphi(0))\right| + \sup_{z\in\mathbb{D}}(1-|z|^2)\left|\psi''(z)\right|\left|f(\varphi(z))\right| \\ &+ \sup_{z\in\mathbb{D}}(1-|z|^2)\left|2\psi'(z)\varphi'(z) + \psi(z)\varphi''(z)\right|\left|f'(\varphi(z))\right| \\ &+ \sup_{z\in\mathbb{D}} \frac{1-|z|^2}{1-|\varphi(z)|^2}\left|\psi(z)(\varphi'(z))^2\right|(1-|\varphi(z)|^2)\left|f''(\varphi(z))\right|. \end{aligned}$$

By Lemma 2.1, we have

$$|f'(\varphi(z))| \leq C \log \frac{2}{1-|\varphi(z)|^2} \|f\|_{\mathscr{Z}} \text{ and } |f(\varphi(z))| \leq \|f\|_{\mathscr{Z}},$$

also

$$|f'(\varphi(0))| \leq C \log \frac{2}{1-|\varphi(0)|^2} \|f\|_{\mathscr{Z}} \text{ and } |f(\varphi(0))| \leq \|f\|_{\mathscr{Z}}.$$

Therefore,

$$\|W_{\psi,\varphi} f\|_{\mathscr{Z}} \leq \left(C_1 + \|\psi\|_{\mathscr{Z}} + \sigma_{\psi,\varphi} + \tau_{\psi,\varphi}\right)\|f\|_{\mathscr{Z}},$$

as desired. To determine a lower bound on $\|W_{\psi,\varphi}\|$, we apply the appropriate test functions. Firstly, if we take the test function $f(z) = 1$, then we have

$$\|W_{\psi,\varphi}\| \geq \|W_{\psi,\varphi}f\|_{\mathscr{Z}} = \|\psi\|_{\mathscr{Z}}.$$

Now we take the test function

$$f(z) = \int_0^z C\log\frac{2}{1-\overline{\varphi(0)}w}dw - \int_0^{\varphi(0)} C\log\frac{2}{1-\overline{\varphi(0)}w}dw$$

for $z \in \mathbb{D}$, where log denotes the principal branch of the logarithm.

If $\varphi(0) = 0$, then $f(z) = C(\log 2)z$ and clearly $f \in \mathscr{Z}$. Also since $f(\varphi(0)) = 0$ and $f'(\varphi(0)) = C\log\frac{2}{1-|\varphi(0)|^2}$, we have

$$\begin{aligned}
\|W_{\psi,\varphi}\| &\geq \|W_{\psi,\varphi}f\|_{\mathscr{Z}} \geq |\psi(0)f(\varphi(0))| + |\psi'(0)f(\varphi(0)) + \psi(0)\varphi'(0)f'(\varphi(0))| \\
&= |\psi(0)\varphi'(0)|C\log\frac{2}{1-|\varphi(0)|^2}.
\end{aligned}$$

If $\varphi(0) \neq 0$, then we have $f(\varphi(0)) = 0$, $f'(z) = C\log\frac{2}{1-\overline{\varphi(0)}z}$ and $f''(z) = \frac{C\overline{\varphi(0)}}{1-\overline{\varphi(0)}z}$. So

$$\begin{aligned}
\|f\|_{\mathscr{Z}} &= |f(0)| + |f'(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)|f''(z)| \\
&\leq C\int_0^{\varphi(0)}\left|\log\frac{2}{1-\overline{\varphi(0)}w}\right|dw + C\log 2 + C\sup_{z\in\mathbb{D}}(1-|z|^2)\frac{|\overline{\varphi(0)}|}{|1-\overline{\varphi(0)}z|} \\
&\leq C\int_0^{\varphi(0)}\left(\log\left|\frac{2}{1-\overline{\varphi(0)}w}\right| + 2\pi\right)dw + C\log 2 + 2C|\varphi(0)| \\
&\leq C\left(\log\frac{2}{1-|\varphi(0)|} + 2\pi\right)\varphi(0) + C\log 2 + 2C|\varphi(0)| < \infty
\end{aligned}$$

which implies that $f \in \mathscr{Z}$. Thus

$$\begin{aligned}
\|W_{\psi,\varphi}\| &\geq \|W_{\psi,\varphi}f\|_{\mathscr{Z}} \geq |\psi(0)f(\varphi(0))| + |\psi'(0)f(\varphi(0)) + \psi(0)\varphi'(0)f'(\varphi(0))| \\
&\geq |\psi(0)\varphi'(0)|C\log\frac{2}{1-|\varphi(0)|^2}.
\end{aligned}$$

Consequently,

$$\|W_{\psi,\varphi}\| \geq \max\left\{|\psi(0)\varphi'(0)|C\log\frac{2}{1-|\varphi(0)|^2}, \|\psi\|_{\mathscr{Z}}\right\}$$

as desired. □

Remark 2.3. *By taking $\psi \equiv 1$, we deduce the estimates on the norm of $C_\varphi = W_{\psi,\varphi}$ as follows:*

$$\max\left\{1, C_2\right\} \leq \|C_\varphi\| \leq \left\{C_2 + 1 + \sigma_\varphi + \tau_\varphi\right\}$$

where $C_2 := |\varphi'(0)|C\log\frac{2}{1-|\varphi(0)|^2}$.

2.1. Norm estimates on multiplication operators

Now we deduce estimates on the norm of the multiplication operator $M_\psi = W_{\psi,\varphi}$ on $\mathscr{Z}$. Before stating the result we note that a Banach space $\mathcal{X}$ of complex-valued functions on a set Ω is said to be a *functional Banach space* if for each $w \in \Omega$, the point evaluation functional $e_w(f) = f(w)$, $f \in \mathcal{X}$, is bounded; that is, there exists a constant $C > 0$ such that $|f(w)| \leq C\|f\|$ for each $f \in \mathcal{X}$.

Lemma 2.4. *Let $\mathcal{X}$ be a functional Banach space on the set Ω and let ψ be a complex-valued function on Ω such that M_ψ maps $\mathcal{X}$ into itself. Then M_ψ is bounded on $\mathcal{X}$ and $|\psi(w)| \leq \|M_\psi\|$ for all $w \in \Omega$. In particular, ψ is bounded.*

Proof. See page 57, Lemma 11 of [12]. □

According to Lemma 2.1, every point-evaluation functionals on $\mathscr{Z}$ are bounded, in fact, we have $|e_z(f)| \leq \|f\|_{\mathscr{Z}}$, for all $z \in \mathbb{D}$. Since constant function is in the space, the second property is also valid and the space $\mathscr{Z}$ is a functional Banach spaces so we have the following inequality:

$$\|\psi\|_\infty \leq \|M_\psi\|.$$

Corollary 2.5. *Suppose ψ is an analytic function on $\mathbb{D}$ inducing a bounded multiplication operator M_ψ on $\mathscr{Z}$. Then*

$$\max\{C_3, \|\psi\|_{\mathscr{Z}}, \|\psi\|_\infty\} \leq \|M_\psi\| \leq \{C_3 + \|\psi\|_{\mathscr{Z}} + \sigma_\psi + \|\psi\|_\infty\}$$

where $C_3 := |\psi(0)|C \log 2$.

Proof. By taking φ to be the identity map, then $\tau_\psi = \|\psi\|_\infty$ and $\sigma_{\psi,\varphi} = \sigma_\psi$. Thus,

$$\max\{C_3, \|\psi\|_{\mathscr{Z}}\} \leq \|M_\psi\| \leq \{C_3 + \|\psi\|_{\mathscr{Z}} + \sigma_\psi + \|\psi\|_\infty\}.$$

Furthermore, by Lemma 2.4, we have $\|\psi\|_\infty \leq \|M_\psi\|$. Therefore

$$\max\{C_3, \|\psi\|_{\mathscr{Z}}, \|\psi\|_\infty\} \leq \|M_\psi\| \leq \{C_3 + \|\psi\|_{\mathscr{Z}} + \sigma_\psi + \|\psi\|_\infty\} \tag{4}$$

as desired. □

We note the following result, which will be used subsequently. By Corollary 2.5, we have

$$\|\psi^n\|_{\mathscr{Z}} \leq \|M_{\psi^n}\| \quad \text{and} \quad \|\psi^n\|_\infty \leq \|M_{\psi^n}\|$$

for all $\psi \in H(\mathbb{D})$ and $n \in \mathbb{N}$.

3. Multiplication operators on the Zygmund space $\mathscr{Z}$

Let X be a complex Banach space, $\mathcal{L}(X)$ the space of bounded linear operators on X, and $T \in \mathcal{L}(X)$. The iterates of T are denoted by $T^n := \underbrace{T \circ T \circ \cdots \circ T}_{n-times}$, $n \in \mathbb{N}$. T is *power bounded* if and only if there is $C > 0$ such that

$$\sup_{n \in \mathbb{N}} \|T^n\|_{\mathcal{L}(X)} \leq C.$$

Given any $T \in \mathcal{L}(X)$, Cešaro means of the sequence $\{T_{[n]}\}_{n\in\mathbb{N}}$ constituted by the iterates of T is defined by

$$T_{[n]} := \frac{1}{n}\sum_{k=1}^{n} T^k, \; n \in \mathbb{N}.$$

The operator T is called *mean ergodic* precisely when $\{T_{[n]}\}_{n\in\mathbb{N}}$ converges for the strong operator topology. If $\{T_{[n]}\}_{n\in\mathbb{N}}$ converges for the norm topology, then T is called *uniformly mean ergodic*.

It is easy to check that for all $n \in \mathbb{N}$, $\frac{1}{n}T^n = T_{[n]} - \frac{n-1}{n}T_{[n-1]}$, where $T_{[0]} := I$ is the identity operator on X. From this, we get if T is mean ergodic then $\lim_{n\to\infty} \frac{1}{n}T^n x = 0$ in X for every $x \in X$ and in the uniform mean ergodic case $\lim_{n\to\infty} \frac{1}{n}T^n = 0$ in $\mathcal{L}(X)$.

Recall that *the resolvent* $\rho(T)$ of a bounded operator $T : X \to X$ is the set of all complex numbers λ such that the operator $T - \lambda I$ is invertible on X, that is, $\rho(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is invertible}\}$, where I is the identity operator. The complement of the resolvent is called *the spectrum* set of T and is denoted by $\sigma(T) = \mathbb{C} \setminus \rho(T)$. The spectrum is a non-empty compact subset of the complex plane (see for more details [28]).

Lemma 3.1. *Let $\{T_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{L}(X)$, where X is Banach space and $T_{(n)} := \frac{1}{n}\sum_{k=1}^{n} T_k$ be Cesàro means of T_n for $n \in \mathbb{N}$. If $T_n \to T$ in the norm topology, then $T_{(n)} \to T$ in the same topology.*

The following lemma provides a necessary condition for multiplication operators to be power bounded, uniform mean ergodic and mean ergodic on $\mathscr{Z}$. A function $\psi \in H(\mathbb{D})$ is said to be *multiplier* of $\mathscr{Z}$ if $M_\psi f \in \mathscr{Z}$ for all $f \in \mathscr{Z}$. An application of the closed graph theorem shows that M_ψ is a bounded linear operator on $\mathscr{Z}$ if and only if ψ is multiplier of $\mathscr{Z}$.

Lemma 3.2. *Let ψ be a multiplier for $\mathscr{Z}$ ($\mathscr{Z}_0$). If the sequence $\{\frac{M_{\psi^n}}{n}\}_{n\in\mathbb{N}}$ is a bounded, then $\|\psi\|_\infty \le 1$. Consequently, if M_ψ is uniformly mean ergodic (and hence mean ergodic) or power bounded on $\mathscr{Z}$ ($\mathscr{Z}_0$), then $\|\psi\|_\infty \le 1$.*

Proof. By Lemma 2.4, we obtain that $|\frac{\psi^n(z)}{n}| \le \|\frac{M_{\psi^n}}{n}\|$ for every $z \in \mathbb{D}$ and $n \in \mathbb{N}$. So there exists a constant $K > 0$ such that $|\psi(z)| \le \sqrt[n]{Kn}$ for every $z \in \mathbb{D}$ and $n \in \mathbb{N}$. Thus for every $z \in \mathbb{D}$, $|\psi(z)| \le 1$ as $n \to \infty$, as desired.

If M_ψ is mean ergodic, for all $f \in \mathscr{Z}$ ($\mathscr{Z}_0$), $\|\frac{M_{\psi^n} f}{n}\|_{\mathscr{Z}} \to 0$ as $n \to \infty$, so by Uniform Boundedness Principle, the sequence $\{\frac{M_{\psi^n}}{n}\}_{n\in\mathbb{N}}$ is bounded. Finally, in the power bounded case, we already know that $\|\frac{M_{\psi^n}}{n}\| \to 0$ as $n \to \infty$, so the proof is complete. □

Lemma 3.3. *Let M_ψ be a bounded multiplication operator on $\mathscr{Z}$ ($\mathscr{Z}_0$). If $\|\psi\|_{\mathscr{Z}} < 1$ then M_{ψ^n} is convergent to zero in operator norm on $\mathscr{Z}$ ($\mathscr{Z}_0$). Moreover, M_ψ is uniformly mean ergodic (and hence mean ergodic) and power bounded on $\mathscr{Z}$ ($\mathscr{Z}_0$).*

Proof. It suffices to consider the estimate in (4). Then for $n \in \mathbb{N}$, we obtain that

$$\begin{aligned}\|M_{\psi^n}\| &\le |\psi^n(0)|C\log 2 + \|\psi^n\|_{\mathscr{Z}} + \sigma_{\psi^n} + \|\psi^n\|_\infty \\ &\le |\psi^n(0)|C\log 2 + \|\psi^n\|_{\mathscr{Z}} + \sup_{z\in\mathbb{D}}(1-|z|^2)n|\psi^{n-1}(z)\psi'(z)|C\log\frac{4}{1-|z|^2} + \|\psi^n\|_\infty \\ &\le |\psi(0)|^n C\log 2 + \|\psi\|_{\mathscr{Z}}^n + n\|\psi\|_\infty^{n-1}\sigma_\psi + \|\psi\|_\infty^n.\end{aligned}$$

Also by Lemma 2.1, we know that $\|\psi\|_\infty < 1$. Combining these facts shows that all the sequences in the right side of the inequality are convergent to zero whenever $\|\psi\|_{\mathscr{Z}} < 1$. The proof of lemma is completed. Lastly by using Lemma 3.1, M_ψ is uniformly mean ergodic and moreover M_ψ is power bounded. □

Theorem 3.4. *Suppose ψ is a analytic function on $\mathbb{D}$ inducing a bounded multiplication operator M_ψ on $\mathscr{Z}$ ($\mathscr{Z}_0$). If $\|\psi\|_\infty < 1$ or $\psi \equiv \gamma$, where $\gamma \in \partial\mathbb{D}$, then $M_\psi : \mathscr{Z}$ ($\mathscr{Z}_0$) $\to \mathscr{Z}$ ($\mathscr{Z}_0$) is power bounded.*

Proof. Assume that $\|\psi\|_\infty < 1$. To show that M_ψ is power bounded, firstly note that $\psi \in \mathscr{Z}$ ($\mathscr{Z}_0$) since $M_\psi f = \psi$ for $f \equiv 1$. Let $f \in \mathscr{Z}$ ($\mathscr{Z}_0$) such that $\|f\|_{\mathscr{Z}} \le 1$. Then

$$\begin{aligned}\|M_{\psi^n} f\|_{\mathscr{Z}} &\le |\psi^n(0)f(0)| + |n\psi^{n-1}(0)\psi'(0)f(0) + \psi^n(0)f'(0)| \\ &+ \sup_{z\in\mathbb{D}}(1-|z|^2)\left|n(n-1)\psi^{n-2}(z)(\psi'(z))^2 f(z)\right| \\ &+ \sup_{z\in\mathbb{D}}(1-|z|^2)\left|n\psi^{n-1}(z)\psi''(z)f(z)\right| \\ &+ 2\sup_{z\in\mathbb{D}}(1-|z|^2)\left|n\psi^{n-1}(z)\psi'(z)f'(z)\right| \\ &+ \sup_{z\in\mathbb{D}}(1-|z|^2)\left|\psi^n(z)f''(z)\right|.\end{aligned}$$

From this, using Lemma 2.1, we have

$$\begin{aligned}\|M_{\psi^n} f\|_{\mathscr{Z}} &\leq \Big(\|\psi\|_\infty^n + n\|\psi\|_\infty^{n-1}|\psi'(0)| + \|\psi\|_\infty^n C\log 2\Big)\|f\|_{\mathscr{Z}}\\ &+ n^2\|\psi\|_\infty^{n-2}\|\psi\|_{\mathscr{Z}}\Big(\sup_{z\in\mathbb{D}}(1-|z|^2)|\psi'(z)|C\log\frac{2}{1-|z|^2}\Big)\|f\|_{\mathscr{Z}}\\ &+ n\|\psi\|_\infty^{n-1}\Big(\sup_{z\in\mathbb{D}}(1-|z|^2)|\psi''(z)|\Big)\|f\|_{\mathscr{Z}}\\ &+ 2n\|\psi\|_\infty^{n-1}\Big(\sup_{z\in\mathbb{D}}(1-|z|^2)|\psi'(z)|C\log\frac{2}{1-|z|^2}\Big)\|f\|_{\mathscr{Z}}\\ &+ \|\psi\|_\infty^n\Big(\sup_{z\in\mathbb{D}}(1-|z|^2)|f''(z)|\Big)\end{aligned}$$

and using the property (3), which is a special case of (2), also we have

$$\begin{aligned}\|M_{\psi^n} f\|_{\mathscr{Z}} &\leq \Big(\|\psi\|_\infty^n + n\|\psi\|_\infty^{n-1}|\psi'(0)| + \|\psi\|_\infty^n C\log 2\Big)\|f\|_{\mathscr{Z}}\\ &+ \Big(n^2\|\psi\|_\infty^{n-2}C\frac{\sigma_\psi}{2}\|\psi\|_{\mathscr{Z}} + n\|\psi\|_\infty^{n-1}\|\psi\|_{\mathscr{Z}} + 2n\|\psi\|_\infty^{n-1}C\frac{\sigma_\psi}{2} + \|\psi\|_\infty^n\Big)\|f\|_{\mathscr{Z}}.\end{aligned}$$

Since each sequence on the right-hand side of the above inequality converges to zero as n tends to infinity, it is a bounded sequence. So there exists a constant $M > 0$ such that

$$\|M_{\psi^n} f\|_{\mathscr{Z}} \leq M\|f\|_{\mathscr{Z}}$$

for all $n \in \mathbb{N}$. Therefore $M_\psi : \mathscr{Z}\ (\mathscr{Z}_0) \to \mathscr{Z}\ (\mathscr{Z}_0)$ is power bounded.

Moreover, if $\psi \equiv \gamma$ where $\gamma \in \partial\mathbb{D}$, M_ψ is an isometry on $\mathscr{Z}\ (\mathscr{Z}_0)$, and for all $n \geq 1$, $\|M_\psi^n\| = 1$, therefore M_ψ is power bounded. □

Conclusion 3.5. *Let M_ψ be a bounded multiplication operator on $\mathscr{Z}\ (\mathscr{Z}_0)$. If $\|\psi\|_\infty < 1$ then M_ψ is uniformly mean ergodic (mean ergodic) on $\mathscr{Z}\ (\mathscr{Z}_0)$.*

Proof. By Theorem 3.4, M_{ψ^n} is convergent to zero in operator norm on $\mathscr{Z}\ (\mathscr{Z}_0)$ whenever $\|\psi\|_\infty < 1$. The result $\|(M_\psi)_{[n]})\| \to 0$ as $n \to \infty$ follows immediately from the Lemma 3.1. □

We continue to investigate under what conditions the operator M_ψ is uniformly mean ergodic.

Proposition 3.6. *Assume that M_ψ is a bounded operator on the Zygmund space (or the little Zygmund space). If $\psi \equiv \gamma$, where $\gamma \in \partial\mathbb{D}$, then M_ψ is uniformly mean ergodic on the Zygmund spaces (or the little Zygmund space).*

Proof. Assume that $\psi \equiv \gamma$, $|\gamma| = 1$. Firstly, if $\gamma = 1$ we obtain $(M_\psi)_{[n]}f = f$ for all $n \in \mathbb{N}$ and $f \in \mathscr{Z}$, then already the operator M_ψ is uniformly mean ergodic on the Zygmund spaces (or the little Zygmund space). Secondly, if $\gamma \neq 1$, then we have

$$M_{\psi^n} f = \psi^n f = \gamma^n f.$$

So

$$(M_\psi)_{[n]}f = \frac{\gamma + \gamma^2 + \cdots + \gamma^n}{n} f = \frac{f}{n}\frac{\gamma(1-\gamma^n)}{1-\gamma}.$$

In that case, for $f \in \mathscr{Z}$ we obtain

$$\begin{aligned}\|(M_\psi)_{[n]}\| &= \sup_{\|f\|_{\mathscr{Z}}\leq 1}\|(M_\psi)_{[n]}f\|_{\mathscr{Z}}\\ &= \frac{1}{n}\frac{|1-\gamma^n|}{|1-\gamma|}\sup_{\|f\|_{\mathscr{Z}}\leq 1}\Big\{|f(0)| + |f'(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)|f''(z)|\Big\}\\ &\leq \frac{2}{n|1-\gamma|}.\end{aligned}$$

Hence $\|(M_\psi)_{[n]}\| \to 0$ as $n \to \infty$ and M_ψ is uniformly mean ergodic on the Zygmund spaces (or the little Zygmund space). It is clear that M_ψ is also mean ergodic. $\square$

The following theorem which is also one of the main results of this study is obtained through a combination and interpretation of Theorem 3.4, Conclusion 3.5 and Proposition 3.6.

Theorem 3.7. *Suppose ψ is a analytic function on $\mathbb{D}$ inducing a bounded multiplication operator M_ψ on $\mathscr{Z}$ ($\mathscr{Z}_0$) and let $\|\psi\|_\infty < 1$ or $\psi \equiv \gamma$ where $\gamma \in \partial\mathbb{D}$, then the following statements are satisfied:*

(i) *$M_\psi : \mathscr{Z}\,(\mathscr{Z}_0) \to \mathscr{Z}\,(\mathscr{Z}_0)$ is power bounded.*

(ii) *$M_\psi : \mathscr{Z}\,(\mathscr{Z}_0) \to \mathscr{Z}\,(\mathscr{Z}_0)$ is mean ergodic.*

(iii) *$M_\psi : \mathscr{Z}\,(\mathscr{Z}_0) \to \mathscr{Z}\,(\mathscr{Z}_0)$ is uniformly mean ergodic.*

Remark 3.8. Observe that this result does not extend to nonconstant functions with $\|\psi\|_\infty = 1$. Indeed, when $\psi(z) = z^2$, the corresponding multiplication operator M_ψ is not power bounded on $\mathscr{Z}\,(\mathscr{Z}_0)$. Likewise, when $\psi(z) = z$, the operator M_ψ is neither mean ergodic nor uniformly mean ergodic on $\mathscr{Z}\,(\mathscr{Z}_0)$, see Counterexamples 3.10 (iv) and (v).

Theorem 3.9. *Suppose $\psi \in H(\mathbb{D})$ and M_ψ is a bounded multiplication operator on the Zygmund spaces $\mathscr{Z}$. Then $\sigma(M_\psi) = \overline{\psi(\mathbb{D})}$, the norm closure of $\psi(\mathbb{D})$.*

Proof. For any $\lambda \in \mathbb{C}$, the operator $M_\psi - \lambda I$ can be reexpressed as a multiplication operator by the function $\psi(z) - \lambda$, denoted as $M_{\psi-\lambda}$. Consequently, λ belongs to the spectrum of M_ψ (i.e., $\lambda \in \sigma(M_\psi)$) if and only if the operator $M_{\psi-\lambda}$ is not invertible. It is clear that if $M^{-1}_{\psi-\lambda}$ exists, it corresponds to the multiplication operator $M_{(\psi-\lambda)^{-1}}$.

Let $\lambda \notin \sigma(M_\psi)$. This implies that $M_{\psi-\lambda}$ is invertible so $M^{-1}_{\psi-\lambda} = M_{(\psi-\lambda)^{-1}}$. This means that the function $(\psi-\lambda)^{-1}$ will not possess a pole at any point of $\mathbb{D}$. Therefore, $\psi(z) - \lambda \neq 0$ for every $z \in \mathbb{D}$, that is, $\lambda \notin \psi(\mathbb{D})$. This line of reasoning establishes that the image set $\psi(\mathbb{D})$ is a subset of M_ψ's spectrum which is a closed set: $\overline{\psi(\mathbb{D})} \subseteq \sigma(M_\psi)$.

Now, assume that $\lambda \notin \overline{\psi(\mathbb{D})}$. Then there exists some $\varepsilon > 0$ such that $|\psi(z) - \lambda| > \varepsilon$ for every $z \in \mathbb{D}$. So $g(z) := 1/(\psi(z) - \lambda) \in H^\infty(\mathbb{D})$. We claim that the multiplicaiton operator M_g is bounded operator on $\mathscr{Z}$. The proof of this claim relies on Theorem 3.1 in [26]. First, we start by noting that $\psi \in \mathscr{Z}$ and $\sigma_\psi < \infty$. Now, applying equation (2), we obtain

$$\sup_{z\in\mathbb{D}}\left((1-|z|^2)|2g'(z)|\log\frac{1}{1-|z|^2}\right) \leq \sup_{z\in\mathbb{D}}\left(\frac{(1-|z|^2)|\psi'(z)|}{|\psi(z)-\lambda|^2}\log\frac{4}{1-|z|^2}\right) \leq \frac{\sigma_\psi}{C\varepsilon^2}.$$

Hence, equation (2) holds. Moreover,

$$\sup_{z\in\mathbb{D}}(1-|z|^2)|g''(z)| \leq \sup_{z\in\mathbb{D}}\frac{(1-|z|^2)|\psi''(z)|}{|\psi(z)-\lambda|^2} + \sup_{z\in\mathbb{D}}\frac{(1-|z|^2)|2\psi'(z)|^2}{|\psi(z)-\lambda|^3} \leq (\frac{1}{\varepsilon^2} + \frac{\sigma_\psi}{\varepsilon^3})\|\psi\|_{\mathscr{Z}},$$

which immediately implies that $g \in \mathscr{Z}$. It then follows that $M_g = M_{(\psi-\lambda)^{-1}}$ is indeed a bounded operator on $\mathscr{Z}$. Thus $\lambda \notin \sigma(M_\psi)$ and consequently $\sigma(M_\psi) \subseteq \overline{\psi(\mathbb{D})}$. $\square$

Counterexamples 3.10. *A few natural questions to consider are the following:*

(i) *Firstly, is the multiplication operator power bounded whenever $\|\psi\|_{\mathscr{Z}} \geq 1$ holds?* The answer to this question is negative.

(a) If we take the function $\psi_1(z) = z$, then $\|\psi_1\|_{\mathscr{Z}} = 1$ and $\psi_1 \in \mathscr{Z}$, which is an analytic function on $\mathbb{D}$ inducing a bounded multiplication operator M_{ψ_1}. But since

$$\begin{aligned}
\|\psi_1^n\|_{\mathscr{Z}} &= |\psi_1^n(0)| + |n\psi_1^{n-1}(0)\psi_1'(0)| \\
&+ \sup_{z\in\mathbb{D}}(1-|z|^2)|n(n-1)\psi_1^{n-2}(z)(\psi_1'(z))^2 + n\psi_1^{n-1}(z)\psi_1''(z)| \\
&\geq n(n-1)\sup_{z\in\mathbb{D}}(1-|z|^2)|z|^{n-2} \\
&= 2(n-1)\Big(1-\frac{2}{n}\Big)^{\frac{n-2}{2}}
\end{aligned}$$

for all $n \in \mathbb{N}$, then $\|\psi_1^n\|_{\mathscr{Z}} \to \infty$ as $n \to \infty$, so M_{ψ_1} may not be power bounded.

(b) If we take the function $\psi_2(z) = e^z$, then since $\sup_{z\in\mathbb{D}}(1-|z|^2)|e^z| < e$ thus $\psi_2 \in \mathscr{Z}$, which is an analytic function on $\mathbb{D}$ inducing a bounded multiplication operator M_{ψ_2}. Clearly $\|\psi_2\|_{\mathscr{Z}} > 1$ and $\psi_2^n(z) = e^{nz}$. But since

$$\begin{aligned}
\|\psi_2^n\|_{\mathscr{Z}} &= |\psi_2^n(0)| + |(\psi_2^n)'(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)|(\psi_2^n)''(z)| \\
&\geq 1 + n + n^2\sup_{z\in\mathbb{D}}(1-|z|^2)|e^{nz}| \\
&\geq n
\end{aligned}$$

for all $n \in \mathbb{N}$, then $\|\psi_2^n\|_{\mathscr{Z}} \to \infty$ as $n \to \infty$, so M_{ψ_2} may not be power bounded.

(ii) *Secondly, is the converse of Lemma 3.3 true? That is, can we infer that $\|\psi\|_{\mathscr{Z}} < 1$ holds if the multiplication operator M_ψ is power bounded?*

The answer to this question is also negative. If we consider the function $\psi(z) = \frac{5}{6}z^2$, then we get $\psi \in \mathscr{Z}$ and $\psi^n(z) = (\frac{5}{6})^n z^{2n}$. Let $\|f\|_{\mathscr{Z}} \leq 1$, from this we obtain that

$$\begin{aligned}
\|M_{\psi^n} f\|_{\mathscr{Z}} &= \sup_{z\in\mathbb{D}}(1-|z|^2)\Big|2n(2n-1)(\frac{5}{6})^n z^{2n-2}f(z) + 4n(\frac{5}{6})^n z^{2n-1}f'(z) + (\frac{5}{6})^n z^{2n}f''(z)\Big| \\
&\leq 4n^2(\frac{5}{6})^n\|f\|_{\mathscr{Z}} + 4nC(\frac{5}{6})^n\|f\|_{\mathscr{Z}} + (\frac{5}{6})^n\|f\|_{\mathscr{Z}} \\
&\leq (4n^2 + 4nC + 1)(\frac{5}{6})^n
\end{aligned}$$

for all $n \in \mathbb{N}$, therefore $\|M_{\psi^n}\| \to 0$ as $n \to \infty$, that is M_ψ is power bounded. But $\|\psi\|_{\mathscr{Z}} = \frac{5}{3} > 1$.

(iii) *Another critical question is as follows: instead of the condition $\|\psi\|_{\mathscr{Z}} < 1$ in Lemma 3.3, does the more general assumption $\|\psi\|_\infty < 1$ imply that the multiplication operator M_ψ is power bounded?* Depending on the Zygmund norm of the ψ function, the answer to this question is positive in either case.

(a) If $\|\psi\|_\infty \leq \|\psi\|_{\mathscr{Z}} < 1$, it is clearly that Lemma 3.3 already states this that M_ψ is power bounded. The choice of the function $\psi(z) = \frac{1}{3}z^2$ such that $\|\psi\|_\infty = \frac{1}{3}$ and $\|\psi\|_{\mathscr{Z}} = \frac{2}{3}$ may be an example of such a situation.

(b) If $\|\psi\|_\infty < 1 \leq \|\psi\|_{\mathscr{Z}}$, as can be seen in the above Theorem 3.4, again M_ψ is power bounded.

(iv) *Another question, does the condition $\|\psi\|_\infty = 1$ entail the power-boundedness of M_ψ?* Let $\psi(z) = z^2$. Thus $\psi \in \mathscr{Z}$ which is an analytic function on $\mathbb{D}$ inducing a bounded multiplication operator M_ψ. On the one hand $\psi^n(z) = z^{2n}$ and $\|\psi\|_\infty = 1$. But on the other hand we have

$$\begin{aligned}
\|\psi^n\|_{\mathscr{Z}} &= |\psi^n(0)| + |(\psi^n)'(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)|(\psi^n)''(z)| \\
&= 2n(2n-1)\sup_{z\in\mathbb{D}}(1-|z|^2)|z^{2n-2}| \\
&= 2(2n-1)(1-\frac{1}{n})^{n-1}
\end{aligned}$$

for all $n \in \mathbb{N}$. From this $\|\psi^n\|_{\mathscr{Z}} \to \infty$ as $n \to \infty$, M_ψ may not be power bounded.

(v) *The final question, which could also help in interpreting Conclusion 3.5, does the condition* $\|\psi\|_\infty = 1$ *imply the uniformly mean ergodic (or mean ergodic) of* M_ψ *on the Zygmund spaces?*

For the analytic function $\psi(z) = z$ with $\|\psi\|_\infty = 1$, the spectrum $\sigma(M_z)$ of the multiplication operator M_z is given by the closed unit disk $\overline{\mathbb{D}}$, see Theorem 3.9. This implies that 1 is an accumulation point of the spectrum, thus by [10, Theorem 3.16] and [9, Theorem], M_z is not uniformly mean ergodic on $\mathscr{Z}$. Moreover, the Cesàro means of M_z are

$$(M_z)_{[n]} = \frac{1}{n}\sum_{k=1}^{n} M_z^k = M_{h_n}, \quad h_n(z) := \frac{z(1-z^n)}{n(1-z)}, \quad n \in \mathbb{N}.$$

If M_z were mean ergodic, the means $(M_z)_{[n]}$ would converge strongly to the projection onto $\ker(I - M_z)$. Since $\ker(I - M_z) = \{0\}$, mean ergodicity would imply $(M_z)_{[n]}f \to 0$ for every $f \in \mathscr{Z}$. We show this fails even for the constant function $f \equiv 1$. Indeed, it is known that

$$\|(M_z)_{[n]}\| \geq \|(M_z)_{[n]}1\|_{\mathscr{Z}} = \|h_n\|_{\mathscr{Z}}.$$

Now, choosing $r_n := 1 - \frac{1}{n}$. Hence we obtain that

$$\sup_{z\in\mathbb{D}}(1-|z|^2)|h_n''(z)| \geq (1-r_n^2)|h_n''(r_n)| \geq \frac{1-r_n^2}{n}\sum_{k=2}^{n} k(k-1)r_n^{k-2} > \frac{n}{6}(1-\frac{1}{n})^n.$$

Consequently $\|(M_z)_{[n]}\| \to \infty$, $(n \to \infty)$. Hence M_z is also not mean ergodic on $\mathscr{Z}$.

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