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Common spectral properties of the operator products AC and BD in quaternionic setting

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Abstract. Let $A, B, C, D : X \to X$ be bounded right linear quaternionic operators acting on a two-sided quaternionic Banach space X, and suppose that

$$ACD = DBD \quad \text{and} \quad DBA = ACA.$$

Let q be a non-zero quaternion. In this paper, we first present a new extension of Jacobson's lemma in the setting of quaternionic two-sided Banach algebras. Building upon this, we analyze the operators $(AC)^2 - 2\,\mathrm{Re}(q)AC + |q|^2\mathcal{I}$ and $(BD)^2 - 2\,\mathrm{Re}(q)BD + |q|^2\mathcal{I}$, where $\mathcal{I}$ denotes the identity operator on X. In particular, we establish the equality

$$\sigma_\star^S(AC) \setminus \{0\} = \sigma_\star^S(BD) \setminus \{0\},$$

where $\sigma_\star^S(\cdot)$ denotes a distinguished subset of the spherical spectrum.

1. Introduction

Let $A, B, C, D : X \to X$ be bounded right linear quaternionic operators acting on a two-sided quaternionic Banach space X, and suppose that

$$ACD = DBD \quad \text{and} \quad DBA = ACA.$$

The spectral properties shared by the operator products AC and BD have been extensively studied in the context of complex Banach spaces; see [1, 4, 8, 12, 13, 16, 17] and the references therein. Early investigations considered the particular case $C = B$ and $D = A$. These studies were later generalized to situations where $D = A$, demonstrating that the operators AC and BA share various spectral properties. Ultimately, these results were extended to the more general setting of operator products AC and BD.

In [3], we established that for $q \in \mathbb{H}\setminus\{0\}$, the operators $(AC)^2 - 2\,\mathrm{Re}(q)AC + |q|^2\mathcal{I}$ and $(BA)^2 - 2\,\mathrm{Re}(q)BA + |q|^2\mathcal{I}$ exhibit several common spectral characteristics under the assumption $ACA = ABA$. The present work builds upon and extends both the results of [3] and those of [12, 13]. Specifically, we investigate the spectral and Fredholm properties of the operators AC and BD under the relations $ACD = DBD$ and $DBA = ACA$.

The paper is organized as follows. Section 2 is devoted to recalling the essential definitions and notations that will be used throughout the paper. Section 3 establishes a new quaternionic extension of Jacobson's lemma. In Section 4, we prove that the operator products AC and BD share several common S-spectral and Fredholm properties.

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2. Preliminaries

In this work, we consider the algebra of quaternions over $\mathbb{R}$, denoted by $\mathbb{H}$, which consists of elements of the form $q = x_0 + x_1 i + x_2 j + x_3 k$, where $x_0, x_1, x_2, x_3 \in \mathbb{R}$. The imaginary units i, j, k satisfy the multiplication rules

$$\begin{cases} i^2 = j^2 = k^2 = -1, \\ ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j. \end{cases}$$

We denote by $\mathrm{Re}(q)$ the real part of q, given by $\mathrm{Re}(q) = x_0$, and by $\mathrm{Im}(q)$ its imaginary part, expressed as $\mathrm{Im}(q) = x_1 i + x_2 j + x_3 k$. The quaternionic conjugate of q is defined by

$$\overline{q} = x_0 - x_1 i - x_2 j - x_3 k,$$

and the norm on $\mathbb{H}$ is given by $|q| = \sqrt{q\overline{q}}$.

Let X be an additive group. We say that X is a quaternionic two-sided vector space if it is equipped with a left scalar multiplication $(q, s) \mapsto qs \in X$ and a right scalar multiplication $(s, q) \mapsto sq \in X$, where $q \in \mathbb{H}$ and $s \in X$, such that the following properties hold:

a) $q(s + t) = qs + qt$ and $(s + t)q = sq + tq$,

b) $(q + p)s = qs + ps$ and $s(q + p) = sq + sp$,

c) $q(ps) = (qp)s$ and $(sp)q = s(pq)$,

d) $(qs)p = q(sp)$,

e) $s = 1s = s1$,

f) $rs = sr$ for all $r \in \mathbb{R}$,

for all $s, t \in X$ and $q, p \in \mathbb{H}$. Moreover, if X is complete with respect to a norm $\|\cdot\|$, then X is called a *two-sided quaternionic Banach space*.

Consider X, a two-sided quaternionic Banach space. A map $T : X \to X$ is called a *right linear operator* if, for all $x, y \in X$ and $q \in \mathbb{H}$, it satisfies

$$T(xq + y) = (Tx)q + Ty.$$

The operator T is said to be *bounded* if there exists a constant $M > 0$ such that

$$\|Tx\| \leq M\|x\| \quad \text{for all } x \in X,$$

and its norm is given by

$$\|T\| = \sup\{\|Tx\| : x \in X, \ \|x\| = 1\} < \infty.$$

The collection of all bounded right linear operators on X is denoted by $\mathcal{B}(X)$. For $T \in \mathcal{B}(X)$ and $q \in \mathbb{H}$, define the quadratic operator

$$\Delta_q(T) = T^2 - 2\,\mathrm{Re}(q)T + |q|^2\mathcal{I},$$

where $\mathcal{I}$ denotes the identity operator on X. The set of quaternions q for which $\Delta_q(T)$ is invertible in $\mathcal{B}(X)$ is called the *spherical resolvent set* of T, denoted $\rho_S(T)$. Its complement,

$$\sigma_S(T) = \mathbb{H} \setminus \rho_S(T),$$

is referred to as the *spherical spectrum* (or S-spectrum) of T. To refine the analysis of invertibility, we also introduce:

- the *right S-spectrum*

$$\sigma_{Sr}(T) = \{q \in \mathbb{H} : \Delta_q(T) \text{ is not right invertible in } \mathcal{B}(X)\},$$

- the *left S-spectrum*

$$\sigma_{Sl}(T) = \{q \in \mathbb{H} : \Delta_q(T) \text{ is not left invertible in } \mathcal{B}(X)\},$$

- the *approximate S-spectrum*

$$\sigma^S_{ap}(T) = \left\{q \in \mathbb{H} : \exists \{x_n\} \subset X,\ \|x_n\| = 1,\ \|\Delta_q(T)x_n\| \to 0\right\},$$

- and the *point S-spectrum*

$$\sigma^S_p(T) = \left\{q \in \mathbb{H} : \ker(\Delta_q(T)) \neq \{0\}\right\}.$$

- and the *surjective S-spectrum*

$$\sigma^S_{sur}(T) = \left\{q \in \mathbb{H} : \operatorname{ran}(\Delta_q(T)) \neq X\right\}.$$

Theorem 2.1 ([5]). *Let $A, B \in \mathcal{B}(X)$. Then the following relations hold:*
(i) $\sigma_S(AB) \setminus \{0\} = \sigma_S(BA) \setminus \{0\}$,
(ii) $\sigma_{Sr}(AB) \setminus \{0\} = \sigma_{Sr}(BA) \setminus \{0\}$,
(iii) $\sigma_{Sl}(AB) \setminus \{0\} = \sigma_{Sl}(BA) \setminus \{0\}$,
(iv) $\sigma^S_p(AB) \setminus \{0\} = \sigma^S_p(BA) \setminus \{0\}$,
(v) $\sigma^S_{ap}(AB) \setminus \{0\} = \sigma^S_{ap}(BA) \setminus \{0\}$.

For additional results on quaternionic Banach spaces and the spectral theory in the quaternionic setting, we refer the reader to [6, 7].

We denote by $\mathcal{R}(T)$ and $\mathcal{N}(T)$ the range and the kernel of an operator T, respectively. Let $\mathbb{N}$ denote the set of nonegative integers. The *nullity* of T, denoted by $\alpha(T)$, is defined as the dimension of its kernel $\alpha(T) = \dim \mathcal{N}(T)$. The *deficiency* of T, denoted by $\beta(T)$, is defined as the codimension of its range $\beta(T) = \dim(X/\mathcal{R}(T))$. Assuming that at least one of $\alpha(T)$ or $\beta(T)$ is finite, the *index* of T is given by $\operatorname{ind}(T) = \alpha(T) - \beta(T)$.

The *ascent* of T, written $\operatorname{asc}(T)$, is the smallest integer $n \in \mathbb{N}$ such that $\mathcal{N}(T^n) = \mathcal{N}(T^{n+1})$, if such an n exists; otherwise, we set $\operatorname{asc}(T) = \infty$. Similarly, the *descent* of T, denoted by $\operatorname{dsc}(T)$, is the smallest integer $n \in \mathbb{N}$ such that $\mathcal{R}(T^n) = \mathcal{R}(T^{n+1})$, with $\operatorname{dsc}(T) = \infty$ if no such n exists. Now, set

$$\begin{array}{lcl}
\Phi_+(X) & = & \{T \in \mathcal{B}(X) : \dim\ker(T) < \infty \text{ and } \operatorname{ran}(T) \text{ is closed}\},\\
\Phi_-(X) & = & \{T \in \mathcal{B}(X) : \operatorname{codim}\operatorname{ran}(T) < \infty \text{ and } \operatorname{ran}(T) \text{ is closed}\},\\
\Phi(X) & = & \Phi_+(X) \cap \Phi_-(X),\\
\Phi_\pm(X) & = & \Phi_+(X) \cup \Phi_-(X),\\
\mathcal{W}_+(X) & = & \{T \in \Phi_+(X) : \operatorname{ind}(T) \leq 0\},\\
\mathcal{W}_-(X) & = & \{T \in \Phi_-(X) : \operatorname{ind}(T) \geq 0\},\\
\mathcal{W}(X) & = & \mathcal{W}_+(X) \cap \mathcal{W}_-(X),\\
\mathcal{B}\Phi_+(X) & = & \{T \in \Phi_+(X) : \operatorname{asc}(T) < \infty\},\\
\mathcal{B}\Phi_-(X) & = & \{T \in \Phi_-(X) : \operatorname{dsc}(T) < \infty\},\\
\mathcal{B}\Phi(X) & = & \{T \in \Phi(X) : \operatorname{asc}(T) = \operatorname{dsc}(T) < \infty\},\\
\mathcal{D}_\ell(X) & = & \left\{T \in \mathcal{B}(X) : \operatorname{asc}(T) < \infty \text{ and } \operatorname{ran}\left(T^{\operatorname{asc}(T)+1}\right) \text{ is closed}\right\},\\
\mathcal{D}_r(X) & = & \left\{T \in \mathcal{B}(X) : \operatorname{dsc}(T) < \infty \text{ and } \operatorname{ran}\left(T^{\operatorname{dsc}(T)}\right) \text{ is closed}\right\},\\
\mathcal{D}(X) & = & \mathcal{D}_\ell(X) \cap \mathcal{D}_r(X).
\end{array}$$

Here:

- $\Phi_+(X), \Phi_-(X), \Phi(X)$, $\Phi_\pm(X)$ denote the classes of upper semi-Fredholm, lower semi-Fredholm, Fredholm, and semi-Fredholm operators, respectively.
- $\mathcal{W}_+(X), \mathcal{W}_-(X), \mathcal{W}(X)$ denote the classes of upper semi-Weyl, lower semi-Weyl, and Weyl operators, respectively.
- $\mathcal{B}_+(X), \mathcal{B}_-(X), \mathcal{B}(X)$ correspond to the upper semi-Browder, lower semi-Browder, and Browder operators, respectively.
- $\mathcal{D}_\ell(X), \mathcal{D}_r(X), \mathcal{D}(X)$ refer to the classes of left Drazin invertible, right Drazin invertible, and Drazin invertible operators, respectively.

3. Extension of Jacobson's Lemma

In this section, let $\mathcal{A}$ be a two-sided quaternionic Banach algebra with unit element e. We define the following subsets of $\mathcal{A}$:

$$\mathcal{A}_l^{-1} = \left\{a \in \mathcal{A} \mid \exists\, x \in \mathcal{A} \text{ such that } xa = e\right\},$$

$$\mathcal{A}_r^{-1} = \left\{a \in \mathcal{A} \mid \exists\, x \in \mathcal{A} \text{ such that } ax = e\right\},$$

$$\mathcal{A}^{-1} = \mathcal{A}_l^{-1} \cap \mathcal{A}_r^{-1}.$$

These denote the sets of left-invertible, right-invertible, and invertible elements of $\mathcal{A}$, respectively. For any $a, b \in \mathcal{A}$ and any nonzero quaternion $q \in \mathbb{H} \setminus \{0\}$, we define the element

$$\Delta_q(ab) := (ab)^2 - 2\ \mathrm{Re}(q)\, ab + |q|^2 e.$$

Proposition 3.1. *Let $a, b, c, d \in \mathcal{R}$ such that $acd = dbd$ and $dba = aca$. Then :*

(i) $db(aca - 2Re(q)a) = (aca - 2Re(q)a)ca,$

(ii) $ac(dbd - 2Re(q)d) = (dbd - 2Re(q)d)bd.$

Proof. We compute

$$\begin{aligned} db(aca - 2\,\mathrm{Re}(q)a) &= dbaca - 2\,\mathrm{Re}(q)dba \\ &= acaca - 2\,\mathrm{Re}(q)aca \\ &= (aca - 2\,\mathrm{Re}(q)a)ca. \end{aligned}$$

Similarly,

$$\begin{aligned} ac(dbd - 2\,\mathrm{Re}(q)d) &= acdbd - 2\,\mathrm{Re}(q)acd \\ &= dbdbd - 2\,\mathrm{Re}(q)dbd \\ &= (dbd - 2\,\mathrm{Re}(q)d)bd. \end{aligned}$$

This completes the proof. □

Theorem 3.2. *Let $a, b, c, d \in \mathcal{A}$ be such that $acd = dbd \quad$ and $\quad dba = aca$, and let $q \in \mathbb{H} \setminus \{0\}$. Then the following hold:*

(i) $\Delta_q(ac) \in \mathcal{A}_l^{-1} \iff \Delta_q(bd) \in \mathcal{A}_l^{-1}.$

(ii) $\Delta_q(ac) \in \mathcal{A}_r^{-1} \iff \Delta_q(bd) \in \mathcal{A}_r^{-1}.$

(iii) $\Delta_q(ac) \in \mathcal{A}^{-1} \iff \Delta_q(bd) \in \mathcal{A}^{-1}.$

Proof. First set

$$d' = dbd - 2\operatorname{Re}(q)d, \quad a' = aca - 2\operatorname{Re}(q)a.$$

(i) Assume that there exists $s \in \mathcal{A}$ such that $s\Delta_q(ac) = e$. Then, we have

$$e - |q|^2 s = s\Big(aca - 2\operatorname{Re}(q)a\Big)c.$$

Therefore

$$\begin{aligned} b\Big(e - |q|^2 s\Big)d' &= bs\Big(aca - 2\operatorname{Re}(q)a\Big)c\Big(dbd - 2\operatorname{Re}(q)d\Big) \\ &= bs(ac - 2\operatorname{Re}(q)e)acd(bd - 2\operatorname{Re}(q)e) \\ &= bs(ac - 2\operatorname{Re}(q)e)dbd(bd - 2\operatorname{Re}(q)e) \\ &= bsd'bd'. \end{aligned}$$

Now compute

$$\begin{aligned} \Big(e - bsd'\Big)\Delta_q(bd) &= \Big(e - bsd'\Big)\Big((bd)^2 - 2\operatorname{Re}(q)bd + |q|^2 e\Big) \\ &= (bd)^2 - 2\operatorname{Re}(q)bd + |q|^2 e - bsd'\Big((bd)^2 - 2\operatorname{Re}(q)bd\Big) - |q|^2 bsd' \\ &= bd' - bsd'bd' - |q|^2 bsd' + |q|^2 e \\ &= b(e - |q|^2 s)d' - bsd'bd' + |q|^2 e \\ &= bsd'bd' - bsd'bd' + |q|^2 e \\ &= |q|^2 e. \end{aligned}$$

Thus,

$$|q|^{-2}\Big(e - bsd'\Big)\Delta_q(bd) = e,$$

which shows that $\Delta_q(bd) \in \mathcal{A}_l^{-1}$.

Conversely, suppose that $\Delta_q(bd) \in \mathcal{A}_l^{-1}$. Then, by [5, Lemma 2.1], it follows that $\Delta_q(db) \in \mathcal{A}_l^{-1}$. Applying the same reasoning as in the direct implication, we deduce that $\Delta_q(ca) \in \mathcal{A}_l^{-1}$, and finally, by [5, Lemma 2.1], we obtain $\Delta_q(ac) \in \mathcal{A}_l^{-1}$.

For (ii), that there exists $s \in \mathcal{A}$ such that $\Delta_q(bd)s = e$. Proceed similarly:

$$\begin{aligned} d' &= \Big(dbd - 2\operatorname{Re}(q)d\Big)\Delta_q(bd)s \\ &= d\Big((bd)^2 - 2\operatorname{Re}(q)bd + |q|^2 e\Big)bds - 2\operatorname{Re}(q)\Big((bd)^2 - 2\operatorname{Re}(q)bd + |q|^2 e\Big)s \\ &= (ac)^2 - 2\operatorname{Re}(q)ac + |q|^2 e)d's \\ &= \Delta_q(ac)d's. \end{aligned}$$

Next, observe

$$\begin{aligned} \Delta_q(ac)\Big(a' - d'sba'\Big)c &= \Delta_q(ac)a'c - \Delta_q(ac)d'sba'c \\ &= \Delta_q(ac)a'c - d'ba'c \\ &= \Delta_q(ac)a'c - a'ca'c. \end{aligned}$$

Since $a'c = \Delta_q(ac) - |q|^2 e$, we find

$$\begin{aligned} \Delta_q(ac)\Big(a' - d'sba'\Big)c &= \Big(\Delta_q(ac) - |q|^2 e\Big)a'c \\ &= |q|^2 a'c \\ &= |q|^2\Big(\Delta_q(ac) - |q|^2 e\Big). \end{aligned}$$

Hence

$$\Delta_q(ac)\Big(e + |q|^{-2}\Big(a' - d'sba'\Big)c\Big)|q|^{-2} = e.$$

This proves that $\Delta_q(ac) \in \mathcal{A}_r^{-1}$.

Conversely, assume that $\Delta_q(bd) \in \mathcal{A}_r^{-1}$. According to [5, Lemma 2.1], this implies that $\Delta_q(db) \in \mathcal{A}_r^{-1}$. Using similar arguments as those employed in the direct implication, it follows that $\Delta_q(ca) \in \mathcal{A}_r^{-1}$. Finally, invoking [5, Lemma 2.1], we conclude that $\Delta_q(ac) \in \mathcal{A}_r^{-1}$.

Part (iii) follows directly from (i) and (ii). $\square$

Remark 3.3. *It should be noted that the results of this theorem extend those in parts (i), (ii), and (iii) of Theorem 2.1.*

4. Commun spectral properties of AC and BD

Throughout the remainder of this work, we assume that X is a two-sided quaternionic Banach space, and fix $q \in \mathbb{H} \setminus \{0\}$. Let $A, B, C, D \in \mathcal{B}(X)$ be bounded right linear operators satisfying the structural relations

$$ACD = DBD \quad \text{and} \quad DBA = ACA.$$

Define $D' = DBD - 2\,\mathrm{Re}(q)D$ and $A' = ACA - 2\,\mathrm{Re}(q)A$. A straightforward computation yields

$$\begin{cases} A'CD' = D'BD', \\ D'BA' = A'CA', \\ \Delta_q(BD) = BD' + |q|^2\mathcal{I}, \\ \Delta_q(AC) = A'C + |q|^2\mathcal{I}. \end{cases} \tag{1}$$

We begin with the following lemmas, which play a fundamental role in establishing the main results of this section.

Lemma 4.1. *Let P be a polynomial with real coefficients. Then the following hold:*

(i) $D\mathcal{R}\Big(P(\Delta_q(BD))\Big) \subseteq \mathcal{R}\Big(P(\Delta_q(AC))\Big)$;

(ii) $D\mathcal{N}\Big(P(\Delta_q(BD))\Big) \subseteq \mathcal{N}\Big(P(\Delta_q(AC))\Big)$;

(iii) $BAC\mathcal{R}\Big(P(\Delta_q(AC))\Big) \subseteq \mathcal{R}\Big(P(\Delta_q(BD))\Big)$;

(iv) $BAC\mathcal{N}\Big(P(\Delta_q(AC))\Big) \subseteq \mathcal{N}\Big(P(\Delta_q(BD))\Big)$.

Proof. (i) Let $y \in \mathcal{R}\Big(P(\Delta_q(BD))\Big)$. Then there exists $x \in X$ such that $y = P(\Delta_q(BD))x$. Using the relation $ACD = DBD$, we compute $Dy = D\Big(P(\Delta_q(BD))x\Big) = P(\Delta_q(AC))Dx$. Thus, $Dy \in \mathcal{R}\Big(P(\Delta_q(AC))\Big)$, and it follows that $D\mathcal{R}\Big(P(\Delta_q(BD))\Big) \subseteq \mathcal{R}\Big(P(\Delta_q(AC))\Big)$.

(ii) Let $x \in \mathcal{N}\Big(P(\Delta_q(BD))\Big)$. Then, by $ACD = DBD$, we obtain that $P(\Delta_q(AC))Dx = DP(\Delta_q(BD))x = D(\{0\}) = 0$, so $Dx \in \mathcal{N}\Big(P(\Delta_q(AC))\Big)$. Consequently, $D\mathcal{N}\Big(P(\Delta_q(BD))\Big) \subseteq \mathcal{N}\Big(P(\Delta_q(AC))\Big)$.

(iii) Let $y \in \mathcal{R}\Big(P(\Delta_q(AC))\Big)$. Then there exists $x \in X$ such that $y = P(\Delta_q(AC))x$. Using the relation $DBA = ACA$, we deduce $BACy = BAC\Big(P(\Delta_q(AC))x\Big) = P(\Delta_q(BD))BACx$. Hence, $BACy \in \mathcal{R}\Big(P(\Delta_q(BD))\Big)$, and we conclude that $BAC\mathcal{R}\Big(P(\Delta_q(AC))\Big) \subseteq \mathcal{R}\Big(P(\Delta_q(BD))\Big)$.

(iv) Let $x \in \mathcal{N}\Big(P(\Delta_q(AC))\Big)$. Then, by $DBA = ACA$, we find $P(\Delta_q(BD))BACx = BAC\Big(P(\Delta_q(AC))y\Big) = BAC(\{0\}) = 0$, so $BACx \in \mathcal{N}\Big(P(\Delta_q(BD))\Big)$. This gives $BAC\mathcal{N}\Big(P(\Delta_q(AC))\Big) \subseteq \mathcal{N}\Big(P(\Delta_q(BD))\Big)$. $\square$

Lemma 4.2. *Let P be a polynomial with real coefficients. Then the following hold:*

(i) $D'\mathcal{R}\Big(P(\Delta_q(BD))\Big) \subseteq \mathcal{R}\Big(P(\Delta_q(AC))\Big)$;

(ii) $D'\mathcal{N}\Big(P(\Delta_q(BD))\Big) \subseteq \mathcal{N}\Big(P(\Delta_q(AC))\Big)$;

(iii) $BA'C\mathcal{R}\Big(P(\Delta_q(AC))\Big) \subseteq \mathcal{R}\Big(P(\Delta_q(BD))\Big)$;

(iv) $BA'C\mathcal{N}\Big(P(\Delta_q(AC))\Big) \subseteq \mathcal{N}\Big(P(\Delta_q(BD))\Big)$.

Proof. The proofs of statements (i) and (ii) follow directly from Lemma 4.1 and the observation that

$$D' = DBD - 2\,\mathrm{Re}(q)D = D\Big(BD - 2\,\mathrm{Re}(q)\mathcal{I}\Big),$$

where the operator $BD - 2\,\mathrm{Re}(q)\mathcal{I}$ commutes with $\Delta_q(BD)$.

Similarly, the proofs of statements (iii) and (iv) follow from Lemma 4.1 and the relation

$$BA'C = BAC\Big(AC - 2\,\mathrm{Re}(q)\mathcal{I}\Big),$$

where the operator $AC - 2\,\mathrm{Re}(q)\mathcal{I}$ commutes with $\Delta_q(AC)$. $\square$

Lemma 4.3. *We have $\Delta_q(AC)$ is injective if and only if $\Delta_q(BD)$ is injective .*

Proof. Suppose that $\Delta_q(AC)$ is injective. Then $\mathcal{N}\Big(\Delta_q(AC)\Big) = \{0\}$. By Theorem 2.1, it follows that $\mathcal{N}\Big(\Delta_q(CA)\Big) = \{0\}$. Applying Lemma 4.1, we deduce that $\mathcal{N}\Big(\Delta_q(DB)\Big) = \{0\}$, and once again, by Theorem 2.1, we conclude that $\mathcal{N}\Big(\Delta_q(BD)\Big) = \{0\}$. Thus, $\Delta_q(BD)$ is injective.

Conversely, assume that $\Delta_q(BD)$ is injective. By Theorem 2.1, it follows that $\Delta_q(DB)$ is also injective. Applying the preceding argument to $\Delta_q(DB)$, we deduce that $\Delta_q(CA)$ is injective. Finally, invoking Theorem 2.1 once more, we conclude that $\Delta_q(AC)$ is injective. $\square$

Lemma 4.4. *We have $\Delta_q(AC)$ is bounded below if and only if $\Delta_q(BD)$ is bounded below.*

Proof. Suppose that $\Delta_q(AC)$ is bounded below. Then there exists a constant $M > 0$ such that

$$\|\Delta_q(AC)\,x\| \geq M\|x\|$$

for all $x \in X$. Consequently, it follows that

$$\|D'x\| \leq M^{-1}\|\Delta_q(AC)\,D'x\| \leq M^{-1}\|D'\|\|\Delta_q(BD)\,x\|.$$

Hence, we deduce

$$\begin{aligned}\|x\| &= \frac{1}{|q|^2}\Big\|\Big(BD' + |q|^2\mathcal{I}\Big)x - BD'x\Big\| \\ &\leq \frac{1}{|q|^2}\Big(\|\Delta_q(BD)\,x\| + \|B\|\,\|D'x\|\Big) \\ &\leq \frac{1}{|q|^2}\Big(1 + M^{-1}\|B\|\,\|D'\|\Big)\|\Delta_q(BD)\,x\|.\end{aligned}$$

This shows that $\Delta_q(BD)$ is bounded below.

Conversely, assume that $\Delta_q(BD)$ is bounded below. By Theorem 2.1, it follows that $\Delta_q(DB)$ is also bounded below. Repeating the above reasoning, one obtains that $\Delta_q(CA)$ is bounded below. Finally, applying Theorem 2.1 once again, we conclude that $\Delta_q(AC)$ is bounded below. $\square$

Lemma 4.5. *We have $\Delta_q(AC)$ is surjective if and only if $\Delta_q(BD)$ is surjective.*

Proof. Assume that $\Delta_q(AC)$ is surjective. Let $x \in X$. By Lemma 4.1 part i) there exist $y, z \in X$ such that

$$Dx = \Delta_q(AC)\,y \quad \text{and} \quad \Big(DBD - 2\,\mathrm{Re}(q)D\Big)x = \Delta_q(AC)\,z.$$

Consider the elements

$$x_1 = B\Big(ACA - 2\,\mathrm{Re}(q)A\Big)Cy \quad \text{and} \quad x_2 = B\Big(DBD - 2\,\mathrm{Re}(q)D\Big)x.$$

By Proposition 3.1, we have

$$\begin{aligned}\Big(DBD - 2\,\mathrm{Re}(q)D\Big)x_1 &= \Big(DBD - 2\,\mathrm{Re}(q)D\Big)B\Big(ACA - 2\,\mathrm{Re}(q)A\Big)Cy\\ &= \Big((AC)^2 - 2\,\mathrm{Re}(q)AC\Big)\Big((AC)^2 - 2\,\mathrm{Re}(q)AC\Big)y\\ &= \Big((AC)^2 - 2\,\mathrm{Re}(q)AC\Big)^2 y,\end{aligned}$$

and

$$\begin{aligned}\Big(DBD - 2\,\mathrm{Re}(q)D\Big)x_2 &= \Big(DBD - 2\,\mathrm{Re}(q)D\Big)B\Big(DBD - 2\,\mathrm{Re}(q)D\Big)x\\ &= \Big((AC)^2 - 2\,\mathrm{Re}(q)AC\Big)\Big(DBD - 2\,\mathrm{Re}(q)D\Big)x.\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}\Delta_q(BD)\Big(x_1 - x_2\Big) &= \Big((BD)^2 - 2\,\mathrm{Re}(q)BD + |q|^2\mathcal{I}\Big)\Big(x_1 - x_2\Big)\\ &= B\Big(DBD - 2\,\mathrm{Re}(q)D\Big)x_1 + |q|^2x_1 - B\Big(DBD - 2\,\mathrm{Re}(q)D\Big)x_2 - |q|^2x_2\\ &= B\Big((AC)^2 - 2\,\mathrm{Re}(q)AC\Big)^2 y + |q|^2B\Big(ACA - 2\,\mathrm{Re}(q)A\Big)Cy\\ &\quad - B\Big((AC)^2 - 2\,\mathrm{Re}(q)AC\Big)\Big(DBD - 2\,\mathrm{Re}(q)D\Big)x\\ &\quad - |q|^2B\Big(DBD - 2\,\mathrm{Re}(q)D\Big)x\\ &= B\Big((AC)^2 - 2\,\mathrm{Re}(q)AC + |q|^2\mathcal{I}\Big)\Big(\Big((AC)^2 - 2\,\mathrm{Re}(q)AC\Big)y - \Big(DBD - 2\,\mathrm{Re}(q)D\Big)x\Big).\end{aligned}$$

Since $\Big(DBD - 2\,\mathrm{Re}(q)D\Big)x = \Delta_q(AC)\,z$, it follows that

$$\Big((AC)^2 - 2\,\mathrm{Re}(q)AC\Big)y - \Big(DBD - 2\,\mathrm{Re}(q)D\Big)x = -|q|^2z.$$

Thus,

$$\begin{aligned}\Delta_q(BD)\Big(x_1 - x_2\Big) &= -|q|^2B\Delta_q(AC)\,z\\ &= -|q|^2B\Big(DBD - 2\,\mathrm{Re}(q)D\Big)x\\ &= -|q|^2\Big(\Delta_q(BD)\,x - |q|^2x\Big).\end{aligned}$$

Hence, we deduce

$$x = \Delta_q(BD)\left(\frac{1}{|q|^4}\Big(x_1 - x_2 + |q|^2x\Big)\right).$$

This proves that $\Delta_q(BD)$ is surjective.

Conversely, suppose that $\Delta_q(BD)$ is surjective. By applying [3, Theorem 3.2], we deduce that $\Delta_q(DB)$ is likewise surjective. Proceeding with a similar argument, it follows that $\Delta_q(CA)$ is surjective. Finally, invoking [3, Theorem 3.2] once more, we conclude that $\Delta_q(AC)$ is also surjective. □

At this point, by means of Theorem 3.2 together with Lemmas 4.3, 4.4 and 4.5, we establish the following result, which represents a novel extension of Jacobson's lemma within the quaternionic framework.

Theorem 4.6. *We have*

(i) $\sigma_S(AC) \setminus \{0\} = \sigma_S(BD) \setminus \{0\}$;
(ii) $\sigma_{Sr}(AC) \setminus \{0\} = \sigma_{Sr}(BD) \setminus \{0\}$;
(iii) $\sigma_{Sl}(AC) \setminus \{0\} = \sigma_{lS}(BD) \setminus \{0\}$;
(iv) $\sigma^S_{sur}(AC) \setminus \{0\} = \sigma^S_{sur}(BD) \setminus \{0\}$;
(v) $\sigma^S_p(AC) \setminus \{0\} = \sigma^S_p(BD) \setminus \{0\}$;
(vi) $\sigma^S_{ap}(AC) \setminus \{0\} = \sigma^S_{ap}(BD) \setminus \{0\}$.

In what follows, we present a series of lemmas that will play a crucial role in the analysis of the operators AC and BD. These results will serve as key tools in establishing the common Fredholm properties of these operators within the quaternionic setting. By laying the groundwork through these intermediate results, we prepare for the proof of the main theorem presented in this section.

Proposition 4.7. *We have $\mathcal{R}(\Delta_q(AC))$ is closed if and only if $\mathcal{R}(\Delta_q(BD))$ is closed.*

Proof. Assume that $\mathcal{R}(\Delta_q(AC))$ is closed. Let $(x_n) \subset \mathcal{R}(\Delta_q(BD))$ be a sequence such that $x_n \to x \in X$. Consequently, $D'x_n \xrightarrow[n\to+\infty]{} D'x$. By Lemma 4.1, we know that $D'x_n \in \mathcal{R}(\Delta_q(AC))$. Since $\mathcal{R}(\Delta_q(AC))$ is closed, it follows that $D'x \in \mathcal{R}(\Delta_q(AC))$. Therefore, there exists $y \in X$ such that $D'x = \Delta_q(AC)y$. Note that $y = \frac{1}{|q|^2}\left(D'x - A'Cy\right)$. Using (1), we compute:

$$\begin{aligned}
|q|^2 x &= \left(BD' + |q|^2\mathcal{I}\right)x - BD'x \\
&= \Delta_q(BD)x - B\Delta_q(AC)y \\
&= \Delta_q(BD)x - \frac{1}{|q|^2}B\left(A'C + |q|^2\mathcal{I}\right)(D'x - A'Cy) \\
&= \Delta_q(BD)x - \frac{1}{|q|^2}\Big(BA'CD'x - BA'CA'Cy + |q|^2BD'x - |q|^2BA'Cy\Big) \\
&= \Delta_q(BD)x - \frac{1}{|q|^2}\Big(BD'BD'x - BD'BA'Cy + |q|^2BD'x - |q|^2BA'Cy\Big) \\
&= \Delta_q(BD)x - \frac{1}{|q|^2}\left(BD' + |q|^2\mathcal{I}\right)(BD'x - BA'Cy) \\
&= \Delta_q(BD)x - \frac{1}{|q|^2}\Delta_q(BD)\,(BD'x - BA'Cy) \\
&= \Delta_q(BD)\left(x - \frac{1}{|q|^2}\,(BD'x - BA'Cy)\right).
\end{aligned}$$

Hence, $x \in \mathcal{R}(\Delta_q(BD))$, and so $\mathcal{R}(\Delta_q(BD))$ is closed.

Conversely, suppose $\mathcal{R}(\Delta_q(BD))$ is closed. By [3, Proposition 3.6], it follows that $\mathcal{R}(\Delta_q(DB))$ is closed. By symmetry, we deduce that $\mathcal{R}(\Delta_q(CA))$ is also closed, and applying [3, Proposition 3.6] once again, we conclude that $\mathcal{R}(\Delta_q(AC))$ is closed. This completes the proof. □

Let $T \in \mathcal{B}(X)$ and $n \in \mathbb{N}$. We define

$$c_n(T) = \dim \frac{\mathcal{R}\left(T^n\right)}{\mathcal{R}\left(T^{n+1}\right)} \quad \text{and} \quad c'_n(T) = \dim \frac{\mathcal{N}\left(T^{n+1}\right)}{\mathcal{N}\left(T^n\right)}.$$

Lemma 4.8. *We have, for all $n \in \mathbb{N}$ $c_n(\Delta_q(AC)) = c_n(\Delta_q(BD))$.*

Proof. Consider the quotient spaces

$$\frac{\mathcal{R}\Big(\Delta_q(BD)^n\Big)}{\mathcal{R}\Big(\Delta_q(BD)^{n+1}\Big)} \quad \text{and} \quad \frac{\mathcal{R}\Big(\Delta_q(AC)^n\Big)}{\mathcal{R}\Big(\Delta_q(AC)^{n+1}\Big)}.$$

Define the right-linear map

$$\Psi_n\colon \frac{\mathcal{R}\Big(\Delta_q(BD)^n\Big)}{\mathcal{R}\Big(\Delta_q(BD)^{n+1}\Big)} \longrightarrow \frac{\mathcal{R}\Big(\Delta_q(AC)^n\Big)}{\mathcal{R}\Big(\Delta_q(AC)^{n+1}\Big)}$$

by $\Psi_n([x]) = [D\,x]$. By Lemma 4.1(i), $D\Big(\mathcal{R}(\Delta_q(BD)^{n+1})\Big) \subseteq \mathcal{R}(\Delta_q(AC)^{n+1})$, so Ψ_n is well defined. To see injectivity, let $x \in \mathcal{R}(\Delta_q(BD)^n)$ satisfy $D\,x \in \mathcal{R}(\Delta_q(AC)^{n+1})$. Then Lemma 4.2(iii) yields $BA'\,C\,D'\,x \in \mathcal{R}(\Delta_q(BD)^{n+1})$. And we have

$$\begin{aligned}
|q|^2 x &= \Big(BD' + |q|^2\mathcal{I}\Big)x - BD'\,x \\
&= \Delta_q(BD)\,x + \frac{1}{|q|^2}\,BA'\,C\,D'\,x - \frac{1}{|q|^2}\,BA'\,C\,D'\,x - BD'\,x \\
&= \Delta_q(BD)\,x + \frac{1}{|q|^2}\,BA'\,C\,D'\,x - \frac{1}{|q|^2}\Big(BD' + |q|^2\mathcal{I}\Big)BD'\,x \\
&= \Delta_q(BD)\Big(x - \frac{1}{|q|^2}BD'\,x\Big) + \frac{1}{|q|^2}\,BA'\,C\,D'\,x \;\in\; \mathcal{R}\Big(\Delta_q(BD)^{n+1}\Big).
\end{aligned}$$

Hence $x \in \mathcal{R}(\Delta_q(BD)^{n+1})$, so $[x] = 0$. Thus, Ψ_n is injective. It follows that

$$c_n\Big(\Delta_q(BD)\Big) \le c_n\Big(\Delta_q(AC)\Big).$$

Using [3, Lemma 3.9], one has

$$c_n\Big(\Delta_q(AC)\Big) = c_n\Big(\Delta_q(CA)\Big) \le c_n\Big(\Delta_q(DB)\Big) = c_n\Big(\Delta_q(BD)\Big).$$

Combining these yields

$$c_n\Big(\Delta_q(AC)\Big) = c_n\Big(\Delta_q(BD)\Big) \quad \text{for all } n \in \mathbb{N},$$

as required. □

Lemma 4.9. *We have, for all $n \in \mathbb{N}$ $c'_n(\Delta_q(AC)) = c'_n(\Delta_q(BD))$.*

Proof. Let us consider the quotient spaces

$$\frac{\mathcal{N}\Big(\Delta_q(BD)^{n+1}\Big)}{\mathcal{N}\Big(\Delta_q(BD)^n\Big)} \quad \text{and} \quad \frac{\mathcal{N}\Big(\Delta_q(AC)^{n+1}\Big)}{\mathcal{N}\Big(\Delta_q(AC)^n\Big)},$$

and define the right-linear map

$$\Phi_n\colon \frac{\mathcal{N}\Big(\Delta_q(BD)^{n+1}\Big)}{\mathcal{N}\Big(\Delta_q(BD)^n\Big)} \longrightarrow \frac{\mathcal{N}\Big(\Delta_q(AC)^{n+1}\Big)}{\mathcal{N}\Big(\Delta_q(AC)^n\Big)}$$

by $\Phi_n\Big([x]\Big) = [D\,x]$. Since Lemma 4.1(ii) ensures $D\Big(\mathcal{N}(\Delta_q(BD)^n)\Big) \subseteq \mathcal{N}(\Delta_q(AC)^n)$, this map is well defined. We now show it is injective. Suppose $x \in \mathcal{N}(\Delta_q(BD)^{n+1})$ and $D\,x \in \mathcal{N}(\Delta_q(AC)^n)$. By Lemma 4.2(iv),

$BA'CD'x \in \mathcal{N}(\Delta_q(BD)^n)$. An argument analogous to that in the proof of Lemma 4.8 gives

$$\begin{aligned}|q|^2 x &= \left(BD' + |q|^2\mathcal{I}\right)x - BD'x\\ &= \Delta_q(BD)\,x + \frac{1}{|q|^2}\,BA'CD'x - \frac{1}{|q|^2}\left(BD' + |q|^2\mathcal{I}\right)BD'x\\ &= \Delta_q(BD)\left(x - \frac{1}{|q|^2}BD'x\right) + \frac{1}{|q|^2}\,BA'CD'x \in \mathcal{N}\left(\Delta_q(BD)^n\right).\end{aligned}$$

Hence $x \in \mathcal{N}(\Delta_q(BD)^n)$, so $[x] = 0$ in the source quotient and Φ_n is injective. It follows that

$$c'_n\left(\Delta_q(BD)\right) \le c'_n\left(\Delta_q(AC)\right).$$

By [3, Lemma 3.9], we have $c'_n(\Delta_q(AC)) = c'_n(\Delta_q(CA))$. By symmetry,

$$c'_n\left(\Delta_q(CA)\right) \le c'_n\left(\Delta_q(DB)\right), \quad \text{and} \quad c'_n\left(\Delta_q(DB)\right) = c'_n\left(\Delta_q(BD)\right).$$

Therefore, for every $n \in \mathbb{N}$,

$$c'_n\left(\Delta_q(AC)\right) = c'_n\left(\Delta_q(BD)\right),$$

as required. □

Lemma 4.10. *We have*

(i) $\alpha(\Delta_q(AC)) = \alpha(\Delta_q(BD))$.
(ii) $\beta(\Delta_q(AC)) = \beta(\Delta_q(BD))$.
(iii) $\mathrm{ind}(\Delta_q(AC)) = \mathrm{ind}(\Delta_q(BD))$.

Proof. (i) Since $\alpha\left(\Delta_q(AC)\right) = c'_0\left(\Delta_q(AC)\right)$, applying Lemma 4.8 yields

$$c'_0\left(\Delta_q(AC)\right) = c'_0\left(\Delta_q(BD)\right).$$

Hence

$$\alpha\left(\Delta_q(AC)\right) = \alpha\left(\Delta_q(BD)\right).$$

(ii) Similarly, using $\beta\left(\Delta_q(AC)\right) = c_0\left(\Delta_q(AC)\right)$ and applying Lemma 4.9, we obtain

$$c_0\left(\Delta_q(AC)\right) = c_0\left(\Delta_q(BD)\right),$$

which implies

$$\beta\left(\Delta_q(AC)\right) = \beta\left(\Delta_q(BD)\right).$$

(iii) From the equalities above, the desired conclusion for the operators AC and BD follows immediately. □

Lemma 4.11. *We have*

(i) $\mathrm{asc}\left(\Delta_q(AC)\right) = \mathrm{asc}\left(\Delta_q(BD)\right)$.
(ii) $\mathrm{dsc}\left(\Delta_q(AC)\right) = \mathrm{dsc}\left(\Delta_q(BD)\right)$.

Proof. (i) Suppose first that $\mathrm{asc}\left(\Delta_q(AC)\right) = \infty$. Then, by definition, for each $n \in \mathbb{N}$, the ascending chain of null spaces does not stabilize, i.e., $\mathcal{N}\left(\Delta_q(AC)^n\right) \subsetneq \mathcal{N}\left(\Delta_q(AC)^{n+1}\right)$, implying that $c'_n\left(\Delta_q(AC)\right) > 0$ for all n. Invoking Lemma 4.9, we conclude that the same holds for $\Delta_q(BD)$, i.e., $c'_n\left(\Delta_q(BD)\right) > 0$ for all n, and thus, $\mathrm{asc}\left(\Delta_q(BD)\right) = \infty$. Conversely, suppose that $\mathrm{asc}\left(\Delta_q(AC)\right) = n < \infty$. Then, by definition,

$\mathcal{N}\Big(\Delta_q(AC)^n\Big) = \mathcal{N}\Big(\Delta_q(AC)^{n+1}\Big)$, which leads to $c'_n\Big(\Delta_q(AC)\Big) = 0$. By Lemma 4.9, we obtain $c'_n\Big(\Delta_q(BD)\Big) = 0$, and therefore $\operatorname{asc}\Big(\Delta_q(BD)\Big) \le n$. On the other hand, from [3, Lemma 3.10, Assertion (i)], we know that $\operatorname{asc}\Big(\Delta_q(BD)\Big) = \operatorname{asc}\Big(\Delta_q(DB)\Big) < \infty$. Applying similar arguments to the operators CA and DB, we get $\operatorname{asc}\Big(\Delta_q(CA)\Big) \le \operatorname{asc}\Big(\Delta_q(BD)\Big)$. Since $n = \operatorname{asc}\Big(\Delta_q(AC)\Big) = \operatorname{asc}\Big(\Delta_q(CA)\Big)$, it follows that $n \le \operatorname{asc}\Big(\Delta_q(BD)\Big)$, which forces equality: $\operatorname{asc}\Big(\Delta_q(BD)\Big) = n$.

For the reverse implication, we have by (i) in [3, Lemma 3.10] that $\operatorname{asc}\Big(\Delta_q(BD)\Big) = \operatorname{asc}\Big(\Delta_q(DB)\Big)$. Following the same argument as above, we obtain $\operatorname{asc}\Big(\Delta_q(DB)\Big) = \operatorname{asc}\Big(\Delta_q(CA)\Big)$, and again, using (i) in [3, Lemma 3.10], we deduce that $\operatorname{asc}\Big(\Delta_q(BD)\Big) = \operatorname{asc}(\Delta_q(DB)) = \operatorname{asc}(\Delta_q(CA)) = \operatorname{asc}\Big(\Delta_q(AC)\Big)$.

(ii) Assume first that $\operatorname{dsc}\Big(\Delta_q(AC)\Big) = \infty$. This means that the sequence of ranges does not stabilize at any finite step; explicitly, for every $n \in \mathbb{N}$, one has $\mathcal{R}\Big(\Delta_q(AC)^n\Big) \subsetneq \mathcal{R}\Big(\Delta_q(AC)^{n+1}\Big)$. As a consequence, the corresponding codimension terms satisfy $c_n\Big(\Delta_q(AC)\Big) > 0 \quad$ for all n. Using Lemma 4.8, it follows that the same holds for the operator $\Delta_q(BD)$, that is, $c_n\Big(\Delta_q(BD)\Big) > 0 \quad$ for all n, which implies $\operatorname{dsc}\Big(\Delta_q(BD)\Big) = \infty$. Now, suppose instead that $\operatorname{dsc}\Big(\Delta_q(AC)\Big) = n < \infty$. By definition, the sequence of ranges stabilizes at step n, so $\mathcal{R}\Big(\Delta_q(AC)^n\Big) = \mathcal{R}\Big(\Delta_q(AC)^{n+1}\Big)$, which is equivalent to having $c_n\Big(\Delta_q(AC)\Big) = 0$. Applying Lemma 4.8 once more, we deduce that $c_n\Big(\Delta_q(BD)\Big) = 0$, and therefore the descent of $\Delta_q(BD)$ satisfies $\operatorname{dsc}\Big(\Delta_q(BD)\Big) \le n$. In addition, by assertion (i) of [3, Lemma 3.10], the equality $\operatorname{dsc}\Big(\Delta_q(DB)\Big) = \operatorname{dsc}\Big(\Delta_q(BD)\Big) < \infty$ holds. Proceeding analogously to the ascent case, one can establish that $\operatorname{dsc}\Big(\Delta_q(CA)\Big) \le \operatorname{dsc}\Big(\Delta_q(BD)\Big)$. Since $n = \operatorname{dsc}\Big(\Delta_q(CA)\Big)$, it follows that $n \le \operatorname{dsc}\Big(\Delta_q(BD)\Big)$, which forces the equality $\operatorname{dsc}\Big(\Delta_q(BD)\Big) = n$.

For the converse implication, it follows from (i) in [3, Lemma 3.10] that $\operatorname{dsc}\Big(\Delta_q(BD)\Big) = \operatorname{dsc}\Big(\Delta_q(DB)\Big)$. By applying the same reasoning as before, we obtain $\operatorname{dsc}\Big(\Delta_q(DB)\Big) = \operatorname{dsc}\Big(\Delta_q(CA)\Big)$. Once again, using (i) in [3, Lemma 3.10], we conclude that $\operatorname{dsc}\Big(\Delta_q(BD)\Big) = \operatorname{dsc}\Big(\Delta_q(DB)\Big) = \operatorname{dsc}\Big(\Delta_q(CA)\Big) = \operatorname{dsc}\Big(\Delta_q(AC)\Big)$, which completes the proof. $\square$

Set

$$\mathcal{T} \in \{\Phi, \Phi_+, \Phi_-, \Phi_\pm, \mathcal{W}, \mathcal{W}_+, \mathcal{W}_-, \mathcal{B}\Phi, \mathcal{B}\Phi_+, \mathcal{B}\Phi_-, \mathcal{D}, \mathcal{D}_\ell, \mathcal{D}_r\}.$$

We are now in a position to state the main result.

Theorem 4.12. *We have $\Delta_q(AC)$ belongs to $\mathcal{T}$ if and only if $\Delta_q(BD)$ belongs to $\mathcal{T}$.*

Proof. To begin with, combining Proposition 4.7 with Lemma 4.10 yields the result for the families

$$\mathcal{T} \in \{\Phi, \Phi_+, \Phi_-, \Phi_\pm, \mathcal{W}, \mathcal{W}_+, \mathcal{W}_-\}.$$

Next, by applying Proposition 4.7 alongside Lemma 4.11, we obtain the result for $\mathcal{T} \in \{\mathcal{B}\Phi, \mathcal{B}\Phi_+, \mathcal{B}\Phi_-, \mathcal{D}, \mathcal{D}_\ell, \mathcal{D}_r\}$. $\square$

We set

$$\sigma^S_{\mathcal{T}}(T) = \{q \in \mathbb{H} : \Delta_q(T) \notin \mathcal{T}\},$$

Corollary 4.13. *We have $\sigma^S_{\mathcal{T}}(AC)\backslash\{0\} = \sigma^S_{\mathcal{T}}(BD)\backslash\{0\}$.*

Proof. The proof is immediately driven by Theorem 4.12. $\square$

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