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On logarithmic q-statistical convergence and Korovkin approximation theorem

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Abstract. Recently, the notion of q-Cesàro summability has been introduced as a new summability method. In this work, we make use of q-calculus and harmonic summability to define logarithmic q-density and logarithmic q-statistical convergence and establish its relation with H_q^1-summability. Further, we discuss how logarithmic q-statistical convergence and $[H_q^1, p]$-summability are related to each other. Finally, we use this new summability method in proving a Korovkin type theorem.

1. Introduction

Statistical summability is an important topic in the field of mathematical analysis, particularly in the study of convergence and summability methods. The notion of statistical convergence was introduced in [9] and [25], independently; which was further studied by several authors, namely, Salat [24], Fridy [10] and Connor [8]. This notion for double sequences was given in 2003 by Móricz [15] and Mursaleen and Edely [17], independently. Recently, a new variant of statistical convergence was given by Aktuğlu and Bekar [2] via q-calculus which was further studied in [18], [19] and [7].

The concepts of logarithmic density and logarithmic statistical convergence was studied in [3]. The idea of q-density and q-statistical convergence [2] and since $\frac{1}{[n]_q} \leq \frac{1}{n}$, this motivates us to define the q-analogs of logarithmic density and logarithmic statistical convergence. The q-logarithmic statistical convergence is a mathematical concept that combines q-statistical convergence with q-logarithmic density. Unlike standard statistical convergence, it uses a different measure of density, where a sequence is q-logarithmically statistically convergent if the proportion of terms within a specified range, weighted by their reciprocals,

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converges to zero. Some observations on generalized logarithmic statistical convergence of order ρ are discussed in [12].

The idea of statistical summability $(C,1)$ was given by Móricz [14] and further generalized to statistical summability $(H,1)$ by replacing Cesàro mean by Harmonic mean [16].

The harmonic mean or $(H,1)$ mean of the sequence (ς_k) is defined by $\mu_n := \frac{1}{l_n}\sum_{k=1}^{n}\frac{\varsigma_k}{k}$, where $l_n := \sum_{k=1}^{n}\frac{1}{k} \approx \log n$. If $\mu_n \to l$ as $n \to \infty$, then (ς_k) is said to be harmonic summable to l.

The concept of q-calculus extends classical analysis by incorporating the deformation parameter q (see [11]). Given a $q > 0$. The q-integer $[r]_q$, the q-factorial $[r]_q!$ and the q-binomial coefficient are defined by

$$[r]_q := \begin{cases} \frac{1-q^r}{1-q}, & \text{if } q \in \mathbb{R}^+ \setminus \{1\} \\ r, & \text{if } q = 1, \end{cases} \quad \text{for } r \in \mathbb{N} \text{ and } [0]_q = 0,$$

$$[r]_q! := \begin{cases} [r]_q[r-1]_q \cdots [1]_q, & r \geq 1, \\ 1, & r = 0, \end{cases}$$

$$\begin{bmatrix} r \\ s \end{bmatrix}_q := \frac{[r]_q!}{[s]_q![r-s]_q!},$$

respectively.

The q-Cesàro matrix of order 1 (see [1]) $C_q^1 = (c_{nk}(q^k))_{n,k=0}^{\infty}$ is defined by

$$c_{nk}(q^k) = \begin{cases} \frac{q^{k-1}}{[n]_q} & \text{if } 1 \leq k \leq n, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

For $q = 1, C_q^1$ reduces to C^1, the Cesàro matrix of order 1. Note that the summability method C^1 is stronger than C_q^1 for $q \geq 1$ [2, Theorem 11].

The sequence $y = (y_n)$ will denote the C_q^1-transform of the sequence $x = (x_k)$ and is given by

$$y_n = (C_q^1 x)_n = \frac{1}{[n]_q}\sum_{k=1}^{n} q^{n-1}x_k, n \in \mathbb{N}.$$

A sequence $x = (x_k)$ is said to be q-Cesàro summable or C_q^1-summable to the number ℓ if $\lim_n y_n = \ell$, and inthis case ℓ is called the C_q^1-limit of x.

Note that the q-analog gives a better rate of convergence, because of the fact that $\frac{1}{[n]_q} < \frac{1}{n}$. For detailed insights into q-integers and their corresponding approximation properties, we refere to [5, 20–22].

The idea of q-density and q-statistically convergence of was given by Aktuğlu and Bekar [2]. For $q \geq 1$, the q-density of $\mathbb{K} \subseteq \mathbb{N}$ is defined by

$$\delta_q(\mathbb{K}) = \delta_{C_q^1}(\mathbb{K}) = \lim\inf_{n\to\infty}(C_q^1\chi_{\mathbb{K}})_n.$$

For the set $\mathbb{K}_\varepsilon = \{k \leq n : |\varsigma_k - L| \geq \varepsilon\}$, if $\delta_q(\mathbb{K}_\varepsilon) = 0$ for every $\varepsilon > 0$, then a sequence $\varsigma = (\varsigma_k)$ refers to q-statistically convergent to L. That is,

$$\lim_n \frac{1}{[n]_q}|\{k \leq n : q^{k-1}|\varsigma_k - L| \geq \varepsilon\}| = 0$$

and we write

$$st_q\text{-}\lim \varsigma_k = L.$$

Denote $st_q := \{\varsigma = (\varsigma_k) : \delta_q(\mathbb{K}_\varepsilon) = 0\}$. For $q = 1$, $\delta_q(\mathbb{K}_\varepsilon) = \delta(\mathbb{K}_\varepsilon)$ (natural density) and $st_q = st$ (statistical convergence).

Every statistically convergent sequence is q-statistically convergent but not conversely (see [2, Example 15]). Recently, in [26, 27] some new sequence spaces have been defined and studied as matrix domains of q-Cesàro matrix.

In this paper, we define q-harmonic mean by applying q-calculus and define q-harmonic summability or H_q^1-summability, and $[H_q^1, p]$-summability. We further define the logarithmic q-density and logarithmic q-statistical convergence and establish its relation with H_q^1-summability. Further, we discuss how logarithmic q-statistical convergence and $[H_q^1, p]$-summability are related to each other. Finally, we use this new summability method in proving a Korovkin type theorem motivated by the fact that the q-Cesàro mean gives a better rate of convergence than the Cesàro mean.

2. Logarithmic q-statistical convergence

The logarithmic q-statistical convergence is an extension of classical notions of convergence, statistical convergence as well as q-statistical convergence.

Recent application of q-calculus was studied in [4] by definining the following which was further applied to prove a variant of Korovkin's theorem.

Definition 2.1. *Let $\rho = (\rho_n)$, where*

$$\rho_n = (C_q^1 \varsigma)_n = \frac{1}{[n]_q} \sum_{k=1}^{n} q^{k-1} \varsigma_k. \tag{2}$$

A sequence $\varsigma = (\varsigma_k)$ is statistically C_q^1-summable to L if

$$st - \lim \rho_n = L.$$

Note that for $(\varsigma_k) \in \ell_\infty$, $st_q - \lim \varsigma_k = L$ implies $st - \lim \rho_n = L$. For $q = 1$, statistically C_q^1-summability becomes simply statistical summability $(C, 1)$ due to Móricz [14].

Example 2.2. *Consider a sequence For $\varsigma = (\varsigma_k) \in \ell_\infty$, $st_q - \lim_{k\to\infty} \varsigma_k = L$ implies $C_q^1 - \lim_{k\to\infty} \varsigma_k = L$ but not conversly, i.e. if for any fixed $q \leq 1$, the divergent sequence $\varsigma = (\varsigma_k)$ is defined by*

$$\varsigma_k := \begin{cases} \frac{1}{q}, & \text{for } k \text{ even}, \\ -\frac{1}{q^2}, & \text{for } k \text{ odd}. \end{cases} \tag{3}$$

Then ς is not q-statistically convergent but C_q^1-summable to 0. Hence statistically C_q^1-summable to 0.

We define the following:

Definition 2.3. *Let $[l_n]_q = \frac{[n]_q}{n} \sum_{k=1}^{n} \frac{q^{k-1}}{[k]_q}$, $(n = 1, 2, 3...)$ and $C_q^1 = (c_{nk}(q^k))_{n,k=0}^{\infty}$ defined by (1). If for every $\epsilon > 0$,*

$$\lim_n \frac{1}{[l_n]_q} |\{1 \leq k \leq n : \frac{q^{k-1}}{[k]_q} |\varsigma_k - L| \geq \epsilon\}| = 0,$$

then $\varsigma = (\varsigma_k)$ refers to be logarithmic q-statistically convergent to $L := st_q^{\ln} - \lim \varsigma$. We designate $st_q^{\ln}$ for the set of all logarithmic q-statistically convergent sequences. When $q \to 1$, logarithmic q-statistical convergence becomes logarithmic statistical convergence [3].

Remark 2.4. *Note that $k = 1$, $[l_n]_q = [n]_q$. Then $st_q^{\ln}$ is reduced to st_q.*

Example 2.5. *Consider the case $k = 1$.*

Define a sequence $\varsigma = (\varsigma_k)$ by $\varsigma_k = \left(\underbrace{1}_{2^0}, \underbrace{0,0}_{2^1}, \underbrace{1,1,1,1}_{2^2}\underbrace{.0.0.,,,,0}_{2^3}, 1, ..., \right)$ and the set $E = \{k \in \mathbb{N} : \varsigma_k = 1\}$. Then $\delta(E)$ does not exist ([10]) and hence $\varsigma = (\varsigma_k)$ is not statistically convergent. But st_q-$\lim \varsigma_k = 0$, since $\left[C_q^1 \chi_E\right]_{2^n-1} \longrightarrow 0$. Hence $st_q^{\ln}$-$\lim \varsigma_k = 0$.

We define the following by considering the q-harmonic mean:

Definition 2.6. *The q-harmonic mean of a sequence (ς_k) is defined by $\nu_n := \frac{1}{[l_n]_q} \sum_{k=1}^{n} \frac{q^{k-1}}{[k]_q} \varsigma_k$. A sequence $\varsigma = (\varsigma_k)$ is q-harmonic summable or H_q^1-summable to L if the sequence $\nu = (\nu_n)$ converges to L, i.e. H_q^1-$\lim \tau = L$.*
A sequence $\varsigma = (\varsigma_k)$ is $[H_q^1, p]$-summable $(0 < p < \infty)$ to L if $\lim\limits_n \frac{1}{[l_n]_q} \sum\limits_{k=1}^{n} \frac{q^{k-1}}{[k]_q} |\varsigma_k - L|^p = 0$, i.e. $\varsigma_k \longrightarrow L[H_q^1, p]$ and $L := [H_q^1, p] - \lim \varsigma$.

Example 2.7. *Let $k = 1$. Define a sequence*

$$\varsigma_k := \begin{cases} 1, & \text{for } k \text{ even,} \\ -1, & \text{for } k \text{ odd.} \end{cases} \tag{4}$$

Then ς is not q-statistically convergent but Cesàro summable to 0. Hence statistically Cesàro summable to 0, because every convergent sequence is also statistically convergent. Therefore in general ς is H_q^1-summable to 0.

Definition 2.8. *A sequence $\varsigma = (\varsigma_k)$ is statistically q-summable $(H, 1)$ (or statistically H_q^1-summable) to L if the sequence $\nu = (\nu_n)$ is q-statistically convergent to L, i.e. $st_q - \lim \nu = L := H(st_q)$-$\lim \varsigma$. The set of all such sequences will be denoted by $H(st_q)$.*

For $q = 1$, we get statistical summability $(H, 1)$ [16].

Example 2.9. *Define the sequence $\varsigma = (\varsigma_k)$ by*

$$\varsigma_k := \begin{cases} 1, & \text{if } k = n^2, \\ 0, & \text{if } k \neq n^2;\ n \in \mathbb{N}. \end{cases} \tag{5}$$

is statistically convergent to 0 as well as Cesàro summable to 0. Hence statistically q-Cesàro summable to 0. That is statistically q-summable $(H, 1)$.

3. Main results

Our main results are concerned with the relationship between our newly defined summability methods.

Theorem 3.1. *Let $\varsigma = (\varsigma_k) \in \ell_\infty$ and logarithmic q-statistically convergent to L. Then it is statistically q-summable $(H, 1)$ (or statistically H_q^1-summable) to L, but not conversely.*

Proof. Write $\mathcal{K}_\epsilon := \{k \in \mathbb{N} : \frac{q^{k-1}}{[k]_q} |\varsigma_k - L| \geq \epsilon\}$. Then, for every $\epsilon > 0$,

$$\lim_n \frac{1}{[l_n]_q} |\{1 \leq k \leq n : \frac{q^{k-1}}{[k]_q} |\varsigma_k - L| \geq \epsilon\}| = 0,$$

$$|\nu_n - L| = \left| \frac{1}{[l_n]_q} \sum_{k=1}^{n} \frac{q^{k-1}}{[k]_q} \varsigma_k - L \right| \leq \frac{1}{[l_n]_q} \sum_{k=1}^{n} \frac{q^{k-1}}{[k]_q} \mid \varsigma_k - L \mid$$

$$\leq \frac{1}{[l_n]_q}\sum_{k\in K_\epsilon}\frac{q^{k-1}}{[k]_q} \mid \varsigma_k - L \mid \leq \frac{1}{[l_n]_q}(\sup_k|\varsigma_k - L|)K_\epsilon \to 0,$$

as $n \to \infty$, which implies that $\nu_n \to L$ as $n \to \infty$. That is, ς is H^1_q-summable to L and hence statistically H^1_q-summable to L.

For converse, consider the case $k = 1$ then $[l_n]_q = [n]_q$. Let the sequence $\varsigma = (\varsigma_k)$ be defined as follows (see [2, Example 15]):

$$\varsigma_k = (1, 0, 0, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, ...)$$

where 1^s occur 2^{2n} times, and 0^s occur 2^{2n-1} times ($n = 0, 1, 2, ...$), respectively. Obviously $\varsigma = (\varsigma_k)$ is not logarithmic q-statistically convergent but is $(H, 1)$-summable to 1 and hence statistically H^1_q-summable to 1. □

Theorem 3.2. *(a) If $0 < p < \infty$ and $\varsigma_k \longrightarrow L[H^1_q, p]$, then $st^{\ln}_q - \lim_k \varsigma_k = L$.*
(b) If $(\varsigma_k) \in \ell_\infty$ and $st^{\ln}_q - \lim_k \varsigma_k = L$. then $\varsigma_k \longrightarrow L[H^1_q, p]$.

Proof. (a) Suppose $0 < p < \infty$ and $\varsigma_k \longrightarrow L[H^1_q, p]$. Then

$$0 \leftarrow \frac{1}{[l_n]_q}\sum_{k=1}^{n}\frac{q^{k-1}}{[k]_q}|\varsigma_k - L|^p \geq \frac{1}{[l_n]_q}\sum_{\substack{k=1\\ |\varsigma_k/[k]_q - L|\geq\epsilon}}^{n}\frac{q^{k-1}}{[k]_q}|\varsigma_k - L|^p$$

$$\geq \frac{1}{[l_n]_q}\epsilon^p \mid \mathcal{K}_\epsilon \mid,$$

as $n \to \infty$. That is, $\lim_{n\to\infty} \frac{1}{[l_n]_q}\epsilon^p \mid \mathcal{K}_\epsilon \mid = 0$, where $\mathcal{K}_\epsilon := \left\{k \leq n : \frac{q^{k-1}}{[k]_q}|\varsigma_k - L| \geq \epsilon\right\}$. Hence $st^{\ln}_q - \lim_k \varsigma_k = L$.

(b) Suppose $(\varsigma_k) \in \ell_\infty$ and $st^{\ln}_q - \lim_k \varsigma_k = L$. Boundedness of ς implies that $(\frac{q^{k-1}\varsigma_k}{[k]_q}) \in \ell_\infty$ and $\frac{q^{k-1}}{[k]_q}|\varsigma_k - L| \leq M$ $(k \in \mathbb{N})$, $M > 0$. Then

$$\begin{aligned}
&\frac{1}{[l_n]_q}\sum_{k=1}^{n}\frac{q^{k-1}}{[k]_q}|\varsigma_k - L|^p \\
&= \frac{1}{[l_n]_q}\sum_{\substack{k=1\\ k\notin\mathcal{K}_\epsilon}}^{n}\frac{q^{k-1}}{[k]_q} \mid \varsigma_k - L|^p + \frac{1}{[l_n]_q}\sum_{\substack{k=1\\ k\in\mathcal{K}_\epsilon}}^{n}\frac{q^{k-1}}{[k]_q} \mid \varsigma_k - L|^p \\
&= \mathbb{T}_1(n) + \mathbb{T}_2(n),
\end{aligned}$$

where

$$\begin{aligned}
\mathbb{T}_1(n) &= \frac{1}{[l_n]_q}\sum_{\substack{k=1\\ k\notin\mathcal{K}_\epsilon}}^{n}\frac{q^{k-1}}{[k]_q} \mid \varsigma_k - L|^p, \\
\mathbb{T}_2(n) &= \frac{1}{[l_n]_q}\sum_{\substack{k=1\\ k\in\mathcal{K}_\epsilon}}^{n}\frac{q^{k-1}}{[k]_q} \mid \varsigma_k - L|^p.
\end{aligned}$$

Now if $k \notin \mathcal{K}_\epsilon$ then $\mathbb{T}_1(n) < \epsilon^p$. For $k \in \mathcal{K}_\epsilon$, we have

$$\begin{aligned}\mathbb{T}_2(n) &\leq (\sup \frac{1}{[l_n]_q} |\frac{q^{k-1}}{[k]_q} \varsigma_k - L|)(|\mathcal{K}_\epsilon| / \frac{1}{[l_n]_q})\\ &\leq M |\mathcal{K}_\epsilon| / \frac{1}{[l_n]_q} \to 0,\end{aligned}$$

as $n \to \infty$, since $st_q^{\ln} - \lim_k \varsigma_k = L$. Hence $\varsigma_k \longrightarrow L[H_q^1, p]$. □

The following result is about the structure of a logrithmic q-statistically convergence which is q-analog of [24].

Theorem 3.3. *For a sequence $\varsigma = (\varsigma_k)$, $st_q^{\ln} - \lim \varsigma = L$ if and only if there exists a set $\mathbb{M} = \{\beta_1 < \beta_2 < \cdots < \beta_n < \cdots\} \subseteq \mathbb{N}$ such that $\delta_q^{\ln}(\mathbb{M}) = 1$ and $\lim \varsigma_{\beta_n} = L$, where $\delta_q^{\ln}(\mathbb{M})$ denotes the logarithmic q-density of the set $\mathbb{M}$.*

Proof. Let $\mathbb{M} = \{\beta_1 < \beta_2 < \cdots < \beta_n < \cdots\} \subseteq \mathbb{N}$ such that $\delta_q^{\ln}(\mathbb{M}) = 1$ and $\lim \varsigma_{\beta_n} = L$. Then there is integer $N > 0$ such that for $n > N$,

$$|\varsigma_{\beta_n} - L| < \epsilon. \tag{6}$$

Put $\mathbb{K}_\epsilon := \{n \in \mathbb{N} : |\varsigma_n - L| \geq \epsilon\}$ and $\mathbb{K}' = \{\beta_{N+1}, \beta_{N+2}, \cdots\}$. Then $\delta_q^{\ln}(\mathbb{K}') = 1$ and $\mathbb{K}_\epsilon \subseteq \mathbb{N} - \mathbb{K}'$ which implies that $\delta_q^{\ln}(\mathbb{K}_\epsilon) = 0$. Hence $st_q^{\ln} - \lim \varsigma = L$.

Conversely, Let $st_q^{\ln} - \lim \varsigma = L$. For $r = 1, 2, 3, \cdots$, put $\mathbb{K}_r := \{n \in \mathbb{N} : |\varsigma_n - L| \geq 1/r\}$ and $\mathbb{M}_r := \{n \in \mathbb{N} : |\varsigma_n - L| < 1/r\}$. Then $\delta_q^{\ln}(\mathbb{K}_r) = 0$ and

$$\mathbb{M}_1 \supset \mathbb{M}_2 \supset \cdots \mathbb{M}_i \supset \mathbb{M}_{i+1} \supset \cdots \tag{7}$$

and

$$\delta_q^{\ln}(\mathbb{M}_r) = 1, \ r = 1, 2, 3, \cdots \tag{8}$$

We need to show that $\varsigma_{\beta_n} \longrightarrow L$ for $n \in \mathbb{M}_r$. Suppose that $\varsigma_{\beta_n} \nrightarrow L$. Therefore there is $\epsilon > 0$ such that $|\varsigma_{\beta_n} - L| \geq \epsilon$ for infinitely many terms. Let $\mathbb{M}_\epsilon := \{n \in \mathbb{N} : |\varsigma_{\beta_n} - L| < \epsilon\}$ and $\epsilon > 1/r$ $(r = 1, 2, 3, \cdots)$. Then

$$\delta_q^{\ln}(\mathbb{M}_\epsilon) = 0, \tag{9}$$

and by (2.2), $\mathbb{M}_r \subset \mathbb{M}_\epsilon$. Hence $\delta_q^{\ln}(\mathbb{M}_r) = 0$, which contradicts (8) and hence $\lim \varsigma_{\beta_n} = L$. □

Similarly we can prove the following dual statement.

Theorem 3.4. *A sequence $\varsigma = (\varsigma_k)$ is statistically q-summable $(H, 1)$ to L if and only if there exists a set $\mathbb{M} = \{\beta_1 < \beta_2 < \cdots < \beta_n < \cdots\} \subseteq \mathbb{N}$ such that $\delta_q(\mathbb{M}) = 1$ and $H_q^1 - \lim \varsigma_{\beta_n} = L$.*

Proof. Instead of (6), we get

$$|\varsigma_n - L| < \epsilon, \tag{10}$$

and the set $\mathbb{K}_\epsilon := \{n \in \mathbb{N} : |\varsigma_{\beta_n} - L| \geq \epsilon\}$.
For converse, take $\mathbb{K}_r := \{n \in \mathbb{N} : |\varsigma_{\beta_n} - L| \geq 1/r\}$ and $\mathbb{M}_r := \{n \in \mathbb{N} : |\varsigma_{\beta_n} - L| < 1/r\}$. Rest of the proof can easily be obtained by using the hypothesis. □

4. An application

Denote $(C[a,b], \|f\|_\infty)$ for the Banach space of all continuous functions on $[a,b]$ with $\|f\|_\infty := \sup_{a\le x\le b} |f(x)|$. The Korovkin approximation theorem [13] states as follows:

Theorem 4.1. *Let $\mathcal{P}_n : C[a,b] \longrightarrow C[a,b]$ be a sequence of positive linear operators and $\lim_n \|\mathcal{P}_n(f_i) - f_i\|_\infty = 0$, where $f_i(x) = x^i$ $(i = 0,1,2)$. Then $\lim_n \|\mathcal{P}_n(\mathrm{F}) - \mathrm{F}\|_\infty = 0$, for all $\mathrm{F} \in C[a,b]$.*

Now, we establish analogous to this theorem by applying the C^1_q-matrix method.

The C^1_q-transform of $x = (x_k)$ is the sequence $y = (y_n)$,

$$y_n = (C^1_q x)_n = \frac{1}{[n]_q} \sum_{k=1}^{n} \frac{q^{n-1}}{[n]_q} x_k, n \in \mathbb{N}.$$

Theorem 4.2. *Let $\mathcal{P}_n : C[a,b] \longrightarrow C[a,b]$ be a sequence of positive linear operators such that*

$$C^1_q - \lim_{n\to\infty} \left\| \mathcal{P}_n(1,\theta) - 1 \right\|_\infty = 0, \tag{11}$$

$$C^1_q - \lim_{n\to\infty} \left\| \mathcal{P}_n(t,\theta) - \theta \right\|_\infty = 0, \tag{12}$$

$$C^1_q - \lim_{n\to\infty} \left\| \mathcal{P}_n(t^2,\theta) - \theta^2 \right\|_\infty = 0. \tag{13}$$

Then

$$C^1_q - \lim_{n\to\infty} \left\| \mathcal{P}_n(\mathrm{F}) - \mathrm{F} \right\|_\infty = 0. \tag{14}$$

for all $\mathrm{F} \in C[a,b]$.

Proof. By standard calculation, for an arbitrary $\epsilon > 0$, we obtain

$$\begin{aligned}
&\mathcal{P}_k(\mathrm{F},\theta) - \mathrm{F}(\theta) < \epsilon \mathcal{P}_k(1,\theta) + \frac{2M}{\delta^2}\{[\mathcal{P}_k(t^2,\theta) - \theta^2] + 2\theta[\mathcal{P}_k(t,\theta) - \theta] \\
&+\theta^2[\mathcal{P}_k(1,\theta) - 1]\} + \mathrm{F}(\theta)(\mathcal{P}_k(1,\theta) - 1) \\
&\qquad\qquad = \epsilon[\mathcal{P}_k(1,\theta) - 1] + \epsilon + \frac{2M}{\delta^2}\{[\mathcal{P}_k(t^2,\theta) - \theta^2] + 2\theta[\mathcal{P}_k(t,\theta) - \theta] \\
&+\theta^2[\mathcal{P}_k(1,\theta) - 1]\} + \mathrm{F}(\theta)(\mathcal{P}_k(1,\theta) - 1).
\end{aligned}$$

Taking C^1_q-transform of $\mathcal{P}_k$, i.e. $S^q_n(\mathrm{F},\theta) = \frac{1}{[n]_q} \sum_{k=1}^{n} \frac{q^{n-1}}{[n]_q} \mathcal{P}_k(\mathrm{F},\theta)$, we get

$$\begin{aligned}
&S^q_n(\mathrm{F},\theta) - \mathrm{F}(\theta) \le \epsilon[S^q_n(1,\theta) - 1] + \frac{2M}{\delta^2}\{[S^q_n(t^2,\theta) - \theta^2] + 2\theta[S^q_n(t,\theta) - \theta] \\
&+\theta^2[S^q_n(1,\theta) - 1]\} + \mathrm{F}(\theta)(S^q_n(1,\theta) - 1),
\end{aligned}$$

and therefore

$$\begin{aligned}
&\left| S^q_n(\mathrm{F},\theta) - \mathrm{F}(\theta) \right| \le \left(\epsilon + \frac{2Mb^2}{\delta^2} + M\right) \left| S^q_n(1,\theta) - 1 \right| \\
&\qquad\qquad + \frac{4Mb}{\delta^2} \left| S^q_n(t,\theta) - \theta \right| + \frac{2M}{\delta^2} \left| S^q_n(t^2,\theta) - \theta^2 \right|.
\end{aligned}$$

Letting $n \to \infty$ and using (11)–(13) yield

$$\lim_{n\to\infty}\left\|S_n^q(\mathrm{F}) - \mathrm{F}\right\|_\infty = 0,$$

i.e.

$$C_q^1 - \lim_{n\to\infty}\left\|\mathcal{P}_n^q(\mathrm{F}) - \mathrm{F}\right\|_\infty = 0.$$

□

The following example demonstrate that the above C_q^1-version is stronger than Theorem 4.1.

Example 4.3. *Consider the q-analog of Bernstein operators which was introduced by Phillips [23] by using the q-binomial coefficients.*

$$\mathbb{B}_n(\psi,\theta) := \sum_{k=0}^{n} \psi\left(\frac{[k]_q}{[n]_q}\right)\begin{bmatrix} n \\ k \end{bmatrix}\theta^k \Pi_{s=0}^{n-k-1}(1-q^s\theta).$$

It was observed that $\mathbb{B}_n(1,\theta)=1, \mathbb{B}_n(t,\theta)=\theta$ *and* $\mathbb{B}_n(t^2,\theta)=\theta^2+\frac{\theta(1-\theta)}{[n]_q}$.
Define $\mathcal{P}_n : C[0,1] \longrightarrow C[0,1]$ *by* $\mathcal{P}_n(\psi,\theta) = (1+\zeta_n)\mathbb{B}_n(\psi,\theta)$, *where the sequence* (ζ_n) *is defined by*

$$\zeta_n := \begin{cases} \frac{1}{q}, & \text{if } n \text{ is even,} \\ -\frac{1}{q^2}, & \text{if } n \text{ is odd.} \end{cases}$$

Then the sequence $(\mathcal{P}_n)$ *satisfies the conditions* (11), (12) *and* (13). *Hence we have*

$$C_q^1 - \lim \|\mathcal{P}_n(\psi) - \psi\|_\infty = 0.$$

On the other hand, we get $\mathcal{P}_n(\psi,0) = (1+\zeta_n)\psi(0)$, *since* $\mathbb{B}_n(\psi,0)=\psi(0)$, *and hence*

$$\|\mathcal{P}_n(\psi) - \psi\|_\infty \geq \mid \mathcal{P}_n(\psi,0) - \psi(0) \mid = \zeta_n \mid \psi(0) \mid .$$

We see that the conditions of Theorem 4.1 are not satisfied by $(\mathcal{P}_n)$, *since* $\lim_{n\to\infty}\zeta_n$ *does not exist.*

Boyanov and Veselinov [6] have proved the Korovkin theorem on $C[0,\infty)$ by using the test functions 1, e^{-x}, e^{-2x}. We state a generalization of the result of Boyanov and Veselinov by using the notion of q-Cesàro summability.

Theorem 4.4. *For a sequence of positive linear operators* $(\mathcal{P}_n) : C(I) \longmapsto C(I)$, $I = [0,\infty)$, *we have*

$$C_q^1 - \lim_{n\to\infty}\left\|\mathcal{P}_n(1,\theta) - 1\right\|_\infty = 0, \tag{15}$$

$$C_q^1 - \lim_{n\to\infty}\left\|\mathcal{P}_n(e^{-t},\theta) - e^{-\theta}\right\|_\infty = 0, \tag{16}$$

$$C_q^1 - \lim_{n\to\infty}\left\|\mathcal{P}_n(e^{-2t},\theta) - e^{-2\theta}\right\|_\infty = 0. \tag{17}$$

Then

$$C_q^1 - \lim_{n\to\infty}\left\|\mathcal{P}_n(\mathrm{F}) - \mathrm{F}\right\|_\infty = 0 \tag{18}$$

for all $\mathrm{F} \in C(I)$.

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