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Relativistic magneto-fluid spacetimes and conformal η-Ricci-Yamabe solitons

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Abstract. The object of the research article is to investigate the geometric properties of a relativistic magneto-fluid spacetime if its metrics admits conformal η-Ricci-Yamabe soliton and gradient conformal η-Ricci-Yamabe soliton of type (κ, l). A $C^{\dagger}$-flat relativistic magneto-fluid spacetime, $\mathcal{P}_1$-flat relativistic magneto-fluid spacetime and T^c-flat relativistic magneto-fluid spacetime filled with a magneto-fluid density ρ, magnetic field strength $\widetilde{\mathcal{H}}$, and magnetic permeability α obeys the Einstein field equation with a concircular vector field are also investigated. Moreover, we illustrates some physical significance of conformal pressure $\tilde{\Omega}$ in view of a conformal η-Ricci-Yamabe soliton of type (κ, l) along with concircular vector field on a relativistic magneto-fluid spacetime. Within this ongoing work, using such soliton, we analyze the various energy conditions, some black holes criteria, and Penrose's singularity theorem on a relativistic magneto-fluid spacetime. In addition of this, we further investigate the generalized Liouville and Poisson equations associated with such soliton on a relativistic magneto-fluid spacetime Moreover, in the context of a relativistic magneto-fluid spacetime, we explore the harmonic aspects of conformal η-Ricci-Yamabe soliton of type (κ, l) on such a relativistic magneto-fluid spacetime. Finally, we discuss some results on different types of magneto-fluid spacetimes coupled with such type of a soliton.

1. Background and Motivations

The study of the theory of Ricci flow by Hamilton ([10], [22], [23]) reached its highest magnitude and popularity soon after Perelman ([31],[32]) successfully applied it to solve the Poincaré conjecture. Ricci solitons were also studied by Hamilton, who viewed them as fixed or stationary points of the Ricci flow in the space of parameterized metrics $g(t)$ on scaling and M modulo diffeomorphisms. Since then, both topics have been extensively explored by numerous mathematicians, including Brendle [3], Cao[11], Chen[9] and many others ([26], [37]).

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A smooth manifold (M^n, g) with a Riemannian metric g is known to be a Ricci soliton if, for some constant λ, there exists a smooth vector field $\mathcal{Z}$ on M^n satisfying

$$\mathcal{R}ic(V_1, V_2) = \lambda g(V_1, V_2) - \frac{1}{2}(\mathfrak{L}_{\mathcal{Z}} g)(V_1, V_2), \tag{1}$$

where $\mathfrak{L}_{\mathcal{Z}}$ denotes the Lie derivative and $\mathcal{R}ic$ is the Ricci tensor. The Ricci soliton is called shrinking, steady and expanding if $\lambda > 0$, $\lambda = 0$ and $\lambda < 0$ respectively.

Hamilton [22] established the Yamabe flow to address the Yamabe problem. This problem primarily involves the quest for a metric on a manifold with a dimension of $n \geq 3$, such that its scalar curvature remains constant. Thus, the Yamabe flow is precisely delineated by the metric $g(t)$ on (M^n, g), which adheres to the condition $\frac{\partial g(t)}{\partial(t)} = -\tau g(t)$, where τ denotes the scalar curvature of (M^n, g).

The conformal Ricci flow is defined on (M^n, g), as a generalization of classical Ricci flow by [16]:

$$\frac{\partial g}{\partial t} = -2\left[\mathcal{R}ic + \frac{g}{n}\right] - \tilde{\Omega} g, \quad \tau(g) = -1, \tag{2}$$

for a dynamically evolving metric g and a scalar non-dynamical field $\tilde{\Omega}$.

The conformal Ricci flow equations are similar to the Navier–Stokes equations of fluid mechanics, that is

$$\frac{\partial v}{\partial t} + \nabla_v v + v \bigtriangleup v = -\operatorname{grad} \tilde{\Omega}, \quad \operatorname{div} v = 0, \tag{3}$$

where $\tilde{\Omega}$ is called the conformal pressure of (M^n, g).

A conformal Ricci soliton on (M^n, g) is defined as follows [4]:

$$(\mathfrak{L}_{\mathcal{Z}} g)(V_1, V_2) + 2\mathcal{R}ic(V_1, V_2) - [(\tilde{\Omega} + \frac{2}{n}) - 2\mu] g(V_1, V_2) = 0, \tag{4}$$

where $\mu \in \mathfrak{R}$ ($\mathfrak{R}$ is the set of real numbers).

A Ricci-Yamabe flow of type (κ, l), which is a scalar combination of Ricci and Yamabe flows, is defined as follows [18]:

$$\frac{\partial}{\partial t} g(t) = 2\kappa \mathcal{R}ic(g(t)) - l\tau(t) g(t), \quad g(0) = g_0, \tag{5}$$

for some scalars κ and l.

A Riemannian manifold is said to have a Ricci-Yamabe soliton of type (κ, l) (briefly, RY-soliton) if [21]

$$(\mathfrak{L}_{\mathcal{Z}} g)(V_1, V_2) + 2\kappa \mathcal{R}ic(V_1, V_2) + (2\mu - l\tau) g(V_1, V_2) = 0, \tag{6}$$

where $l, \kappa, \mu \in \mathfrak{R}$.
According to [50], the conformal Ricci-Yamabe soliton (briefly, CRY-soltion) of type (κ, l) on (M^n, g) is given by

$$(\mathfrak{L}_{\mathcal{Z}} g)(V_1, V_2) + 2\kappa \mathcal{R}ic(V_1, V_2) + [2\mu - l\tau - (\tilde{\Omega} + \frac{2}{n})] g(V_1, V_2) = 0. \tag{7}$$

In this follow-up, the conformal η-Ricci-Yamabe soliton (briefly, CERY-soliton) of type (κ, l) on (M^n, g) is defined by [20]

$$(\mathfrak{L}_{\mathcal{Z}} g)(V_1, V_2) + 2\kappa \mathcal{R}ic(V_1, V_2) + [2\mu - l\tau - (\tilde{\Omega} + \frac{2}{n})] g(V_1, V_2) + 2\nu\, \eta(V_1)\eta(V_2) = 0, \tag{8}$$

where $l, \kappa, \mu, \nu \in \mathfrak{R}$. If $\mathcal{Z}$=$grad(f)$, then (8) is called a gradient conformal η-Ricci-Yamabe soliton (briefly, GCERY-soliton) of type (κ, l) and given by

$$\nabla^2 f + \kappa \mathcal{R}ic(V_1, V_2) + [\mu - \frac{l\tau}{2} - \frac{1}{2}(\tilde{\Omega} + \frac{2}{n})] g(V_1, V_2) + \nu\, \eta(V_1)\eta(V_2) = 0, \tag{9}$$

where $\nabla^2 f$ is said to be the Hessian of f.

A CERY-soliton (or GCERY-soliton) is said to shrink, steady or expand if $\mu < 0$, $= 0$ or > 0, respectively.

In this scenario, several mathematicians and physicists investigated the geometrical and physical features of spacetime in terms of solitons ([10], [14], [6],[25],[40],[41], [42],[43],[44],[45],[46],[47],[48]) and others. Very recently, authors([35],[33]) have studied relativistic magneto-fluid spacetime (MFST) if its metric is Einstein soliton and solitonic aspect under some specific vector fields. Motivated by this recent work, we discuss relativistic magneto-fluid spacetime (briefly, MFST) using CERY-soliton (or GCERY-soliton) of type (κ, l) and we obtain many geometrical and physically relevant findings.

2. Relativistic Magneto-fluid spacetime MFST

The magnetic type matter tensor coupled with spacetime, is called as a relativistic magneto-fluid spacetime (briefly, MFST)[27], are consider in this studied. We begin our studies with the following definition.

Definition 2.1. *[12] A generalized quasi-Einstein manifold (briefly, $G(QE)_n$) on (M^n, g), $(n > 2)$ is defined by*

$$\mathcal{R}ic = a_1 g(V_1, V_2) + a_2\eta(V_1)\eta(V_2) + a_3\omega(V_1)\omega(V_2), \tag{10}$$

where a_1, a_2 and a_3 are non-zero scalars and η, ω are non-zero 1-forms.

In a relativistic MFST the magnetic energy momentum tensor (EMT) $\widetilde{T}$ has the form ([27],[28],[34]):

$$\begin{aligned}\widetilde{\mathcal{T}}(V_1, V_2) = pg(V_1, V_2) \quad &+ \quad (p+\rho)\eta(V_1)\eta(V_2)\\ &+ \quad \alpha\left\{\widetilde{\mathcal{H}}[\eta(V_1)\eta(V_2) + \frac{1}{2}g(V_1, V_2)] - \omega(V_1)\omega(V_2)\right\},\end{aligned} \tag{11}$$

where ρ defines magneto-fluid density, p denotes the pressure, α defines magnetic permeability, ω is the magnetic flux, $\widetilde{\mathcal{H}}$ is the strength of magnetic field such that $\eta(V_1) = g(V_1, \xi)$ and $g(V_2, \zeta) = \omega(V_2)$ are two non-zero 1-form.

Also, ξ is a unit time-like vector field $g(\xi, \xi) = -1$ and ζ is a unit space-like vector field $g(\zeta, \zeta) = 1$ and $g(\xi, \zeta) = 0$.

The Einstein's field equation (briefly, EFE)[28]:

$$\mathcal{R}ic(V_1, V_2) = \lambda\widetilde{\mathcal{T}}(V_1, V_2) - [\beta - \frac{\tau}{2}]g(V_1, V_2), \tag{12}$$

where β leads to cosmological constant, λ remits to gravitational constant.

Therefore, from (11) and (12), the EFE for a MFST

$$\mathcal{R}ic(V_1, V_2) = [-\beta + \frac{\tau}{2} + \lambda(p + \frac{\widetilde{\mathcal{H}}\alpha}{2})]g(V_1, V_2) + [\lambda\alpha\widetilde{\mathcal{H}} + \lambda(p+\rho)]\eta(V_1)\eta(V_2) - \alpha\lambda\omega(V_1)\omega(V_2). \tag{13}$$

On contracting (13), we yield

$$\tau = 4\beta - \lambda[\alpha(\widetilde{\mathcal{H}} - 1) + 3p - \rho]. \tag{14}$$

Applying (14) in (13), it follows

$$\mathcal{R}ic(V_1, V_2) = \left\{\beta - \frac{\lambda}{2}[p - (\alpha + \rho)]\right\}g(V_1, V_2) + [\lambda\alpha\widetilde{\mathcal{H}} + \lambda(p+\rho)]\eta(V_1)\eta(V_2) - \alpha\lambda\omega(V_1)\omega(V_2). \tag{15}$$

Thus, a relativistic MFST is a G(QE)-sapcetime with $\beta - \frac{\lambda}{2}[p - (\alpha + \rho)]$, $\lambda\alpha\widetilde{\mathcal{H}} + \lambda(p+\rho)$ and $-\alpha\lambda$ are scalars associated with 1-forms η and ω. As a result we state that.

Theorem 2.2. *A relativistic magneto-fluid spacetime coupled with magneto-fluid density ρ, magnetic field strength $\widetilde{\mathcal{H}}$, and magnetic permeability α fulfilling EFE represents a G(QE)-spacetime.*

Corollary 2.3. *In a relativistic MFST, fulfilling EFE the scalar curvature is given by*

$$\tau = 4\beta - \lambda[\alpha(\widetilde{\mathcal{H}} - 1) + 3p - \rho].$$

Also, from (14), we present the result

$$p = \frac{1}{3}[\rho - \alpha(\widetilde{\mathcal{H}} - 1) + \frac{1}{\lambda}(\tau - 4\beta)]. \tag{16}$$

Thus, we gain the outcome

Corollary 2.4. *If a relativistic MFST, fulfilling EFE with constant scalar curvature, the equation of state (EoS) is determine by*

$$p = \frac{\rho}{3} - \frac{[\lambda\alpha(\widetilde{\mathcal{H}} - 1) - (\tau - 4\beta)]}{3\lambda}.$$

In particular if, $\alpha = 0,\ \tau = 4\beta$ or $\widetilde{\mathcal{H}} = 1,\ \tau = 4\beta$.Then from (16), we get EoS = $\frac{1}{3}$, which represents the matter source is of the radiation type. We declare the outcome

Corollary 2.5. *Let a relativistic MFST coupled with magnetic field strength $\mathcal{H}$, and magnetic permeability α fulfilling EFE and $\alpha = 0$, $\tau = 4\beta$ or $\widetilde{\mathcal{H}} = 1$,$\tau = 4\beta$. Then EoS displays the radiation era.*

Also, if the matter source is of phantom barrier, $\rho = -p$, then from (16), we obtain

$$p = \frac{[(\tau - 4\beta) - \alpha\lambda(\widetilde{\mathcal{H}} - 1)]}{4\lambda}. \tag{17}$$

Therefore, we may conclude that

Corollary 2.6. *If the source of matter in MFST is of phantom barrier type. Then, the pressure p and magneto-fluid density ρ is given by*

$$\rho = -p = \frac{[(\tau - 4\beta) - \alpha\lambda(\widetilde{\mathcal{H}} - 1)]}{4\lambda}.$$

3. Relativistic Magneto-fluid spacetime admitting conformal η-Ricci-Yamabe soliton

In this section, we contract the equation (8) and using (14), we obtain

$$\mu = -\frac{(\kappa - 2l)}{4}[4\beta - \lambda\alpha(\widetilde{\mathcal{H}} - 1) + \lambda\rho - 3p\lambda] + \frac{1}{8}[4\tilde{\Omega} - div(\mathcal{Z}) + 2\nu + 2], \tag{18}$$

where, div($\mathcal{Z}$) is the divergence of the vector field $\mathcal{Z}$. Thus we have the outcome

Theorem 3.1. *If a relativistic magneto-fluid spacetime obeying EFE, admits CERY-soliton of type (κ, l), then the soliton is expanding, steady or shrinking according to*

$$\beta \gtreqless \frac{1}{4(2\kappa - 4l)}[4\tilde{\Omega} - div(\mathcal{Z}) + 2\nu + 2] + \frac{1}{4}[\lambda(3p - \rho) + \lambda\alpha(\widetilde{H} - 1)].$$

In view of the Theorem 3.1, we state the outcome

Corollary 3.2. *If a relativistic magneto-fluid spacetime obeying EFE, admits CERY-soliton of type (κ, l), then the vector field $\mathcal{Z}$ is solenoidal if and only if*

$$\mu = -\frac{(\kappa - 2l)}{4}[4\beta - \lambda\alpha(\widetilde{\mathcal{H}} - 1) + \lambda(\rho - 3p)] + \frac{1}{4}[2\tilde{\Omega} + \nu + 1].$$

Also the soliton is expanding, steady or shrinking according

$$\beta \gtreqless \frac{1}{4}[\frac{2\tilde{\Omega} - \nu - 1}{(\kappa - 2l)} + \lambda\alpha(\widetilde{H} - 1) - \lambda(\rho - 3p)].$$

According to [49], a quasi-conformal curvature tensor is defined by

$$\begin{aligned} C^{\dagger}(V_1,V_2)V_3 = \ & a^{\dagger} \quad \mathcal{R}(V_1,V_2)V_3 + b^{\dagger}[\mathcal{R}ic(V_2,V_3)V_1 - \mathcal{R}ic(V_1,V_3)V_2 \\ & + \quad g(V_2,V_3)QV_1 - g(V_1,V_3)QV_2] \\ & - \quad \frac{\tau}{n}\left[\frac{a^{\dagger}}{n-1} + 2b^{\dagger}\right][g(V_2,V_3)V_1 - g(V_1,V_3)V_2], \end{aligned} \tag{19}$$

where $a^{\dagger}$ and $b^{\dagger}$ are constants and τ is the scalar curvature. If $a^{\dagger}$=1 and $b^{\dagger}$=-$\frac{1}{n-2}$, then (19) reduces to the conformal curvature tensor (or, Weyl tensor).

Definition 3.3. *A spacetime is said to be quasi-conformally flat if $C^{\dagger} = 0$ for $n > 3$ vanishes identically.*

The quasi-conformal curvature tensor have been studied by several authors in various ways such as ([1],[14],[17],[24],[13]) and many others. The quasi-conformal curvature tensor has significant applications in general relativity, providing valuable insights into the geometric structure and physical properties of space-time. A quasi-conformally flat perfect fluid spacetime is both a de Sitter spacetime and a Robertson–Walker spacetime.

Let a quasi-conformally flat MFST that obeys EFE. Then from (19) , we have

$$\begin{aligned} a^{\dagger}\mathcal{R}'(V_1,V_2,V_3,V_4) = \ & - \quad b^{\dagger}[\mathcal{R}ic(V_2,V_3)g(V_1,V_4) - \mathcal{R}ic(V_1,V_3)g(V_2,V_4) \\ & + \quad g(V_2,V_3)\mathcal{R}ic(V_1,V_4) - g(V_1,V_3)\mathcal{R}ic(V_2,V_4)] \\ & + \quad \frac{\tau}{4}\left[\frac{a^{\dagger}}{3} + 2b^{\dagger}\right][g(V_2,V_3)g(V_1,v_4) - g(V_1,V_3)g(V_2,V_4)]. \end{aligned} \tag{20}$$

After contraction over V_1 and V_4, we get from (20) that

$$(a^{\dagger} + 2b^{\dagger})\mathcal{R}ic(V_2,V_3) = (a^{\dagger} + 2b^{\dagger})\frac{\tau}{4}g(V_2,V_3). \tag{21}$$

Utilizing (14) and (21) in (20), one can get

$$\mathcal{R}'(V_1,V_2,V_3,V_4) = \frac{4\beta - \lambda[\alpha(\widetilde{\mathcal{H}} - 1) + 3p - \rho]}{12}[g(V_2,V_3)g(V_1,V_4) - g(V_1,V_3)g(V_2,V_4)]. \tag{22}$$

Therefore, we state the following result

Theorem 3.4. *A quasi-conformally flat relativistic magneto-fluid spacetime attached with magneto-fluid density ρ, magnetic field strength $\widetilde{\mathcal{H}}$, and magnetic permeability α that obeys EFE is an Einstein spacetime, provided $a^{\dagger}+2b^{\dagger} \neq 0$.*

Corollary 3.5. *A quasi-conformally flat relativistic magneto-fluid spacetime coupled with magneto-fluid density ρ, magnetic field strength $\widetilde{\mathcal{H}}$, and magnetic permeability α that obeys EFE, is a spacetime of constant curvature, provided $a^{\dagger} + 2b^{\dagger} \neq 0$.*

Again, we consider a relativistic MFST with $C^{\dagger} = 0$ obeying EFE without cosmological constant. Then from (14), we have

$$\tau = -\lambda[\alpha(\widetilde{\mathcal{H}} - 1) + 3p - \rho]. \tag{23}$$

From (21) and (23), we get

$$\mathcal{R}ic(V_2,V_3) = \frac{-\lambda[\alpha(\widetilde{\mathcal{H}} - 1) + 3p - \rho]}{4}g(V_2,V_3). \tag{24}$$

Let Q be the Ricci operator given by $g(QV_2,V_3) = \mathcal{R}ic(V_2,V_3)$ and $\mathcal{R}ic(QV_2,V_3) = \mathcal{R}ic^2(V_2,V_3)$. Then we have $\eta(QV_2) = g(QV_2,\xi) = \mathcal{R}ic(V_2,\xi)$. Therefore, we yield from (24) that

$$\mathcal{R}ic(QV_2,V_3) = \frac{\lambda^2[\alpha(\widetilde{\mathcal{H}} - 1) + 3p - \rho]^2}{16}g(V_2,V_3). \tag{25}$$

After contraction (25) over V_2 and V_3, we yield

$$||Q||^2 = \frac{\lambda^2[\alpha(\widetilde{\mathcal{H}}-1)+3p-\rho]^2}{4}. \tag{26}$$

Hence we reflect the theorem.

Theorem 3.6. *If a quasi-conformally flat relativistic magneto-fluid spacetime that obeys EFE without cosmological constant, then the square of the length of the Ricci operator of the spacetime is given by*

$$||Q||^2 = \frac{\lambda^2[\alpha(\widetilde{\mathcal{H}}-1)+3p-\rho]^2}{4}.$$

Again, after contraction over V_1 and V_2, equation(12) implies that

$$\tau = -\lambda \mathrm{T}, \tag{27}$$

where $\mathrm{T} = \mathrm{trace}\,\widetilde{\mathcal{T}}$. So, equation (12) takes the form

$$\mathcal{R}ic(V_1,V_2) = \lambda[\widetilde{\mathcal{T}}(V_1,V_2) - \frac{\mathrm{T}}{2}]g(V_1,V_2). \tag{28}$$

According to [2], EFE without cosmological constant for a purely electromagnetic distribution is given by

$$\mathcal{R}ic(V_1,V_2) = \lambda\widetilde{\mathcal{T}}(V_1,V_2). \tag{29}$$

In view of (28) and (29), we get $\mathrm{T} = 0$. So from (27), we yield $\tau = 0$. Thus equation (22) implies that $\mathcal{R}'(V_1,V_2,V_3,V_4) = 0$, which means that the MFST is flat. So, we reflect the outcome

Theorem 3.7. *A quasi-conformally flat relativistic magneto-fluid spacetime satisfying EFE without cosmological constant for a purely electromagnetic distribution is an Euclidean space.*

Next, combing (23) and (27), one can obtain

$$\mathrm{T} = [3p+\rho-\alpha(\widetilde{\mathcal{H}}-1)]. \tag{30}$$

Since for a purely electromagnetic distribution $\mathrm{T} = 0$. So, from (30), we yield

$$\alpha(\widetilde{H}-1) = 3p+\rho. \tag{31}$$

Thus, we point out the result.

Theorem 3.8. *In a purely electromagnetic distribution the relativistic MFST satisfying EFE without cosmological constant with vanishing quasi-conformal curvature tensor satisfies the following relation*

$$\alpha(\widetilde{\mathcal{H}}-1) = 3p+\rho.$$

Also, using (21), equation (8) takes the form

$$(\mathfrak{L}_{\mathcal{Z}}g)(V_2,V_3) + [\frac{\kappa\tau}{2} + 2\mu - l\tau - (\tilde{\Omega}+\frac{1}{2})]g(V_2,V_3) + 2\nu\,\eta(V_2)\eta(V_3) = 0. \tag{32}$$

According to [7], if the vector field $\mathcal{Z}$ is concircular vector field, then

$$(\mathfrak{L}_{\mathcal{Z}}g)(V_2,V_3) = 2\psi g(V_2,V_3). \tag{33}$$

From (33) and (32), we achieve

$$\tau = \frac{1}{(\kappa-2l)}[-4\psi - 4\mu + 2\tilde{\Omega} + 4\nu + 1]. \tag{34}$$

With the help of (14) and (34), we get

$$\mu = -\psi - \frac{(\kappa-2l)}{4}[4\beta - \lambda\{\alpha(\widetilde{\mathcal{H}}-1)+3p-\rho\}] + \frac{1}{4}[2\tilde{\Omega}+4\nu+4]. \tag{35}$$

Thus we can state the results.

Corollary 3.9. *A quasi-conformally flat relativistic magneto-fluid spacetime obeying EFE admit CERY-soliton of type (κ, l), where $\mathcal{Z}$ is a concircular vector field, then the soliton is expanding, steady or shrinking according as*

$$\beta \gtreqless \frac{\lambda}{4}[\alpha(\widetilde{\mathcal{H}}-1)+3p-\rho]+\frac{(\tilde{\Omega}+2\nu-2\psi+2)}{2(\kappa-2l)}.$$

In 1973, G. P. Pokhariyal [30] introduced a new curvature tensor defined from W_3-curvature tensor is called $\mathcal{P}_1$-curvature tensor and on (M^n, g), is defined as follows

$$\begin{aligned}\mathcal{P}_1(V_1,V_2,V_3,V_4) &= \frac{1}{2}[W_3(V_1,V_2,V_3,V_4)-W_3(V_2,V_1,V_3,V_4)]\\ &= \mathcal{R}(V_1,V_2,V_3,V_4)+\frac{1}{2(n-1)}[g(V_2,V_3)\mathcal{R}ic(V_1,V_4)\\ &- g(V_2,V_4)\mathcal{R}ic(V_1,V_3)-g(V_1,V_3)\mathcal{R}ic(V_2,V_4)+g(V_1,V_4)\mathcal{R}ic(V_2,V_3)],\end{aligned} \tag{36}$$

where W_3-curvature tensor given by

$$W_3(V_1,V_2,V_3,V_4)=\mathcal{R}'(V_1,V_2,V_3,V_4)+\frac{1}{(n-1)}[g(V_2,V_3)\mathcal{R}ic(V_1,V_4)-g(V_2,V_4)\mathcal{R}ic(V_1,V_3)], \tag{37}$$

for all vector fields V_1, V_2, V_3, V_4 and $\mathcal{R}'(V_1,V_2,V_3,V_4)=g(\mathcal{R}(V_1,V_2)V_3,V_4)$.

Definition 3.10. *A spacetime is said to be $\mathcal{P}_1$-flat if its $\mathcal{P}_1$-curvature tensor vanishes identically.*

We assume that $\mathcal{P}_1$-flat relativistic MFST that obeys EFE. Then from (36), we have

$$\begin{aligned}\mathcal{R}(V_1,V_2,V_3,V_4) = &-\frac{1}{2(n-1)}[g(V_2,V_3)\mathcal{R}ic(V_1,V_4)-g(V_2,V_4)\mathcal{R}ic(V_1,V_3)\\ &- g(V_1,V_3)\mathcal{R}ic(V_2,V_4)+g(V_1,V_4)\mathcal{R}ic(V_2,V_3)].\end{aligned} \tag{38}$$

On putting $V_1 = V_4 = e_i$ in (38) and summing over $i = 1,2,3,4$ we obtain

$$\mathcal{R}ic(V_2,V_3)=-\frac{\tau}{8}g(V_2,V_3). \tag{39}$$

Using (39), equation (8) reduces to

$$(\mathfrak{L}_{\mathcal{Z}}g)(V_2,V_3)+[2\mu-\frac{1}{4}(\kappa+4l)\tau-(\tilde{\Omega}+\frac{1}{2})]g(V_2,V_3)+2\nu\,\eta(V_2)\eta(V_3)=0. \tag{40}$$

In view of (33) and (40), we get

$$\tau=\frac{2}{(4l+\kappa)}[4(\mu+\psi-\tilde{\Omega})-(\nu+2)]. \tag{41}$$

From (14) and (41), con can obtain

$$\mu=-\psi+\frac{(4l+\kappa)}{8}[4\beta-\lambda\{\alpha(\widetilde{\mathcal{H}}-1)+3p-\rho\}+(\tilde{\Omega}+\frac{1}{2})+\frac{\nu}{4}]. \tag{42}$$

This reflects the following outcome

Theorem 3.11. *If a $\mathcal{P}_1$-flat relativistic MFST that obeys EFE admits CERY-soliton of type (κ, l), where $\mathcal{Z}$ is a concircular vector field, then the soliton is expanding, steady or shrinking according as*

$$\beta \gtreqless \frac{1}{(4l+\kappa)}[\psi-(2\tilde{\Omega}+1)-\frac{\nu}{2}]+\frac{1}{4}[\lambda\{\alpha(\widetilde{\mathcal{H}}-1)-3p+\rho\}].$$

According to [38], the T^c-curvature tensor of type $(0,4)$ on (M^n, g) is defined as follows

$$\begin{aligned} T^c(V_1,V_2,V_3,V_4) = \ & \gamma_0 \ \mathcal{R}'(V_1,V_2,V_3,V_4) + \gamma_1 \mathcal{R}ic(V_2,V_3)g(V_1,V_4) \\ + & \ \gamma_2 \mathcal{R}ic(V_1,V_3)g(V_2,V_4) + \gamma_3 \mathcal{R}ic(V_1,V_3)g(V_2,V_4) \\ + & \ \gamma_4 g(V_2,V_3)\mathcal{R}ic(V_1,V_4) + \gamma_5 g(V_1,V_3)\mathcal{R}ic(V_2,V_4) \\ + & \ \gamma_6 g(V_1,V_2)\mathcal{R}ic(V_3,V_4) \\ + & \ \tau\gamma_7[g(V_2,V_3)g(V_1,V_4) - g(V_1,V_3)g(V_2,V_4)], \end{aligned} \tag{43}$$

where $V_1, V_2, V_3 \in \chi(M)$, $\gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6$ and γ_7 are smooth functions on (M^n, g) and $\mathcal{R}'(V_1,V_2,V_3,V_4) = g(\mathcal{R}(V_1,V_2)V_3,V_4)$. The T^c-curvature tensor reduces to many other curvature tensors for different values of $\gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6$ and γ_7, respectively.

Definition 3.12. *A spacetime is called T^c-flat if the T^c-curvature tensor of type $(0,4)$ satisfies the relation $T^c(V_1,V_2,V_3,V_4) = 0$ on (M^n, g) for any $V_1, V_2, V_3 \in \chi(M)$.*

Let a T^c-flat relativistic MFST that obeys EFE. Then from (43), we have

$$\begin{aligned} 0 = \ & \gamma_0 \ \mathcal{R}'(V_1,V_2,V_3,V_4) \\ + & \ \gamma_1 \mathcal{R}ic(V_2,V_3)g(V_1,V_4) + \gamma_2 \mathcal{R}ic(V_1,V_3)g(V_2,V_4) + \gamma_3 \mathcal{R}ic(V_1,V_3)g(V_2,V_4) \\ + & \ \gamma_4 g(V_2,V_3)\mathcal{R}ic(V_1,V_4) + \gamma_5 g(V_1,V_3)\mathcal{R}ic(V_2,V_4) + \gamma_6 g(V_1,V_2)\mathcal{R}ic(V_3,V_4) \\ + & \ \tau\gamma_7[g(V_2,V_3)g(V_1,V_4) - g(V_1,V_3)g(V_2,V_4)]. \end{aligned} \tag{44}$$

After, taking contraction (44) over V_1 and V_4, we yield

$$\mathcal{R}ic(V_2,V_3) = \varrho g(V_2,V_3), \tag{45}$$

where $\varrho = -[\frac{\tau(\gamma_4+3\gamma_7)}{(\gamma_0+4\gamma_1+\gamma_2+\gamma_3+\gamma_5+\gamma_6)}]$.

It is clearly that, if $(\gamma_0 + 4\gamma_1 + \gamma_2 + \gamma_3 + \gamma_5 + \gamma_6) \neq 0$ then a spacetime is an Einstein spacetime. Thus we reflect that

Theorem 3.13. *If $(\gamma_0 + 4\gamma_1 + \gamma_2 + \gamma_3 + \gamma_5 + \gamma_6) \neq 0$, then a T^c-flat magneto-fluid spacetime is an Einstein spacetime.*

Also, using (45), the equation (44) implies that

$$\begin{aligned} \mathcal{R}'(V_1,V_2,V_3,V_4) = \ & - \left[\frac{(\gamma_1+\gamma_4)\sigma + \tau\gamma_7)}{\gamma_0}\right][g(V_2,V_3)g(V_1,V_4) \\ & + \left[\frac{\tau\gamma_7 - (\gamma_2+\gamma_5)\sigma}{\gamma_0}\right]g(V_1,V_3)g(V_2,V_4)] \\ & - \left[\frac{\sigma(\gamma_3+\gamma_6)}{\gamma_0}\right]g(V_1,V_2)g(V_3,V_4). \end{aligned} \tag{46}$$

Thus from, (46) it is obvious that if $\gamma_0 \neq 0$, $\gamma_3 + \gamma_6 = 0$, $(\gamma_1 + \gamma_2 + \gamma_4 + \gamma_5) = 0$ and $(\gamma_0 + 4\gamma_1 + \gamma_2 + \gamma_3 + \gamma_5 + \gamma_6) \neq 0$ then (46) takes the form

$$\mathcal{R}'(V_1,V_2,V_3,V_4) \ = \ \left[\frac{(\gamma_1+\gamma_4)\sigma + \tau\gamma_7)}{\gamma_0}\right]\{g(V_1,V_3)g(V_2,V_4) - g(V_2,V_3)g(V_1,V_4\}. \tag{47}$$

Therefore, a T^c-flat MFST is a spacetime of constant curvature. So, our finding is that

Theorem 3.14. *If $\gamma_0 \neq 0$, $\gamma_3 + \gamma_6 = 0$, $(\gamma_1 + \gamma_2 + \gamma_4 + \gamma_5) = 0$ and $(\gamma_0 + 4\gamma_1 + \gamma_2 + \gamma_3 + \gamma_5 + \gamma_6) \neq 0$, then a T^c-flat MFST is a spacetime of constant curvature.*

Moreover, let a relativistic MFST with T^c-flat obeying EFE without cosmological constant. Then from (14), equation (45) reduces to

$$\mathcal{R}ic(V_2, V_3) = \left[\frac{-\lambda\{\alpha(\widetilde{\mathcal{H}} - 1) + 3p - \rho\}(\gamma_4 + 3\gamma_7)}{(\gamma_0 + 4\gamma_1 + \gamma_2 + \gamma_3 + \gamma_5 + \gamma_6)}\right] g(V_2, V_3). \tag{48}$$

If Q be the Ricci operator then $g(QV_2, V_3) = \mathcal{R}ic(V_2, V_3)$ and $\mathcal{R}ic(QV_2, V_3) = \mathcal{R}ic^2(V_2, V_3)$. So from (48), we get

$$\mathcal{R}ic(QV_2, V_3) = \left[\frac{\lambda^2\{\alpha(\widetilde{\mathcal{H}} - 1) + 3p - \rho\}^2(\gamma_4 + 3\gamma_7)^2}{(\gamma_0 + 4\gamma_1 + \gamma_2 + \gamma_3 + \gamma_5 + \gamma_6)^2}\right] g(V_2, V_3). \tag{49}$$

After contraction (49) both side over V_2 and V_3 we yield

$$||Q||^2 = \left[\frac{4\lambda^2(\alpha(\widetilde{\mathcal{H}} - 1) + 3p - \rho)^2(\gamma_4 + 3\gamma_7)^2}{(\gamma_0 + 4\gamma_1 + \gamma_2 + \gamma_3 + \gamma_5 + \gamma_6)^2}\right]. \tag{50}$$

So, we conclude the result.

Theorem 3.15. *If a* T^c*-flat relativistic magneto-fluid spacetime coupled with magneto-fluid density* ρ*, magnetic field strength* $\widetilde{\mathcal{H}}$ *and magnetic permeability* α *fulfilling EFE without cosmological constant, then*

$$||Q||^2 = \left[\frac{4\lambda^2(\alpha(\widetilde{\mathcal{H}} - 1) + 3p - \rho)^2(\gamma_4 + 3\gamma_7)^2}{(\gamma_0 + 4\gamma_1 + \gamma_2 + \gamma_3 + \gamma_5 + \gamma_6)^2}\right],$$

where Q *is the Ricci operator.*

Again, using (45) in (8), we have

$$(\mathfrak{L}_{\mathcal{Z}}g)(V_2, V_3) + [2\sigma\kappa + 2\mu - l\tau - (\tilde{\Omega} + \frac{1}{2})]g(V_2, V_3) + 2\nu\, \eta(V_2)\eta(V_3) = 0. \tag{51}$$

In view of (33), we get from (51) that

$$\tau = \frac{1}{l}[\mu + 2(\psi + \kappa\sigma) - \frac{1}{2}(2\tilde{\Omega} + \nu + 1)]. \tag{52}$$

Now, from (14) and (52), we can obtain

$$\mu = 4l\beta - l\lambda[\alpha(\widetilde{\mathcal{H}} - 1) + 3p - \rho] - 2(\psi + \kappa\sigma) + \frac{1}{2}(2\tilde{\Omega} + \nu + 1). \tag{53}$$

This leads to the following

Theorem 3.16. *If a* T^c*-flat relativistic MFST that admits CERY-soliton of type* (κ, l)*, where* $\mathcal{Z}$ *is a concircular vector field, then the soliton is expanding, steady or shrinking according as*

$$\beta \gtreqless \frac{1}{8l}[4(\psi + \kappa\sigma) - (2\tilde{\Omega} + \nu + 1)] + \frac{\lambda}{4l}[\alpha(\widetilde{\mathcal{H}} - 1) + 3p - \rho].$$

4. Physical significance of conformal pressure on relativistic MFST

The time-dependent scalar field, $\tilde{\Omega}$ defined in Eq.(2) is true physical pressure in fluid mechanics(called, conformal pressure). According to [4], outside of an equilibrium point the conformal pressure $\widetilde{p}$ is negative and zero inside. Also, the metric g is an equilibrium point or Einstein, providing a nonlinear restoring force. Then from Eq (8) and (15), we have

$$\begin{aligned}(\mathfrak{L}_{\mathcal{Z}}g)(V_1,V_2) &+ \Big[2\kappa\beta-\kappa\lambda(p-\alpha-\rho)+2\mu-l\tau-(\tilde{\Omega}+\frac{1}{2})\Big]g(V_1,V_2)\\ &+ \Big[2\kappa\alpha\lambda\widetilde{\mathcal{H}}+2\kappa\lambda(p+\rho)+2\nu\Big]\eta(V_1)\eta(V_2)\\ &- 2\alpha\kappa\lambda\omega(V_1)\omega(V_2)=0. \end{aligned} \tag{54}$$

Using equation (33) in (54), the conformal pressure.

$$\tilde{\Omega}=2(\kappa\beta+\mu+\psi)-\lambda\kappa[\alpha(2\widetilde{\mathcal{H}}-1)+3p+\rho]-\frac{(2l\mu+4\nu+1)}{2}. \tag{55}$$

Thus we state our findings

Theorem 4.1. *If a relativistic magneto-fluid spacetime coupled with magneto-fluid density ρ, magnetic field strength $\widetilde{\mathcal{H}}$ and magnetic permeability α admits CERY-soliton of type (κ, l), where $\mathcal{Z}$ is a concircular vector field then, the conformal pressure is*

$$\tilde{\Omega}=2(\kappa\beta+\mu+\psi)-\lambda\kappa[\alpha(2\widetilde{\mathcal{H}}-1)+3p+\rho]-\frac{(2l\mu+4\nu+1)}{2}.$$

Since inside the equilibrium point $\tilde{\Omega}=0$. So from (55), we reflect the outcome

Corollary 4.2. *If a relativistic magneto-fluid spacetime attached with magneto-fluid density ρ, magnetic field strength $\widetilde{\mathcal{H}}$ and magnetic permeability α admits CERY-soliton of type (κ, l), where $\mathcal{Z}$ is a concircular vector field then, the metric g is an equilibrium point or Einstein, if and only if*

$$\mu=\frac{\lambda\kappa}{2}[\alpha(2\widetilde{\mathcal{H}}-1)+3p+\rho]+\frac{(2l\mu+4\nu+1)}{2}-(\kappa\beta+\psi).$$

As per [16], if the metric g is an equilibrium point or Einstein it provide a nonlinear restoring force. So we entail the following result regrading the dynamical system.

Corollary 4.3. *If a relativistic magneto-fluid spacetime admits CERY-soliton of type (κ, l), where $\mathcal{Z}$ is a concircular vector field then, the metric g is an equilibrium point and acts as a nonlinear restoring force.*

5. Energy conditions relativistic magneto-fluid spacetime coupled with conformal η-Ricci-Yamabe soliton of type (κ, l)

According to [36] if the Ricci tensor $\mathcal{R}ic$ of type $(0,2)$ of the spacetime satisfies the condition

$$\mathcal{R}ic(V_1,V_1)>0, \tag{56}$$

for every timelike vector field V_1, then equation (56) is called the timelike convergence condition.

With the help of (33) and (14), equation (8) for fix V_1=V_2=ξ, takes the form

$$\mathcal{R}ic(\xi,\xi)=\frac{1}{\kappa}[\psi+\mu-2l\beta+\frac{l\lambda}{2}\{\alpha(\widetilde{\mathcal{H}}-1)+3p-\rho\}-\frac{1}{2}(\tilde{\Omega}+\frac{1}{2})-\nu]. \tag{57}$$

Since the spacetime admit CERY-soliton type (κ, l), where $\mathcal{Z}$ is a concircular vector field obeys the TCC, if $\mathcal{R}ic(V_1,V_1)>0$. Then from (57), one can get

$$[\psi+\mu-2l\beta+\frac{l\lambda}{2}\{\alpha(\widetilde{\mathcal{H}}-1)+3p-\rho\}]>[\frac{1}{2}(\tilde{\Omega}+\frac{1}{2})+\nu]. \tag{58}$$

Thus we can state

Theorem 5.1. *If a relativistic magneto-fluid spacetime admits a CERY-soliton of type* (κ, l)*, where* $\mathcal{Z}$ *is a concircular vector field and satisfies TCC, then the relation* (58) *holds.*

According to Hawking and Ellis [19], the time-like convergence condition(TCC) $\Rightarrow$ the cosmological strong energy condition (SEC), TCC $\Rightarrow$ the null convergence condition (NCC), SEC $\Rightarrow$ the null energy condition (NEC). Consequently, TCC$\Rightarrow$ NCC as well. Thus using this fact and Theorem 5.1, we reflect the outcome

Corollary 5.2. *If a relativistic magneto-fluid spacetime admits a CERY-soliton of type* (κ, l) *with* $\mathcal{Z}$ *is a concircular vector field and satisfies TCC then such a spacetime satisfies SEC.*

Corollary 5.3. *If a relativistic magneto-fluid spacetime coupled with a CERY-soliton of type* (κ, l)*, where* $\mathcal{Z}$ *is a concircular vector field and satisfies TCC then such a spacetime satisfies satisfies NCC.*

Corollary 5.4. *If a relativistic magneto-fluid spacetime admits a CERY-soliton of type* (κ, l) *with* $\mathcal{Z}$ *is a concircular vector field and satisfies TCC and if the spacetime satisfies SEC, then the Ricci tensor* $\mathcal{R}ic$ *is of the second Segre type [19].*

6. Application of Singularity Theorem in magneto-fluid spacetime coupled with conformal η-Ricci-Yamabe soliton of type (κ, l)

According to Penrose's singularity theorem, in [39] authors have shown that the MFST obeys null convergence condition (NCC), represents the presence of black holes and a trapped surface outside these black holes within the spacetime.

Utilizing above facts and Theorem 5.1 and Corollary 5.3, we conclude the upcoming result.

Theorem 6.1. *If a relativistic magneto-fluid spacetime with magneto-fluid density* ρ*, magnetic field strength* $\widetilde{\mathcal{H}}$ *and magnetic permeability* α *admits CERY-soliton of type* (κ, l)*, where* $\mathcal{Z}$ *is a concircular vector field and satisfies NCC, then the MFST include black holes with a trapped surface which is outside the black holes.*

Corollary 6.2. *If a relativistic magneto-fluid spacetime with magneto-fluid density* ρ*, magnetic field strength* $\widetilde{\mathcal{H}}$ *and magnetic permeability* α *admits CERY-soliton of type* (κ, l)*, where* $\mathcal{Z}$ *is a Killing vector field and satisfies NCC, then the MFST includes black holes with a trapped surface outside these black holes.*

7. Generalized Liouville equation on relativistic magneto-fluid spacetime admitting CERY-soliton of type (κ, l)

Let a relativistic MFST admit CERY-soliton of type (κ, l). Then from (8) and (15), we get

$$\begin{aligned} g(\nabla_{V_1}\xi, V_2) + g(V_1, \nabla_{V_2}\xi) &+ [2\kappa\beta - \alpha\lambda\{p - (\alpha + \rho)\} + 2\mu - l\tau - (\tilde{\Omega} + \frac{1}{2})]g(V_1, V_2) \\ &+ [2\kappa\lambda\alpha\widetilde{\mathcal{H}} + 2\kappa\lambda(p + \rho) + 2\nu]\eta(V_1)\eta(V_2) \\ &- 2\kappa\alpha\lambda\omega(V_1)\omega(V_2) = 0, \end{aligned} \tag{59}$$

for any $V_1, V_2 \in \chi(\mathrm{M}^4)$. On contracting (59), we obtain

$$\mathrm{D}iv(\xi) = -2(2\mu + 2\kappa\beta - l\tau) + \lambda\kappa(3p - \rho - 3\alpha + \widetilde{\mathcal{H}}) + (2\tilde{\Omega} + 1). \tag{60}$$

Now, for a smooth function $\psi^\star \in C^\infty(\mathrm{M}^4, g)$ and a vector field V_1, we have

$$\mathrm{D}iv(\psi^\star\xi) = v(d\psi^\star) + \psi^\star\mathrm{D}iv(\xi), \tag{61}$$

the function $\psi^\star$ is referred to as last Lagrange multiplier of the vector field V_1 corresponding to g if $\mathrm{div}(\psi^\star\xi) = 0$. So the concerning equation gives

$$\xi(d\log\psi^\star) = -\mathrm{D}iv(\xi), \tag{62}$$

is known as generalized Liouville equation of the vector V_1 with respect to g [29].
Therefore, from (60) and (62) we state the result

Theorem 7.1. *Let a relativistic magneto-fluid spacetime with magneto-fluid density ρ, magnetic field strength $\widetilde{\mathcal{H}}$ and magnetic permeability α admit CERY-soliton of type (κ, l), where ξ with a unit time-like vector field and $\psi^\star$ is the last multiplier of ξ, and if η be the g-dual 1-form of the vector field ξ, then the generalized Liouville equation is given by*

$$\xi(d\log\psi^\star) = 2(2\mu + 2\kappa\beta - l\tau) - \lambda\kappa(3p - \rho - 3\alpha + \widetilde{\mathcal{H}}) - (2\tilde{\Omega} + 1).$$

In particular, if the vector field ξ is incompressible or Killing, then from (60), we have a result

Corollary 7.2. *Let a relativistic magneto-fluid spacetime admit CERY-soliton of type (κ, l), where ξ with a unit time-like vector field and $\psi^\star$ is the last multiplier of ξ, and if η be the g-dual 1-form of the vector field ξ. If the field ξ is incompressible or Killing, then the soliton is expanding, steady, and shrinking as*

$$\left(\frac{2\tilde{\Omega} + 1}{2}\right) \begin{smallmatrix}>\\=\\<\end{smallmatrix} \beta - l\tau - \frac{\lambda\alpha}{2}[3p - 3\alpha - \rho + \widetilde{\mathcal{H}}].$$

8. Harmonic aspect of CERY-Soliton of type (κ, l) on relativistic magneto-fluid spacetime

In this section, we discuss CERY-soliton of type (κ, l) on MFST by considering the g-gual form ξ, the 1-form η is a harmonic or Schrödinger-Ricci harmonic.

Let η is a g-dual 1-form of ξ such that $g(V_1, \xi)$=$\eta(V_1)$ and $g(\xi, \xi)$=-1, then ξ is called a solution of the Schrödinger-Ricci equation if it satisfies

$$\mathrm{D}iv(\mathfrak{L}_\xi g) = 0, \tag{63}$$

where $(\mathfrak{L}_\xi g)$ is Lie derivative for the vector field ξ. According to [8], Chow et al. the divergence of Lie derivative as given bellows

$$\mathrm{D}iv(\mathfrak{L}_\xi g) = (\Theta + \mathcal{R}ic)(\xi) + d(\mathrm{D}iv(\xi)), \tag{64}$$

where Θ is the Laplace-Hodge operator with respect to the metric g and $\mathcal{R}ic$ is the Ricci curvature tensor field. So, from (8), we have

$$\mathfrak{L}_\xi g + 2\kappa\mathcal{R}ic + [2\mu - l\tau - (\tilde{\Omega} + \frac{1}{2})]g + 2\nu\,\eta\otimes\eta = 0. \tag{65}$$

Taking trace of (65) and using (14) we yield

$$\mathrm{D}iv(\xi) + [8\mu + (2\kappa - 4l)(4\beta - \lambda\{\alpha(\widetilde{\mathcal{H}} - 1) + 3p - \rho\} - (4\tilde{\Omega} + 2)] + 2\nu|\xi|^2 = 0. \tag{66}$$

As per, direct calculation, we have

$$\mathrm{D}iv(\eta\otimes\eta) = \mathrm{D}iv(\xi)\eta + \nabla_\xi\eta. \tag{67}$$

After taking the divergence of (65) and using (67), we obtain

$$\mathrm{D}iv(\mathfrak{L}_\xi g) + (2\kappa - l)d(3p - \rho) + 2\nu[\mathrm{D}iv(\xi)\eta + \nabla_\xi\eta] = 0. \tag{68}$$

If the 1-form η is a solution of the Schrödinger-Ricci equation then

$$(\Theta + \mathcal{R}ic)(\xi) + d(\mathrm{D}iv(\xi)) = 0. \tag{69}$$

Thus, we state the outcome

Theorem 8.1. *Let a relativistic magneto-fluid spacetime coupled with magneto-fluid density ρ, magnetic field strength $\widetilde{\mathcal{H}}$ and magnetic permeability α admit CERY-soliton of type (κ, l), where η is the g-dual of the vector field ξ. Then η is the solution of Schrödinger-Ricci equation if and only if*

$$d(3p - \rho) = \frac{2\nu}{((2\kappa - l)}[\{8\mu - (4l - 2\kappa)(4\beta - \lambda[\alpha(\widetilde{\mathcal{H}} - 1) + 3p - \rho]) - (4\widetilde{p}\tilde{\Omega} + 2) + 2\nu\}\eta - \nabla_\xi\eta]. \tag{70}$$

Proof. With the help of (63),(64) (66),(68) and (14) it follows that η is a solution of the Schrödinge-Ricci equation if and only if (74) holds □

Finally, for Schrödinger-Ricci harmonic form, let the 1-form η is a Schrödinger-Ricci harmonic form if [5]

$$(\Theta + \mathcal{R}ic)(\xi) = 0. \tag{71}$$

Beside this if ν=0, which yield the conformal Ricci-Yamabe soliton of type (κ, l), or

$$\nabla_\xi \eta = [8\mu - (4l - 2\kappa)(4\beta - \lambda\{\alpha(\widetilde{\mathcal{H}} - 1) + 3p - \rho\}) - (4\tilde{\Omega} + 2)]\eta,$$

which implies that

$$\nu = 8\mu - (4l - 2\kappa)(4\beta - \lambda\{\alpha(\widetilde{\mathcal{H}} - 1) + 3p - \rho\}) - (4\tilde{\Omega} + 2).$$

As per above consequences, we get the outcome

Theorem 8.2. *Let a relativistic magneto-fluid spacetime with magneto-fluid density ρ, magnetic field strength $\widetilde{\mathcal{H}}$ and magnetic permeability α admit CERY-soliton of type (κ, l), where η is the g-dual of the vector field ξ. Then η is the solution of Schrödinger-Ricci harmonic form if and only if ν=0, which produce conformal Ricci-Yamabe soliton of type (κ, l) or*

$$\nabla_\xi \eta = [8\mu - (4l - 2\kappa)(4\beta - \lambda\{\alpha(\widetilde{\mathcal{H}} - 1) + 3p - \rho\}) - (4\tilde{\Omega} + 2)]\eta.$$

9. Gradient conformal η-Ricci-Yamabe soliton on relativistic magneto-fluid spacetime

Let the soliton vector $\mathcal{Z}$=$\mathcal{D}f$, where f is a smooth function and $\mathcal{D}$ stands for gradient operator of g on relativistic MSFT. So equation (9) implies that

$$\nabla^2 f = -\kappa Q - [\mu - \frac{l\tau}{2} - \frac{1}{2}(\tilde{\Omega} + \frac{1}{2})]\mathcal{I} - \nu\, \eta \otimes \xi, \tag{72}$$

After, contracting (72) and using (14), we yield

$$\nabla^2 f = -4\mu - (\kappa - 2l)[4\beta - \lambda\{\alpha(\widetilde{\mathcal{H}} - 1) + 3p - \rho\}] + (2\tilde{\Omega} + 1) + \nu. \tag{73}$$

Thus, we state the outcome

Theorem 9.1. *If a relativistic magneto-fluid spacetime attached with magneto-fluid density ρ, magnetic field strength $\widetilde{H}$ and magnetic permeability α admits GCERY-soliton of type (κ, l), then the potential function f of the soliton satisfies Poisson's equation*

$$\nabla^2 f = -4\mu - (\kappa - 2l)[4\beta - \lambda\{\alpha(\widetilde{\mathcal{H}} - 1) + 3p - \rho\}] + (2\tilde{\Omega} + 1) + \nu.$$

A smooth function χ^* on a Riemannian manifold (M^n, g) of dim $M = n \geq 3$ is said to be harmonic, strictly super-harmonic and strictly sub-harmonic if $\nabla\chi^* = 0$, $\nabla\chi^* < 0$ and $\nabla\chi^* > 0$. So from above facts and (73), we reveal the corollary

Corollary 9.2. *Let a relativistic magneto-fluid spacetime admit GCERY-soliton of type (κ, l), then the soliton function f is strictly sub-harmonic, harmonic or, strictly supper-harmonic according to as*

$$\mu \gtreqless \frac{1}{4}[(\tilde{\Omega} + 1) - (\kappa - 2l)\{4\beta - \lambda[\alpha(\widetilde{\mathcal{H}} - 1) + 3p - \rho]\} + \nu].$$

Corollary 9.3. *Let a relativistic magneto-fluid spacetime admit GCERY-soliton of type (κ, l) and if the soliton function f is harmonic, then the soliton is expanding, steady, or shrinking as*

$$(\tilde{\Omega} + 1) \gtreqless (\kappa - 2l)[4\beta - \lambda\{\alpha(\widetilde{\mathcal{H}} - 1) + 3p - \rho\}] - \nu.$$

10. Applications of relativistic magneto-fluid spacetime

We demonstrate some different types of relativistic MFST on behalf of its EMT and deduce the results, when the MFST satisfying the new types of EFE, coupled with such a soliton. So, we reflect the following cases

Case-I: $p = 0$, then from the Theorem 8.1 we state the outcome

Theorem 10.1. *Let a pressureless relativistic MFST admit CERY-soliton of type* (κ, l), *where* η *is the* g*-dual of the vector field* ξ*. Then* η *is the solution of Schrödinger-Ricci equation if and only if*

$$d(\rho) = \frac{-2\nu}{(2\kappa - l)}[\{8\mu - (4l - 2\kappa)(4\beta - \lambda[\alpha(\widetilde{\mathcal{H}} - 1) - \rho]) - (4\tilde{\Omega} + 2) + 2\nu\}\eta - \nabla_\xi \eta].$$

Again, from (73) we state that

Theorem 10.2. *If a pressureless relativistic MFST admits GCERY-soliton of type* (κ, l), *then the potential function* f *of the soliton satisfies Poisson's equation*

$$\nabla^2 f = -4\mu - (\kappa - 2l)[4\beta - \lambda\{\alpha(\widetilde{\mathcal{H}} - 1) - \rho\}] + (2\tilde{\Omega} + 1) + \nu.$$

Also, from the Theorem 7.1 we state the result.

Theorem 10.3. *Let a pressureless relativistic MFST admit CERY-soliton of type* (κ, l), *where* ξ *with a unit time-like vector field and* $\psi^\star$ *is the last multiplier of* ξ, *and if* η *be the* g*-dual* 1*-form of the vector field* ξ, *then the generalized Liouville equation is determine by*

$$\xi(d \log \psi^\star) = 2(2\mu + 2\kappa\beta - l\tau) + \lambda\kappa(\rho + 3\alpha - \widetilde{\mathcal{H}}) - (2\tilde{\Omega} + 1).$$

Case-II: $p + \rho = 0$, i.e., $\rho = -p$, then we state the result from the Theorem 8.1 that

Theorem 10.4. *Let a relativistic MFST admit CERY-soliton of type* (κ, l), *where* η *is the* g*-dual of the vector field* ξ*. If the sum of the pressure and density of the fluid is zero, then* η *is the solution of Schrödinger-Ricci equation if and only if*

$$d(p) = \frac{2\nu}{4(2\kappa - l)}[\{8\mu - (4l - 2\kappa)(4\beta - \lambda[\alpha(\widetilde{\mathcal{H}} - 1) + 4p]) - (4\tilde{\Omega} + 2) + 2\nu\}\eta - \nabla_\xi \eta].$$

Moreover, from (73), we yield the findings

Theorem 10.5. *Let a relativistic MFST admit GCERY-soliton of type* (κ, l)*. If the sum of the pressure and density of the fluid is zero, then the potential function* f *of the soliton satisfies the following Poisson's equation*

$$\nabla^2 f = -4\mu - (\kappa - 2l)[4\beta - \lambda\{\alpha(\widetilde{\mathcal{H}} - 1) + 4p\}] + (2\tilde{\Omega} + 1) + \nu.$$

Once again, one can state from the Theorem 7.1 that

Theorem 10.6. *Let a relativistic MFST admits CERY-soliton of type* (κ, l), *where* ξ *with a unit time-like vector field and* $\psi^\star$ *is the last multiplier of* ξ, *and if* η *be the* g*-dual* 1*-form of the vector field* ξ*. If the sum of the pressure and density of the fluid is zero, then the generalized Liouville equation is given by*

$$\xi(d \log \psi^\star) = 2(2\mu + 2\kappa\beta - l\tau) - \lambda\kappa(4p - 3\alpha + \widetilde{\mathcal{H}}) - (2\tilde{\Omega} + 1).$$

Case-III: $\rho = 3p$, then in view of the Theorem 8.1 we have

Theorem 10.7. *Let a relativistic magneto-fluid spacetime admit CERY-soliton of type* (κ, l), *where* η *is the* g*-dual of the vector field* ξ*. If the density of the fluid is thrice of the pressure, then* η *is the solution of Schrödinger-Ricci equation if and only if*

$$\nabla_\xi \eta = [8\mu - (4l - 2\kappa)(4\beta - \lambda\alpha(\widetilde{\mathcal{H}} - 1)) + 3p - (4\tilde{\Omega} + 2) + 2\nu]\eta.$$

Also, from (73), we state that

Theorem 10.8. *Let a relativistic MFST admit GCERY-soliton of type (κ, l). If the density of the fluid is thrice of the pressure, then the potential function f of the soliton satisfies the following Poisson's equation*

$$\nabla^2 f = -4\mu - (\kappa - 2l)[4\beta - \lambda\alpha(\widetilde{\mathcal{H}} - 1)] + (2\tilde{\Omega} + 1) + \nu.$$

Beside this, one can get from the Theorem7.1 that

Theorem 10.9. *Let a relativistic MFST admit CERY-soliton of type (κ, l) coupled with density of the fluid is thrice of the pressure, where ξ with a unit time-like vector field and $\psi^\star$ is the last multiplier of ξ, and if η be the g-dual 1-form of the vector field ξ. If the sum of the pressure and density of the fluid is zero, then the generalized Liouville equation is given by*

$$\xi(d\log\psi^\star) = 2(2\mu + 2\kappa\beta - l\tau) + \lambda\kappa(3\alpha - \widetilde{\mathcal{H}}) - (2\tilde{\Omega} + 1).$$

Case-IV: $\alpha = 0$, then from (11), we have the magnetic EMT on MFST

$$\widetilde{\mathcal{T}}(V_1, V_2) = pg(V_1, V_2) + (p + \rho)\eta(V_1)\eta(V_2),$$

which represents the EMT of perfect fluid spacetime. So, we reflect the finding

Theorem 10.10. *Let a relativistic MFST obeying EFE. If in the fluid, the magnetic permeability vanishes, then the spacetime reduces to a perfect fluid spacetime(PFST).*

11. Conclusion

This research focuses on identifying significant results and properties that emerge from the interaction of a relativistic MFST and CERY-soliton (or GCERY-soliton) under some specific curvature conditions. A key aspect of this study is the demonstration of geometrical and physical characteristics of MFST. In particular, the MFST under such a soliton provide insight into the underlying geometry and the dynamic behavior of the spacetime, contributing to our understanding of the spacetime physical and mathematical properties. Apart from this, We have explored that the spacetime satisfied the strong energy condition and the null convergence condition in term a conformal η-Ricci-Yamabe soliton of type (κ, l). Also, this study demonstrate the implications of singularity theorem and the occurrence of a black hole of spacetime, admitting the such type soliton. Finally, we extend our analysis by examining harmonic, the modified versions of the Liouville and Poisson equations in the context of such soliton on MFST. Finally, we discuss some results on different types of magneto-fluid spacetime coupled with such type of soliton. These results play a critical role in defining the geometric and physical behavior of the system, and our findings help bridge the gap between relativistic MFST and PFST.

Data Availability. This study did not gather or produce any underlying data.
Conditions of Interest. All authors have no conditions of interest.
Research ethics. This study does not have ethical issues.

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