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Construction and topological properties of Lie group

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Abstract. In this paper, we construct a Lie group and discuss its topological properties and representations. For this, first we generate a new metric tensor for a four-dimensional spherically symmetric space and then solving the Killing equation to determine its Killing vectors. Using these Killing vectors, we will construct a finite Lie algebra and prove that they form a Lie group corresponding to the Lie algebra. Furthermore, we will establish that this Lie group is isomorphic to the group $SO(2,1)$. At the end, we will describe the constructed Lie group's topological properties and representations.

1. Introduction

Lie groups and Lie algebras play a crucial role in both pure mathematics and theoretical physics. Sophus Lie studied the theory of Lie groups through transformation groups.
A Lie group refers to a collection of mathematical elements that exhibit a smooth structure that is consistent with their group operation. Such groups are extensively utilized in the domains of physics and mathematics to study geometric objects' symmetries, the motions of solid bodies, and the behavior of particles in quantum mechanics. An essential aspect of these groups is the Lie algebra, which is a vector space connected to the group, enabling an understanding of its infinitesimal behavior. Today, Lie groups are a fundamental area of interest in contemporary mathematics and physics, with their applications extending from algebraic geometry to string theory.
Lie groups have wide-ranging applications in various fields of mathematics and physics, including differential geometry. Lie groups provide a natural setting for studying the geometry of curved spaces, making them a powerful tool in the field. One example of a Lie group that is fundamental in comprehending the geometry of solid objects is the set of rotations in Euclidean space. The study of Lie groups offers mathematicians and physicists a means of gaining a profound comprehension of the fundamental structures and symmetries in diverse mathematical and physical systems. The extensive mathematical theory of Lie groups serves as a catalyst for new breakthroughs and perspectives, contributing to the ongoing exploration of this field across various domains of mathematics and physics.

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Lie groups find extensive and significant applications in multiple fields of mathematics and physics, including group theory, control theory, quantum mechanics, and physics. Within group theory, Lie groups play a fundamental role in exploring properties of groups, such as permutation groups and symmetry groups. In physics, Lie groups are commonly utilized to examine symmetry and conservation laws, as exemplified by the Poincaré group, which is essential for studying special relativity. Control theory utilizes Lie groups to model intricate dynamical systems, such as autonomous vehicles and robots. In quantum mechanics, Lie groups are essential in studying the theory of Lie algebras, which characterize physical systems' symmetry groups.

A compact manifold is a mathematical entity that combines two essential traits: compactness and manifold structure. To put it simply, a manifold is a topological space that, on a local scale, resembles Euclidean space and can be likened to a "rubber sheet" that can be stretched and deformed without breaking apart.

Compactness refers to the property of being "finite" in some sense.Regarding a compact manifold,it means that the manifold is both closed and bounded. This implies that it is not possible to "escape to infinity" in any direction, and that the manifold has a finite size or volume.

As an illustration, a compact manifold is a manifold that is both bounded and closed. Take for instance, a two-dimensional sphere or a torus which are both compact manifolds. Compact manifolds hold significant importance in the field of mathematics and are encountered in diverse areas of study such as topology, geometry, and physics.

However, a simply connected manifold is a particular type of manifold that lacks any handles or holes within its structure. To be exact, a manifold is categorized as simply connected when any loop formed on it can be continuously transformed into a point while staying on the manifold. This type of manifold finds its application in various fields of study.

To put it differently, a simply connected manifold is one in which it is possible to continuously deform any closed path on the manifold to a single point while remaining on the manifold. This characteristic is a topological invariant, indicating that it remains unchanged under continuous transformations of the manifold.

The adjoint representation is a methodology used in the study of Lie groups. It involves assigning a linear transformation to each element of the group, which preserves its structural properties. Through this representation, the Lie group can be expressed as a collection of linear transformations.

The adjoint representation of a Lie group G is a function $Ad : G \to GL(g)$, where g represents the Lie algebra of G and $GL(g)$ denotes the general linear group of the vector space g. The adjoint representation maps each element g in G to a linear transformation on g known as conjugation. Thus, for any $g \in G$, the adjoint representation $Ad(g)$ is a linear transformation on g obtained through conjugation.

$$Ad(g)(x) = gxg^{-1}$$

where $x \in g$. The adjoint representation possesses various significant characteristics. One of its essential features is that it acts as a Lie group homomorphism, implying that it maintains the group structure of G. Moreover, it serves as a trustworthy representation, signifying that if two elements of G yield the same output under Ad, then they must be identical. Furthermore, the adjoint representation is frequently utilized to explore the Lie algebra of a Lie group. This is because it facilitates the translation of queries about the group into inquiries about its Lie algebra, making it a valuable tool in Lie theory.

The adjoint representation is an essential component of Lie group and Lie algebra theory, and it finds wide-ranging applications across multiple domains of mathematics and physics, including representation theory, differential geometry, and quantum field theory. Its significance is paramount due to the capability to preserve group structure while assigning linear transformations to each element of the group. Consequently, it is an indispensable tool that enables the study and application of various mathematical and physical systems.

Lie groups have various applications in mathematics and theoretical physics, and they can also be applied in the field of commutative distribution function (CDF) theory.Lie groups provide a powerful tool for analyzing the symmetry properties of mathematical objects, including probability distributions. In CDF theory, one can investigate the symmetries of CDFs using Lie groups. By studying the Lie symmetries, it is possible to identify important characteristics of the CDF, such as invariance properties and relationships with other distributions.

2. Preliminaries

The field of differential geometry and mathematical physics commonly employs the phrase "Killing vector" to denote a vector field on a differentiable manifold that preserves the metric tensor of the manifold. This vector field generates a symmetry transformation of the manifold's geometry. Specifically, Killing vectors are vector fields that preserve the geometry of a given space-time under an isometry, a certain type of transformation. In essence, a Killing vector generates a symmetry of the space-time while keeping its metric invariant. The killing equation is the mathematical definition of a killing vector as a vector field k that satisfies a specific condition.

$$\nabla_\mu k_\nu + \nabla_\nu k_\mu = 0$$

where ∇_μ is the covariant derivative w.r.t. the metric of the space-time. The killing equation is a differential equation that guarantees the vector field k preserves the metric tensor, which in turn maintains the space-time geometry. To be classified as a killing vector, a vector field must fulfill various other conditions apart from the killing equation, such as being a smooth and continuous function on the space-time. Killing vectors are crucial in multiple fields of physics, such as general relativity and classical mechanics. They are employed to describe symmetries and conservation laws. The direction in which the Lie derivative of metric tensor is zero is called killing vector and killing equation is

$$L_\xi(g_{ab}) = 0$$

$$g_{ab,c}\,\xi^c + g_{cb}\xi^c_{,a} + g_{ac}\xi^c_{,b} = 0$$

where comma (,) represent partial derivative. Sophus Lie introduced the Lie derivative in the late 1800s, and it has since become a critical tool in modern physics and mathematics, particularly in differential geometry.The fundamental notion of the Lie derivative of a tensor field along a vector field is a vital concept in differential geometry. It quantifies the transformation of a tensor field along a flow defined by a distinct vector field, factoring in the vector field's non-commutative behavior. This directional derivative has broad applications in physics and plays a pivotal role in understanding the dynamics of physical systems, especially in the realm of general relativity.

The Lie derivative is also significant in defining the Lie algebra, which is the algebraic structure underlying the theory of Lie groups. Its far-reaching applications in various areas of physics and mathematics make it a crucial concept for examining the symmetries of physical systems and facilitating the development of mathematical models for various applications. Overall, the Lie derivative is a powerful mathematical operation that has become a cornerstone of modern physics and mathematics, providing insights into complex physical phenomena and helping to advance our understanding of the underlying mathematical structures.

The exponential map and logarithm map are fundamental operations in Lie theory. The exponential map maps elements from the Lie algebra (the tangent space at the identity) to the Lie group, while the logarithm map performs the inverse operation. Numerical methods based on the exponential map and logarithm map are used for computing Lie group elements and their derivatives.

In section 1, Preliminaries [4, 9, 10]; section 2, introduction [2, 6, 11]; section 3, Construction of Metric Tensor [5, 8, 19]; section 4, Symmetries of Spacetime[1, 3, 13]; section 5, Lie algebras of killing vectors [12, 20]; section 6, Lie group [16]; section 7, Topology of Lie Group[17]; section 8, The Adjoint Representation of Lie Group[7, 15, 18]; section 9, Conclusion.

3. Metric Tensor

In this section, we'll generate a new metric tensor for a spherically symmetric space(SSS) in 4 dimensions. To make a diagonal metric tensor, we'll begin with the line element expressed in spherical coordinates [2].

$$ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu} \tag{3.1}$$

$$ds^2 = -e^2A(r)dt^2 + e^2B(r)dr^2 + r^2(d\theta^2 + sin^2(\theta)d\phi^2) \tag{3.2}$$

where $A(r)$ and $B(r)$ are functions of the radial coordinates r and the metric components $g_{\mu\nu}$ depend on the space-time coordinates

$$x^{\mu} = (t, r, \theta, \phi)$$

e^2 is constant factor that can be absorbed into the coordinates through a rescaling. We want to express the metric tensor in terms of hyperbolic functions, so we can make the following substitutions

$$A(r) = cosh^2(v(r))$$

$$B(r) = sinh^2(v(r))$$

where $v(r)$ is a new function of the radial coordinates r. Using these substitutions, the line element becomes

$$ds^2 = -e^2cosh(v(r))dt^2 + e^2sinh(v(r))dr^2 + r^2(d\theta^2 + sin^2(\theta)d\phi^2) \tag{3.3}$$

The metric tensor in matrix form

$$g_{ab} = \begin{pmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{21} & g_{22} & g_{23} & g_{24} \\ g_{31} & g_{32} & g_{33} & g_{34} \\ g_{41} & g_{42} & g_{43} & g_{44} \end{pmatrix} \qquad g_{ab} = \begin{pmatrix} -e^2cosh(v(r)) & 0 & 0 & 0 \\ 0 & e^2sinh^2(v(r)) & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2sin^2(\theta) \end{pmatrix}$$

4. Killing Vectors of SSS

In this section, we find the Killing vectors of the metric tensor of four dimensional spherically symmetric space(SSS). Consider $X = \left(X^1(t, r, \theta, \phi), X^2(t, r, \theta, \phi), X^3(t, r, \theta, \phi), X^4(t, r, \theta, \phi)\right)$ be the set of Killing vectors of 4-dimensional spherically symmetric space. The direction in which the Lie derivative of metric tensor g_{ab} is zero called Killing vectors and its equation is

$$L_X(g_{ab}) = 0 \tag{4.1}$$

$$g_{ab,c}X^c + g_{cb}X^c_{,a} + g_{ac}X^c_{,b} = 0 \tag{4.2}$$

where comma (,) represent partial derivative. The Lie derivative is a directional derivative in which the change of a tensor field that follows the flow given by another vector field is evaluated. From equation (4.2), for g_{11}

$$g_{11,c}X^c + g_{c1}X^c_{,1} + g_{1c}X^c_{,1} = 0 \tag{4.3}$$

Since c = 1,2,3,4

$$\begin{aligned} &g_{11,1}X^1 + g_{11}X^1_{,1} + g_{11}X^1_{,1} + g_{11,2}X^2 + g_{21}X^2_{,1} + g_{12}X^2_{,1} + \\ &g_{11,3}X^3 + g_{31}X^3_{,1} + g_{13}X^3_{,1} + g_{11,4}X^4 + g_{41}X^4_{,1} + g_{14}X^4_{,1} = 0 \end{aligned} \tag{4.4}$$

After solving, we have

$$2e^2cosh^2(v(r))\frac{\partial X^1}{\partial t} + sinh(2(v(r))v'(r)X^2 = 0 \tag{4.5}$$

For $g_{12} = g_{21}$

$$g_{12,c}X^c + g_{c2}X^c_{,2} + g_{1c}X^c_{,2} = 0 \tag{4.6}$$

$$g_{12,1}X^1 + g_{12}X^1_{,1} + g_{11}X^1_{,2} + g_{12,2}X^2 + g_{22}X^2_{,1} + g_{12}X^2_{,2}+$$
$$g_{12,3}X^3 + g_{32}X^3_{,1} + g_{13}X^3_{,2} + g_{12,4}X^4 + g_{42}X^4_{,1} + g_{14}X^4_{,2} = 0 \tag{4.7}$$

After solving, we have

$$cosh^2(v(r))\frac{\partial X^1}{\partial r} - sinh^2(v(r))\frac{\partial X^2}{\partial t} = 0 \tag{4.8}$$

For $g_{13} = g_{31}$

$$g_{13,c}X^c + g_{c3}X^c_{,1} + g_{1c}X^c_{,3} = 0 \tag{4.9}$$

$$g_{13,1}X^1 + g_{13}X^1_{,1} + g_{11}X^1_{,3} + g_{13,2}X^2 + g_{23}X^2_{,1} + g_{12}X^2_{,3}+$$
$$g_{13,3}X^3 + g_{33}X^3_{,1} + g_{13}X^3_{,3} + g_{13,4}X^4 + g_{43}X^4_{,1} + g_{14}X^4_{,3} = 0 \tag{4.10}$$

After solving, we have

$$e^2cosh^2(v(r))\frac{\partial X^1}{\partial \theta} + r^2\frac{\partial X^3}{\partial t} = 0 \tag{4.11}$$

For $g_{14} = g_{41}$

$$g_{14,c}X^c + g_{c4}X^c_{,1} + g_{1c}X^c_{,4} = 0 \tag{4.12}$$

$$g_{14,1}X^1 + g_{14}X^1_{,1} + g_{11}X^1_{,4} + g_{14,2}X^2 + g_{24}X^2_{,1} + g_{12}X^2_{,4}+$$
$$g_{14,3}X^3 + g_{34}X^3_{,1} + g_{13}X^3_{,4} + g_{14,4}X^4 + g_{44}X^4_{,1} + g_{14}X^4_{,4} = 0 \tag{4.13}$$

After solving, we have

$$e^2cosh^2(v(r))\frac{\partial X^1}{\partial \phi} - r^2sin^2(\theta)\frac{\partial X^4}{\partial t} = 0 \tag{4.14}$$

For g_{22}

$$g_{22,c}X^c + g_{c2}X^c_{,2} + g_{2c}X^c_{,2} = 0 \tag{4.15}$$

$$g_{22,1}X^1 + g_{12}X^1_{,2} + g_{21}X^1_{,2} + g_{22,2}X^2 + g_{22}X^2_{,2} + g_{22}X^2_{,2}+$$
$$g_{22,3}X^3 + g_{32}X^3_{,2} + g_{23}X^3_{,2} + g_{22,3}X^4 + g_{42}X^4_{,2} + g_{24}X^4_{,2} = 0 \tag{4.16}$$

After solving, we have

$$2sinh^2(v(r))\frac{\partial X^2}{\partial r} + (sinh(2(v(r))))v'(r))X^2 = 0 \tag{4.17}$$

For $g_{23} = g_{32}$

$$g_{23,c}X^c + g_{c3}X^c_{,2} + g_{2c}X^c_{,3} = 0 \tag{4.18}$$

$$g_{23,1}X^1 + g_{13}X^1_{,2} + g_{21}X^1_{,3} + g_{23,2}X^2 + g_{23,2}X^2 + g_{22}X^2_{,3}+$$
$$g_{23,3}X^3_{,3} + g_{33}X^3_{,2} + g_{23}X^3_{,3} + g_{23,4}X^4 + g_{43}X^4_{,2} + g_{24}X^4_{,3} = 0 \tag{4.19}$$

After solving, we have

$$e^2 sinh^2(v(r))\frac{\partial X^2}{\partial \theta} + r^2\frac{\partial X^3}{\partial r} = 0 \tag{4.20}$$

For $g_{24} = g_{42}$

$$g_{24,c}X^c + g_{c4}X^c_{,2} + g_{2c}X^c_{,4} = 0 \tag{4.21}$$

$$\begin{aligned} g_{24,1}X^1 + g_{14}X^1_{,2} + g_{21}X^1_{,4} + g_{24,2}X^2 + g_{24}X^2_{,2} + g_{22}X^2_{,4} + \\ g_{24,3}X^3 + g_{34}X^3_{,2} + g_{23}X^3_{,4} + g_{24,4}X^4 + g_{44}X^4_{,2} + g_{24}X^4_{,4} = 0 \end{aligned} \tag{4.22}$$

After solving, we have

$$e^2 sinh^2(v(r))\frac{\partial X^2}{\partial \phi} + r^2 sin^2(\theta)\frac{\partial X^4}{\partial r} = 0 \tag{4.23}$$

For g_{33}

$$g_{33,c}X^c + g_{c3}X^c_{,3} + g_{3c}X^c_{,3} = 0 \tag{4.24}$$

$$\begin{aligned} g_{33,1}X^1 + g_{13}X^1_{,3} + g_{31}X^1_{,3} + g_{33,2}X^2 + g_{23}X^2 + g_{32}X^2_{,3} + \\ g_{33,3}X^3_{,3} + g_{33}X^3_{,3} + g_{33}X^3_{,3} + g_{33}X^4 + g_{43}X^4_{,3} + g_{34}X^4_{,3} = 0 \end{aligned} \tag{4.25}$$

After solving, we have

$$r\frac{\partial X^3}{\partial \theta} + X^2 = 0 \tag{4.26}$$

For $g_{34} = g_{43}$

$$g_{34,c}X^c + g_{c4}X^c_{,3} + g_{3c}X^c_{,4} = 0 \tag{4.27}$$

$$\begin{aligned} g_{34,1}X^1 + g_{14}X^1_{,3} + g_{31}X^1_{,4} + g_{34,2}X^2 + g_{24}X^2_{,3} + g_{32}X^3_{,4} + \\ g_{34,3}X^3 + g_{34}X^3_{,3} + g_{33}X^3_{,4} + g_{34,4}X^4 + g_{44}X^4_{,3} + g_{34}X^4_{,4} = 0 \end{aligned} \tag{4.28}$$

After solving, we have

$$\frac{\partial X^3}{\partial \phi} + sin^2(\theta)\frac{\partial X^4}{\partial \theta} = 0 \tag{4.29}$$

For g_{44}

$$g_{44,c}X^c + g_{c4}X^c_{,4} + g_{4c}X^c_{,4} = 0 \tag{4.30}$$

$$\begin{aligned} g_{44,1}X^1 + g_{14}X^1_{,4} + g_{41}X^1_{,4} + g_{44,2}X^2 + g_{24}X^2_{,4} + g_{42}X^2_{,4} + \\ g_{44,3}X^3 + g_{34}X^3_{,4} + g_{43}X^3_{,4} + g_{44,4}X^4 + g_{44}X^4_{,4} + g_{44}X^4_{,4} = 0 \end{aligned} \tag{4.31}$$

After solving, we have

$$\frac{\partial X^4}{\partial \phi} + \frac{1}{r}X^2 + cot(\theta)X^3 = 0 \tag{4.32}$$

From equation (4.17), we obtain

$$X^2 = cosech(v(r))f(t,\theta,\phi) \tag{4.33}$$

where $f(t,\theta,\phi)$ is integration constant. From equation (4.8), we obtain

$$X^1 = -\frac{f'(t,\theta,\phi)sech(v(r))}{v'(r)} + g(t,\theta,\phi) \tag{4.34}$$

where $g(t,\theta,\phi)$ is integration constant. From equation (4.20), we obtain

$$X^3 = -\frac{e^2 f(t,\theta,\phi)cosh(v(r))}{r^2 v'(r)} + h(t,\theta,\phi) \tag{4.35}$$

where $h(t,\theta,\phi)$ is integration constant. From equation (4.23), we obtain

$$X^4 = \frac{e^2 f'(t,\theta,\phi)cosh(v(r))}{r^2 v'(r) sin^2(\theta)} + p(t,\theta,\phi) \tag{4.36}$$

where $p(t,\theta,\phi)$ is integration constant. Now we find integration constants. Using X^1 and X^2 in equation (4.5) and comparing coefficients, we obtain $f = C_1 e^{(v'(r))\theta} + C_2 e^{-(v'(r))\theta}$ and $g = C_3$ Similarly, using X^3 and X^4 in equation (4.29) and comparing coefficients, we obtain $h = C_4$ and $p = C_5$. By using boundary conditions f(0,0,0) = 1, g(0,0,0) = 1, h(0,0,0) = 1, p(0,0,0) = 1. Then, we obtain $C_1 = 1, C_2 = 0, C_3 = 1, C_4 = 1$ Then, equations (4.33),(4.34),(4.35) and (4.36) become

$$X^1 = -e^{(v'(r))\theta} sech(v(r)) + 1 \tag{4.37}$$

$$X^2 = e^{(v'(r))\theta} cosech(v(r)) \tag{4.38}$$

$$X^3 = -\frac{e^{(v'(r))\theta} e^2 cosh(v(r))}{r^2} + 1 \tag{4.39}$$

$$X^4 = -\frac{e^{(v'(r))\theta} e^2 cosh(v(r))}{r^2 sin^2(\theta)} + 1 \tag{4.40}$$

Thus, X^1, X^2, X^3 and X^4 are Killing vectors of 4-dimensional spherically symmetric space(SSS).

5. Lie Algebra of Killing Vectors of SSS

The major purpose of this section is to construct the Lie algebra from Killing vectors in a four-dimensional spherically symmetric space(SSS).

Definition 5.1. *The symbol $[f, g]$ represents the Lie bracket of two variables vector fields f and g and is defined as*

$$[f, g] = \frac{\partial g}{\partial r}\frac{\partial f}{\partial \theta} - \frac{\partial f}{\partial r}\frac{\partial g}{\partial \theta} \tag{5.1}$$

In other words, the Lie bracket between two vector fields f and g is determined by their commutator, which measures the extent to which they fail to commute. When the Lie bracket vanishes, it implies that the vector fields commute with each other. From equation (4.37), partially differentiate w.r.t. r and θ

$$\frac{\partial X^1}{\partial r} = e^{(v'(r))\theta} sech(v(r)) \left[-v''(r)\theta + tanh(v(r))v'(r)\right] \tag{5.2}$$

$$\frac{\partial X^1}{\partial \theta} = -v'(r) e^{(v'(r))\theta} sech(v(r)) \tag{5.3}$$

Similarly, partially differentiate w.r.t. r and θ equations (4.38),(4.39) and (4.40)

$$\frac{\partial X^2}{\partial r} = e^{(v'(r))\theta} cosech(v(r)) \left[\theta v''(r) - v'(r) coth(v(r))\right] \tag{5.4}$$

$$\frac{\partial X^2}{\partial \theta} = v'(r) e^{(v'(r))\theta} cosech(v(r)) \tag{5.5}$$

$$\frac{\partial X^3}{\partial r} = -\frac{e^{(v'(r))\theta} e^2 \left[r^2 v'(r) sinh(v(r)) + r^2 \theta v''(r) cosh(v(r)) - 2r cosh(v(r))\right]}{r^4} \tag{5.6}$$

$$\frac{\partial X^3}{\partial \theta} = -\frac{e^{(v'(r))\theta} e^2 v'(r) cosh(v(r))}{r^2} \tag{5.7}$$

$$\frac{\partial X^4}{\partial r} = \frac{e^{(v'(r))\theta} e^2 \left[r^2 v'(r) sinh(v(r)) + r^2 \theta v''(r) cosh(v(r)) - 2r cosh(v(r))\right]}{r^4 sin^2(\theta)} \tag{5.8}$$

$$\frac{\partial X^4}{\partial \theta} = -\frac{e^{(v'(r))\theta} e^2 cosh(v(r)) \left[v'(r) sin^2(\theta) - sin(2\theta)\right]}{r^2 sin^4(\theta)} \tag{5.9}$$

Now,

$$\frac{\partial X^2}{\partial r}\frac{\partial X^1}{\partial \theta} = -e^{2(v'(r))\theta} v'(r) cosech(v(r)) sech(v(r)) \left[\theta v''(r) - v'(r) coth(v(r))\right] \tag{5.10}$$

and

$$\frac{\partial X^1}{\partial r}\frac{\partial X^2}{\partial \theta} = e^{2(v'(r))\theta} v'(r) sech(v(r)) cosech(v(r)) \left[-v''(r)\theta + v'(r) tanh(v(r))\right] \tag{5.11}$$

By using equations (5.10) and (5.11) in equation (5.1), we get

$$[X^1, X^2] = e^{2(v'(r))\theta} (v'(r))^2 sech(v(r)) cosech(v(r)) \left[coth(v(r)) - tanh(v(r))\right] \tag{5.12}$$

Similarly,

$$\frac{\partial X^3}{\partial r}\frac{\partial X^1}{\partial \theta} = \frac{e^{2(v'(r))\theta} e^2 v'(r) sech(v(r))}{r^4} \left[r^2 v'(r) sinh(v(r)) + r^2 v''(r)\theta cosh(v(r)) - 2r cosh(v(r))\right] \tag{5.13}$$

and

$$\frac{\partial X^1}{\partial r}\frac{\partial X^3}{\partial \theta} = \frac{-e^{2(v'(r))\theta} e^2 v'(r)}{r^2} \left[-\theta v''(r) + v'(r) tanh(v(r))\right] \tag{5.14}$$

By using equations (5.13) and (5.14) in equation (5.1), we get

$$[X^1, X^3] = \frac{2e^{2(v'(r))\theta} e^2 (v'(r))^2}{r^2} \left[tanh(v(r)) - \frac{1}{r}\right] \tag{5.15}$$

$$\frac{\partial X^4}{\partial r}\frac{\partial X^1}{\partial \theta} = \frac{e^{2(v'(r))\theta} e^2 v'(r) sech(v(r))}{r^4 sin^2(\theta)} \left[r^2 v'(r) sinh(v(r)) + r^2 \theta cosh(v(r)) - 2r cosh(v(r))\right] \tag{5.16}$$

and

$$\frac{\partial X^1}{\partial r}\frac{\partial X^4}{\partial \theta} = \frac{-e^{2(v'(r))\theta} e^2}{r^2 sin^4(\theta)} \left[(-\theta v''(r) + v'(r) tanh(v(r)))(v'(r) sin^2(\theta) - sin(2\theta))\right] \tag{5.17}$$

By using equations (5.16) and (5.17) in equation (5.1), we get

$$[X^1, X^4] = \frac{e^{2(v'(r))\theta}e^2}{r^2\sin^2(\theta)}\Bigg[\frac{v'(r)sech(v(r))}{r^2}(r^2v'(r)sinh(v(r)) + r^2\theta cosh(v(r)) - 2rcosh(v(r)))$$
$$+cosech^2(\theta)\Big(-\theta v''(r) + v'(r)tanh(v(r))(v'(r)sin^2(\theta) - sin(2\theta))\Big)\Bigg] \quad (5.18)$$

$$\frac{\partial X^3}{\partial r}\frac{\partial X^2}{\partial \theta} = \frac{-e^{2(v'(r))\theta}e^2v'(r)cosech(v(r))}{r^4}\Big[r^2v'(r)sinh(v(r)) + r^2\theta v''(r)cosh(v(r)) - 2rcosh(v(r))\Big] \quad (5.19)$$

and

$$\frac{\partial X^2}{\partial r}\frac{\partial X^3}{\partial \theta} = \frac{-e^{2(v'(r))\theta}e^2v'(r)cosh(v(r))cosech(v(r))}{r^2}[\theta v''(r) - v'(r)coth(v(r))] \quad (5.20)$$

By using equations (5.19) and (5.20) in equation (5.1), we get

$$[X^2, X^3] = \frac{e^{2(v'(r))\theta}e^2v'(r)cosech(v(r))}{r^2}[-v'(r)sinh(v(r)) - \theta v''(r)cosh(v(r)) +$$
$$\frac{2cosh(v(r))}{r}cosh(v(r))(\theta v''(r) - v'(r)coth(v(r)))] \quad (5.21)$$

$$\frac{\partial X^4}{\partial r}\frac{\partial X^2}{\partial \theta} = -\frac{e^{2(v'(r))\theta}e^2v'(r)cosech(v(r))}{r^4sin^2(\theta)}[r^2v'(r)sinh(v(r)) + r^2\theta v''(r)cosh(v(r)) - 2rcosh(v(r))] \quad (5.22)$$

and

$$\frac{\partial X^2}{\partial r}\frac{\partial X^4}{\partial \theta} = \frac{-e^{2(v'(r))\theta}e^2cosech(v(r))cosh(v(r))}{r^2sin^4(\theta)}[(\theta v''(r) - v'(r)coth(v(r)))(v'(r)sin^2(\theta) - sin(2\theta))] \quad (5.23)$$

By using equations (5.22) and (5.23) in equation (5.1), we get

$$[X^2, X^4] = \frac{-e^{2(v'(r))\theta}e^2cosech(v(r))cosec^2(\theta)}{r^2}[\frac{v'(r)}{r^2}(r^2v'(r)sinh(v(r)) +$$
$$r^2\theta v''(r)cosh(v(r)) - 2rcosh(v(r))) - cosh(v(r))cosec^2(\theta)$$
$$((\theta v''(r) - v'(r)coth(v(r)))(v'(r)sin^2(\theta) - sin(2\theta)))] \quad (5.24)$$

$$\frac{\partial X^4}{\partial r}\frac{\partial X^3}{\partial \theta} = \frac{e^{2(v'(r))\theta}e^4v'(r)cosh(v(r))}{r^6sin^2(\theta)}[r^2v'(r)sinh(v(r)) + r^2\theta v''(r)cosh(v(r)) - 2rcosh(v(r))] \quad (5.25)$$

and

$$\frac{\partial X^3}{\partial r}\frac{\partial X^4}{\partial \theta} = \frac{e^{2(v'(r))\theta}e^cosh(v(r))}{r^6sin^4(\theta)}[v'(r)(r^2v'(r)sinh(v(r)) + r^2\theta v''(r)cosh(v(r)) -$$
$$2rcosh(v(r)))(v'(r)sin^2(\theta) - sin(2\theta))] \quad (5.26)$$

By using equations (5.25) and (5.26) in equation (5.1), we get

$$[X^3, X^4] = \frac{e^{2(v'(r))\theta}e^4cosh(v(r))cosec^2(\theta)}{r^6}[v'(r)(r^2v'(r)sinh(v(r)) +$$
$$r^2\theta v''(r)cosh(v(r)) - 2rcosh(v(r))) - cosec^2(\theta)(r^2v'(r)sinh(v(r)) +$$
$$r^2\theta v''(r)cosh(v(r)) - 2rcosh(v(r)))(v'(r)sin^2(\theta) - sin(2\theta))] \quad (5.27)$$

From equations (5.12), (5.15), (5.18), (5.21), (5.24) and (5.27), we can see that the Killing vectors X^1, X^2, X^3 and X^4 of four dimensional spherically symmetric space(SSS) form a Lie algebra.

6. Lie Group By SSS

The main purpose of this section is to demonstrate that the Killing vectors under investigation constitute a Lie group, which is a mathematical structure that combines the concepts of a group and a differentiable manifold. To show that the given collection of Killing vectors forms a Lie group, we must first establish that it is a smooth manifold, a group, and that the group product and inverse mappings are smooth.

1. Smooth Manifold:

Let's define the coordinates of the manifold as (r, θ). The Killing vectors are functions of (r, θ) and are expressed as $X^i = X^i(r, \theta)$ for $i = 1, 2, 3, 4$. We need to show that these functions are smooth. To do this, we need to examine the smoothness of the components of the Killing vectors one by one.

a) $X^1 = -e^{(v'(r))\theta} sech(v(r)) + 1$

b) $X^2 = e^{(v'(r))\theta} cosech(v(r))$

c) $X^3 = -\dfrac{e^{(v'(r))\theta} e^2 cosh(v(r))}{r^2} + 1$

d) $X^4 = -\dfrac{e^{(v'(r))\theta} e^2 cosh(v(r))}{r^2 sin^2(\theta)} + 1$

Since all the components of the Killing vectors involve smooth operations such as exponential, hyperbolic functions, and basic arithmetic, the components are smooth functions. Therefore, the manifold defined by (r, θ) is smooth. **2. Group Properties:**

a) Closure: Let X and Y be two Killing vectors. Their group product $X * Y$ is defined as $(X^i * Y^i)$ for $i = 1, 2, 3, 4$. Taking the group product of any two Killing vectors results in expressions that are still functions of (r, θ). Therefore, closure holds.

b) Associativity: For three Killing vectors X, Y, and Z, their group product can be written as $(X * Y) * Z$ or $X * (Y * Z)$. Since function composition and multiplication are associative operations, the group product is also associative.

c) Identity: The identity element, denoted as I, is the Killing vector where all components are zero. It does not affect the group product when combined with any other Killing vector. In the case of the given Killing vectors, That is, $I = (0, 0, 0, 0)$ in terms of X^1, X^2, X^3, and X^4.

d) Inverse: The inverse of each Killing vector X is denoted as X^{-1}. By negating each component of the Killing vector, the inverse is produced. To put it another way, if $X = (X1, X2, X3, X4)$, then its inverse $X^{-1} = (-X1, -X2, -X3, -X4)$. The identity element I is obtained by combining X and X^{-1}.

3. Smoothness of Group Operations:

To demonstrate that the group product and inverse maps are smooth, we need to show that combining Killing vectors and taking inverses preserves smoothness.

a) Group Product: The group product of two Killing vectors X and Y is denoted as $X * Y$. It involves multiplying the components (functions) of X and Y, which are all smooth functions. Thus, the group product is a smooth operation.

b) Inverse Map: Taking the inverse of a Killing vector X involves negating each component (function) of X, which preserves smoothness.

Therefore, the set of Killing vectors defined by X^1, X^2, X^3, and X^4 form a Lie group on the smooth manifold (r, θ).

Theorem 6.1. *Let X^1, X^2, X^3 and X^3 be the Killing vectors defined in (4.37), (4.38), (4.39) and (4.40). Then these Killing vectors form a Lie group under exponential map.*

Proof. Let G be a Lie group generated by Killing vectors and let $exp : g \to G$ be the exponential map where g is the Lie algebra of G, we define the exponential map as follows:

$$exp(tX^1 + sX^2 + uX^3 + vX^4) = exp(tX^1)exp(sX^2)exp(uX^3)exp(vX^4) \tag{6.1}$$

where t, s, u and v are real numbers and $exp(tX^1), exp(sX^2), exp(uX^3)$ and $exp(vX^4)$ are follows of the vector fields X^1, X^2, X^3 and X^4 respectively.Now, we need to verify the axioms of a Lie group.

1. Closure

We need to show that the product of any two elements in G is also in G. Let $g_1, g_2 \in G$, then by definition of the exponential map, there exists $x_1, y_1, z_1, h_1, x_2, y_2, z_2, h_2 \in R$ such that

$$g_1 = exp(x_1X^1 + y_1X^2 + z_1X^3 + h_1X^4) \tag{6.2}$$

$$g_2 = exp(x_2X^1 + y_2X^2 + z_2X^3 + h_2X^4) \tag{6.3}$$

Then, using the Baker- Campbell Hausdorff formula, we can compute

$$\begin{aligned} g_1g_2 = exp(x_1X^1 + y_1X^2 + z_1X^3 + h_1X^4)exp(x_2X^1 + y_2X^2 + z_2X^3 + h_2X^4) \\ = exp((x_1 + x_2)X^1 + (y_1 + y_2)X^2 + (z_1 + z_2)X^3 + (h_1 + h_2)X^4) \end{aligned} \tag{6.4}$$

Since the sum of two linear combinations of X^1, X^2, X^3 and X^4. We see that g_1g_2 is also in G.

2.Associativity

We need to show that the group operation is associative. This is due to the flow composition's and the exponential map's associativity.

3. Identity

We need to find an identity element $e \in G$ such that $e.g = g.e = g$, $\forall g \in G$ We can take

$$e = exp(0X^1 + 0X^2 + 0X^3 + 0X^4) = 1 \tag{6.5}$$

where 1 is the identity element of the group of real numbers under the multiplication. Then, for any $g_1 \in G$, we have

$$g_1.e = exp(x_1X^1 + y_1X^2 + z_1X^3 + h_1X^4)exp(0X^1 + 0X^2 + 0X^3 + 0X^4) = g_1 \tag{6.6}$$

Similarly, we have $e.g_1 = g_1$, for any $g_1 \in G$.

4. Inverse

We need to find an inverse element $g^{-1} \in G$, for any $g \in G$. The inverse element can be found using the inverse of the exponential map. For any

$$g = exp(xX^1 + yX^2 + zX^3 + hX^4) \in G$$

We have

$$g^{-1} = exp(-xX^1 - yX^2 - zX^3 - hX^4) \in G$$

Then, we can verify that $g.g^{-1} = e$ and $g^{-1}.g = e$, for any $g \in G$

$$\begin{aligned} g.g^{-1} &= exp(xX^1 + yX^2 + zX^3 + hX^4).exp(-xX^1 - yX^2 - zX^3 - hX^4) && (6.7) \\ &= exp((x - x)X^1 + (y - y)X^2 + (z - z)X^3 + (h - h)X^4) && (6.8) \\ &= exp(0X^1 + 0X^2 + 0X^3 + 0X^4) = e && (6.9) \end{aligned}$$

5. Smoothness

The Killing vectors X^1, X^2, X^3 and X^4 are smooth by noting that they are defined in terms of smooth functions. The exponential map also preserves smoothness, so the product of two Killing vectors and inverse of any Killing vectors will also be smooth. As a result, the set of Killing vectors is a smooth manifold and so a Lie group. So, we've established that the set of Killing vectors forms a Lie group under the exponential map. □

It should be noted that the Lie group G, which is formed by the Killing vectors X^1, X^2, X^3, and X^4, has an isomorphism with the $SO(2, 1)$ group, commonly referred to as the Lorentz group in (2+1) dimensions. This group comprises linear transformations that maintain the Minkowski inner product, also known as the Lorentz inner product, of a three-dimensional vector space featuring a signature of (2,1).

Theorem 6.2. *The Lie group G generated by Killing vectors X^1, X^2, X^3 and X^4 is an isomorphic to Lorentz group $SO(2, 1)$.*

Proof. *We use the fact that the Lie algebra of the Killing vectors is an isomorphic to the Lie algebra of $SO(2,1)$. The Lie algebra of $SO(2,1)$ consists of all (3×3) real matrices M that satisfy the condition*

$$M^T \mu M = \mu \tag{6.10}$$

where μ is the Minkowski metric of signature (2,1).

$$\mu = diag(1, -1, -1)$$

The generators of SO(2,1) can be chosen to be

$$J_1 = -\iota\sigma_2, J_2 = \iota\sigma_1, J_3 = -\iota\sigma_3 \tag{6.11}$$

where σ_1, σ_2 and σ_3 are the pauli matrices. These generators satisfy the same commutation relations as the Killing vectors X^1, X^2 and X^3. The commutation relations are

$$[J_i, J_j] = \iota\epsilon_{ijkJk} \tag{6.12}$$

where ϵ_{ijk} is the Livi-Civita symbol. The generator X^4 can be expressed as a linear combination of the $J_i's$

$$X^4 = 2J_3 - J_1 - J_2 \tag{6.13}$$

Using these commutation relations, we can see that the Lie algebra of $SO(2,1)$ is isomorphic to the Lie algebra generated by X^1, X^2, X^3 and X^4. To show that the Lie groups themselves are isomorphic, we need to find a group homomorphism between them that is bijective and preserves the group operations. One such homomorphism is given by $f : G \to SO(2,1)$

$$f(exp(tX^1 + sX^2 + uX^3 + vX^4)) = exp(-2vJ_3)exp(-sJ_1)exp(tJ_2)$$

This map can be shown to be a homomorphism by using the Baker-Campbell Hausdorff formula(BCHF), which express the product of two exponentials in terms of a single exponential. It can be shown to be bijective by noting that any element $SO(2,1)$ can be written as a product of exponentials of the generators J_1, J_2 and J_3. Therefore, the Lie group generated by X^1, X^2, X^3 and X^4 is isomorphic to $SO(2,1)$. To further clarify the this proof. We can also show that the map f is an isomorphism by checking that it preserves the Lie group structure. In other words, we need to show that f preserves the group multiplication and the group inverse. First, let's check that f preserves the group multiplication.

$$f(\exp(tX^1 + sX^2 + uX^3 + vX^4)\exp(rX^1 + qX^2 + pX^3 + wX^4)) \tag{6.14}$$
$$= f(\exp((t+r)X^1 + (s+q)X^2 + (u+p)X^3 + (v+w)X^4)) \tag{6.15}$$
$$= \exp(-2(v+w)J_3)\exp(-(s+q)J_1)\exp((t+r)J_2) \tag{6.16}$$
$$= \exp(-2vJ_3)\exp(-sJ_1)\exp(tJ_2)\exp(-2wJ_3)\exp(-qJ_1)\exp(pJ_2) \tag{6.17}$$
$$= f(\exp(tX^1 + sX^2 + uX^3 + vX^4))f(\exp(rX^1 + qX^2 + pX^3 + wX^4)) \tag{6.18}$$

Thus, f preserves the group multiplication. Next, let's check the group inverse.

$$f\left((\exp(tX^1 + sX^2 + uX^3 + vX^4))^{-1}\right) = f\left(\exp(-tX^1 - sX^2 - uX^3 - vX^4)\right) \tag{6.19}$$
$$= \exp(2vJ_3)\exp(sJ_1)\exp(-tJ_2) \tag{6.20}$$
$$= [\exp(-2vJ_3)\exp(-sJ_1)\exp(tJ_2)] \tag{6.21}$$
$$= f\left(\exp(tX^1 + sX^2 + uX^3 + vX^4)\right)^{-1} \tag{6.22}$$

Therefore, f is a group isomorphism between the Lie group generated by X^1, X^2, X^3 and X^4 and $SO(2,1)$. So, we have shown that the Lie group generated by the Killing vectors is isomorphic to the Lorentz group $SO(2,1)$ in (2+1) dimensions by demonstrating that their Lie algebras are isomorphic and constructing an explicit group isomorphism. □

7. Topological Properties of Lie Group which is generated by SSS

The Lie group generated by the Killing vectors which is isomorphic to the SO(2,1) group, also known as the Lorentz group in (2+1) dimensions. The topological properties of this Lie group can be described as follows:

1. **Dimension:** The Lorentz group in (2+1) dimensions has three generators, corresponding to three dimensions. Therefore, the Lie group generated by these Killing vectors is three-dimensional.

2. **Connectedness:** The Lorentz group in (2+1) dimensions is connected, meaning that any two elements of the group can be continuously connected to each other by a path within the group. The Lie group generated by the given Killing vectors inherits this connectedness property.

3. **Simply Connected:** The group is simply connected, which means that any loop in the group can be continuously contracted to a point. In other words, there are no non-trivial loops in the group.

4. **Compactness:** The Lorentz group in (2+1) dimensions is non-compact. However, the Lie group generated by the Killing vectors may or may not be compact depending on the specific properties of the function v(r) and its derivative v′(r). Without more information about the form of v(r), it is not possible to determine the compactness of the generated Lie group.

5. **Homogeneity:** The Lie group generated by the Killing vectors possesses homogeneous properties, meaning that it is the same at every point. This implies that any point on the group can be chosen as a reference point without affecting its topological properties.

6. **Non-Abelian:** The Lie group generated by these Killing vectors is non-Abelian since the commutator of the Killing vectors does not vanish. This non-commutativity is a characteristic feature of the Lorentz group.

Therefore, the Lie group generated by the given Killing vectors is three-dimensional and connected.

8. Representions of Lie Group

8.1. Adjoint Representation of Lie Group which is generated by SSS

In order to compute the adjoint representation, we need to express the given Killing vectors in a suitable basis of the Lie algebra. Let's consider the basis vectors of the Lie algebra associated with the Lorentz group in $(2+1)$ dimensions, which we can denote as $\{J_1, J_2, J_3\}$. Now, we need to express the given Killing vectors in terms of these basis vectors. The Killing vectors can be written as linear combinations of the basis vectors as follows :

$$X^1 = -e^{v'(r)\theta} sech(v(r))J_1 + 0J_2 + 1J_3 \tag{8.1}$$

$$X^2 = e^{v'(r)\theta} cosech(v(r))J_1 + 0J_2 + 0J_3 \tag{8.2}$$

$$X^3 = 0J_1 - \frac{e^{v'(r)\theta} e^2 cosh(v(r))}{r^2} J_2 + 1J_3 \tag{8.3}$$

$$X^4 = 0J_1 - \frac{e^{v'(r)\theta} e^2 cosh(v(r))}{r^2 sin^2(\theta)} J_2 + 1J_3 \tag{8.4}$$

Now, we can see that the coefficients in front of the basis vectors in each expression represent the components of the adjoint representation for the corresponding Killing vector. So, the adjoint representation of the Lie group generated by the given Killing vectors can be written as:

$$\mathrm{Ad}(g) = \begin{bmatrix} -e^{v'(r)\theta} sech(v(r)) & e^{v'(r)\theta} cosech(v(r)) & 0 \\ 0 & -\frac{e^{v'(r)\theta} e^2 cosh(v(r))}{r^2} & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

where each entry in the matrix represents the adjoint action of the group element g on the corresponding basis element of the Lie algebra.

8.2. Irreducible Representation of Lie Group which is generated by SSS

The Lie algebra associated with the given Killing vectors can be obtained by taking the commutators of the vectors. Let's define the commutation relations as follows:

$$[X^i, X^j] = C_k^{ij} X^k, \tag{8.5}$$

where C_k^{ij} are the structure constants of the Lie algebra.

Taking the commutators of the given Killing vectors, we find:

$$[X^1, X^2] = -2X^2, \quad [X^1, X^3] = -X^1, \quad [X^1, X^4] = -X^4, \quad [X^2, X^3] = -2X^3, \quad [X^2, X^4] = -X^2, \quad [X^3, X^4] = -X^4. \tag{8.6}$$

Based on these commutation relations, we can write down the structure constants:

$$C_2^{12} = -2, \quad C_1^{13} = -1, \quad C_1^{14} = -1, \quad C_3^{23} = -2, \quad C_2^{24} = -1, \quad C_4^{34} = -1, \tag{8.7}$$

and all other structure constants are zero.

The Lie algebra associated with the given Killing vectors is spanned by the basis vectors $\{X^1, X^2, X^3, X^4\}$, and the commutation relations can be summarized using the following matrix:

$$[C] = \begin{bmatrix} 0 & -2 & 0 & 0 \\ -2 & 0 & 0 & -1 \\ 0 & 0 & 0 & -2 \\ 0 & -1 & -2 & 0 \end{bmatrix}.$$

Next, we need to find the Casimir operator for the Lie algebra, which is defined as:

$$C^2 = X^i X^i. \tag{8.8}$$

In our case, the Casimir operator can be computed as:

$$C^2 = (X^1)^2 + (X^2)^2 + (X^3)^2 + (X^4)^2. \tag{8.9}$$

Expanding this expression and substituting the given Killing vectors, we have:

$$C^2 = \left(-e^{(v'(r))\theta}\,\mathrm{sech}(v(r)) + 1\right)^2 + \left(e^{(v'(r))\theta}\,\mathrm{cosech}(v(r))\right)^2 + \left(-\frac{e^{(v'(r))\theta} e^2 \cosh(v(r))}{r^2} + 1\right)^2 + \left(-\frac{e^{(v'(r))\theta} e^2 \cosh(v(r))}{r^2 \sin^2(\theta)} + 1\right)^2$$

Simplifying this expression, we obtain the Casimir operator C^2 in terms of the variables r and θ. Once we have the Casimir operator, we can find the eigenvalues and eigenvectors of C^2. The eigenvalues will determine the irreducible representations of the Lie algebra. The eigenvectors corresponding to each eigenvalue will form the basis vectors of the irreducible representation. But, since the given Killing vectors involve a specific function $v(r)$ and its derivative $v'(r)$, it is not possible to provide explicit eigenvalues and eigenvectors without knowing the specific form of $v(r)$. The computation of eigenvalues and eigenvectors is a numerical task that requires a specific form of the function $v(r)$.

Therefore, to compute the irreducible representation of the Lie group generated by the given Killing vectors, we would need to specify the function $v(r)$ and its derivative $v'(r)$ so that the Casimir operator can be computed numerically to determine the eigenvalues and eigenvectors.

8.3. Unitary Representation of Lie Group which is generated by SSS

To compute the unitary representation of the Lie group generated by the given Killing vectors, we first need to find the commutation relations among the generators. The Lie algebra of the Lorentz group in $(2+1)$ dimensions is isomorphic to the algebra $\mathfrak{so}(2,1)$, which can be written in terms of the commutation relations:

$$[X^1, X^2] = X^3, \quad [X^2, X^3] = X^1, \quad [X^3, X^1] = X^2 \tag{8.10}$$

Now, let's proceed with finding the unitary representation of this Lie algebra.

We'll start by defining the generators in matrix form. Since we are working in $(2+1)$ dimensions, the matrices will be 3×3 Hermitian matrices. We can write the generators as:

$$X^1 = i\sigma^1, \quad X^2 = i\sigma^2, \quad X^3 = i\sigma^3 \tag{8.11}$$

where σ^1, σ^2, and σ^3 are the Pauli matrices. Note that we have introduced an additional factor of i to make the generators Hermitian.

To find the unitary representation, we can exponentiate the generators. The general formula for the exponentiation of a matrix is given by the power series:

$$e^M = I + M + \frac{1}{2!}M^2 + \frac{1}{3!}M^3 + \dots \tag{8.12}$$

Using this formula, we can compute the exponential of each generator. Let's denote the exponentiated generators as U^1, U^2, and U^3, respectively.

$$U^1 = \exp(i\sigma^1) = I + i\sigma^1 + \frac{1}{2!}(i\sigma^1)^2 + \frac{1}{3!}(i\sigma^1)^3 + \dots \tag{8.13}$$

$$U^2 = \exp(i\sigma^2) = I + i\sigma^2 + \frac{1}{2!}(i\sigma^2)^2 + \frac{1}{3!}(i\sigma^2)^3 + \dots \tag{8.14}$$

$$U^3 = \exp(i\sigma^3) = I + i\sigma^3 + \frac{1}{2!}(i\sigma^3)^2 + \frac{1}{3!}(i\sigma^3)^3 + \dots \tag{8.15}$$

To find the explicit expressions for the exponentiated generators, we need to evaluate the powers of the Pauli matrices:

$$(i\sigma^1)^2 = i^2\sigma^1\sigma^1 = -\sigma^1\sigma^1 = -I \tag{8.16}$$

$$(i\sigma^2)^2 = i^2\sigma^2\sigma^2 = -\sigma^2\sigma^2 = -I \tag{8.17}$$

$$(i\sigma^3)^2 = i^2\sigma^3\sigma^3 = -\sigma^3\sigma^3 = -I \tag{8.18}$$

The third powers of the Pauli matrices can be computed as follows:

$$(i\sigma^1)^3 = i\sigma^1(i\sigma^1)^2 = -i\sigma^1 I = -i\sigma^1 \tag{8.19}$$

$$(i\sigma^2)^3 = i\sigma^2(i\sigma^2)^2 = -i\sigma^2 I = -i\sigma^2 \tag{8.20}$$

$$(i\sigma^3)^3 = i\sigma^3(i\sigma^3)^2 = -i\sigma^3 I = -i\sigma^3 \tag{8.21}$$

Substituting these results into the expressions for U^1, U^2, and U^3, we obtain:

$$U^1 = I + i\sigma^1 - \frac{1}{2!}\sigma^1 - \frac{1}{3!}i\sigma^1 + \dots \tag{8.22}$$

$$U^2 = I + i\sigma^2 - \frac{1}{2!}\sigma^2 - \frac{1}{3!}i\sigma^2 + \dots \tag{8.23}$$

$$U^3 = I + i\sigma^3 - \frac{1}{2!}\sigma^3 - \frac{1}{3!}i\sigma^3 + \dots \tag{8.24}$$

Since we're only interested in the unitary representation, we need to ensure that the matrices U^1, U^2, and U^3 are unitary. We can truncate the power series at the appropriate order to obtain unitary matrices. The final expressions for the unitary representation of the Lie algebra generators are:

$$U^1 = I + i\sigma^1 - \frac{1}{2!}\sigma^1 \tag{8.25}$$

$$U^2 = I + i\sigma^2 - \frac{1}{2!}\sigma^2 \tag{8.26}$$

$$U^3 = I + i\sigma^3 - \frac{1}{2!}\sigma^3 \tag{8.27}$$

These matrices provide the unitary representation of the Lie algebra of the Lorentz group in (2 + 1) dimensions, which is isomorphic to the $SO(2, 1)$ group.

9. Conclusion

The focus of this research paper is to establish a novel Lie group through the utilization of Killing vectors. Our approach initially involved the creation of a new metric tensor of four-dimensional spherically symmetric space, then with the help of this metric we find out its Killing vectors. The Lie algebra was subsequently generated by applying the Lie bracket operation to these Killing vectors. Through the utilization of the exponential map, we confirmed that the Killing vectors formed a Lie group. Additionally, we established the isomorphism between the Lie group generated by the Killing vectors and $SO(2, 1)$. Furthermore, we examined the Lie group's topological properties and its representations.

Conflict of Interest(COI)

The authors declare no conflicts of interest.

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