Filomat 40:10 (2026), 3619–3633
https://doi.org/10.2298/FIL2610619S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

The pseudo S-spectrum and the numerical ranges in a right quaternionic Hilbert space

Bilel Saadaoui[a]

[a]*Institut supérieur des sciences sociales et de l'éducation de Gafsa, University of Gafsa, 2119, Route de Tozeur, Gafsa- Tunisia*

Abstract. We applied our work to finite-dimensional approximations of an operator in an infinite-dimensional right quaternionic Hilbert space, studying the convergence of various quaternionic approximations. Our results demonstrate that the pseudo S-spectrum of the linear operator is influenced by the intersection of sets expressed in quaternionic numerical range terms.

1. Introduction

The rigorous proof of the spectral theorem for quaternionic linear operators has remained an open challenge since as early as 1936, when Birkhoff and von Neumann demonstrated that quantum mechanics could be described using real, complex, and quaternionic numbers. The primary difficulty stemmed from the lack of a precise definition for the quaternionic spectrum of a linear operator. It was not until 2006 that F. Colombo and I. Sabadini introduced the concept of the S-spectrum, which paved the way for a comprehensive development of quaternionic operator theory. Subsequently, the spectral theorem based on the S-spectrum required several additional years, and in 2016, D. Alpay, F. Colombo, and D.P. Kimsey provided a complete proof of this fundamental theorem for both bounded and unbounded operators (see [1]). The development of the S-functional calculus, particularly in relation to spectral theory on the S-spectrum, began in earnest in 2006. The discovery of the S-functional calculus and the S-spectrum is comprehensively elucidated in [11], along with a comprehensive list of references. The text also clarifies how hypercomplex analysis techniques were instrumental in defining the concept of quaternionic spectrum, a concept crucial for the existence of quaternionic quantum mechanics.

The spectral theory concerning the S-spectrum, explored across various papers, has been systematically organized with historical insights in the books [11] and [12]. For a detailed discussion on the spectral theory regarding the S-spectrum for Clifford operators, the book [1] is recommended.

Recently, as evidenced in [10], the spectral theorem based on the S-spectrum for quaternionic normal operators has been established. These developments have significantly spurred investigations into the perturbation theory of quaternionic linear operators, a domain that has been extensively studied in classical complex analysis, as seen in [9].

2020 *Mathematics Subject Classification*. Primary 47A30.
Keywords. numerical ranges, quaternionic Hilbert space, pseudo S-spectra.
Received: 25 April 2024; Revised: 06 December 2025; Accepted: 10 February 2026
Communicated by Snežana Č. Živković-Zlatanović
Email address: saadaoui.bilel@hotmail.fr (Bilel Saadaoui)
ORCID iD: https://orcid.org/0000-0002-9037-9282 (Bilel Saadaoui)

The concept of holomorphicity in vector fields is closely linked with quaternionic-valued functions and more broadly with Clifford algebra-valued functions. The Fueter-Sce-Qian theorem, also known as the Fueter-Sce-Qian construction, provides two distinct theories of hyperholomorphic functions, as elaborated in [13]. The theory of holomorphic functions plays a central role in operator theory. The Cauchy integral formula plays a crucial role in establishing the holomorphic functional calculus in Banach spaces, as discussed in [10]. Additionally, the presence of S-resolvent operators is instrumental in studying quaternionic Schur applications, which have seen rapid development in Schur analysis within the framework of hyperholomorphic settings, as detailed in the book [3]. The quaternionic spectral theorem for quaternionic operators was finally demonstrated in [3], with a separate treatment for unitary operators provided in [2].

In the seminal paper [14], F. Colombo, J. Gantner, and S. Pinton addressed the perturbation of quaternionic operators using the concept of the S-spectrum and the hyperholomorphicity of S-resolvent operators.

The spectral theorem for quaternionic Hilbert spaces, as discussed in [28], lacks a systematic realization of the continuous functional calculus. Recently, spectral theory has been employed in quantum theories through functional calculus, as seen in [5, 8]. However, various challenges arise with quaternionic linear operators due to their non-commutativity with quaternions, hindering the straightforward application of results derived in the complex set to the quaternionic space.

The issue of defining the spectrum has been a longstanding challenge. Many authors have attempted to prove the quaternionic spectral theorem over the past decades, as documented in works like [28]. However, the concept of spectrum remained elusive. The difficulty lies in adapting the classical notion of spectrum to quaternionic linear operators, which leads to ambiguity. In Bilel Saadaoui work [6, 7, 23, 24], the concept of pseudospectrum was proposed as a potential solution, offering insights into the essential pseudospectra of closed linear relations.

Due to non-commutativity in quaternionic spaces, there exist three types of Hilbert spaces: two-sided, left, and right, depending on how scalar multiplication interacts with vectors. However, this concept can lead to various issues. For instance, in a one-sided quaternionic Hilbert space, when approaching a linear operator A and a quaternion $\mathbf{q} \in \mathbb{H}$, we observe that the conjugate of $\mathbf{q}$ times A is not equal to the adjoint of $\mathbf{q}A$.

To address this, a notion of multiplication can be introduced on both sides using a fixed arbitrary Hilbert basis of $\mathbb{H}$. This approach allows us to achieve a linear structure in combining linear operators, thus establishing a coherent theory. Consequently, this article focuses on a right quaternionic Hilbert space with left multiplication, achieved through the establishment of a Hilbert basis. We aim to contribute to the operator theory of Fredholm and explore the essential pseudospectrum, an aspect yet to be thoroughly studied in the context of quaternionic spaces. Our investigation begins with fundamental questions concerning the concept of essential pseudospectrum of an operator on a quaternionic Hilbert space.

In a Hilbert space $\mathbb{H}$, a bounded linear operator A in the complex setting fails to be invertible if it lacks lower boundedness. The set of approximate eigenvalues, denoted by $\lambda \in \mathbb{C}$ and defined as $A - \lambda\mathbb{I}_{\mathfrak{h}}$, where $\mathbb{I}_{\mathfrak{h}}$ is the identity operator on $\mathbb{H}$, is not lower bounded. Similarly, the set of $\mathbb{C}$ for which there exists a sequence of unit vectors $\lambda \in \mathbb{C}$ such that $\lim_{n\to\infty} \|A\phi_n - \lambda\phi_n\| = 0$, is also unbounded. The collection of approximate eigenvalues is famous as the approximate spectrum. Let $V_{\mathbb{H}}^{R}$ be a separable right Hilbert space, A be a bounded right linear operator, and $R_{\mathbf{q}}(A) = A^2 - 2Re(\mathbf{q})A + |\mathbf{q}|^2\mathbb{I}_{\mathcal{V}_{\mathbb{H}^{\mathcal{R}}}}$, with $\mathbf{q} \in \mathbb{H}$, in proposition 4.5 [16] the collection of commonalty quaternions, the collection of right eigenvalues of $R_{\mathbf{q}}(A)$ correspond with the point S-spectrum. For the first time, it is appropriate to examine and define the quaternionic approximate S-point spectrum as the quaternions for which $R_{\mathbf{q}}(A)$ is not lower bounded. This concept aligns with the introduction of pseudospectrum in the context of linear operators by Varah in [27]. In recent years, Wilkinson's interpretation [29] extended it to a spot matrix norm derived from a vector norm. Many researchers, including those in [15], have explored the concept of approximation pseudospectrum for linear operators. Particularly noteworthy is Trefethen's contribution [26], who expanded this idea to matrices and operators, applying it to investigate intriguing problems in mathematical physics. The pseudo S-spectrum of a right linear operator A, denoted by $\sigma_{\varepsilon}^{S}(A)$ for every $\varepsilon > 0$, is defined as follows:

$$\sigma_{\varepsilon}(A) := \sigma^{S}(A) \bigcup \left\{\mathbf{q} \in \mathbb{H} : \|R_{\mathbf{q}}(A)^{-1}\| > \frac{1}{\varepsilon}\right\},$$

where $\varepsilon > 0$ and $\sigma^S(A)$ is the spectrum of a right linear operator A. By agreement we record $\|R_{\mathbf{q}}(A)^{-1}\| = \infty$ if $R_{\mathbf{q}}(A)^{-1}$ is unbounded or nonexistent, i.e., if $\mathbf{q}$ is in $\sigma^S(A)$.

Quaternions: Let $\mathbb{H}$ denote the field of all quaternions and $\mathbb{H}^*$ the group (under quaternionic multiplication) of all invertible quaternions. A general quaternion can be written as

$$\mathbf{q} = \mathbf{q}_0 + \mathbf{q}_1 i + \mathbf{q}_2 j + \mathbf{q}_3 k, \qquad \mathbf{q}_0, \mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3 \in \mathbb{R}$$

where i, j, k are the three quaternionic imaginary units, satisfying $i^2 = j^2 = k^2 = -1$ and $ij = k = -ji$, $jk = i = -kj$, $ki = j = -ik$. The quaternionic conjugate of $\mathbf{q}$ is

$$\overline{\mathbf{q}} = \mathbf{q}_0 - \mathbf{q}_1 i - \mathbf{q}_2 j - \mathbf{q}_3 k,$$

while $|q| = (q\overline{q})^{1/2}$ denotes the usual norm of the quaternion q. If q is non-zero element, it has inverse $q^{-1} = \frac{\overline{q}}{|q|^2}$. Finally, the set

$$\mathbb{S} = \left\{ I = x_1 i + x_2 j + x_3 k : x_1, x_2, x_3 \in \mathbb{R} \text{ and } x_1^2 + x_2^2 + x_3^2 = 1 \right\}$$

contains all the elements whose square is -1. It is a 2-dimensional sphere in $\mathbb{H}$ identified with $\mathbb{R}^4$.

Quaternionic Hilbert spaces: In this section, we delve into the realm of right quaternionic Hilbert spaces. For a deeper understanding, we recommend consulting [28].
Right quaternionic Hilbert Space: Let $V_{\mathbb{H}}^R$ be a vector space under right multiplication by quaternions. For $\phi, \psi, \omega \in V_{\mathbb{H}}^R$ and $\mathbf{q} \in \mathbb{H}$, the inner product

$$\langle \cdot \mid \cdot \rangle_{V_{\mathbb{H}}^R} : V_{\mathbb{H}}^R \times V_{\mathbb{H}}^R \longrightarrow \mathbb{H}$$

satisfies the following properties

1. $\overline{\langle \phi \mid \psi \rangle_{V_{\mathbb{H}}^R}} = \langle \psi \mid \phi \rangle_{V_{\mathbb{H}}^R}$
2. $\|\phi\|^2_{V_{\mathbb{H}}^R} = \langle \phi \mid \phi \rangle_{V_{\mathbb{H}}^R} > 0$ unless $\phi = 0$, a real norm
3. $\langle \phi \mid \psi + \omega \rangle_{V_{\mathbb{H}}^R} = \langle \phi \mid \psi \rangle_{V_{\mathbb{H}}^R} + \langle \phi \mid \omega \rangle_{V_{\mathbb{H}}^R}$
4. $\langle \phi \mid \psi \mathbf{q} \rangle_{V_{\mathbb{H}}^R} = \langle \phi \mid \psi \rangle_{V_{\mathbb{H}}^R} \mathbf{q}$
5. $\langle \phi \mathbf{q} \mid \psi \rangle_{V_{\mathbb{H}}^R} = \overline{\mathbf{q}} \langle \phi \mid \psi \rangle_{V_{\mathbb{H}}^R}$, where $\overline{\mathbf{q}}$ stands for the quaternionic conjugate.

We always assume that the space $V_{\mathbb{H}}^R$ is complete and separable under the given norm. This, along with $\langle \cdot \mid \cdot \rangle V_{\mathbb{H}}^R$, constitutes a right quaternionic Hilbert space. Quaternionic Hilbert spaces exhibit many standard properties akin to complex Hilbert spaces.
A mapping $A : \mathcal{D}(A) \subseteq V_{\mathbb{H}}^R \longrightarrow \mathbb{H}$is right $\mathbb{H}$-linear if, $f(u + v) = f(u) + f(v)$ and $f(u\mathbf{q}) = f(u)\mathbf{q}$ for every $u, v \in V_{\mathbb{H}}^R$ and $\mathbf{q} \in \mathbb{H}$. Let $V_{\mathbb{H}}^R$ be the topological dual space of $V_{\mathbb{H}}^R$ consisting of all continuous right $\mathbb{H}$-linea functions on $V_{\mathbb{H}}^R$.
The upcoming result follows directly from the quaternionic Hahn-Banach Theorem, which is discussed in detail in [21, Section 2.10] and [21, Section 4.10.1].

Proposition 1.1. [21, Proposition 2.1.] *Let $O = \{\varphi_k : k \in \mathbb{N}\}$ be an orthonormal subset of $V_{\mathbb{H}}^R$. As a result, the following conditions are mutually equivalent:*

1. *The closure of the linear combinations of elements from O using right-sided coefficients is $V_{\mathbb{H}}^R$.*
2. *For every $\phi, \psi \in V_{\mathbb{H}}^R$, the series $\sum_{k \in \mathbb{N}} \langle \phi | \psi \rangle_{V_{\mathbb{H}}^R} \langle \varphi_k | \psi \rangle_{V_{\mathbb{H}}^R}$ converges absolutely and it holds:*

$$\langle \phi | \varphi_k \rangle_{V_{\mathbb{H}}^R} = \sum_{k \in \mathbb{N}} \langle \phi | \varphi_k \rangle_{V_{\mathbb{H}}^R} \langle \varphi_k | \psi \rangle_{V_{\mathbb{H}}^R}.$$

3. *For every $\phi \in V_{\mathbb{H}}^R$, it holds:*

$$\|\phi\|^2_{V_{\mathbb{H}}^R} = \sum_{k\in\mathbb{N}} |\langle \varphi_k|\phi\rangle_{V_{\mathbb{H}}^R}|^2.$$

4. $O^{\perp} = \{0\}$.

Definition 1.2. *If the set O satisfies the equivalent conditions outlined in Proposition 1.1, it is termed a Hilbert basis of $V_{\mathbb{H}}^R$.*

Remark 1.3. *Every quaternionic Hilbert space $V_{\mathbb{H}}^R$ possesses a Hilbert basis, and all Hilbert bases of $V_{\mathbb{H}}^R$ have the same cardinality. Moreover, if O serves as a Hilbert basis for $V_{\mathbb{H}}^R$, then every $\phi \in V_{\mathbb{H}}^R$ can be decomposed uniquely as follows:*

$$\phi = \sum_{k\in\mathbb{N}} \varphi_k \langle \varphi_k|\phi\rangle_{V_{\mathbb{H}}^R},$$

where the series $\sum_{k\in\mathbb{N}} \varphi_k \langle \varphi_k|\phi\rangle_{V_{\mathbb{H}}^R}$ converges absolutely in $V_{\mathbb{H}}^R$.
The quaternion field $\mathbb{H}$ can be transformed into a left quaternionic Hilbert space by introducing the inner product $\langle \mathbf{q}, \mathbf{q}'\rangle = \mathbf{q}\overline{\mathbf{q}'}$ or into a right quaternionic Hilbert space with $\langle \mathbf{q}, \mathbf{q}'\rangle = \overline{\mathbf{q}}\mathbf{q}'$.

The article is organized as follows: the second section covers essential details about quaternionic Hilbert spaces and right linear operators. In section 3, we introduce the concept of quaternionic numerical ranges for the shifted inverse of a quaternionic linear operator, aiming to establish an enclosure for its pseudo S-spectrum. Section 4 presents a pseudo S-spectrum enclosure for the entire operator A, based on quaternionic numerical ranges of its approximating quaternionic counterpart A_n. This approach facilitates the computation of the enclosure using numerical methods. Finally, the last section introduces additional assumptions regarding the quaternionic approximations A_n of the operator A, enabling us to estimate the initial index n_0 for which the pseudo S-spectrum enclosures hold.

2. Preliminary results

2.1. Right quaternionic linear operators and some basic properties

In this subsection, we will define right-linear operators and review some fundamental properties, many of which are widely recognized.

Definition 2.1. *Consider $V_{\mathbb{H}}^R$ as a right quaternionic Hilbert space. A right $\mathbb{H}$-linear operator, which we'll refer to as a right linear operator for simplicity, is a mapping $A : \mathcal{D}(A) \subseteq V_{\mathbb{H}}^R \longrightarrow V_{\mathbb{H}}^R$ such that*

$$A(\phi\mathbf{a} + \psi\mathbf{b}) = (A\phi)\mathbf{a} + (A\psi)\mathbf{b} \ \text{ and } \ \mathbf{a}, \mathbf{b} \in \mathbb{H}$$

where the domain $\mathcal{D}(A)$ of A is a right $\mathbb{H}$-linear subspace of $V_{\mathbb{H}}^R$.

The collection of all right linear operators from $V_{\mathbb{H}}^R$ to $U_{\mathbb{H}}^R$ will be denoted by $\mathcal{L}(V_{\mathbb{H}}^R, U_{\mathbb{H}}^R)$ and the identity linear operator on $V_{\mathbb{H}}^R$ will be denoted by $\mathbb{I}_{\mathbb{H}}^R$. For a given $A \in \mathcal{L}(V_{\mathbb{H}}^R, U_{\mathbb{H}}^R)$, The image and kernel will be

$$\begin{aligned} R(A) &= \left\{\psi \in U_{\mathbb{H}}^R : A\phi = \psi \text{ for } \phi \in \mathcal{D}(A)\right\} \\ N(A) &= \left\{\phi \in \mathcal{D}(A) : A\phi = 0\right\}. \end{aligned}$$

We call an operator $A \in \mathcal{B}(V_{\mathbb{H}}^R, U_{\mathbb{H}}^R)$ bounded if

$$\|A\| = \sup_{\|\phi\|_{V_{\mathbb{H}}^R}=1} \|A\phi\|_{U_{\mathbb{H}}^R} < \infty,$$

or equivalently, there exist $K \geq 0$ such that $||A\phi||_{U^R_{\mathbb{H}}} \leq K||\phi||_{V^R_{\mathbb{H}}}$ for all $\phi \in \mathcal{D}(A)$. The set of all bounded right linear operators from $V^R_{\mathbb{H}}$ to $U^R_{\mathbb{H}}$ will be denoted by $\mathcal{B}(V^R_{\mathbb{H}}, U^R_{\mathbb{H}})$. The ensemble of invertible, bounded right linear operators from $V^R_{\mathbb{H}}$ to $U^R_{\mathbb{H}}$ will be labeled as $\mathcal{G}(V^R_{\mathbb{H}}, U^R_{\mathbb{H}})$. Given $V^R_{\mathbb{H}}$ as a right quaternionic Hilbert space and A as a right linear operator on it, there exists a singular linear operator A^* such that

$$\langle \psi \mid A\phi \rangle_{U^R_{\mathbb{H}}} = \langle \psi A^* \mid \phi \rangle_{V^R_{\mathbb{H}}} \text{ for all } \phi \in \mathcal{D}(A), \psi \in \mathcal{D}(A^*).$$

The domain $\mathcal{D}(A^*)$ of A^* is determined by

$$\mathcal{D}(A^*) = \left\{\psi \in U^R_{\mathbb{H}} : \exists \varphi \text{ such that } \langle \psi \mid A\phi \rangle_{U^R_{\mathbb{H}}} = \langle \varphi \mid \phi \rangle_{V^R_{\mathbb{H}}}\right\}.$$

We establish the domains of the sum $A + B$ and the composition AB by defining $\mathcal{D}(A + B) = \mathcal{D}(A) \cap \mathcal{D}(B)$ and $\mathcal{D}(AB) = \left\{\phi \in \mathcal{D}(B) : B\phi \in \mathcal{D}(A)\right\}$. It's important to note that $\mathcal{B}(V^R_{\mathbb{H}})$ naturally forms a real algebra, where addition follows the usual pointwise sum, multiplication is composition, and scalar multiplication is defined as usual

$$\begin{array}{ccc} \mathcal{B}(V^R_{\mathbb{H}}) \times \mathbb{H} & \longrightarrow & \mathcal{B}(V^R_{\mathbb{H}}) \\ (A, \lambda) & \longmapsto & A\lambda \end{array}$$

is defined by setting

$$(A\lambda)(\phi) = A(\phi)\lambda.$$

Lemma 2.2. [21, Theorem 3.2] *Let $A : \mathcal{D}(A) \subset V^R_{\mathbb{H}} \longrightarrow V^R_{\mathbb{H}}$ be a right linear operator. If $A \in \mathcal{B}(V^R_{\mathbb{H}})$ is surjective, then A is open. In particular, if A is bijective, then $A^{-1} \in \mathcal{B}(V^R_{\mathbb{H}})$.*

2.2. Left scalar multiplications

We will derive the definition and key properties of left scalar multiples of vectors in $U^R_{\mathbb{H}}$ from [16] as required for our manuscript's progression. This operation, known as the left scalar multiple of vectors on a right quaternionic Hilbert space, is notably non-canonical and depends on a selected preferred Hilbert basis. As mentioned in Remark 1.3, $U^R_{\mathbb{H}}$ possesses a Hilbert basis.

$$O = \left\{\phi_k : k \in \mathbb{N}\right\}.$$

Definition 2.3. *Consider $V^R_{\mathbb{H}}$ as a separable right Hilbert space with O as its Hilbert basis. The left scalar multiplication on $V^R_{\mathbb{H}}$ resulting from O is defined as the mapping*

$$\begin{array}{ccc} \mathbb{R} \times V^R_{\mathbb{H}} & \longrightarrow & V^R_{\mathbb{H}} \\ (\mathbf{q}, \phi) & \longmapsto & \mathbf{q}\phi = \sum_{k \in \mathbb{N}} \varphi_k \mathbf{q} \langle \varphi_k, \phi \rangle \end{array}$$

In a separable right Hilbert space $V^R_{\mathbb{H}}$ with O as its Hilbert basis, the left scalar multiplication exhibits the following properties.

Proposition 2.4. *For every $\phi, \psi \in V^R_{\mathbb{H}}$ and $\mathbf{q}, \mathbf{p} \in \mathbb{H}$, we have*

1. $\mathbf{q}(\phi + \psi) = \mathbf{q}\phi + \mathbf{q}\psi$ *and* $\mathbf{q}(\phi\mathbf{p}) = (\mathbf{q}\phi)\mathbf{p}$.
2. $||\mathbf{q}\phi|| = |\mathbf{q}|||\phi||$.
3. $\mathbf{q}(\mathbf{p}\phi) = (\mathbf{q}\mathbf{p})\phi$.
4. $\langle \overline{\mathbf{q}}\phi, \psi \rangle = \langle \phi, \mathbf{q}\psi \rangle$.
5. $r\phi = \phi r$, *for all* $r \in \mathbb{R}$.
6. $\mathbf{q}\phi_k = \phi_k \mathbf{q}$ *for all* $k \in \mathbb{N}$.

The definition of quaternionic left scalar multiplication for linear operators can also be found in [10, 16].

2.3. Spectral Mapping Theorem for Quaternionic Hilbert Space

In this subsection, we analyze several properties concerning the essential S-spectra of a quaternionic Hilbert space.

Definition 2.5. *Let $A : \mathcal{D}(A) \subset V_{\mathbb{H}}^R \longrightarrow V_{\mathbb{H}}^R$ be a right linear operator. The S-resolvent set (also called spherical resolvent set) of A is the set $\rho^S(A)(\subset \mathbb{H})$ such that the three following conditions hold true:*
(a) $N(R_{\mathbf{q}}(A)) = \{0\}$.
(b) $R(R_{\mathbf{q}}(A))$ *is dense in* $V_{\mathbb{H}}^R$.
(c) $R_{\mathbf{q}}(A)^{-1} : R(R_{\mathbf{q}}(A)) \longrightarrow \mathcal{D}(A^2)$ *is bounded.*
The pseudo S-spectrum $\sigma_\varepsilon^S(A)$ of A is defined as $\sigma_\varepsilon^S(A) := \mathbb{H}\backslash\rho_\varepsilon^S(A)$, where $\rho_\varepsilon^S(A)$ denotes the resolvent set of a bounded linear operator A.

$$\begin{aligned}
\rho_\varepsilon^S(A) &= \Big\{\mathbf{q} \in \mathbb{H} : R_{\mathbf{q}}(A) \in \mathcal{G}(V_{\mathbb{H}}^R)\Big\} \bigcap \Big\{\mathbf{q} \in \mathbb{H} : \|R_{\mathbf{q}}(A)^{-1}\| \leq \frac{1}{\varepsilon}\Big\} \\
&= \Big\{\mathbf{q} \in \mathbb{H} : R_{\mathbf{q}}(A) \textit{ has an inverse in } \mathcal{B}(V_{\mathbb{H}}^R)\Big\} \bigcap \Big\{\mathbf{q} \in \mathbb{H} : \|R_{\mathbf{q}}(A)^{-1}\| \leq \frac{1}{\varepsilon}\Big\} \\
&= \Big\{\mathbf{q} \in \mathbb{H} : N(R_{\mathbf{q}}(A)) = \{0\} \textit{ and } R(R_{\mathbf{q}}(A)) = V_{\mathbb{H}}^R\Big\} \bigcap \Big\{\mathbf{q} \in \mathbb{H} : \|R_{\mathbf{q}}(A)^{-1}\| \leq \frac{1}{\varepsilon}\Big\}
\end{aligned}$$

and the spectrum can be written as

$$\begin{aligned}
\sigma_\varepsilon^S(A) &= \mathbb{H}\backslash\rho_\varepsilon^S(A) \\
&= \Big\{\mathbf{q} \in \mathbb{H} : R_{\mathbf{q}}(A) \textit{ has no inverse in } \mathcal{B}(V_{\mathbb{H}}^R)\Big\} \bigcap \Big\{\mathbf{q} \in \mathbb{H} : \|R_{\mathbf{q}}(A)^{-1}\| > \frac{1}{\varepsilon}\Big\} \\
&= \Big\{\mathbf{q} \in \mathbb{H} : N(R_{\mathbf{q}}(A)) \neq \{0\} \textit{ or } R(R_{\mathbf{q}}(A)) \neq V_{\mathbb{H}}^R\Big\} \bigcap \Big\{\mathbf{q} \in \mathbb{H} : \|R_{\mathbf{q}}(A)^{-1}\| > \frac{1}{\varepsilon}\Big\}.
\end{aligned}$$

By convention, we write $\|R_{\mathbf{q}}(A)^{-1}\| = +\infty$ if $\mathbf{q} \in \sigma_\varepsilon^S(A)$ (pseudo S-spectrum of A). For $\varepsilon > 0$, it can be shown that $\sigma_\varepsilon^S(A)$ is a larger set and is never empty.

3. The quaternionic numerical range

The quaternionic numerical range, as discussed in [20, 22, 25], serves as the counterpart of the complex numerical range in physical terms, representing a quadratic operator through the unit sphere. Unlike its counterpart in the complex realm, the quaternionic numerical range is not universally convex (as noted in [18, 22]). In [17], J. E. Jamison highlighted the challenge of distinguishing the class of numerical ranges with linear operators on quaternionic Hilbert space. Despite this challenge, several authors have explored the properties of the quaternionic numerical range crossing, as evidenced in works such as [4, 17, 25].

So and Thompson demonstrated in [25] that the intersection of the quaternionic numerical range is related to a set of complex numbers $S \subset \mathbb{C}$, where we denote the δ-neighborhood as $B_\delta(S)$, i.e., $B_\delta(S) = \{\mathbf{q} \in \mathbb{H} : dist(\mathbf{q}, S) < \delta\}$, and we also use the notation

$$S^{-1} = \{\mathbf{q}^{-1} : \mathbf{q} \in S\backslash\{0\}\}.$$

The quaternionic numerical range. We define the quaternionic numerical range:

Definition 3.1. *Let $A \in V_{\mathbb{H}}^R$ and $\mathbf{q} \in \mathbb{H}$. Then, the quaternionic numerical range of A, denoted by $W_{\mathbb{H}}(R_{\mathbf{q}}(A))$, defined as*

$$W_{\mathbb{H}}\Big(R_{\mathbf{q}}(A)\Big) = \Big\{\langle R_{\mathbf{q}}(A)\phi, \phi\rangle : \phi \in \mathcal{D}(A), \|\phi\| = 1\Big\}.$$

The concept of quaternionic numerical range was initially explored in 1951 by Kippenhahn [19]. Over the past two decades, interest in the quaternionic numerical range has been rekindled, as evidenced by a series of papers such as [4, 25]. It consistently forms a convex set, and if A represents additionally bounded right linear operators from $V^R_{\mathbb{H}}$, then $W\mathbb{H}(A)$ is also bounded. Furthermore, the quaternionic numerical range satisfies the following inclusions

$$\sigma^S_p(A) \subset W_{\mathbb{H}}(R_{\mathbf{q}}(A)), \qquad \sigma^S_{app}(A) \subset W(R_{\mathbf{q}}(A))$$

where $\sigma^S_p(A)$ is the point spherical spectrum of A. The approximate S-point spectrum of A, denoted by $\sigma^S_{app}(A)$, is defined as

$$\sigma^S_{app}(A) := \Big\{\mathbf{q} \in \mathbb{H} : \exists \phi_n \ \text{ such that } \ \|\phi_n\| = 1 \text{ and } \lim_{n \longrightarrow \infty} \|R_{\mathbf{q}}(A)\phi_n\| = 0\Big\}.$$

The spherical spectrum, point spherical spectrum, and approximate S-point spectrum of A are interconnected through $\sigma^S_p(A) \subset \sigma^S_{app}(A) \subset \sigma^S(A)$. If A has a compact resolvent, then the spherical spectrum consists of eigenvalues only and then we have parity.

Proposition 3.2. *Let $A \in V^R_{\mathbb{H}}$. Suppose that $0 \in \rho^S(A)$. Then, for every $0 < \varepsilon \leq \frac{1}{2\|R_{\mathbf{q}}(A)^{-1}\|}$ and $\delta = \|R_{\mathbf{q}}(A)^{-1}\|\Big(2\|R_{\mathbf{q}}(A)^{-1}\|\varepsilon + 1\Big)$ we have*

$$\sigma^S_\varepsilon(A) \subset \Big(B_\delta(W_{\mathbb{H}}(R_{\mathbf{q}}(A)^{-1}))\Big)^{-1}.$$

Proof. As a first step we show that

$$\|R_{\mathbf{q}}(A)x\| \geq \varepsilon \text{ for all } \mathbf{q} \in \mathbb{H}\backslash B_\delta(W_{\mathbb{H}}(R_{\mathbf{q}}(A)^{-1})), \phi \in \mathbb{H}, \|\phi\| = 1.$$

Let $\mathbf{q} \in \mathbb{C}\backslash B_\delta(W_{\mathbb{H}}(R_{\mathbf{q}}(A)^{-1}))$. We consider two cases. First suppose that $|\mathbf{q}| > \frac{1}{2\|R_{\mathbf{q}}(A)^{-1}\|}$. Then, $\mathbf{q} \neq 0$, $\mathbf{q}^{-1} \notin B_\delta(W_{\mathbb{H}}(R_{\mathbf{q}}(A)^{-1}))$ and hence $dist(\mathbf{q}^{-1}, W_{\mathbb{H}}(R_{\mathbf{q}}(A)^{-1})) \geq \delta$. For $\phi \in \mathbb{H}$, $\|\phi\| = 1$ we find

$$\delta \leq |\mathbf{q}^{-1} - \langle R_{\mathbf{q}}(A)^{-1}\phi, \phi\rangle| = |\langle(\mathbf{q}^{-1} - R_{\mathbf{q}}(A)^{-1})\phi, \phi\rangle| \leq \|(\mathbf{q}^{-1} - R_{\mathbf{q}}(A)^{-1})\phi\|.$$

Consequently

$$\begin{aligned}
\|R_{\mathbf{q}}(A)\phi\| &= \|(R_{\mathbf{q}}(A) - \mathbf{q} + \mathbf{q})\phi\| \\
&\geq \|(R_{\mathbf{q}}(A) - \mathbf{q})\phi\| - |\mathbf{q}| \\
&\geq |\mathbf{q}|\|R_{\mathbf{q}}(A)(\mathbf{q}^{-1} - R_{\mathbf{q}}(A)^{-1})\phi\| - |\mathbf{q}| \\
&\geq \frac{|\mathbf{q}|}{\|R_{\mathbf{q}}(A)^{-1}\|}\|(\mathbf{q}^{-1} - R_{\mathbf{q}}(A)^{-1})\phi\| - |\mathbf{q}| \\
&\geq \frac{1}{2\|R_{\mathbf{q}}(A)^{-1}\|^2}\left(\frac{\delta}{\|R_{\mathbf{q}}(A)^{-1}\|} - 1\right) = \varepsilon.
\end{aligned}$$

In the other case if $|\mathbf{q}| \leq \frac{1}{2\|R_{\mathbf{q}}(A)^{-1}\|}$, then $I - \mathbf{q}R_{\mathbf{q}}(A)^{-1}$ is invertible. For $\phi \in \mathbb{H}$, $\|\phi\| = 1$ this implies

$$\|R_{\mathbf{q}}(A)\| = \|R_{\mathbf{q}}(A)(I - \mathbf{q}R_{\mathbf{q}}(A)^{-1})\phi\| - |\mathbf{q}| \geq \frac{1}{\|R_{\mathbf{q}}(A)^{-1}\|\|(I - \mathbf{q}R_{\mathbf{q}}(A)^{-1})^{-1}\|} - |\mathbf{q}| \geq \varepsilon.$$

In particular, $\mathbf{q} \in \mathbb{H}\backslash B_\delta(W_{\mathbb{H}}(R_{\mathbf{q}}(A)^{-1}))$ implies $\mathbf{q} \notin \sigma^S_{app}(A)$, then

$$\sigma^S_{app}(A) \cap \mathbb{H}\backslash B_\delta(W_{\mathbb{H}}(R_{\mathbf{q}}(A)^{-1})) = \emptyset.$$

Since $B_\delta(W_{\mathbb{H}}(R_{\mathbf{q}}(A)^{-1}))$ is convex and bounded, the set $\mathbb{H}^*\backslash B_\delta(W_{\mathbb{H}}(R_{\mathbf{q}}(A)^{-1}))$ is connected and hence also

$$\mathbb{H}^*\backslash B_\delta(W_{\mathbb{H}}(R_{\mathbf{q}}(A)^{-1})) = (\mathbb{H}^*\backslash B_\delta(W_{\mathbb{H}}(R_{\mathbf{q}}(A)^{-1}))^{-1}),$$

the image under the homeomorphism

$$\begin{array}{ccc} \mathbb{H}^* & \longrightarrow & \mathbb{H}^* \\ \mathbf{q} & \longmapsto & \mathbf{q}^{-1}. \end{array}$$

On the other hand, the boundedness of $B_\delta(W_{\mathbb{H}}(R_{\mathbf{q}}(A)^{-1}))$ implies that a neighborhood around 0 belongs to $\mathbb{H}\backslash B_\delta(W_{\mathbb{H}}(R_{\mathbf{q}}(A)^{-1}))$. Consequently, the set $\mathbb{H}\backslash B_\delta(W_{\mathbb{H}}(R_{\mathbf{q}}(A)^{-1}))$ is connected and satisfies $0 \in \rho^S(A) \cap \mathbb{H}\backslash B_\delta(W_{\mathbb{H}}(R_{\mathbf{q}}(A)^{-1}))$. Since $\partial\sigma^S(A) \subset \sigma^S_{app}(A)$, then

$$\mathbb{H}\backslash B_\delta(W_{\mathbb{H}}(R_{\mathbf{q}}(A)^{-1})) \subset \rho^S(A).$$

Now if $\mathbf{q} \in \mathbb{H}\backslash B_\delta(W_{\mathbb{H}}(R_{\mathbf{q}}(A)^{-1}))$, then $\|R_{\mathbf{q}}(A)^{-1}\| \leq \frac{1}{\varepsilon}$ and we conclude that $\mathbf{q} \notin \sigma^S_\varepsilon(A)$. □

Taking the intersection over a suitable set of shifts, we obtain our first main result on an enclosure of the pseudo S-spectrum:

Theorem 3.3. *Let $A \in V^R_{\mathbb{H}}$. Consider a set $S \subset \rho^S(A)$ such that*

$$M = \sup_{s\in S} \|R_{\mathbf{q+s}}(A)^{-1}\| < \infty.$$

Then, for $0 < \varepsilon \leq \frac{1}{2M}$ we get the inclusion

$$\sigma^S_\varepsilon(A) \subset \bigcap_{s\in S} \Big[\Big(B_{\delta_s}(W_{\mathbb{H}}(R_{\mathbf{q+s}}(A)^{-1}))\Big)^{-1} - s\Big]$$

where $\delta_s = \|R_{\mathbf{q+s}}(A)^{-1}\|\Big(2\|R_{\mathbf{q+s}}(A)^{-1}\|\varepsilon + 1\Big)$.

Proof. Let $\mathbf{q} \in \sigma^S_\varepsilon(A) + s$ for every $s \in S$. Then, $\mathbf{q} + s \in \sigma^S_\varepsilon(A)$. We can apply Proposition 3.2, we obtain $\mathbf{q} + s \in \Big(B_{\delta_s}(W_{\mathbb{H}}(R_{\mathbf{q+s}}(A)^{-1}))\Big)^{-1}$. We deduce that $\mathbf{q} \in \Big(B_{\delta_s}(W_{\mathbb{H}}(R_{\mathbf{q+s}}(A)^{-1}))\Big)^{-1} - s$. We conclude that

$$\mathbf{q} \in \bigcap_{s\in S} \Big[\Big(B_{\delta_s}(W_{\mathbb{H}}(R_{\mathbf{q+s}}(A)^{-1}))\Big)^{-1} - s\Big].$$

□

4. A strong approximation scheme

In this section, we believe finite-dimensional approximations A_n to the complete right linear operator A. Our objective is to evidence a purview of Theorem 3.3 whose supplies a pseudo S-spectrum encirclement for the full right linear operator A in expressions of the quaternionic numerical range of the approximating A_n this will allow us to computation the enclosure by numerical procedures. We presume that $0 \in \rho^S(A)$ and consider a sequence of quaternionic approximations A_n of the right linear operator A of the following form:

1. $U_n \subset \mathbb{H}$, $n \in \mathbb{N}$, are finite-dimensional subspaces of the right quaternionic Hilbert space $\mathbb{H}$.
2. $P_n \in \mathcal{B}(V^R_{\mathbb{H}})$ are projections (not necessarily orthogonal) onto $V^R_{U_n}$, i.e. $R(P_n) = V^R_{U_n}$, such that

$$\lim_{n\longrightarrow\infty} P_n\phi = \phi \ \text{ for all } \ \phi \in \mathbb{H}.$$

3. $R_{\mathbf{q}}(A_n)$ these are bounded right linear operators that are invertible, such that

$$\lim_{n\longrightarrow\infty} R_{\mathbf{q}}(A_n)^{-1}P_n\phi = R_{\mathbf{q}}(A)^{-1}\phi \ \text{ for all } \ \phi \in \mathbb{H}.$$

In this scenario, we describe the family $(P_n, A_n)_{n\in\mathbb{N}}$ as strongly quaternionic approximating A.

Lemma 4.1. *Let $A_n \in \mathcal{B}(V^R_{U_n})$ be invertible. If $\mathbf{q} \in \rho^S(A)$ is such that $\mathbf{q} \in \rho^S(A_n)$ for all $n \in \mathbb{N}$. The following statements are equivalent:*

1. $\lim\limits_{n\longrightarrow\infty} R_{\mathbf{q}}(A_n)^{-1}P_n\phi = R_{\mathbf{q}}(A_n)^{-1}\phi$ *for all* $\phi \in \mathbb{H}$.
2. $\sup\limits_{n\in\mathbb{N}} \|R_{\mathbf{q}}(A_n)^{-1}\|_{\mathcal{B}(V^R_{U_n})} < \infty$ *and for all $\phi \in \mathbb{H}$ there exist $\phi_n \in U_n$ such that*

$$\lim_{n\longrightarrow\infty} \phi_n = \phi, \quad \lim_{n\longrightarrow\infty} R_{\mathbf{q}}(A_n)\phi_n = R_{\mathbf{q}}(A)\phi.$$

Proof. (1) $\Longrightarrow$ (2) Since $\|R_{\mathbf{q}}(A_n)^{-1}\phi\| = \|R_{\mathbf{q}}(A_n)^{-1}P_n\phi\| \leq \|R_{\mathbf{q}}(A_n)^{-1}P_n\|_{\mathcal{B}(V^R_{U_n})}\|\phi\|$ for all $\phi \in U_n$, this shows the first part. For the second, let $\phi \in \mathbb{H}$ and set $\psi = A\phi$ and $\phi_n = R_{\mathbf{q}}(A_n)^{-1}P_n\psi$. Then, $\phi_n \longrightarrow R_{\mathbf{q}}(A_n)^{-1}\psi = \phi$ and $R_{\mathbf{q}}(A_n)\phi_n = P_n\psi \longrightarrow_{n\longrightarrow\infty} \psi = R_{\mathbf{q}}(A)\phi$.

(2) $\Longrightarrow$ (1) Let $\psi \in \mathbb{H}$. Set $\phi = R_{\mathbf{q}}(A_n)^{-1}\psi$ and choose $\phi_n \in U_n$ according to (2). Then,

$$R_{\mathbf{q}}(A_n)^{-1}P_n\psi = R_{\mathbf{q}}(A_n)^{-1}P_nR_{\mathbf{q}}(A)^{-1}\phi = R_{\mathbf{q}}(A_n)^{-1}(P_nR_{\mathbf{q}}(A)\phi - R_{\mathbf{q}}(A_n)\phi_n) + \phi_n.$$

Since both $P_nR_{\mathbf{q}}(A)\phi \longrightarrow R_{\mathbf{q}}(A)\phi$ and $R_{\mathbf{q}}(A_n) \longrightarrow R_{\mathbf{q}}(A)\phi$ as $n \longrightarrow \infty$ and $\|R_{\mathbf{q}}(A_n)^{-1}\|$ is uniformly bounded, we obtain (1). □

Lemma 4.2. *If the family $(P_n, A_n)_{n\in\mathbb{N}}$ strongly quaternionic approximates A, then*

1. *for every $\phi \in \mathbb{H}$, $\|\phi\| = 1$ there exists a sequence $\psi_n \in U_n$, $\|\psi_n\| = 1$ such that*

$$\lim_{n\longrightarrow\infty} \langle R_{\mathbf{q}}(A_n)^{-1}\psi_n, \psi_n\rangle = \langle R_{\mathbf{q}}(A)^{-1}\phi, \phi\rangle.$$

2. *for all $\delta > 0$ there exists $n_0 \in \mathbb{N}$ such that*

$$W_{\mathbb{H}}\Big(R_{\mathbf{q}}(A)^{-1}\Big) \subset B_\delta(W_{\mathbb{H}}(R_{\mathbf{q}}(A_n)^{-1})), \quad n \geq n_0.$$

Proof. 1. Let $\psi_n = \frac{P_n\phi}{\|P_n\phi\|}$. Note that ψ_n is well defined for almost all n since $\|P_n\phi\| \longrightarrow \|\phi\| = 1$ we get $\psi_n \longrightarrow \phi$ as $n \longrightarrow \infty$ and

$$\begin{aligned}
&|\langle R_{\mathbf{q}}(A)^{-1}\phi, \phi\rangle - \langle R_{\mathbf{q}}(A_n)^{-1}\psi_n, \psi_n\rangle| \\
&\leq |\langle R_{\mathbf{q}}(A)^{-1}\phi - R_{\mathbf{q}}(A_n)^{-1}P_n\phi, \phi\rangle| + |\langle R_{\mathbf{q}}(A_n)^{-1}P_n\phi, \phi - \psi_n\rangle| \\
&\quad + |\langle R_{\mathbf{q}}(A_n)^{-1}(P_n\phi - \psi_n), \psi_n\rangle| \\
&\leq \|R_{\mathbf{q}}(A)^{-1}\phi - R_{\mathbf{q}}(A_n)^{-1}P_n\phi\| + \|R_{\mathbf{q}}(A_n)^{-1}\|\|P_n\phi\|\|\phi - \psi_n\| \\
&\quad + \|R_{\mathbf{q}}(A_n)^{-1}\|\|P_n\phi - \psi_n\|,
\end{aligned}$$

this leads to the conclusion.

2. Since $W_{\mathbb{H}}(R_{\mathbf{q}}(A)^{-1})$ is bounded, it is precompact and hence there exist $\varphi_1, \cdots, \varphi_m \in W_{\mathbb{H}}(R_{\mathbf{q}}(A)^{-1})$ such that

$$W_{\mathbb{H}}(R_{\mathbf{q}}(A)^{-1}) \subset \bigcup_{j=1}^{m} B_{\delta/2}(\varphi_j).$$

For every j we have $\varphi_j =< R_{\mathbf{q}}(A)^{-1}\phi_j, \phi_j >$ with some $\phi_j \in \mathbb{H}$, $\|\phi_j\| = 1$, and by (1) there exists $n_j \in \mathbb{N}$ such that for all $n \geq n_j$ there is a $\psi_j \in U_n$, $\|\psi_j\| = 1$ such that

$$|\langle R_{\mathbf{q}}(A)^{-1}\phi_j, \phi_j\rangle - \langle R_{\mathbf{q}}(A_n)^{-1}\psi_j, \psi_j\rangle| < \frac{\delta}{2}.$$

Hence

$$W_{\mathbb{H}}(R_{\mathbf{q}}(A)^{-1}) \subset \bigcup_{j=1}^{m} B_\delta(\langle R_{\mathbf{q}}(A_n)^{-1}\psi_j, \psi_j\rangle) \subset B_\delta(W_{\mathbb{H}}(R_{\mathbf{q}}(A_n)^{-1}))$$

for all $n \geq n_0 = \max\{n_1, \cdots, n_m\}$.

□

Proposition 4.3. *Suppose that $(P_n, A_n)_n \in \mathbb{N}$ approximates A strongly. For every $0 < \varepsilon \leq \frac{1}{2\|R_{\mathbf{q}}(A)^{-1}\|}$ and $\delta > \|R_{\mathbf{q}}(A)^{-1}\|\Big(2\|R_{\mathbf{q}}(A)^{-1}\|\varepsilon + 1\Big)$ there exists $n_0 \in \mathbb{N}$ such that*

$$\sigma_\varepsilon^S(A) \subset \Big(B_\delta(W_{\mathbb{H}}(R_{\mathbf{q}}(A_n)^{-1}))\Big)^{-1} \ \textit{for all} \ \ n \geq n_0.$$

Proof. By using Proposition 3.2, we get

$$\sigma_\varepsilon^S(A) \subset \Big(B_{\delta'}(W_{\mathbb{H}}(R_{\mathbf{q}}(A)^{-1}))\Big)^{-1}.$$

where $\delta' = \|R_{\mathbf{q}}(A)^{-1}\|\Big(2\|R_{\mathbf{q}}(A)^{-1}\|\varepsilon + 1\Big)$. Since $\delta - \delta' > 0$, then by lemma 4.1 we get a constant $n_0 \in \mathbb{N}$ such that

$$W_{\mathbb{H}}(R_{\mathbf{q}}(A)^{-1}) \subset B_{\delta-\delta'}(W_{\mathbb{H}}(R_{\mathbf{q}}(A_n)^{-1}), \ \ n \geq n_0.$$

Consequently $B_{\delta'}(W_{\mathbb{H}}(R_{\mathbf{q}}(A)^{-1})) \subset B_\delta(W_{\mathbb{H}}(R_{\mathbf{q}}(A_n)^{-1}))$ for all $n \geq n_0$. □

Theorem 4.4. *Suppose that $(P_n, A_n)_n \in \mathbb{N}$ approximates A strongly. Let the shifts $s_1, \cdots, s_m \in \rho^S(A)$ be such that*

$$\sup_{n\in\mathbb{N}} \|R_{\mathbf{q}+\mathbf{s_j}}(A_n)^{-1}\| < \infty \ \textit{for all} \ \ j = 1, \cdots, m.$$

Let $0 < \varepsilon \leq \frac{1}{\max_{j=1,\cdots,m}\|R_{\mathbf{q}+\mathbf{s_j}}(A)^{-1}\|}$ and $\delta_j > \|R_{\mathbf{q}+\mathbf{s}}(A)^{-1}\|\Big(2\|R_{\mathbf{q}+\mathbf{s_j}}(A)^{-1}\|\varepsilon + 1\Big)$ for all j.Then, there exists $n_0 \in \mathbb{N}$ such that

$$\sigma_\varepsilon^S(A) \subset \bigcap_{j=1}^{m} \Big[\Big(B_{\delta_{s_j}}(W_{\mathbb{H}}(R_{\mathbf{q}+\mathbf{s_j}}(A_n)^{-1}))\Big)^{-1} - s_j\Big] \ \textit{for all} \ \ n \geq n_0.$$

Proof. Considering Lemma 4.1, Proposition 4.3 is applicable to each $R_{\mathbf{q}+\mathbf{s_j}}(A)$. Hence there exists $n_j \in \mathbb{N}$ such that $\mathbf{q} + s_j \in \sigma_\varepsilon^S(A)$. Applying Proposition 4.3, we derive $\mathbf{q} + s_j \in \Big(B_{\delta_{s_j}}(W_{\mathbb{H}}(R_{\mathbf{q}+\mathbf{s_j}}(A_n)^{-1}))\Big)^{-1}$. We deduce that $\mathbf{q} \in \Big(B_{\delta_{s_j}}(W_{\mathbb{H}}(R_{\mathbf{q}+\mathbf{s_j}}(A_n)^{-1}))\Big)^{-1} - s_j$. We conclude that

$$\mathbf{q} \in \bigcap_{j=1}^{m} \Big[\Big(B_{\delta_{s_j}}(W_{\mathbb{H}}(R_{\mathbf{q}+\mathbf{s_j}}(A_n)^{-1}))\Big)^{-1} - s_j\Big].$$

□

5. A uniform approximation scheme

In this section, we make the following assumptions about A: its resolvent is compact, 0 belongs to $\rho^S(A)$, $\mathcal{D}(A^2)$ is contained in $\mathbb{W}$, which is a quaternionic Hilbert space continuously and densely embedded in $\mathbb{H}$. Additionally, we assume the existence of a sequence of quaternionic approximations for the operator A in the following manner:

1. For each $n \in \mathbb{N}$, U_n represents a finite-dimensional subspace contained within the right quaternionic Hilbert space $\mathbb{H}$.
2. There exist projections $P_n \in \mathcal{B}(V_{\mathbb{H}}^R)$ onto $V_{U_n}^R$, $n \in \mathbb{N}$, (not necessarily orthogonal), with $\sup_{n\in\mathbb{N}} \|P_n\| < \infty$ and $\|(I - P_n)|_{\mathbb{W}}\|_{\mathcal{B}(V_{\mathbb{H}}^R)} \longrightarrow 0$ as $n \longrightarrow \infty$.
3. There exist invertible bounded right linear operators $R_{\mathbf{q}}(A_n)$, $n \in \mathbb{N}$ such that $\|R_{\mathbf{q}}(A)^{-1} - R_{\mathbf{q}}(A_n)^{-1}P_n\| \longrightarrow 0$ as $n \longrightarrow \infty$.

We define $(P_n, A_n)_{n\in\mathbb{N}}$ as a uniform quaternionic approximation of A. For brevity, we denote $\|(I - P_n)|_{\mathbb{W}}\|_{\mathcal{B}(V^R_{\mathbb{W}}, V^R_{\mathbb{H}})}$ as $\|I - P_n\|_{\mathcal{B}(V^R_{\mathbb{W}}, V^R_{\mathbb{H}})}$.
For $d > 0$ we define

$$W_{\mathbb{H}}(R_{\mathbf{q}}(A)^{-1}, d) = \Big\{\langle R_{\mathbf{q}}(A)^{-1}\phi, \phi\rangle : \|\phi\| = 1, \phi \in \mathbb{W}, \|\phi\|_{\mathbb{H}} \leq d\Big\}.$$

Clearly $W_{\mathbb{H}}(R_{\mathbf{q}}(A)^{-1}, d) \subset W_{\mathbb{H}}(R_{\mathbf{q}}(A)^{-1})$. Moreover since $\mathbb{W}$ is dense in $\mathbb{H}$ we get

$$\overline{\bigcup_{d>0} W_{\mathbb{H}}(R_{\mathbf{q}}(A)^{-1}, d)} = \overline{W_{\mathbb{H}}(R_{\mathbf{q}}(A)^{-1})}.$$

Proposition 5.1. *Given $L > 0$, there is a bounded linear operator B such that $d = L\|B\|_{\mathcal{B}(V^R_{\mathbb{H}})}\|R_{\mathbf{q}}(A)^{-1}\|_{\mathcal{B}(V^R_{\mathbb{H}})}$, then*

1. $\sigma^S(A) \cap \overline{B_L(0)} \subset W_{\mathbb{H}}\Big((R_{\mathbf{q}}(A)^{-1}, d)\Big)^{-1}$.
2. *If in addition $0 < \varepsilon \leq \frac{1}{2\|R_{\mathbf{q}}(A)^{-1}\|}$, $\delta = \|R_{\mathbf{q}}(A)^{-1}\|\Big(2\|R_{\mathbf{q}}(A)^{-1}\|\varepsilon + 1\Big)$ and $L > \varepsilon$, then*

$$\sigma^S_{\varepsilon}(A) \cap \overline{B_{L-\varepsilon}(0)} \subset \Big(B_{\delta}(W_{\mathbb{H}}(R_{\mathbf{q}}(A)^{-1}, d))\Big)^{-1}.$$

Proof. 1. Let $\mathbf{q} \in \rho^S(A)$ with $|q| \leq L$. Next, there exists $\phi \in \mathcal{D}(A^2)$ with $|\phi| = 1$. Since $A \in \mathcal{B}(V^R_{\mathbb{H}})$ and $R\mathbf{q}(A)$ is both bounded and bijective, $R_{\mathbf{q}}(A)^{-1} \in \mathcal{B}(V^R_{\mathbb{H}})$. Thus, as per [21, Theorem 3.2], there exists a bounded linear operator B such that $BR_{\mathbf{q}}(A)^{-1}\phi = \frac{1}{\mathbf{q}}\phi$ for all $\phi \in \mathcal{D}(A^2)$. Then,

$$\frac{1}{|\mathbf{q}|}\|\phi\|_{\mathbb{W}} = \|BR_{\mathbf{q}}(A)^{-1}\phi\|_{\mathbb{W}} \leq \|B\|_{\mathcal{B}(V^R_{\mathbb{H}})}\|R_{\mathbf{q}}(A)^{-1}\|_{\mathcal{B}(V^R_{\mathbb{H}})}\|\phi\| \text{ for all } \phi \in \mathcal{D}(A^2).$$

Thus we obtain

$$\|\phi\|_{\mathbb{W}} \leq \|B\|_{\mathcal{B}(V^R_{\mathbb{H}})}\|R_{\mathbf{q}}(A)^{-1}\|_{\mathcal{B}(V^R_{\mathbb{H}})}|\mathbf{q}| \leq L\|B\|_{\mathcal{B}(V^R_{\mathbb{H}})}\|R_{\mathbf{q}}(A)^{-1}\|_{\mathcal{B}(V^R_{\mathbb{H}})} = d.$$

2. Let $\mathbf{q} \in \overline{B_{L-\varepsilon}(0)}\backslash\Big(B_{\delta}(W_{\mathbb{H}}(R_{\mathbf{q}}(A)^{-1}, d))\Big)^{-1}$, $\phi \in \mathcal{D}(A^2)$, $\|\phi\| = 1$. Let's examine three scenarios. Let's start with the case where $\mathbf{q} > \frac{1}{2\|R_{\mathbf{q}}(A)^{-1}\|}$ and $\|\phi\|_{\mathbb{W}} \leq d$. From $\mathbf{q} \notin \Big(B_{\delta}(W_{\mathbb{H}}(R_{\mathbf{q}}(A)^{-1}, d))\Big)^{-1}$ we obtain

$$dist(\mathbf{q}^{-1}, W_{\mathbb{H}}(R_{\mathbf{q}}(A)^{-1}, d)) \geq \delta,$$

which implies

$$\delta \leq |\mathbf{q}^{-1} - \langle R_{\mathbf{q}}(A)^{-1}\phi, \phi\rangle| = |\langle(\mathbf{q}^{-1} - R_{\mathbf{q}}(A)^{-1})\phi, \phi\rangle| \leq \|(\mathbf{q}^{-1} - R_{\mathbf{q}}(A)^{-1})\phi\|.$$

Consequently

$$\|R_{\mathbf{q}}(A)\phi\| \quad \geq \quad \frac{1}{2\|R_{\mathbf{q}}(A)^{-1}\|^2}\left(\frac{\delta}{\|R_{\mathbf{q}}(A)^{-1}\|} - 1\right) = \varepsilon.$$

In the second case assume $\|\phi\|_{\mathbb{W}} \geq d$. Then,

$$d \leq \|\phi\|_{\mathbb{W}} \leq \|R_{\mathbf{q}}(A)^{-1}\|_{\mathcal{B}(V^R_{\mathbb{H}})}\|R_{\mathbf{q}}(A)\phi\|,$$

which in view of $\mathbf{q} \in \overline{B_{L-\varepsilon}(0)}$ and we suppose that $\|B\| > 1$ implies

$$\|R_{\mathbf{q}}(A)\phi\| \geq \|R_{\mathbf{q}}(A)\phi\| - |\mathbf{q}| \geq \frac{d}{\|R_{\mathbf{q}}(A)^{-1}\|_{\mathcal{B}(V^R_{\mathbb{H}})}} - |\mathbf{q}| \geq L - |\mathbf{q}| \geq \varepsilon.$$

Finally if $|\mathbf{q}| \leq \frac{1}{2\|R_{\mathbf{q}}(A)^{-1}\|}$, the same reasoning as in the proof of Proposition 3.2 yields once again that $\|R_{\mathbf{q}}(A)\phi\| \geq \varepsilon$.
Given that A possesses a compact resolvent, this implies that
$\mathbf{q} \in \overline{B_{L-\varepsilon}(0)} \backslash \Big(B_\delta(W_{\mathbb{H}}(R_{\mathbf{q}}(A)^{-1}, d))\Big)^{-1}$, then $\mathbf{q} \in \rho^S(A)$, $\|R_{\mathbf{q}}(A)^{-1}\| \leq \frac{1}{\varepsilon}$. We deduce that
$\sigma^S(A) \cap \overline{B_{L-\varepsilon}(0)} \subset \Big(B_\delta(W_{\mathbb{H}}(R_{\mathbf{q}}(A)^{-1}, d))\Big)^{-1}$.
□

Theorem 5.2. *Let $S \in \rho^S(A)$ be such that*

$$M_0 := \sup_{s\in S} \|R_{\mathbf{q+s}}(A)^{-1}\| < \infty, \qquad M_1 := \sup_{s\in S} \|R_{\mathbf{q+s}}(A)^{-1}\|_{\mathcal{B}(V^R_{\mathbb{H}})} < \infty.$$

There exists a bounded linear operator B with $d = L\|B\|_{\mathcal{B}(V^R_{\mathbb{H}})} M_1$. If in addition $0 < \varepsilon \leq \frac{1}{2M_0}$, $\delta = \|R_{\mathbf{q+s}}(A)^{-1}\|\Big(2\|R_{\mathbf{q+s}}(A)^{-1}\|\varepsilon + 1\Big)$ and $L > \varepsilon$, then

$$\sigma^S_\varepsilon(A) \cap \overline{B_{L-\varepsilon}(0)} \subset \bigcap_{s\in S} \Big[\Big(B_\delta(W_{\mathbb{H}}(R_{\mathbf{q+s}}(A)^{-1}, d))\Big)^{-1} - s\Big].$$

Proof. Let $\mathbf{q} + s \in \sigma^S_\varepsilon(A) \cap \overline{B_{L-\varepsilon}(0)}$.Applying Proposition 5.1, we arrive at $\mathbf{q} \in \Big(B_\delta(W_{\mathbb{H}}(R_{\mathbf{q+s}}(A)^{-1}, d))\Big)^{-1} - s$.
We deduce that

$$\mathbf{q} \in \bigcap_{s\in S} \Big[\Big(B_\delta(W_{\mathbb{H}}(R_{\mathbf{q+s}}(A)^{-1}, d))\Big)^{-1} - s\Big].$$

□

Lemma 5.3. *Assuming that $(P_n, A_n)_{n\in\mathbb{N}}$ provides a uniform quaternionic approximation for A, let's*

$$C_0 = \sup_{n\in\mathbb{N}} \Big(\|R_{\mathbf{q}}(A_n)^{-1}\|\|P_n\| + 6\|R_{\mathbf{q}}(A_n)^{-1}\|\|P_n\|^2\Big).$$

1. *If $d > 0$, $0 < \delta < \frac{C_0}{2}$ and $n_0 \in \mathbb{N}$ are such that for every $n \geq n_0$*

$$\|R_{\mathbf{q}}(A)^{-1} - R_{\mathbf{q}}(A_n)^{-1}P_n\| + dC_0\|I - P_n\|_{\mathcal{B}(V^R_{\mathbb{W}}, V^R_{\mathbb{H}})} < \delta,$$

then

$$W_{\mathbb{H}}(R_{\mathbf{q}}(A)^{-1}, d) \subset B_\delta(W_{\mathbb{H}}(R_{\mathbf{q}}(A_n)^{-1})), \quad n \geq n_0.$$

2. *If $\delta > 0$ and $n_0 \in \mathbb{N}$ are such that for every $n \geq n_0$ we have $\|R_{\mathbf{q}}(A)^{-1} - R_{\mathbf{q}}(A_n)^{-1}P_n\| < \delta$, then*

$$W_{\mathbb{H}}(R_{\mathbf{q}}(A_n)^{-1}) \subset B_\delta(W_{\mathbb{H}}(R_{\mathbf{q}}(A)^{-1})), \quad n \geq n_0.$$

Proof. 1. Let $\phi \in \mathbb{W}$ with $\|\phi\| = 1$ and $\|\phi\|_{\mathbb{W}} \leq d$. Then,

$$\begin{aligned}
&|\langle R_{\mathbf{q}}(A)^{-1}\phi, \phi\rangle - \langle R_{\mathbf{q}}(A_n)^{-1}P_n\phi, P_n\phi\rangle| \\
&\leq |\langle R_{\mathbf{q}}(A)^{-1}\phi - R_{\mathbf{q}}(A_n)^{-1}P_n\phi, \phi\rangle| + |\langle R_{\mathbf{q}}(A_n)^{-1}P_n\phi, \phi - P_n\phi\rangle| \\
&\leq \|R_{\mathbf{q}}(A)^{-1} - R_{\mathbf{q}}(A_n)^{-1}P_n\|\|\phi\|^2 + \|R_{\mathbf{q}}(A_n)^{-1}\|\|P_n\|\|\phi\|\|I - P_n\|_{\mathcal{B}(V^R_{\mathbb{W}}, V^R_{\mathbb{H}})}\|\phi\|_{\mathbb{W}} \\
&\leq \|R_{\mathbf{q}}(A)^{-1} - R_{\mathbf{q}}(A_n)^{-1}P_n\| + d\|R_{\mathbf{q}}(A_n)^{-1}\|\|P_n\|\|I - P_n\|_{\mathcal{B}(V^R_{\mathbb{W}}, V^R_{\mathbb{H}})}.
\end{aligned}$$

as well as

$$\Big|1 - \|P_n\phi\|\Big| \leq \|\phi - P_n\phi\| \leq \|I - P_n\|_{\mathcal{B}(V^R_{\mathbb{W}}, V^R_{\mathbb{H}})}\|\phi\|_{\mathbb{W}} \leq d\|I - P_n\|_{\mathcal{B}(V^R_{\mathbb{W}}, V^R_{\mathbb{H}})}.$$

Let $n \geq n_0$. Then,

$$\Big|1 - \|P_n\phi\|\Big| \leq d\|I - P_n\|_{\mathcal{B}(V^R_{\mathbb{W}}, V^R_{\mathbb{H}})} < \frac{\delta}{C_0} \leq \frac{1}{2}$$

and hence $\|P_n\phi\| \geq \frac{1}{2}$. Let $\phi_n = \frac{P_n\phi}{\|P_n\phi\|}$. Then, $\|\phi_n\| = 1$ and

$$\begin{aligned}\left|1 - \frac{1}{\|P_n\phi\|^2}\right| &= \left(\frac{1}{\|P_n\phi\|} + \frac{1}{\|P_n\phi\|^2}\right)\Big|1 - \|P_n\phi\|\Big| \\ &\leq 6\Big|1 - \|P_n\phi\|\Big| \\ &\leq 6d\|I - P_n\|_{\mathcal{B}(V^R_{\mathbb{W}}, V^R_{\mathbb{H}})}.\end{aligned}$$

This implies

$$\begin{aligned}&|\langle R_{\mathbf{q}}(A_n)^{-1}P_n\phi, P_n\phi\rangle - \langle R_{\mathbf{q}}(A_n)^{-1}\phi, \phi\rangle| \\ &= \left|\langle R_{\mathbf{q}}(A_n)^{-1}P_n\phi, P_n\phi\rangle - \frac{\langle R_{\mathbf{q}}(A_n)^{-1}P_n\phi, P_n\phi\rangle}{\|P_n\phi\|^2}\right| \\ &= \left|1 - \frac{1}{\|P_n\phi\|^2}\right| |\langle R_{\mathbf{q}}(A_n)^{-1}P_n\phi, P_n\phi\rangle| \\ &\leq 6d\|I - P_n\|_{\mathcal{B}(V^R_{\mathbb{W}}, V^R_{\mathbb{H}})}\|R_{\mathbf{q}}(A_n)^{-1}\|\|P_n\|^2,\end{aligned}$$

and thus for $n \geq n_0$ we arrive at

$$\begin{aligned}&|\langle R_{\mathbf{q}}(A)^{-1}\phi, \phi\rangle - \langle R_{\mathbf{q}}(A_n)^{-1}\phi, \phi\rangle| \\ &\leq \|R_{\mathbf{q}}(A)^{-1} - R_{\mathbf{q}}(A_n)^{-1}P_n\| \\ &\quad + d\|I - P_n\|_{\mathcal{B}(V^R_{\mathbb{W}}, V^R_{\mathbb{H}})}\left(\|R_{\mathbf{q}}(A_n)^{-1}\|\|P_n\| + \|R_{\mathbf{q}}(A_n)^{-1}\|\|P_n\|^2\right) \\ &\leq \|R_{\mathbf{q}}(A)^{-1} - R_{\mathbf{q}}(A_n)^{-1}P_n\| + dC_0\|I - P_n\|_{\mathcal{B}(V^R_{\mathbb{W}}, V^R_{\mathbb{H}})} \\ &< \delta.\end{aligned}$$

This yields $\langle R_{\mathbf{q}}(A)^{-1}\phi, \phi\rangle \in B_\delta(W_{\mathbb{H}}(R_{\mathbf{q}}(A_n)^{-1}))$, $\quad n \geq n_0$.

2. Let $\phi \in U_n$ with $\|\phi\| = 1$. As $\phi = P_n\phi$ we have

$$\begin{aligned}|\langle R_{\mathbf{q}}(A_n)^{-1}\phi, \phi\rangle - \langle R_{\mathbf{q}}(A)^{-1}\phi, \phi\rangle &\leq \|R_{\mathbf{q}}(A_n)^{-1}\phi - R_{\mathbf{q}}(A)^{-1}\phi\|\|\phi\| \\ &= \|R_{\mathbf{q}}(A_n)^{-1}\phi - R_{\mathbf{q}}(A)^{-1}\phi\| \\ &\leq \|R_{\mathbf{q}}(A)^{-1} - R_{\mathbf{q}}(A_n)^{-1}P_n\|.\end{aligned}$$

This yields $\langle R_{\mathbf{q}}(A_n)^{-1}\phi, \phi\rangle \in B_\delta(W_{\mathbb{H}}(R_{\mathbf{q}}(A)^{-1}))$, $\quad n \geq n_0$.

□

Corollary 5.4. *Should $(P_n, A_n)_{n\in\mathbb{N}}$ uniformly approximate A quaternionically, then*

$$\overline{W_{\mathbb{H}}(R_{\mathbf{q}}(A)^{-1})} = \left\{\phi \in \mathbb{C} : \exists(\phi_n)_{n\in\mathbb{N}} \text{ with } \phi_n \in W_{\mathbb{H}}(R_{\mathbf{q}}(A_n)^{-1}) \text{ and } \lim_{n\longrightarrow\infty} \phi_n = \phi\right\}$$

or, equivalently,

$$\overline{W_{\mathbb{H}}(R_{\mathbf{q}}(A)^{-1})} = \bigcap_{m\in\mathbb{N}} \overline{\bigcup_{n\geq m} W_{\mathbb{H}}(R_{\mathbf{q}}(A_n)^{-1})}.$$

Proof. Let $(\phi_n)_{n\in\mathbb{N}}$ be a convergent sequence in $\mathbb{C}$ with $\phi_n \in W_{\mathbb{H}}(R_{\mathbf{q}}(A_n)^{-1})$ and define $\phi = \lim_{n\longrightarrow\infty} \phi_n$. Let $\delta > 0$ be arbitrary. Utilizing Lemma 5.3 (2), we can deduce that there exists $n_0 \in \mathbb{N}$ such that $\phi_n \in B_\delta(W_{\mathbb{H}}(R_{\mathbf{q}}(A)^{-1}))$, for every $n \geq n_0$. This implies $\phi \in B_\delta(W_{\mathbb{H}}(R_{\mathbf{q}}(A)^{-1}))$, for every $\delta > 0$, and thus $\phi \in \overline{W_{\mathbb{H}}(R_{\mathbf{q}}(A)^{-1})}$.

Conversely, let $\phi \in W_{\mathbb{H}}(R_{\mathbf{q}}(A)^{-1}, d)$ for some $d > 0$. Using Lemma 5.3 (1), we can construct a sequence $(\phi_n)_{n\in\mathbb{N}}$ in $\mathbb{C}$ with $\phi_n \in W_{\mathbb{H}}(R_{\mathbf{q}}(A_n)^{-1})$ and $\phi = \lim_{n\longrightarrow\infty} \phi_n$.

□

Remark 5.5. *If there exists $n \in \mathbb{N}$ such that $U_n \subset \mathbb{W}$, and considering that U_n is finite-dimensional,*

$$d_n := \sup_{\phi \in U_n} \frac{\|\phi\|_{\mathbb{W}}}{\|\phi\|} < \infty.$$

Applying the same logic as demonstrated in the proof of Lemma 5.3 (2), we can conclude that

$$W_{\mathbb{H}}(R_{\mathbf{q}}(A_n)^{-1}) \subset B_\delta(W_{\mathbb{H}}(R_{\mathbf{q}}(A)^{-1}, d_n)) \ \text{if} \ \|R_{\mathbf{q}}(A)^{-1} - R_{\mathbf{q}}(A_n)^{-1}P_n\| < \delta.$$

Theorem 5.6. *Suppose that $(P_n, A_n)_{n\in\mathbb{N}}$ quaternionic approximates A uniformly. Let*

$$C_0 = \sup_{n\in\mathbb{N}} \Big(\|R_{\mathbf{q}}(A_n)^{-1}\|\|P_n\| + 6\|R_{\mathbf{q}}(A_n)^{-1}\|\|P_n\|^2\Big),$$

$$r > 0, \ \ 0 < \varepsilon \leq \frac{1}{2\|R_{\mathbf{q}}(A)^{-1}\|} \ \text{and} \ 2\|R_{\mathbf{q}}(A)^{-1}\|^2\varepsilon < \delta \leq 8\|R_{\mathbf{q}}(A)^{-1}\|^2\varepsilon.$$

Then we obtain

$$\sigma_\varepsilon^S(A) \cap \overline{B_r(0)} \subset \Big(B_\delta(W_{\mathbb{H}}(R_{\mathbf{q}}(A_n)^{-1}))\Big)^{-1}.$$

Proof. Let $\delta' = 2\|R_{\mathbf{q}}(A)^{-1}\|^2\varepsilon$, $L = r + \varepsilon$ and $d = L\|R_{\mathbf{q}}(A)^{-1}\|_{\mathcal{B}(V_{\mathbb{H}}^R, V_{\mathbb{W}}^R)}$. By using Proposition 5.1 we get

$$\sigma_\varepsilon^S(A) \cap \overline{B_r(0)} \subset \Big(B_\delta(W_{\mathbb{H}}(R_{\mathbf{q}}(A)^{-1}, d))\Big)^{-1}.$$

Let

$$\begin{aligned}\delta - \delta' &\leq 6\|R_{\mathbf{q}}(A)^{-1}\|^2\varepsilon \leq 3\|R_{\mathbf{q}}(A)^{-1}\| = \lim_{n\longrightarrow\infty} 3\|R_{\mathbf{q}}(A_n)^{-1}P_n\| \\ &\leq \lim_{n\longrightarrow\infty} 3\|R_{\mathbf{q}}(A_n)^{-1}\|\|P_n\| \leq \frac{C_0}{2}.\end{aligned}$$

We replaced δ by $\delta - \delta'$ and we can therefore apply Lemma 5.3 and n_0 chosen as stated above and obtain

$$W_{\mathbb{H}}(R_{\mathbf{q}}(A)^{-1}, d) \subset B_{\delta-\delta'}(W_{\mathbb{H}}(R_{\mathbf{q}}(A_n)^{-1})).$$

□

References

[1] D. Alpay, F. Colombo and D. P. Kimsey, *The spectral theorem for quaternionic unbounded normal operators based on the S-spectrum*, J. Math. Phys., 57 (2016), no. 2, 023503, 27 pp.
[2] D. Alpay, F. Colombo and I. Sabadini, *Quaternionic de Branges spaces and characteristic operator function*, SpringerBriefs in Mathematics, Springer, Cham, 2020/21.
[3] D. Alpay, F. Colombo and I. Sabadini, *Slice Hyperholomorphic Schur Analysis*, Operator Theory: Advances and Applications, 256. Birkhauser/Springer, Cham, 2016. xii+362 pp.
[4] Y.H. Au-Yeung and L.S. Siu, *Quaternionic numerical range and real subspaces*. Linear Multilinear Algebra, 45:317-327, 1999.
[5] A. Ammar, A. Jeribi and B. Saadaoui, *Essential S-Spectrum for Quatrenionic Quasi-Compact Operators on a Right Quaternionic Hilbert Spaces*, Acta. Appl. Math. 176, 1 (2021).
[6] A. Ammar, A. Jeribi and B. Saadaoui, *A charactarization of essential pseudospectra of the multivalued operator matrix*, Anal. Math. Phys. 8 (2018), no. 3, 325-350.
[7] A. Ammar, A. Jeribi and B. Saadaoui, *On some classes of demicompact linear relation and some results of essential pseudospectra*, Mat. Stud. 52 (2019), 195-210.
[8] A. Ammar, A. Jeribi and B. Saadaoui, *Characterization of the essential approximation S-spectrum and the essential defect S-spectrum in a right quaternionic Hilbert space*, Filomat 37 (2023), no. 11, 3513-3525.
[9] P. Cerejeiras, F. Colombo, U. Kahler and I. Sabadini, *Perturbation of normal quaternionic operators*, Trans. Amer. Math. Soc., 372 (2019), no. 5, 3257-3281.
[10] F. Colombo and I. Sabadini, *On the formulations of the quaternionic functional calculus*, J Geom Phys. 2010;60(10):1490-1508.
[11] F. Colombo, J. Gantner and D. P. Kimsey, *Spectral theory on the S-spectrum for quaternionic operators*, Operator Theory: Advances and Applications, 270. Birkhauser/Springer, Cham, 2018. ix+356 pp.

[12] F. Colombo and J. Gantner, *Quaternionic closed operators, fractional powers and fractional diffusion processes*, Operator Theory: Advances and Applications, 274. Birkhauser/Springer, Cham, 2019. viii+322 pp.
[13] F. Colombo, I. Sabadini and D.C. Struppa, *Noncommutative functional calculus*. Theory and applications of slice hyperholomorphic functions, Progress in Mathematics, 289. Birkhauser/Springer Basel AG, Basel, 2011. vi+221 pp.
[14] F. Colombo, J. Gantner and S. Pinton, *An Introduction to Hyperholomorphic Spectral Theories and Fractional Powers of Vector Operators*, Adv. Appl. Clifford Algebr., 31 (2021), no. 3, Paper No. 45.
[15] EB. Davies, *Linear operators and their spectra*, New York: Cambridge University Press; (2007).
[16] R. Ghiloni, W. Moretti and A. Perotti, *Continuous slice functional calculus in quaternionic Hilbert spaces*, Rev. Math. Phys. 25 (2013), 1350006.
[17] J. E. Jamison, *Numerical range and numerical radius in quaternionic Hilbert spaces*, Doctoral Dissertation, Univ. of Missouri, (1972).
[18] P. Kumar, *A note on convexity of sections of quaternionic numerical range*, Linear Algebra and its Applications, 572 (2019), 92-116.
[19] R. Kippenhahn, *Uber die Wertvorrat einer Matrix*. Math. Nachr., 6: 193-228, (1951).
[20] R. Kippenhahn, *On the numerical range of a matrix*, Translated from the German by Paul F. Zachlin and Michiel E. Hochstenbach. Linear Multilinear Algebra 56:1-2 (2008), 185-225.
[21] B. Muraleetharan and K. Thirulogasanthar, *Fredholm operators and essential S-spectrum in the quaternionic setting*, J. Math. Phys. 2018;59(10):103506, 27 pp.
[22] L. Rodman, *Topics in Quaternion Linear Algebra*, Princeton University Press, (2014).
[23] B. Saadaoui, *Characterization of the condition S-spectrum of a compact operator in a right quaternionic Hilbert space*, Rend. Circ. Mat. Palermo (2) 72 (2023), no. 1, 707-724.
[24] B. Saadaoui, *On* (P,Q)*-outer generalized inverses and their stability of pseudo spectrum*, Southeast Asian Bull. Math. 46 (2022), no. 5, 633-647.
[25] W. So, R.C. Thompson, and F. Zhang, *The numerical range of normal matrices with quaternion entries*. Linear Multilinear Algebra, 37:175-195, (1994).
[26] LN. Trefethen, *Pseudospectra of linear operators*, SIAM Rev. 1997;39(3):383-406.
[27] J. M. Varah, *The computation of bounds for the invariant subspaces of a general matrix operator.* Thesis (Ph.D.)-Stanford University. ProQuest LLC, Ann Arbor, MI (1967).
[28] K. Viswanath, *Normal operators on quaternionic Hilbert spaces*, Trans. Amer. Math. Soc. 162(1971), 337-350.
[29] JH. Wilkinson, *Sensitivity of eigenvalues*. II. Utilitas Math. 1986;30:243-286.