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New results on the solvability of Sylvester-type operator equations

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Abstract. This paper investigates several forms of Sylvester-type operator equations in infinite-dimensional Hilbert spaces, focusing on both the classical equation $AX - XB = C$ and its generalized version $AX - YB = C$, which involves two unknowns. We establish new necessary and sufficient conditions for the existence of solutions by employing generalized inverses under novel structural assumptions. Special attention is given to the behavior of these equations when restricted to subspaces such as $\ker(A + I)$ and $\ker(B + I)$, and to cases involving two distinct subspaces. The study highlights how operator properties-such as involution and pseudo-inverses-govern solvability and solution structure. The results offer a unified theoretical framework that encompasses both classical and generalized operator equations, with potential applications in control theory, perturbation analysis, and related areas. Illustrative examples are provided to demonstrate the applicability and relevance of the theoretical developments.

1. Introduction

The concept of invertibility plays a fundamental role across various branches of mathematics, including algebra, numerical analysis, and spectral theory. A wide range of problems can be formulated in the form of an operator equation $BY = E$, where B represents a given transformation often a matrix or a bounded linear operator. When B is invertible, the equation has a unique solution given by $Y = B^{-1}E$. However, in many practical and theoretical contexts, B may fail to be invertible. This non-invertibility introduces significant challenges, which motivated the development of an extended notion of inverse-known as the generalized inverse or pseudo-inverse to facilitate solutions in such cases.

The origins of the generalized inverse can be traced back to Fredholm in 1903, who introduced a specific type of pseudoinverse in the context of integral operators that are not classically invertible. A year later, in 1904, Hilbert extended this idea through the introduction of the generalized Green's function, which corresponds to the integral kernel of the pseudoinverse of a differential operator. Subsequently, in 1912, Hurwitz characterized the class of all pseudoinverses. Using the finite-dimensionality of the null spaces of Fredholm operators, he provided a clear algebraic framework for constructing generalized inverses.

Let $\mathcal{H}$ be an infinite-dimensional separable Hilbert space, and let $\mathcal{B}(\mathcal{H})$ denote the algebra of bounded linear operators on $\mathcal{H}$. For any operator $\mathcal{B}(\mathcal{H})$, we denote its spectrum by $\sigma(T)$.

Let $T \in \mathcal{B}(\mathcal{H})$. The concept of a generalized inverse was introduced in [17] as an element $T^p \in \mathcal{B}(\mathcal{H})$ that satisfies the following conditions:

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1. $TT^pT = T$,
2. $T^pTT^p = T^p$,
3. $(TT^p)^* = TT^p$,
4. $(T^pT)^* = T^pT$.

In 1947, Lyapunov discovered a profound relationship between the stability of solutions to systems of linear differential equations and the existence of a positive definite solution to the matrix equation $AX + XA^* = -C$, where C is any positive definite matrix. This result, now known as Lyapunov's theorem, has inspired extensive research into related operator equations particularly the Sylvester equation $AX - XB = C$. This equation has been studied not only in the context of finite-dimensional matrices but also within the broader framework of bounded and unbounded operators on infinite-dimensional spaces.

Many authors have studied the equation (see [10–13]). Equation (1) was first examined in the finite-dimensional case, where a foundational result was established by Sylvester in 1884 [11]. Remarkably, analyzing the conditions for the existence of solutions to the equation $AXB - EXD = C$ leads to significant insights across a wide range of topics, including similarity transformations, commutativity of operators, hyperinvariant subspaces, spectral operators, and differential equations. Some of these topics are discussed below. We also derive several distinct explicit forms of the solution and illustrate their effectiveness in applications such as perturbation theory. In addition, special attention is given to the operator equation $AXB - EXD = C$, which presents further theoretical interest and practical relevance.

In 1987, J. Bevis, F. Hall, and R. Hartwig established the following Theorem:

Theorem 1.1. *[6] Let $\widetilde{X}$ denote the matrix whose entries are complex-conjugates of the entries of $X \in \mathbb{C}_{m,m}$. Then $AX - \widetilde{X}B = C$ over $\mathbb{C}$ has a solution if and only if*

$$\overline{S^{-1}}\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}, \; S = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix},$$

for some nonsingular S.

Let $\mathcal{H}$ be an infinite-dimensional separable Hilbert space and let $\mathcal{B}(\mathcal{H})$ denotes the algebra of bounded linear operators acting on $\mathcal{H}$. If $T \in \mathcal{B}(\mathcal{H})$, we write $\sigma(T)$ for the spectrum of T.

As highlighted in [6, 8, 10], a more comprehensive study of the Sylvester operator equation is warranted. In particular, we focus on analyzing the existence and properties of solutions X and Y to the equation $AX - YB = C$ for given operators A, B, and C in $\mathcal{B}(\mathcal{H})$. This formulation also encompasses the classical Sylvester equation $AX - XB = C$, which has been previously examined in works such as [7, 8].

In this paper, we investigate necessary and sufficient conditions for the existence of solutions to Sylvester-type equations, with a particular emphasis on operator theoretic settings,

$$AX - XB = C, \tag{1.1}$$

$$AXB - EXD = C, \tag{1.2}$$

$$AXB - XD = C, \tag{1.3}$$

and

$$AXB - X = C, \tag{1.4}$$

receptively, where $A, B, C \in B(H)$ are given. Moreover, we also explore a more generalized form of the following Sylvester equations:

$$AX - YB = C, \tag{1.5}$$

$$AXB - EYD = C, \tag{1.6}$$

$$AXB - YD = C, \tag{1.7}$$

and

$$AXB - Y = C, \tag{1.8}$$

receptively.

The structure of this paper is organized as follows. In Section 3, we study the classical Sylvester operator equation of the form $AX - XB = C$, highlighting fundamental properties and solution conditions. Section 4 extends this framework to a generalized version, $AX - YB = C$, and introduces a coupled system involving two unknown operators. Subsection 4 focuses on analyzing the equation $AX - XB = C$ when restricted to two subspaces, providing additional insights into its behavior under decomposition. Sections 4.1 and 4.2 are dedicated to studying the same operator equation on the subspaces $\ker(A + I)$ and $\ker(B + I)$, respectively, revealing structural conditions that affect solvability. In Section 5, we present illustrative examples to demonstrate the applicability of the theoretical results. Finally, Section 6 concludes the paper with a summary of the main findings and possible directions for future research.

2. Sylvester operator equation $AX - XB = C$

Let $A, B, C, D, E \in B(H)$, where $C \neq 0$, let X be a solution to the system of Sylvester operator equations (1.1)–(1.4), which arise in the study of certain control problems [2,4,6,9]. These equations have been investigated in various contexts by numerous authors [9, 13, 14]. In this section, we examine several properties of such a solution X. In particular, we establish the existence of a solution X to the system (1.1)–(1.4) under appropriate conditions. Recall that A^p denotes the pseudo-inverse of the operator A.

Theorem 2.1. *Let $A, B, C \in \mathcal{B}(\mathcal{H})$ be given.*

i. If $AB = BA$ and A^* is injective. Then $A^pB = BA^p$.

ii. If $AB = BA$ and B^* is injective. Then $AB^p = B^pA$.

Proof. i. We have

$$\begin{aligned} AB &= AA^pAB \\ &= AA^pBA, \end{aligned}$$

and

$$\begin{aligned} BA &= BAA^pA \\ &= ABA^pA. \end{aligned}$$

Also,

$$AA^pBA = ABA^pA.$$

Since A^* is injective, we get

$$A^pB = BA^p.$$

ii. We have

$$\begin{aligned} AB &= ABB^pB, \\ &= BAB^pB, \end{aligned}$$

and

$$\begin{aligned} BA &= BB^pBA, \\ &= BB^pAB. \end{aligned}$$

So,

$$BAB^pB = BB^pAB.$$

Since B^* is injective, we obtain

$$AB^p = B^pA.$$

Therefore, the desired results are achieved. □

Theorem 2.2. *Let $A, B, C \in \mathcal{B}(\mathcal{H})$ satisfy all the conditions stated in [12, 15, 16]. Under these assumptions, we conclude the following result: If $AB = BA$ then $A^pB^p = B^pA^p$.*

Proof. Following [12, 16], we have $(AB)^p = B^pA^p$. So,

$$\begin{aligned} (BA)^p &= A^pB^p, \\ (AB)^p &= B^pA^p, \end{aligned}$$

$$(AB)^p = (BA)^p, \quad \textit{because} \quad AB = BA.$$

Thus,

$$A^pB^p = B^pA^p.$$

□

Theorem 2.3. *Let $A, B, C \in \mathcal{B}(\mathcal{H})$ such that $BC = CB$ and $AC = CA$. Then the operator equation* (1.1) *has a solution:*

1. *$X = B^pC$ if and only if*

$$I - AB^p + B^pB = 0.$$

2. *$X = A^pC$ if and only if*

$$I - AA^p + A^pB = 0.$$

where A^p and B^p are the pseudo-inverse of A and B, respectively.

Proof. 1. Suppose that $I - AB^p + B^pB = 0$ where B^p is the pseudo-inverse of B. Then we multiply C on the left

$$\begin{aligned} (I - AB^p + B^pB)C &= 0, \\ C - AB^pC + B^pBC &= 0, \\ AB^pC - B^pBC &= C, \\ AB^pC - B^pCB &= C. \end{aligned}$$

Taking $X = B^pC$ where $BC = CB$, we get the desired result.

2. Suppose that $I - AA^p + A^pB = 0$ where A^p is the pseudo-inverse of A, then we multiply C on the left

$$\begin{aligned} (I - AA^p + A^pB)C &= 0, \\ C - AA^pC + A^pBC &= 0, \\ AA^pC - A^pBC &= C, \\ AA^pC - A^pCB &= C. \end{aligned}$$

Taking $X = A^pC$ where $AC = CA$, we get the desired result.
□

Remark 2.4. *Similarly, if we multiply C on the right, we obtain*

- *If* $C(I - AB^p + B^pB) = 0$ *taking* $X = CB^p$, *we get the desired result.*
- *If* $C(I - AA^p + A^pB) = 0$ *taking* $X = CA^p$, *we get the desired result.*

Corollary 2.5. *Let* $A, B, C \in B(H)$.

1 If A is invertible, then the operator equation (1.1) has a solution

$$X = A^{-1}C. \tag{2.1}$$

2 If B is invertible, then the operator equation (1.1) has a solution

$$X = B^{-1}C. \tag{2.2}$$

Proof. 1. If A is invertible, then $A^{-1} = A^p$. Hence we obtain the same result as in the case where AA is invertible.

$$I - AA^p + A^pB = 0.$$

By applying equation (2.1), it follows from Theorem 2.3 that the operator equation (1.1) admits a solution.

2. If B is invertible, then $B^{-1} = B^p$. Proceeding as in the case when A is invertible, we arrive at equation (2.2).
□

Corollary 2.6. *Let* $A, B, C \in \mathcal{B}(\mathcal{H})$ *be given. If A or B is is involution, then the operator equation* (1.1) *has a solution.*

Proof. If A or B is involution, that is, $A^2 = I$, then

$$AA^p = I = A^pA.$$

Hence

$$A^p = A^{-1}$$

and hence A is invertible.

Similarly, if $B^2 = I$, it follows that $BB^p = I = B^pB$ and so $B^p = B^{-1}$. Thus, the Corollary 2.5 confirms that the operator equation (1.1) has a solution. □

Theorem 2.7. *Let* $A, B, C, D, E \in \mathcal{B}(\mathcal{H})$ *be given B commutes with C, E, and D, E commutes with A, and D, C commutes with D,* E^{p*} *and* B^{p*} *are injectives. Then the operator equation*(1.2) *has a solution*

$$X = E^pEA^pCB^p$$

if and only if

$$(I - E^pAA^pE + A^pEDB^p)E^pCB^p = 0,$$

where A^p, B^p *and* E^p *are the pseudo-inverse of* A, B *and* E*, respectively.*

Proof. Suppose that

$$(I - E^pAA^pE + A^pEDB^p)E^pCB^p = 0,$$

where A^p, B^p and E^p denote the pseudo-inverses of A, B and E, respectively. Then,

$$\begin{aligned} (I - E^pAA^pE + A^pEDB^p)E^pCB^p &= 0,\\ E^pCB^p - E^pAA^pEE^pCB^p + A^pEDB^pE^pCB^p &= 0,\\ E^pAA^pEE^pCB^p - A^pEDB^pE^pCB^p &= E^pCB^p. \end{aligned}$$

Since $B^{p*}$is injective and from Theorem 2.1, we get

$$\begin{aligned} E^pAA^pEE^pCB^pB - A^pEDB^pE^pC &= E^pC,\\ E^pAE^pEA^pCB^pB - E^pEA^pCB^pD &= E^pC. \end{aligned}$$

Also, since $E^{p*}$is injective and from Theorem 2.1, we obtain

$$AE^pEA^pCB^pB - EE^pEA^pCB^pD = C. \tag{2.3}$$

Since equation (2.3) is of the form (1.2), a solution is given by

$$X = E^pEA^pCB^p.$$

□

Corollary 2.8. *Let* $A, B, C, D, E \in B(H)$ *be given such that:* B *commutes with* C, E*, and* D*,* E *commutes with* D*,* C *commutes with* D*. The adjoint pseudo-inverses* E^{p*} *and* B^{p*} *are injectives, if and only if*

$$(I - E^pAB^pE + B^pEDB^p)E^pCB^p = 0.$$

In this case, the operator equation (1.2) *has a solution if and only if*

$$X = E^pECB^pB^p$$

where A^p *is the pseudo-inverse of* A*,* B^p *is the pseudo-inverse of* B*,* E^p *is the pseudo-inverse of* E*.*

Proof. Assume that $(I - E^pAB^pE + B^pEDB^p)E^pCB^p = 0$ where B^p is the pseudo-inverse of B. Then,

$$\begin{aligned} (I - E^pAB^pE + B^pEDB^p)E^pCB^p &= 0,\\ E^pCB^p - E^pAB^pEE^pCB^p + B^pEDB^pE^pCB^p &= 0,\\ E^pAB^pEE^pCB^p - B^pEDB^pE^pCB^p &= E^pCB^p. \end{aligned}$$

Since B^{p*} is injective, we get

$$E^pAB^pEE^pCB^pB - B^pEDB^pE^pC = E^pC.$$

The fact that E^{p*} is injective and for Theorem 2.1 give

$$AE^pEB^pCB^pB - EE^pEB^pCB^pD = C. \tag{2.4}$$

Since equation (2.4) is of the same form as (1.2), a solution is given by

$$X = E^pECB^pB^p.$$

□

Corollary 2.9. *Let $A, B, C, D \in \mathcal{B}(\mathcal{H})$, assume that E^{p*} and B^{p*} are injectives. Then,*

1 *If A is invertible,then the operator equation* (1.2) *has a solution*

$$X = E^p E A^{-1} C B^p.$$

2 If B is invertible,then the operator equation (1.2) has a solution

$$X = E^p E A^p C B^{-1} \text{or } X = E^p E C (B^{-1})^2.$$

3 if E is invertible,then the operator equation (1.2) has a solution

$$X = A^p C B^p \text{or } X = C (B^p)^2.$$

Proof. It follows from Theorem 2.7 and corollary 2.8 that the operator equation (1.2) has a solution

$$(I - E^p A A^p E + A^p E D B^p) E^p C B^p = 0 \text{ or } (I - E^p A B^p E + B^p E D B^p) E^p C B^p = 0.$$

Taking

$$\begin{aligned} \text{If } A \text{ is invertible, then } A^{-1} &= A^p \Rightarrow X = E^p E A^{-1} C B^p \\ \text{If } B \text{ is invertible, then } B^{-1} &= B^p \Rightarrow X = E^p E A^p C B^{-1} \text{or } X = E^p E C (B^{-1})^2 \\ \text{If } E \text{ is invertible, then } E^{-1} &= E^p \Rightarrow X = A^p C B^p \text{or } X = C (B^p)^2. \end{aligned}$$

□

Corollary 2.10. *Let $A, B, C \in B(H)$ be given,E^{p*} and B^{p*} are injectives. If A or B is is involution, then the operator equation* (1.2) *has a solution.*

Proof. If A or B is involution ie $A^2 = I$,it follows that $AA^p = I = A^p A$ and so $A^p = A^{-1}$. Hence A is invertible. Similarly, $B^2 = I$,it follows that $BB^p = I = B^p B$ and so $B^p = B^{-1}$Then we get that it follows from corollary 2.9 that the operator equation (1.2) has a solution . □

Remark 2.11. *Let $A, B, C \in B(H)$ be given,E^{p*} and B^{p*} are injectives. If A and B are invertible et $A^{-1} = B^{-1}$, then the operator equation* (1.2) *has a solution*

$$X = E^p E A^{-1} C A^{-1} \textit{or } X = E^p E C (A^{-1})^2.$$

Remark 2.12. *Let $A, B, C \in B(H)$ be given,E^{p*} and B^{p*} are injectives. If A and B are invertible et $A^{-1} = B^{-1} = E^{-1}$, then the operator equation* (1.2) *has a solution*

$$X = A^{-1} C A^{-1} \textit{or } X = C (A^{-1})^2.$$

Remark 2.13. *Let $A, B, C \in B(H)$ be given. If B and E are invertible, then the operator equation*

$$AXB - EXD = C \Rightarrow E^{-1} A X - X D B^{-1} = E^{-1} C B^{-1}.$$

in the form of equation (1.1), then the solution has a solution

$$X = A^p C B^{-1} \text{or } X = (C B^{-1})^2.$$

Proposition 2.14. *Let $A, B, C, D \in B(H)$ such that $DB = BD$,$DC = CD$, B^{p*} is injective. Then the operator equation*(1.3)*, has a solution*

$$X = A^p C B^p$$

if and only if

$$(I - AA^p + A^pDB^p)CB^p = 0$$

where A^p and B^p are the pseudo-inverse of A and B, respectively.

Proof. Suppose that $(I - AA^p + A^pDB^p)CB^p = 0$ where A^p is the pseudo-inverse of A

$$\begin{aligned} (I - AA^p + A^pDB^p)CB^p &= 0, \\ CB^p - AA^pCB^p + A^pDB^pCB^p &= 0, \\ AA^pCB^p - A^pDB^pCB^p &= CB^p. \end{aligned}$$

Since $B^{p*}$is injective

$$AA^pCB^pBB^p - A^pDB^pCB^p = CB^p$$

and for Theorem 2.1

$$AA^pCB^pB - A^pCB^pD = C. \tag{2.5}$$

Since the equation (2.5) is of the form (1.3), then the solution taking $X = A^pCB^p$. □

Corollary 2.15. *Let $A, B, C, D \in B(H)$ such that $DB = BD, DC = CD$, B^{p*} is injective if and only if*

$$(I - AB^p + B^pDB^p)CB^p = 0.$$

Then the operator equation (1.3) *has a solution*

$$X = B^pCB^p$$

where B^p are the pseudo-inverse of B.

Proof. Suppose that $(I - AB^p + B^pDB^p)CB^p = 0$ where B^p is the pseudo-inverse of B

$$\begin{aligned} (I - AB^p + B^pDB^p)CB^p &= 0, \\ CB^p - AB^pCB^p + B^pDB^pCB^p &= 0, \\ AB^pCB^p - B^pDB^pCB^p &= CB^p, \\ AB^pCB^pBB^p - B^pDB^pCB^p &= CB^p. \end{aligned}$$

The fact that B^{p*} is injective

$$AB^pCB^pB - B^pCB^pD = C$$

and Theorem 2.1 give

$$ACB^pB^pB - CB^pB^pD = C \tag{2.6}$$

Since the eqeation (2.6) is of the form (1.3), then the solution taking $X = C(B^p)^2$. □

Corollary 2.16. *Let $A, B, C, D \in B(H)$, $B^{p*}$is injective be given.*

1. *If A is invertible, then the operator equation* (1.3) *has a solution*
$$X = A^{-1}CB^p. \tag{2.7}$$
2. *If B is invertible, then the operator equation* (1.3) *has a solution of the form*
$$X = A^pCB^{-1} \text{ or } X = C(B^{-1})^2. \tag{2.8}$$

Proof. 1. If A is invertible, then $A^{-1} = A^p$. Hence

$$(I - AA^p + A^p DB^p)CB^p = 0 \text{ or } (I - AB^p + B^p DB^p)CB^p = 0.$$

It follows from Proposition 2.14 and Corollary 2.15 that (2.7) solves the operator equation(1.3).

2. If B is invertible and $B^{-1} = B^p$, then the same reasoning used for the case of A invertible leads to equation (2.8).
□

Corollary 2.17. *Let $A, B, C \in \mathcal{B}(\mathcal{H})$, assume that B^{p*} is injective. If A or B is is involution, then the operator equation* (1.3) *has a solution.*

Proof. If A or B is involution that is $A^2 = I$, it follows that $AA^p = I = A^pA$ and so $A^p = A^{-1}$. Hence A is invertible.

Similarly, if $B^2 = I$, it follows that $BB^p = I = B^pB$ and so $B^p = B^{-1}$. Following Corollary 2.16, we deduce that the operator equation(1.3) has a solution. □

Remark 2.18. *Let $A, B, C \in \mathcal{B}(\mathcal{H})$, assume that B^{p*} is injective. If A and B are invertible and $A^{-1} = B^{-1}$, then the operator equation* (1.3) *has a solution of the form*

$$X = A^{-1}CA^{-1} \;\; or \;\; X = C(B^{-1})^2.$$

Remark 2.19. *Let $A, B, C \in Ł(\mathcal{H})$. If B is invertible, then the operator equation*

$$AXB - XD = C \Rightarrow AX - XDB^{-1} = CB^{-1}$$

has a solution

$$X = A^pCB^{-1} or\ X = C(B^{-1})^2.$$

Remark 2.20. *Let $A, B, C, D, E \in \mathcal{B}(\mathcal{H})$. The operator equation* (1.2) *reduces to equation* (1.3) *if $E = I$.*

Proposition 2.21. *Let $A, B, C \in \mathcal{B}(\mathcal{H})$, assume that $BC = CB$, B^{p*} is injective. Then the operator equation* (1.4) *has a solution $X = A^pCB^p$, if and only if*

$$(I - AA^p + A^pB^p)CB^p = 0,$$

where A^p and B^p are the pseudo-inverse of A and B, respectively.

Proof. Suppose that $(I - AA^p + A^pB^p)CB^p = 0$ where A^p is the pseudo-inverse of A

$$\begin{aligned} (I - AA^p + A^pB^p)CB^p &= 0, \\ CB^p - AA^pCB^p + A^pB^pCB^p &= 0, \\ AA^pCB^p - A^pB^pCB^p &= CB^p, \\ AA^pCB^pBB^p - A^pB^pCB^p &= CB^p. \end{aligned}$$

Since B^{p*} is injective

$$AA^pCB^pB - A^pCB^p = C. \tag{2.9}$$

Given that equation (2.9) has the same structure as equation (1.4), it admits the solution $X = A^pCB^p$. □

Corollary 2.22. *Let $A, B, C \in \mathcal{B}(\mathcal{H})$. Assume that $BC = CB$. So, the pseudo-inverse B^{p*} is injective if and only if,*

$$(I - AB^p + (B^p)^2)CB^p = 0.$$

In this case, the operator equation (1.4) *has a solution*

$$X = B^pCB^p,$$

where B^p *are the pseudo-inverse of* B.

Proof. Suppose that $(I - AB^p + (B^p)^2)CB^p = 0$ where B^p is the pseudo-inverse of B

$$\begin{aligned} (I - AB^p + B^pB^p)CB^p &= 0, \\ CB^p - AB^pCB^p + B^pB^pCB^p &= 0, \\ AB^pCB^p - B^pB^pCB^p &= CB^p, \\ AB^pCB^pBB^p - B^pB^pCB^p &= CB^p. \end{aligned}$$

Since B^{p*} is injective

$$AB^pCB^pB - B^pCB^p = C. \tag{2.10}$$

Since the equation (2.10) takes the form (1.4), then $X = B^pCB^p$ solves it. $X = B^pCB^p$. □

Corollary 2.23. *Let* $A, B, C \in \mathcal{B}(\mathcal{H})$. *We have*

1. *If* A *is invertible and* B^{p*} *is injective, then the operator equation* (1.4) *has a solution*

$$X = A^{-1}CB^p. \tag{2.11}$$

2. *If* B *is invertible and* B^{p*} *is injective, then the operator equation* (1.4) *has a solution*

$$X = A^pCB^{-1}, \quad or \quad X = B^{-1}CB^{-1}. \tag{2.12}$$

Proof. 1. If A is invertible, then $A^{-1} = A^p$. Then

$$(I - AA^p + A^pB^p)CB^p = 0 \text{ or } (I - AB^p + B^pB^p)CB^p = 0.$$

Proposition 2.14 and Corollary 2.22 confirm that the operator equation (1.4) has a solution of the form (2.11).

2. If B is invertible with $B^{-1} = B^p$, then the same reasoning used for the invertibility of A leads to equation (2.12).

□

Corollary 2.24. *Let* $A, B, C \in \mathcal{B}(\mathcal{H})$. *If either* A *or* B *is an involution, then the operator equation* (1.4) *has a solution.*

Proof. If A or B is involution, it follows that $AA^p = I = A^pA$ and so $A^p = A^{-1}$. Hence A is invertible.
Similarly, if $B^2 = I$, then B is an involution, which implies $BB^p = I = B^pB$ and hence $B^p = B^{-1}$. It then follows from the corollary that the operator equation (1.4) admits a solution. □

Remark 2.25. *Let* $A, B, C \in \mathcal{B}(\mathcal{H})$. *If* B *is invertible, then the operator equation*

$$AXB - X = C \Rightarrow AX - XB^{-1} = CB^{-1}$$

witch takes the form (1.1) has a solution

$$X = A^pCB^{-1} \text{or } X = B^{-1}CB^{-1}.$$

Remark 2.26. *Let* $A, B, C, D, E \in \mathcal{B}(\mathcal{H})$. *Then*

1. *The operator equation* (1.3) *in the form of equation* (1.1) *if* $D = I$.
2. *The operator equation* (1.2) *in the form of equation* (1.4) *if* $E = D = I$.

3. Sylvester operator equation of the form $AX - YB = C$

Let $A, B, C, D, E \in \mathcal{B}(\mathcal{H})$ with $C \neq 0$. Let X, Y be a solution pair to the system of generalized operator equations (1.5), (1.6), (1.7), and (1.8), which arise in the study of certain control problems [11, 17]. This system has been investigated by several authors in various contexts [6, 8]. In particular, [9] established necessary and sufficient conditions for the existence of a solution pair (X, Y) to the generalized Sylvester equation (1.5).

In this section, we explore further properties of such a solution pair (X, Y). Specifically, we prove the existence of solutions to the system (1.5)-(1.8) under suitable assumptions. Recall that B^p denotes the pseudo-inverse of the operator B.

Theorem 3.1. *Let $A, B, C, D, E \in \mathcal{B}(\mathcal{H})$ E^{p*} and B^{p*} are injective be given. Then the operator equation* (1.6) *has a solution pair*

$$(X, Y) = (A^pEE^PCB^p, \ \ -E^p(I - AA^PEE^p)CB^PBD^PD$$

if and only if

$$(I - E^pAA^pE)E^pCB^p(I - BD^PDB^P) = 0,$$

where A^p is the pseudo-inverse of A, B^p is the pseudo-inverse of B, D^p is the pseudo-inverse of D, E^p is the pseudo-inverse of E.

Proof. Assume that

$$(I - E^pAA^pE)E^pCB^p(I - BD^PDB^P) = 0,$$

where A^p, B^p, D^p, and E^p denote the pseudo-inverses of the operators A, B, D, and E, respectively.

So,

$$\begin{aligned}
&(I - E^pAA^pE)E^pCB^p(I - BD^PDB^P) = 0,\\
&(E^pCB^p - E^pAA^pEE^pCB^p)(I - BD^PDB^P) = 0,\\
&E^pCB^p - E^pAA^pEE^pCB^p - E^pCB^pBD^PDB^P + E^pAA^pE^pCB^pBD^PDB^P = 0,\\
&E^pAA^pEE^pCB^p + E^p(I - AA^PEE^p)CB^PBD^PDB^P = E^pCB^p.
\end{aligned}$$

Since $E^{p*}$and B^{p*} are injective

$$AA^pEE^pCB^pB + EE^p(I - AA^PEE^p)CB^PBD^PD = C. \tag{3.1}$$

Since equation (3.1) is of the form (1.6), a corresponding solution pair (X, Y) is given by

$$(X, Y) = (A^pEE^PCB^p, -E^p(I - AA^PEE^p)CB^PBD^PD.$$

□

Corollary 3.2. *Let $A, B, C, D, E \in B(H)$, and suppose that $E^{p*}$and B^{p*} are injective.*

1. *If A is invertible, then the operator equation* (1.6) *admits a solution pair*

$$(X, Y) = (A^{-1}EE^PCB^p, -E^p(I - EE^p)CB^PBD^PD).$$

2. *If B is invertible, then the operator equation* (1.6) *admits a solution pair*

$$(X, Y) = (A^pEE^PCB^{-1}, -E^p(I - AA^PEE^p)CD^PD)$$

3. *If D is invertible, then the operator equation* (1.6) *admits a solution pair*

$$(X, Y) = (A^p E E^P C B^p, -E^p(I - AA^P EE^p)CB^P B)$$

4. *If E is invertible, then the operator equation* (1.6) *admits a solution pair*

$$(X, Y) = (A^p C B^p, -E^{-1}(I - AA^P)CB^P BD^P D).$$

Here, A^p, B^p, D^p, and E^p denote the pseudo-inverses of the operators A, B, D, and E, respectively.

Proof. It follows from Theorem 3.1 that the operator equation (1.6) has a solution if and only if

$$(I - AA^p)CB^p(I - BD^P DB^P) = 0.$$

Note that

$$\begin{aligned}
\text{If } A \text{ is invertible, then } A^{-1} &= A^p \Rightarrow (X, Y) = (A^{-1}EE^P CB^p, -E^p(I - EE^p)CB^P BD^P D).\\
\text{If } B \text{ is invertible, then } B^{-1} &= B^p \Rightarrow (X, Y) = (A^p EE^P CB^{-1}, -E^p(I - AA^P EE^p)CD^P D).\\
\text{If } D \text{ is invertible, then } D^{-1} &= D^p \Rightarrow (X, Y) = (A^p EE^P CB^p, -E^p(I - AA^P EE^p)CB^P B).\\
\text{If } E \text{ is invertible, then } E^{-1} &= E^p \Rightarrow (X, Y) = (A^p CB^p, -E^{-1}(I - AA^P)CB^P BD^P D).
\end{aligned}$$

□

Remark 3.3. *If A and E are invertibles, then the operator equation* (1.6) *has a solution pair* $(X, Y) = (A^{-1}CB^p, 0)$.

Corollary 3.4. *Let $A, B, C, D, E \in B(H)$ be given. If A or B or D or E is involution, then the operator equation* (1.6) *has a solution.*

Proof. Suppose that any one of the operators A, B, D, or E is an involution, i.e., satisfies $A^2 = I$, $B^2 = I$, $D^2 = I$, or $E^2 = I$, respectively.

It follows that $AA^p = I = A^p A$ and so $A^p = A^{-1}$. Hence A is invertible. By Corollary 3.2, the operator equation (1.6) has a solution.

Similarly, $B^2 = I$, it follows that $BB^p = I = B^p B$ and so $B^p = B^{-1}$. Then we get the same proof of B invertible.

The same reasoning applies if $D^2 = I$ then $DD^p = I = D^p D$ and so $D^p = D^{-1}$, showing that D is invertible.

Finally, if $E^2 = I$, we conclude that $EE^p = I = E^p E$ and so $E^p = E^{-1}$, and E is invertible. □

Remark 3.5. *Let $A, B, C, D \in B(H)$. If B and E are invertible, then the operator equation*

$$AXB - EYD = C \Rightarrow E^{-1}AX - YDB^{-1} = E^{-1}CB^{-1}$$

takes the form of equation (1.5), *(see, [9]).*

Proposition 3.6. *Let $A, B, C, D, E \in B(H)$, The following statements hold.*

If (1.6) *has a solution pair* (X, Y) *, then*

$$\left(\begin{pmatrix} A & C \\ 0 & D \end{pmatrix}, \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}, \begin{pmatrix} E & 0 \\ 0 & I \end{pmatrix}\right)$$

$$\text{and} \left(\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}, \begin{pmatrix} E & 0 \\ 0 & I \end{pmatrix}\right)$$

are equivalent on $\mathcal{H} \oplus \mathcal{H}$.

Proof. Let (1.6) has a solution pair (X, Y) , then

$$\begin{pmatrix} I & EY \\ 0 & I \end{pmatrix}\begin{pmatrix} A & C \\ 0 & D \end{pmatrix}, = \begin{pmatrix} A & C+EYD \\ 0 & D \end{pmatrix}$$

$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}\begin{pmatrix} I & XB \\ 0 & D \end{pmatrix}, = \begin{pmatrix} A & AXB \\ 0 & D \end{pmatrix}$$

$$\begin{pmatrix} A & C+YD \\ 0 & D \end{pmatrix} = \begin{pmatrix} A & AXB \\ 0 & D \end{pmatrix}$$

and

$$\begin{pmatrix} I & X \\ 0 & I \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}, = \begin{pmatrix} I & XB \\ 0 & B \end{pmatrix}$$

$$\begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} I & XB \\ 0 & I \end{pmatrix}, = \begin{pmatrix} I & XB \\ 0 & B \end{pmatrix}$$

and

$$\begin{pmatrix} I & EY \\ 0 & I \end{pmatrix}\begin{pmatrix} E & 0 \\ 0 & I \end{pmatrix}, = \begin{pmatrix} E & EY \\ 0 & I \end{pmatrix}$$

$$\begin{pmatrix} E & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} I & Y \\ 0 & I \end{pmatrix}, = \begin{pmatrix} E & EY \\ 0 & I \end{pmatrix}$$

□

Proposition 3.7. *Let $A, B, C \in \mathcal{B}(\mathcal{H})$. Suppose that (M, W, L) and (N, W, L) are equivalent on $H \oplus H$, meaning there exist invertible operators U and V such that $UM = NV$, $UW = WV$ and $UL = LV$ with the additional condition that $NV = VN$. Then the operator equation* (1.6) *admits a solution pair* (X, Y).

Proof. Suppose that

$$\left[M = \begin{pmatrix} A & C \\ 0 & I \end{pmatrix}, W = \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}, L = \begin{pmatrix} E & 0 \\ 0 & I \end{pmatrix}\right]$$

and

$$\left[N = \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}, W = \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}, L = \begin{pmatrix} E & 0 \\ 0 & I \end{pmatrix}\right]$$

are equivalent on $H \oplus H$. Then there exist invertible operator matrices $U = \begin{pmatrix} U_1 & U_2 \\ U_3 & U4 \end{pmatrix}$ and $V = \begin{pmatrix} V_1 & V_2 \\ V_3 & V_4 \end{pmatrix}$ on $H \oplus H$ such that

$$\begin{aligned} UM &= NV \\ \begin{pmatrix} U_1 & U_2 \\ U_3 & U4 \end{pmatrix}\begin{pmatrix} A & C \\ 0 & D \end{pmatrix} &= \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}\begin{pmatrix} V_1 & V_2 \\ V_3 & V_4 \end{pmatrix} \\ UW &= WV \\ \begin{pmatrix} U_1 & U_2 \\ U_3 & U4 \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix} &= \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} V_1 & V_2 \\ V_3 & V_4 \end{pmatrix} \\ UL &= LV \\ \begin{pmatrix} U_1 & U_2 \\ U_3 & U4 \end{pmatrix}\begin{pmatrix} E & 0 \\ 0 & I \end{pmatrix} &= \begin{pmatrix} E & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} V_1 & V_2 \\ V_3 & V_4 \end{pmatrix} \end{aligned}$$

This implies

$$\begin{cases} U_1A = AV_1 \\ U_1C + U_2D = AV_2 \\ U_3A = DV_3 \\ U_3C + U_4D = DV_4 \end{cases}$$

and

$$\begin{cases} U_1 = V_1 \\ U_2B = V_2 \\ U_3 = BV_3 \\ U_4B = V_4 \end{cases}$$

and

$$\begin{cases} U_1E = EV_1 \\ U_2 = EV_2 \\ U_3E = V_3 \\ U_4 = V_4 \end{cases}$$

then

$$\begin{aligned} U_1C + U_2D &= AU_2B \\ U_1C &= AU_2B - EV_2D \\ U_3C + U_4D &= V_4\,D \text{ where } NV = VN \\ U_1C &= AU_2B - EV_2D. \end{aligned}$$

Since U is invertible, then$U_1$is invertible , it follows that

$$C = AU_1{}^{-1}U_2B - EU_1{}^{-1}V_2D \tag{3.2}$$

Since the equation (3.2) takes the form (1.6), then the solution take

$$X = U_1{}^{-1}U_2\,;\, Y = U_1{}^{-1}V_2.$$

□

Proposition 3.8. *Let $A, B, C, D \in \mathcal{B}(\mathcal{H})$,B^{p*} is injective. Then the operator equation* (1.7) *has a solution pair*

$$(X, Y) = (A^pCB^p, -(I - AA^p)CB^PBD^p$$

if and only if

$$(I - AA^p)CB^p(I - BD^PDB^P) = 0$$

where A^p is the pseudo-inverse of A, B^p is the pseudo-inverse of B, D^p is the pseudo-inverse of D.

Proof. Suppose that $(I - AA^p)CB^p(I - BD^PDB^P) = 0$,where A^p is the pseudo-inverse of A, B^p is the pseudo-inverse of B, D^p is the pseudo-inverse of D.

Then

$$\begin{aligned}(I - AA^p)CB^p(I - BD^PDB^P) &= 0\\ (CB^p - AA^pCB^p)(I - BD^PDB^P) &= 0\\ CB^p - AA^pCB^p - CB^PBD^PDB^P + AA^pCB^PBD^PDB^P &= 0\\ AA^pCB^p + (I - AA^P)CB^PBD^PDB^P &= CB^p.\end{aligned}$$

Since B^{p*} is injective, we get

$$AA^pCB^pB + (I - AA^P)CB^PBD^PD = C. \tag{3.3}$$

Since the equation (3.3) is of the form (1.7), then the solution pair takes the form

$$(X, Y) = (A^pCB^p, -(I - AA^p)CB^PBD^p).$$

□

Corollary 3.9. *Let $A, B, C, D \in \mathcal{B}(\mathcal{H})$, $B^{p*}$is injective be given,*

1. *If A is invertible, then the operator equation* (1.7) *has a solution pair*

$$(X, Y) = (A^{-1}CB^p, 0)\ . \tag{3.4}$$

2. *IfA is invertible, then the operator equation* (1.7) *has a solution pair*

$$(X, Y) = (A^pCB^{-1}, -(I - AA^P)CD^PD). \tag{3.5}$$

3. *If D is invertible, then the operator equation* (1.7) *has a solution pair*

$$(X, Y) = (A^pCB^p, -(I - AA^P)CB^PB) \tag{3.6}$$

where A^p is the pseudo-inverse of A, B^p is the pseudo-inverse of B, D^p is the pseudo-inverse of D.

Proof. 1 If A is invertible , then $A^{-1} = A^p$. Then we get that

$$(I - AA^p)CB^p(I - BD^PDB^P) = 0.$$

It follows from Proposition 3.1 that the operator equation (1.7) has a solution Taking (3.4).

2 If B is invertible , then $B^{-1} = B^p$. Then we get that (3.5) .

3 If D is invertible , then $D^{-1} = D^p$. Then we get that (3.6).
□

Corollary 3.10. *Let $A, B, C \in \mathcal{B}(\mathcal{H})$ be given. If A or B is is involution, then the operator equation* (1.7) *has a solution.*

Proof. If A or B is involution ie $A^2 = I$, it follows that $AA^p = I = A^pA$ and so $A^p = A^{-1}$. Hence A is invertible. It then follows from Corollary 3.9 that the operator equation (1.7) admits a solution.

Similarly, if $B^2 = I$, then B is an involution and hence $BB^p = I = B^pB$, which implies $B^p = B^{-1}$, so B is invertible. Thus, the same conclusion as in the case of A applies.

Likewise, if $D^2 = I$, then $DD^p = I = D^pD$, and so $D^p = D^{-1}$ Again, the same reasoning as for A holds.

Finally, if $E^2 = I$, then $EE^p = I = E^pE$ and so $E^p = E^{-1}$, and similarly The same conclusion follows. □

Remark 3.11. *Let $A, B, C, D \in \mathcal{B}(\mathcal{H})$ be given If B is invertible, then the operator equation*

$$AXB - YD = C \Rightarrow AX - YDB^{-1} = CB^{-1}$$

in the form of equation (1.5) ,in article[9] .

Remark 3.12. *Let $A, B, C, D, E \in \mathcal{B}(\mathcal{H})$. The operator equation* (1.6) *takes the form of equation* (1.7) *if $E = I$.*

Proposition 3.13. *Let $A, B, C \in \mathcal{B}(\mathcal{H})$, The following statements hold.*
If (1.7) *has a solution pair (X, Y) , then*

$$\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} and \begin{pmatrix} A & C \\ 0 & I \end{pmatrix}$$

are equivalent on $H \oplus H$.

Proof. Let (1.7) has a solution pair (X, Y), then

$$\begin{pmatrix} I & YD \\ 0 & I \end{pmatrix}\begin{pmatrix} A & C \\ 0 & I \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} A & XB \\ 0 & I \end{pmatrix}$$
$$\begin{pmatrix} A & C+YD \\ 0 & I \end{pmatrix} = \begin{pmatrix} A & AXB \\ 0 & I \end{pmatrix}.$$

□

Corollary 3.14. *Let $A, B, C \in \mathcal{B}(\mathcal{H})$. If* (1.7) *has a solution pair (X, Y) , then*

$$\left(\begin{pmatrix} A & C \\ 0 & D \end{pmatrix}, \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}\right) and \left(\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}\right)$$

are equivalent on $H \oplus H$.

Proof. Let (1.7) has a solution pair (X, Y) , then

$$\begin{pmatrix} I & Y \\ 0 & I \end{pmatrix}\begin{pmatrix} A & C \\ 0 & D \end{pmatrix}, = \begin{pmatrix} A & C+YD \\ 0 & D \end{pmatrix}$$
$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}\begin{pmatrix} I & XB \\ 0 & D \end{pmatrix}, = \begin{pmatrix} A & AXB \\ 0 & D \end{pmatrix}$$
$$\begin{pmatrix} A & C+YD \\ 0 & D \end{pmatrix} = \begin{pmatrix} A & AXB \\ 0 & D \end{pmatrix}$$

and

$$\begin{pmatrix} I & X \\ 0 & I \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}, = \begin{pmatrix} I & XB \\ 0 & B \end{pmatrix}$$
$$\begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} I & XB \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & XB \\ 0 & B \end{pmatrix}.$$

□

Proposition 3.15. *Let $A, B, C \in \mathcal{B}(\mathcal{H})$. If (M, W) and (N, W) are equivalent on $H \oplus H$ such that $UM = NV$ and $UW = WV$ where U and V are invertible, and $NV = VN$ also $(U_1 + U_3)$ is invertible, then* (1.7) *has a solution pair (X, Y).*

Proof. Suppose that

$$\left[M = \begin{pmatrix} A & C \\ 0 & D \end{pmatrix}, W = \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}\right] \text{and} \left[N = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, W = \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}\right]$$

are equivalent on $H \oplus H$. Then there exist invertible operator matrices $U = \begin{pmatrix} U_1 & U_2 \\ U_3 & U4 \end{pmatrix}$ and $V = \begin{pmatrix} V_1 & V_2 \\ V_3 & V_4 \end{pmatrix}$ on $H \oplus H$ such that

$$\begin{array}{rcl} UM & = & NV \\ \begin{pmatrix} U_1 & U_2 \\ U_3 & U4 \end{pmatrix}\begin{pmatrix} A & C \\ 0 & D \end{pmatrix} & = & \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}\begin{pmatrix} V_1 & V_2 \\ V_3 & V_4 \end{pmatrix} \\ UW & = & WV \\ \begin{pmatrix} U_1 & U_2 \\ U_3 & U4 \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix} & = & \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} V_1 & V_2 \\ V_3 & V_4 \end{pmatrix}. \end{array}$$

This implies

$$\begin{cases} U_1A = AV_1 \\ U_1C + U_2D = AV_2 \\ U_3A = DV_3 \\ U_3C + U_4D = DV_4 \end{cases}$$

and

$$\begin{cases} U_1 = V_1 \\ U_2B = V_2 \\ U_3 = BV_3 \\ U_4B = V_4 \end{cases}$$

then

$$\begin{array}{rcl} U_1C + U_2D & = & AU_2B \\ U_3C + U_4D & = & V_4\,D, \quad \text{where } NV = VN \\ (U_1 + U_3)C & = & AU_2B - (U_2 + U_4 - V_4)D \end{array}$$

Since $(U_1 + U_3)$ is invertible, it follows that

$$C = A(U_1 + U_3)^{-1}U_2B - (U_1 + U_3)^{-1}(U_2 + U_4 - V_4)D. \tag{3.7}$$

Since the equation (3.7) takes the form (1.7), then the solution is given by

$$X = (U_1 + U_3)^{-1}U_2\ ;\ Y = (U_1 + U_3)^{-1}(U_2 + U_4 - V_4).$$

□

Proposition 3.16. *Let $A, B, C \in \mathcal{B}(\mathcal{H})$, B^{p*} injective be given. Then the operator equation* (1.8) *has a solution pair*

$$(X, Y) = (A^pCB^p, -(I - AA^p)C(B^P)^2$$

if and only if

$$(I - AA^p)CB^p(I - (B^P)^2) = 0$$

where A^p is the pseudo-inverse of A, B^p is the pseudo-inverse of B.

Proof. Suppose that $(I - AA^p)CB^p(I - (B^P)^2) = 0$,where A^p is the pseudo-inverse of A, B^p is the pseudo-inverse of B.

Hence

$$\begin{aligned} (I - AA^p)CB^p(I - (B^P)^2) &= 0, \\ (CB^p - AA^pCB^p)(I - (B^P)^2) &= 0, \\ CB^p - AA^pCB^p - C(B^P)^3 + AA^pC(B^P)^3 &= 0, \\ AA^pCB^pBB^p + (I - AA^P)C(B^P)^3 &= CB^p. \end{aligned}$$

Since B^{p*} injective

$$AA^pCB^pB + (I - AA^P)C(B^P)^2 = C \tag{3.8}$$

Since the equation (3.8) is of the form (1.8), then the solution is given by $(X, Y) = (A^pCB^p, -(I-AA^p)C(B^P)^2$. □

Corollary 3.17. *Let $A, B, C \in \mathcal{B}(\mathcal{H})$ be given:*

1. *If A is invertible, then the operator equation* (1.8) *has a solution pair*

$$(X, Y) = (A^{-1}CB^p, 0) \,. \tag{3.9}$$

2. *If B is invertible, then the operator equation* (1.8) *has a solution pair*

$$(X, Y) = (A^pCB^{-1}, -(I - AA^p)C(B^{-1})^2), \tag{3.10}$$

where A^p is the pseudo-inverse of A, B^p is the pseudo-inverse of B.

Proof. 1 If A is invertible, then $A^{-1} = A^p$. Thus,

$$(I - AA^p + A^pB^p)CB^p = 0$$

it follows from Proposition 3.16 that the operator equation (1.8) has a solution Taking (3.9) .

2 If B is invertible, then $B^{-1} = B^p$. Then we get that (3.10).
□

Corollary 3.18. *Let $A, B, C \in \mathcal{B}(\mathcal{H})$ be given. If A or B is is involution, then the operator equation* (1.8) *has a solution.*

Proof. If A or B is involution ie $A^2 = I$,it follows that $AA^p = I = A^pA$ and so $A^p = A^{-1}$. Hence A is invertible. Similarly, $B^2 = I$,it follows that $BB^p = I = B^pB$ and so $B^p = B^{-1}$Then we get that it follows from corollary 3.17 that the operator equation (1.8) has a solution . □

Remark 3.19. *Let $A, B, C \in \mathcal{B}(\mathcal{H})$ be givenIf B is invertible, then the operator equation*

$$AXB - Y = C \Rightarrow AX - YB^{-1} = CB^{-1}$$

in the form of equation (1.5) ,in article[9] .

Remark 3.20. *Let A, B, C, D and $E \in \mathcal{B}(\mathcal{H})$ be given.*

1. the operator equation (1.7) in the form of equation (1.8) if $D = I$.
2. the operator equation (1.6) in the form of equation (1.8) if $E = D = I$.

Proposition 3.21. *Let $A, B, C \in \mathcal{B}(\mathcal{H})$, The following statements hold.*

If (1.8) *has a solution pair* (X, Y) *, then*

$$\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} \text{and} \begin{pmatrix} A & C \\ 0 & I \end{pmatrix}$$

are equivalent on $H \oplus H$.

Proof. Let (1.8) has a solution pair (X, Y) , then

$$\begin{pmatrix} I & Y \\ 0 & I \end{pmatrix}\begin{pmatrix} A & C \\ 0 & I \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} A & XB \\ 0 & I \end{pmatrix}$$
$$\begin{pmatrix} A & C+Y \\ 0 & I \end{pmatrix} = \begin{pmatrix} A & AXB \\ 0 & I \end{pmatrix}.$$

□

Proposition 3.22. *Let* $A, B, C \in \mathcal{B}(\mathcal{H})$, *The following statements hold*
if (M, W) *and* (N, W) *are equivalent on* $H \oplus H$ *such that* $UM = NV$ *and* $UW = WV$ *where* U *and* V *are invertible,and* $(U_1 + U_3)$ *is invertible, then* (1.8) *has a solution pair* (X, Y).

Proof. Suppose that

$$\left[M = \begin{pmatrix} A & C \\ 0 & I \end{pmatrix}, W = \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}\right] \text{and} \left[N = \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}, W = \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}\right]$$

are equivalent on $H \oplus H$. Then there exist invertible operator matrices $U = \begin{pmatrix} U_1 & U_2 \\ U_3 & U4 \end{pmatrix}$ and $V = \begin{pmatrix} V_1 & V_2 \\ V_3 & V_4 \end{pmatrix}$ on $H \oplus H$ such that

$$UM = NV$$
$$\begin{pmatrix} U_1 & U_2 \\ U_3 & U4 \end{pmatrix}\begin{pmatrix} A & C \\ 0 & I \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} V_1 & V_2 \\ V_3 & V_4 \end{pmatrix}$$
$$UW = WV$$
$$\begin{pmatrix} U_1 & U_2 \\ U_3 & U4 \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} V_1 & V_2 \\ V_3 & V_4 \end{pmatrix}$$

This implies

$$\begin{cases} U_1A = AV_1 \\ U_1C + U_2 = AV_2 \\ U_3A = V_3 \\ U_3C + U_4 = V_4 \end{cases}$$

and

$$\begin{cases} U_1 = V_1 \\ U_2B = V_2 \\ U_3 = BV_3 \\ U_4B = V_4 \end{cases}.$$

So

$$\begin{aligned} U_1C + U_2 &= AU_2B \\ U_3C + U_4 &= V_4 \\ (U_1 + U_3)C &= AU_2B - (U_2 + U_4 - V_4). \end{aligned}$$

Since $(U_1 + U_3)$ is invertible, it follows that

$$C = A(U_1 + U_3)^{-1}U_2B - (U_1 + U_3)^{-1}(U_2 + U_4 - V_4). \tag{3.12}$$

Since the equation (3.12) is of the form (1.8), then the solution take

$$X = (U_1 + U_3)^{-1}U_2, \quad Y = (U_1 + U_3)^{-1}(U_2 + U_4 - V_4).$$

□

4. Operator equation $AX - XB = C$ on two subspace ker$(A + I)$ and ker$(B + I)$

4.1. *Operator equation $AX - XB = C$ on the subspace* ker$(A + I)$

If A, B and $C \in \mathcal{B}(\mathcal{H})$, such that $\sigma(A) \cap \sigma(B) = \oslash$ can be put in the form of

$$(A + I)X - X(B + I) = C.$$

Now let us consider $x \in \ker(A + I)$. So,

$$\begin{aligned} -xX(B + I) &= xC, \\ xX &= -xC(B + I)^{-1}. \end{aligned}$$

Thus, the solution of the equation (1.1) on the subspace ker$(A + I)$ coincides with $-C(B + I)^{-1}$ that is,

$$X\,|_{\ker(A+I)} = -C(B + I)^{-1}\,|_{\ker(A+I)}$$

if A, B , C, D and $E \in \mathcal{B}(\mathcal{H})$, such that $\sigma(A) \cap \sigma(D) = \oslash$. Also,

$$(A + I)X(B + I) - (E + I)X(D + I) = C.$$

Consider $x \in \ker(A + I)$. So,

$$\begin{aligned} -x(E + I)X(D + I) &= xC, \\ X &= -(E + I)^{-1}C(D + I)^{-1}. \end{aligned}$$

Hence the solution of the equation (1.2) on the subspace ker$(A + I)$ ccoincides with $-(E + I)^{-1}C(D + I)^{-1}$ that is,

$$X\,|_{\ker(A+I)} = -(E + I)^{-1}C(D + I)^{-1}\,|_{\ker(A+I)}\,.$$

4.2. *Operator equation $AX - XB = C$ on the subspace* ker$(B + I)$

If A, B and $C \in \mathcal{B}(\mathcal{H})$, such that $\sigma(A) \cap \sigma(B) = \oslash$ then

$$(A + I)X - X(B + I) = C.$$

Let us consider $x \in \ker(B + I)$ then

$$\begin{aligned} (A + I)Xx &= Cx \\ Xx &= (A + I)^{-1}Cx. \end{aligned}$$

Hence the solution of the equation (1.1) on the subspace ker$(B + I)$ Coincides with $(A + I)^{-1}C$ that is, we have

$$X\,|_{\ker(B+I)} = (A + I)^{-1}C\,|_{\ker(B+I)}\,.$$

If A, B , C, D and $E \in \mathcal{B}(\mathcal{H})$, such that $\sigma(A) \cap \sigma(D) = \emptyset$. then

$$(A+I)X(B+I) - (E+I)X(D+I) = C.$$

Consider $x \in \ker(B+I)$ then

$$\begin{aligned} -x(E+I)X(D+I) &= Cx, \\ X &= -(E+I)^{-1}C(D+I)^{-1}. \end{aligned}$$

Thus, the solution of the equation (1.2) on the subspace $\ker(B+I)$ coincides with $-(E+I)^{-1}C(D+I)^{-1}$ that is,

$$X\,|_{\ker(B+I)} = -(E+I)^{-1}C(D+I)^{-1}\,|_{\ker(B+I)}\ .$$

5. Examples

Example 5.1. *Let us begin with the following example to illustrate Theorem 2.1. To this end, Let*

$$A = \begin{pmatrix} \frac{I}{3} & 0 \\ 0 & \frac{I}{3} \end{pmatrix} \text{ and } B = \begin{pmatrix} 2I & 0 \\ I & I \end{pmatrix}.$$

So,

$$A^p = \begin{pmatrix} 3I & 0 \\ 0 & 3I \end{pmatrix} \text{ and } B^p = \begin{pmatrix} \frac{1}{2}I & 0 \\ \frac{1}{2}I & I \end{pmatrix}.$$

Also,

$$A^* = \begin{pmatrix} \frac{I}{3} & 0 \\ 0 & \frac{I}{3} \end{pmatrix}$$

is injective. Moreover,

$$AB = BA = \begin{pmatrix} \frac{2}{3}I & 0 \\ \frac{1}{3}I & \frac{1}{3}I \end{pmatrix},$$

and

$$AB^p = B^pA = \begin{pmatrix} \frac{1}{6}I & 0 \\ \frac{1}{6}I & \frac{1}{3}I \end{pmatrix}.$$

Example 5.2. *This example illustrates the results of Theorem 2.2. To this end, let*

$$A = \begin{pmatrix} I & I \\ I & I \end{pmatrix} \text{ and } B = \begin{pmatrix} 2I & 0 \\ 0 & 2I \end{pmatrix}.$$

So,

$$A^p = \frac{1}{4}\begin{pmatrix} I & I \\ I & I \end{pmatrix} \text{ and } B^p = \begin{pmatrix} \frac{1}{2}I & 0 \\ 0 & \frac{1}{2}I \end{pmatrix}.$$

Also,

$$AB = BA = 2\begin{pmatrix} I & I \\ I & I \end{pmatrix}.$$

Moreover,

$$A^pB^p = B^pA^p = \frac{1}{8}\begin{pmatrix} I & I \\ I & I \end{pmatrix}.$$

Example 5.3. *The following example serves to illustrate the results of Theorem 2.3. Tn order to achieve this goal, let*

A, B and C on $\ell^2(N)$

$$A(e_n) = 2\lambda_n e_n, \text{ for all } n \in N \text{ with } \lambda_n \neq 0,$$
$$B(e_n) = \lambda_n e_n, \text{ for all } n \in N \text{ with } \lambda_n \neq 0,$$
$$C(e_n) = \lambda_n^2 e_n, \text{ for all } n \in N \text{ with } \lambda_n \neq 0.$$

So,

$$A^p(e_n) = \frac{1}{2\lambda_n} e_n, \text{ for all } n \in N \text{ with } \lambda_n \neq 0,$$
$$B^p(e_n) = \frac{1}{\lambda_n} e_n, \text{ for all } n \in N \text{ with } \lambda_n \neq 0.$$

Also,

$$BC(e_n) = CB(e_n) = \lambda_n^3 e_n, \text{ for all } n \in N \text{ with } \lambda_n \neq 0,$$
$$AB^p(e_n) = 2e_n, \text{ for all } n \in N,$$
$$BB^p(e_n) = e_n, \text{ for all } n \in N.$$

Hence

$$(I - AB^p + BB^p) = 0.$$

This gives

$$X = B^p C = \lambda_n e_n, \text{ for all } n \in N \text{ with } \lambda_n \neq 0.$$

Example 5.4. *This example serves to illustrate the conclusions of Corollary 2.5. To that end, let us define the following:*

$$B = \begin{pmatrix} 2I & I \\ 0 & 2I \end{pmatrix}, \ A = \begin{pmatrix} 4I & 2I \\ 0 & 4 \end{pmatrix}, \ \text{and } C = \begin{pmatrix} S_1 & S_2 \\ 0 & S_1 \end{pmatrix}.$$

So,

$$B^p = B^{-1} = \begin{pmatrix} \frac{1}{2}I & -\frac{1}{4}I \\ 0 & \frac{1}{2}I \end{pmatrix}$$

and

$$BC = CB = \begin{pmatrix} 2S_1 & 2S_2 + S_1 \\ 0 & 2S_1 \end{pmatrix}.$$

Hence

$$X = B^p C = \begin{pmatrix} \frac{S_1}{2} & \frac{S_2}{2} - \frac{S_1}{4} \\ 0 & \frac{S_1}{2} \end{pmatrix}$$

and

$$C^* = \begin{pmatrix} S_1 & 0 \\ S_2 & S_1 \end{pmatrix}$$

is injective. Then

$$X = B^p C = CB^p = \begin{pmatrix} \frac{S_1}{2} & \frac{S_2}{2} - \frac{S_1}{4} \\ 0 & \frac{S_1}{2} \end{pmatrix}.$$

Example 5.5. *This example serves to illustrate the results of Proposition 2.14. To that end, let $A, B, C, D \in \ell^2$ such* *that*

$$\begin{aligned}A(x_1, x_2, x_3, \cdots) &= (x_1, 0, 0, \cdots),\\ B(x_1, x_2, x_3, \cdots) &= (0, x_2, 0, 0, \cdots),\\ C(x_1, x_2, x_3, \cdots) &= (x_1 + x_2, 0, 0, \cdots),\\ D(x_1, x_2, x_3, \cdots) &= (x_1, x_2, 0, 0, \cdots).\end{aligned}$$

So,

$$\begin{aligned}A^p(x_1, x_2, x_3, \cdots) &= (x_1, 0, 0, \cdots),\\ B^p(x_1, x_2, x_3, \cdots) &= (0, x_2, 0, 0, \cdots).\end{aligned}$$

Therefore,

$$\begin{aligned}AA^p(x_1, x_2, x_3, \cdots) &= A(x_1, 0, 0, \cdots) = (x_1, 0, 0, \cdots),\\ CB^p(x_1, x_2, x_3, \cdots) &= C(0, x_2, 0, 0, \cdots) = (x_2, 0, 0, \cdots).\end{aligned}$$

Also,

$$A^pDB^p(x_1, x_2, x_3, \cdots) = A^pD(0, x_2, 0, 0, \cdots) = A^p(0, x_2, 0, 0, \cdots) = 0.$$

Hence

$$(I - AA^p)(x_1, x_2, x_3, \cdots) = (0, x_2, x_3, \cdots)$$

and hence

$$(I - AA^p)CB^p(x_1, x_2, x_3, \cdots) = 0.$$

Thus,

$$A^pDB^pCB^p(x_1, x_2, x_3, \cdots) = 0.$$

Also,

$$(I - AA^p + A^pDB^p)CB^p = 0.$$

Consequently,

$$X = (x_2, 0, 0, \cdots).$$

Example 5.6. *This example serves to illustrate the results of Proposition 2.21. To that end, let us define the following:*

$$A = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad C = \begin{pmatrix} -I & I \\ 0 & 0 \end{pmatrix}.$$

So,

$$A^p = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B^p = \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}.$$

Hence

$$(B^p)^* = \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}$$

is injective. Also, we have

$$AA^p = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad CB^p = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence

$$A^pB^p = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

This gives,

$$I - AA^p + A^pB^p = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$$

and

$$(I - AA^p + A^pB^p)CB^p = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} = 0.$$

Thus,

$$X = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}.$$

Example 5.7. *This example serves to illustrate the results of Proposition Proposition 3.16. To that end, let us consider the following*

$$B = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}, \ A = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \ C = \begin{pmatrix} 0 & S \\ 0 & 0 \end{pmatrix}.$$

Hence

$$A^p = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \ \textit{and} \ B^p = \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix},$$

and hence

$$AA^p = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \textit{and} \ CB^p = \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix}.$$

Also,

$$(B^p)^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

This gives

$$(I - AA^p)CB^p(I - (B^p)^2) = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} = 0.$$

Thus,

$$X = \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix}, \ Y = 0.$$

Example 5.8. *This example is presented to clarify the outcome of Proposition 3.16. To proceed, let $A, B, C \in \ell^2$, such that:*

$$A(x_1, x_2, x_3, \cdots) = (x_1, 0, 0, \cdots),$$
$$B(x_1, x_2, x_3, \cdots) = (0, x_2, 0, 0, \cdots),$$
$$C(x_1, x_2, x_3, \cdots) = (0, c(x_2)^2, 0, \cdots).$$

So,

$$A^p(x_1, x_2, x_3, \cdots) = (x_1, 0, 0, \cdots),$$
$$B^p(x_1, x_2, x_3, \cdots) = (0, x_2, 0, 0, \cdots),$$
$$AA^p(x_1, x_2, x_3, \cdots) = A(x_1, 0, 0, \cdots) = (x_1, 0, 0, \cdots),$$
$$CB^p(x_1, x_2, x_3, \cdots) = C(0, x_2, 0, 0, \cdots) = (0, c(x_2)^2, 0, 0, \cdots),$$
$$B^pB^p(x_1, x_2, x_3, \cdots) = B^p(0, x_2, 0, 0, \cdots) = (0, x_2, 0, 0, \cdots).$$

From this, we conclude that

$$(I - AA^p)(x_1, x_2, x_3, \cdots) = (0, x_2, x_3, \cdots),$$
$$(I - (B^p)^2)(x_1, x_2, x_3, \cdots) = (x_1, 0, x_3, x_4, \cdots),$$
$$(I - AA^p)CB^p(I - (B^p)^2) = 0.$$

Consequently,

$$X = 0, \quad Y = (0, -c(x_2)^2, 0, 0, \cdots).$$

Example 5.9. *In support of Proposition Corollary 3.17, we offer the following illustrative example. Let*

$$A = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \quad C = \begin{pmatrix} 2S & 0 \\ 0 & 0 \end{pmatrix}.$$

So,

$$A^p = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad B^p = B^{-1} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

Also,

$$AA^p = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad CB^p = \begin{pmatrix} 2S & 0 \\ 0 & 0 \end{pmatrix}.$$

This gives

$$(B^p)^2 = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

From this, we conclude that

$$(I - AA^p)CB^p(I - (B^p)^2) = 0.$$

Therefore

$$X = \begin{pmatrix} 2S & 0 \\ 0 & 0 \end{pmatrix}, \quad Y = 0.$$

6. Conclusion

In this work, we have studied various forms of the Sylvester-type operator equations in infinite-dimensional Hilbert spaces, including the classical equation $AX - XB = C$ and its generalized version $AX - YB = C$, which introduces additional complexity due to the presence of two unknowns. We established new necessary and sufficient conditions for the solvability of these equations by employing generalized inverses under novel structural assumptions. Our analysis extended to the behavior of these equations on specific subspaces, such as $\ker(A + I)$ and $\ker(B + I)$, as well as on pairs of distinct subspaces. These investigations highlighted how properties such as involution and pseudo-inverses influence the existence and structure of solutions. The findings provide a unified perspective on classical and generalized operator equations, enriching the theoretical framework and suggesting potential applications in areas such as control theory, perturbation analysis, and operator theory. The illustrative examples further confirm the practical relevance of the results and open the door to future research in more specialized settings.

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