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On approximation of certain fuzzy linear operators

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Abstract. The present work introduces a new positive fuzzy linear operator by means of fractional Kantorovich–rational operator of Riemann Liouville type. Further, investigations related to fuzzy Korovkin theorem, fuzzy Voronovskaya, and Grüss Voronovskaya theorem have been carried out via deferred A-statistical convergence of order γ. Results including A-deferred statistical fuzzy rates of convergence have also been established using the modulus of smoothness of higher order, Peetre's K–functional, Lipschitz maximal functions etc.

1. Introductions, definitions and prerequisites

Approximation theories enriched with recent advancement of mathematics, science, and engineering. These developed concept has always associated with certain operators and demonstrate itself as a massive and achieved research area in diverse domain such as Sequence space and Summabilities, differential equations, Stochastic analysis, Neural netrwork, fuzzy logic, image processing and machine learning and many more (see [7], [20], [18], [45] etc.). Meanwhile, fractional calculus is implemented with some potentially active and emerging source of applications, which can be surged from several articles concerning robustness, modelling of real life problem, various inference system, time delay problem, and many more (see [8], [12], [42]).

While surveying the existing literatures, we found that the notion of fractional calculus in the domain of operator theories is capable to approximate the continuous functions. However, Starting with Karl Weirstrass[46], in 1885, the famous Weierstrass approximations theorem enriched with the approximation of the trigonometric polynomial. Since then, various researchers have been influenced by this idea, and incorporated it in various disciplines (see [10], [15], [16], [44], etc.).

Being particular, Popovicu [36], Bohman [9], and Korovkin [21] developed the notion of Bohman–Korovkin type theorem respectively, in the year 1951, 1952 and 1953, which is stated as below:

Theorem 1.1. *(Bohman–Korovkin theorem) Suppose that $\{\bar{\mathcal{L}}_m\}_{m\in\mathbb{N}_0}$ is a sequence of linear operators, which is positive from $C[a,b]$ into itself having uniform convergence for $\bar{\mathcal{L}}_m$ i.e., $\lim_{m\to\infty}\bar{\mathcal{L}}_m e_i = e_i$ (for $i = 0,\ 1,\ 2$). Then,*

$$\lim_{m\to\infty}\bar{\mathcal{L}}_m g^* = g^*,$$

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converges uniformly in $[a, b]$.

Later on, several extensions of the above theorem have been characterized on certain well known positive linear operators, for instance, Bernstein operator(see [1], [3]), Balazs operator[23], Szàsz operator (see [22], [26], [43]), and many more. Moreover, some convergence analysis in order to establish the rate of approximation, local and global estimates for error bound have also been conducted not only in the classical sense but also in the sense of statistical, A-statistical, $\lambda-$ statistical, deferred $A-$ statistical convergence, etc.

Additionally, Statistical convergence, initiated with the greatest discovery of Fast [13], that unifies the notion of classical convergence and focuses on the majority of terms in the sequence that converge to a point. In brief, we may say that a sequence $x = (x_\xi)$ is statistical convergence to $l \in \mathbb{R}$ or $\mathbb{C}$, if for given $\epsilon > 0$,

$$\lim_{m\to\infty} \frac{1}{m} |\{\xi \leq m : \ |x_\xi - l| \geq \epsilon\}| = 0.$$

As evidenced, plenty of articles including Schoenberg [40], Steinhaus [41], Miller and Orhan [24], Fridy [14], Kadak [19], Nayak et al. [31], [32], Saini et al. [38] were available providing their truthful investigations and key contributions towards various properties and applications of statistical convergence.
The ideas as discussed earlier are not only limited to the classical theory of summability over real or complex fields, but also extended in fuzzy aspects. However, in 1965, Zadeh [47] discovered the fuzzy set and operations in set theory. Subsequently, Savas [43], Nuray and Savas [33], Duman [11], Anastassiou [4], Mohiuddine et al. [25], Rahaman and Mursaleen [37], etc. had implemented the fuzzy theoretic approach in sequence spaces, Summability, approximation theory, topology, and many more. By a fuzzy number we mean the function $\mathfrak{I} : \mathbb{R} \to [0, 1]$, such that the following conditions hold:

1. There exists $r \in \mathbb{R}$ with $\mathfrak{I}(r) = 1$ (Normal),
2. $\mathfrak{I}(t) \geq \min_{t\in(a,b)}\{\mathfrak{I}(a),\ \mathfrak{I}(b)\}$(Fuzzy convex),
3. $\mathfrak{I}$ is semi-continuous (upper),
4. $\overline{supp(\mathfrak{I})} = \overline{\{r' \in \mathbb{R} : \mathfrak{I}(r') > 0\}}$, is compact(– stands for closure of the set).

It is pertinent to note the fact from [5], [23] is that the Hausdorff metric

$$d(\mathfrak{I}_1, \mathfrak{I}_2) = \sup_{\alpha\in[0,1]} d_H([\mathfrak{I}_1]_\alpha,\ [\mathfrak{I}_2]_\alpha) = \sup_{\alpha\in[0,1]} \max\left\{[\mathfrak{I}_1]^-_\alpha - [\mathfrak{I}_2]^-_\alpha, [\mathfrak{I}_1]^+_\alpha - [\mathfrak{I}_2]^+_\alpha\right\},$$

forms a complete metric space $(\mathbb{R}_\mathcal{F}, d)$, where $\mathbb{R}_\mathcal{F}$ is the set of all fuzzy number on the set of real number. In addition, we may denote the set $[\mathfrak{I}_1]_\alpha = \{r \in \mathbb{R} : \mathfrak{I}_1(r) \geq \alpha\}$ as $\alpha-$level cut of fuzzy number $\mathfrak{I}_1$, $[\mathfrak{I}_1]^-_\alpha$ as lower bound of $[\mathfrak{I}_1]_\alpha$, and $[\mathfrak{I}_1]^+_\alpha$ as upper bound of $[\mathfrak{I}_1]_\alpha$. By a fuzzy continuous function, we mean a fuzzy valued function $g : [c_1, c_2] \to \mathbb{R}_\mathcal{F}$ where for $x_0 \in [c_1, c_2]$, such that $d(g(x_\xi), g(x_0)) \to 0$, as $\xi \to \infty$, if and only if $x_\xi \to x_0$.

Throughout this paper we denote the set $C_\mathcal{F}[a, b]$ by the collections of fuzzy continuous functions from $[a, b]$. We represent $[u_1 \bigoplus u_2]_\alpha = [u_1]_\alpha + [u_2]_\alpha$, and $[u_1 \bigodot u_2]_\alpha = [u_1]_\alpha.[u_2]_\alpha$ as fuzzy sum and fuzzy product, respectively. The operator $\vartheta : C_\mathcal{F}[c_1, c_2] \to C_\mathcal{F}[c_1, c_2]$, is said to be fuzzy linear if

$$\vartheta(\kappa_1 \bigodot \bar{f_1} \bigoplus \kappa_2 \bigodot \bar{f_2}; \mathfrak{I}) = \kappa_1 \bigodot \vartheta(\bar{f_1}) \bigoplus \kappa_2 \bigodot \vartheta(\bar{f_2}), \quad \text{for } \kappa_1, \kappa_2 \in \mathbb{R}, \text{ and } \bar{f_1}, \bar{f_2} \in C_\mathcal{F}[a, b].$$

If $\vartheta(\bar{f_1}, \mathfrak{I}) \leq \vartheta(\bar{f_2}; \mathfrak{I})$, if $\bar{f_1}(\mathfrak{I}) \leq \bar{f_2}(\mathfrak{I})$, then we call the fuzzy linear operator to be positive.

Looking further, In 2022, Nayak et al. [31] introduced the notion of *deferred A−* statistical convergence of order $\bar{\gamma}$ in fuzzy sequence $\mathfrak{I} = (\mathfrak{I}_\xi)_{\xi\in\mathbb{N}_0}$ to a fuzzy number $\mathfrak{I}_0$, which is defined as follows: i.e., for each $\epsilon > 0$,

$$\lim_{m\to\infty} \frac{1}{(q_m - p_m)^{\bar{\gamma}}} \left| \left\{ p_m \leq \xi \leq q_m : d\left(\sum_{\xi=p_m+1}^{q_m} \tilde{a}_{m\xi} \mathfrak{I}_\xi, \mathfrak{I}_0 \right) \geq \epsilon \right\} \right| = 0,$$

where, $A = (\tilde{a}_{nk})$, denote a regular infinite matrix. For simplicity, we may denote this as $St_A^D - \lim_{m\to\infty} \mathfrak{I}_m = \mathfrak{I}_0$. In brief, by a conservative infinite matrix, we mean $Ax \in c$, for all $x \in c$, where c stands for the space of all convergent sequences. Furthermore, by a regular infinite matrix, we mean A is conservative and $\lim Ay = \lim y$ (see [7]). Regarding their work, author used the deffero Cesáro summability mean which was described by Agnew [2] (see also Nayak et al. [30]) via the sequence $\mathfrak{I} = (\mathfrak{I}_\xi)$ is $\frac{1}{q_m - p_m} \sum_{\xi=p_{m+1}}^{q_m} \mathfrak{I}_\xi$, where (p_m), and (q_m) are the sequence of nonnegative integer such that $p_m < q_m$, and $\lim_{m\to\infty} q_m = +\infty$, for all $m \in \mathbb{N}_0$. Consequently, the fact of mark is that we can explore large number of literatures, such as [29], [27],[28], [28], [6], [39], etc., related to various properties of positive linear operators along with some interesting results like Korovkin theorem, Voronovskaya theorem, Grüss Voronovskaya inequalities, etc. Further, few observation concerning local and global direct estimations for the rate of approximations with respect to modulus of smoothness have been discussed for the order up to 2 over the real or complex field. But, in the later, fewer work have been conducted except the Bernstein operator in the direction of fuzzy contrast.

Being motivated by all the prospectives of authors, cited above, we aim to investigate fuzzy theoretic treatment of the positive linear operator, which was examined by Özkan [35], by expanding with the help of Riemann-Liouville fractional operator encompasses with some Stancu variant. Especially, we focuss on performing the rate of approximation via higher order modulus of smoothness along with the notion of deffered $A-$ statistical convergence. Additionally, we also establish certain results related to the Korovkin type theorem, the Voronovskaya theorem, and inequalities in fuzzy contrast. Furthermore, we examined the rate of convergence via Peetre's K-functional Lipschitz maximal functions also, and carried out few bound estimations for our newly developed operator in particular cases.

In contrast, study of Özkan [35], involves the construction of operator $K_m^G(f;y)$, for $f \in C[0,\infty)$, which is characterized as follows:

$$K_m^G(f;y) = \tau_m \sum_{\kappa=0}^{m} \binom{m}{\kappa} \frac{(\sigma_m)^{m-\kappa}(\rho_m y)^\kappa}{(\sigma_m + \rho_m y)^m} \int_{\frac{\kappa}{\tau_m}}^{\frac{\kappa+1}{\tau_m}} f(t)dt, \tag{1}$$

where $y > 0$ and $\lim_{m\to\infty} \rho_m = 0$, $\lim_{m\to\infty} \sigma_n = 1$, $\tau_m = m\rho_m$, and $\lim_{m\to\infty} \tau_m = \infty$.

Now, for $\bar{f} \in C_{\mathcal{F}}[0,1]$, we construct the fuzzy linear operator $\mathcal{A}_n^{\nu_1,\nu_2}(\bar{f};x)$, by combining the Riemann Liouville fractional operator with Stancu variant $\nu_1, \nu_2 \in \mathbb{N}_0$, by

$$\mathcal{A}_n^{\nu_1,\nu_2}(\bar{f};y) = \bigoplus_{k=0}^{n} \binom{n}{k} \frac{(\sigma_n + \nu_1)^{n-k}(\rho_n y)^k}{(\sigma_n + \rho_n y + \nu_1)^n} \bigotimes \Gamma(\beta+1) \int_0^1 \frac{(1-t)^{\beta-1}}{\Gamma(\beta)} \bigotimes \bar{f}\left(\frac{k+t}{\tau_m + \nu_2}\right) dt, \tag{2}$$

provided $St_A^D - \lim_{n\to\infty} \rho_n = 0$, $St_A^D - \lim_{n\to\infty} \sigma_n = 1$, $\tau_n = n\rho_n$, and $St_A^D - \lim_{n\to\infty} \tau_n = \infty$, in fuzzy sense. It is obvious that $\bigoplus$ and $\bigotimes$ stand for fuzzy sum and product, respectively for the fuzzy functions. However, these fuzzy operations can be easy to handle in normal arithmetic operation by crisping it into the $\alpha-$level cut, for different $\alpha \in [0,1]$. Therefore, we first crisped our proposed fuzzy operator as in equation (2) to the following form as in equation (3) form, i.e.

$$\tilde{\mathcal{A}}_n^{\nu_1,\nu_2}(\bar{f}_\alpha^{\pm};y) = \sum_{k=0}^{n} \binom{n}{k} \frac{(\sigma_n + \nu_1)^{n-k}(\rho_n y)^k}{(\sigma_n + \rho_n y + \nu_1)^n} \Gamma(\beta+1) \int_0^1 \frac{(1-t)^{\beta-1}}{\Gamma(\beta)} \bar{f}_\alpha^{\pm}\left(\frac{k+t}{\tau_m + \nu_2}\right) dt. \tag{3}$$

Being particular, $\nu_1 = \nu_2 = 0$, and $\beta = 1$, this operator reduces to the original operator defined as in [35], if the function takes the values between $[0,1]$.

Definition 1.2. *Let $T = (T_\xi)_{\xi\in\mathbb{N}_0}$ be the sequence of fuzzy linear operators. Then, the deferred $A-$ statistical convergence of order $\bar{\gamma}$, for the sequence of operators (T_ξ), i.e., to a fuzzy operator T_0, is defined as follows: i.e., for each $\epsilon > 0$,*

$$\lim_{m\to\infty} \frac{1}{(q_m - p_m)^{\bar{\gamma}}} \left| \left\{ p_m \le \xi \le q_m : d\left(\sum_{\xi=p_m+1}^{q_m} \tilde{a}_{m\xi} T_\xi(f;y), T_0(f;y) \right) \ge \epsilon \right\} \right| = 0,$$

where A, q_m, p_m, are as stated above.

2. Main results

Theorem 2.1. *Let us consider $f(t) = t^{\hat{j}}$ ($\hat{j} \in \mathbb{N}_0$). Then, we obtain the following identity:*

$$\tilde{\mathcal{A}}_{m'}^{\nu_1,\nu_2}(e_{\hat{j}}; y) = \sum_{k=0}^{m'} \frac{(\sigma_{m'} + \nu_1)^{m'-k}(\rho_{m'} y)^k}{(\sigma_{m'} + \nu_1 + \rho_{m'})^{m'}} \frac{\Gamma(\bar{\beta}+1)}{(\tau_{m'} + \nu_2)^{\hat{j}}} \sum_{\iota=0}^{\hat{j}} \binom{\hat{j}}{\iota} k^{\iota} \frac{\Gamma(\hat{j} - i + 1)}{\Gamma(\bar{\beta} + \hat{j} - \iota + 1)} \tag{4}$$

Proof. The proof proceed like this

$$\begin{aligned}
\tilde{\mathcal{A}}_{m'}^{\nu_1,\nu_2}(e_{\hat{j}}; y) &= \sum_{\kappa=0}^{m'} \frac{(\sigma_{m'} + \nu_1)^{m'-\kappa}(\rho_{m'} y)^\kappa}{(\sigma_{m'} + \nu_1 + \rho_{m'})^{m'}} \Gamma(\bar{\beta}+1) \int_0^1 \frac{(1-t)^{\bar{\beta}-1}}{\Gamma(\bar{\beta})} \left(\frac{\kappa + t}{\tau_{m'} + \nu_2}\right)^{\hat{j}} dt \\
&= \sum_{\kappa=0}^{m'} \frac{(\sigma_{m'} + \nu_1)^{m'-\kappa}(\rho_{m'} y)^\kappa}{(\sigma_{m'} + \nu_1 + \rho_{m'})^{m'}} \frac{\Gamma(\bar{\beta}+1)}{(\tau_{m'} + \nu_2)^{\hat{j}}} \int_0^1 \frac{(1-t)^{\bar{\beta}-1}}{\Gamma(\bar{\beta})} (\kappa + t)^{\hat{j}} dt \\
&= \sum_{\kappa=0}^{m'} \frac{(\sigma_{m'} + \nu_1)^{m'-\kappa}(\rho_{m'} y)^\kappa}{(\sigma_{m'} + \nu_1 + \rho_{m'})^{m'}} \frac{\Gamma(\bar{\beta}+1)}{(\tau_{m'} + \nu_2)^{\hat{j}}} \int_0^1 \frac{(1-t)^{\bar{\beta}-1}}{\Gamma(\bar{\beta})} \sum_{i=0}^{\hat{j}} \binom{\hat{j}}{i} \kappa^{\iota} t^{\hat{j}-\iota} dt \\
&= \sum_{\kappa=0}^{m'} \frac{(\sigma_{m'} + \nu_1)^{m'-\kappa}(\rho_{m'} y)^\kappa}{(\sigma_{m'} + \nu_1 + \rho_{m'})^{m'}} \frac{\Gamma(\bar{\beta}+1)}{(\tau_{m'} + \nu_2)^{\hat{j}}} \sum_{\iota=0}^{\hat{j}} \binom{\hat{j}}{\iota} \kappa^{\iota} \int_0^1 \frac{(1-t)^{\bar{\beta}-1}}{\Gamma(\bar{\beta})} \times t^{\hat{j}-\iota} dt \\
&= \sum_{\kappa=0}^{m'} \frac{(\sigma_{m'} + \nu_1)^{m'-\kappa}(\rho_{m'} y)^\kappa}{(\sigma_{m'} + \nu_1 + \rho_{m'})^{m'}} \frac{\Gamma(\bar{\beta}+1)}{(\tau_{m'} + \nu_2)^{\hat{j}} \Gamma(\bar{\beta})} \sum_{\iota=0}^{\hat{j}} \binom{\hat{j}}{\iota} \kappa^{\iota} B(\bar{\beta}, \hat{j} - \iota + 1) \\
&= \sum_{\kappa=0}^{m'} \frac{(\sigma_{m'} + \nu_1)^{m'-\kappa}(\rho_{m'} y)^\kappa}{(\sigma_{m'} + \nu_1 + \rho_{m'})^{m'}} \frac{\Gamma(\bar{\beta}+1)}{(\tau_{m'} + \nu_2)^{\hat{j}}} \sum_{\iota=0}^{\hat{j}} \binom{\hat{j}}{\iota} \kappa^{\iota} \frac{\Gamma(\hat{j} - \iota + 1)\Gamma(\bar{\beta})}{\Gamma(\bar{\beta} + \hat{j} - \iota + 1)\Gamma(\bar{\beta})} \\
&= \sum_{\kappa=0}^{m'} \frac{(\sigma_{m'} + \nu_1)^{m'-\kappa}(\rho_{m'} y)^\kappa}{(\sigma_{m'} + \nu_1 + \rho_{m'})^{m'}} \frac{\Gamma(\bar{\beta}+1)}{(\tau_{m'} + \nu_2)^{\hat{j}}} \sum_{\iota=0}^{\hat{j}} \binom{\hat{j}}{\iota} \kappa^{\iota} \frac{\Gamma(\hat{j} - \iota + 1)}{\Gamma(\bar{\beta} + \hat{j} - \iota + 1)}
\end{aligned}$$

□

Following to the above Theorem 2.1, we have an immediate lemma.

Lemma 2.2.

$$\begin{aligned}
\tilde{\mathcal{A}}_{m'}^{\nu_1,\nu_2}(1; y) &= 1 \\
\tilde{\mathcal{A}}_{m'}^{\nu_1,\nu_2}(t; y) &= \frac{1}{(\bar{\beta}+1)(\tau_{m'} + \nu_2)} + \frac{\tau_n y}{(\tau_{m'} + \nu_2)(\sigma_{m'} + \rho_{m'} y + \nu_1)} \\
\tilde{\mathcal{A}}_{m'}^{\nu_1,\nu_2}(t^2; y) &= \frac{2!}{(\bar{\beta}+2)(\bar{\beta}+1)(\tau_{m'} + \nu_2)^2} + \frac{m'(m'-1)\rho_{m'}^2 y^2}{(\sigma_{m'} + \rho_{m'} y + \nu_1)^2(\tau_{m'} + \nu_2)^2} \\
&+ \frac{\tau_{m'} y}{(\sigma_{m'} + \rho_{m'} y + \nu_1)(\tau_{m'} + \nu_2)^2} \left(1 + \frac{2}{\bar{\beta}+1}\right) \\
\tilde{\mathcal{A}}_{m'}^{\nu_1,\nu_2}(t^3; y) &= \frac{3!}{(\bar{\beta}+3)(\bar{\beta}+2)(\bar{\beta}+1)(\tau_{m'} + \nu_2)^3} + \frac{\tau_{m'}^3 y^3 (1 - \frac{1}{m'})(1 - \frac{2}{m'})}{(\tau_{m'} + \nu_2)^3(\sigma_{m'} + \rho_{m'} y + \nu_1)^3} \\
&+ \frac{3\bar{\beta}(1 - \frac{1}{m'})\tau_{m'}^2 y^2}{(\tau_{m'} + \nu_2)^3(\bar{\beta}+1)(\sigma_{m'} + \rho_{m'} y + \nu_1)^2}
\end{aligned}$$

$$+\frac{\tau_{m'}y}{(\tau_{m'}+\nu_1)^3(\sigma_{m'}+\rho_{m'}y+\nu_1)}\left(\frac{6}{(\bar{\beta}+2)(\bar{\beta}+1)-2+\frac{3}{\bar{\beta}+1}}\right)$$

$$\tilde{\mathcal{A}}_{m'}^{\nu_1,\nu_2}(t^4;y)=\frac{4!}{(\tau_{m'}+\nu_2)^4(\bar{\beta}+4)(\bar{\beta}+3)(\bar{\beta}+2)(\bar{\beta}+1)}+\frac{1}{\tau_{m'}+\nu_2)^4}\frac{(1-\frac{1}{m'})(1-\frac{2}{m'})(1-\frac{3}{m'})\tau_{m'}^4}{(\tau_{m'}^4+\nu_2)^4(\sigma_{m'}+\rho_{m'}y+\nu_1)^4}$$
$$+\left(\frac{6}{(\tau_{m'}+\nu_1)^4}+\frac{4}{(\tau_{m'}+\nu_2)^4(\bar{\beta}+1)}\right)\frac{m'(m'-1)(m'-2)\rho_{m'}^3y^3}{(\sigma_{m'}+\rho_{m'}y+\nu_1)^3}$$
$$+\Big(\frac{-11}{(\tau_{m'}+\nu_2)^4}+\frac{3}{(\tau_{m'}+\nu_2)^4}\Big(6+\frac{4}{\bar{\beta}+1}\Big)+\frac{12}{(\tau_{m'}+\nu_2)^4(\bar{\beta}+2)(\bar{\beta}+1)}\Big)\frac{m'(m'-1)(\rho_{m'}y)^2}{(\sigma_{m'}+\rho_{m'}y+\nu_1)^2}$$
$$+\Big(\frac{6}{(\tau_{m'}+\nu_2)^4}+-2\Big(\frac{-6}{(\tau_{m'}+\nu_2)^4+\frac{4}{(\tau_{m'}+\nu_2)^4(\bar{\beta}+1)}}\Big)+\frac{24}{(\tau_{m'}+\nu_2)^4}$$
$$+\Big(\frac{-11}{(\tau_{m'}+\nu_2)^4}+\frac{3}{(\tau_{m'}+\nu_2)^4}\Big(6+\frac{4}{\bar{\beta}+1}\Big)+\frac{12}{(\tau_{m'}+\nu_2)^4(\bar{\beta}+2)(\bar{\beta}+1)}\Big)\Big)\frac{m'\rho_{m'}y}{\Sigma_{m'}+\rho_{m'}+\nu_1}$$

As consequence of the above Lemma, we have an immediate remark.

Remark 2.3. *The operator $\tilde{\mathcal{A}}_{m'}^{\nu_1,\nu_2}$, forms a convex operator if $f_\alpha^\pm$ is constant.*

This can be countered by following example. Suppose that $\lambda\in[0,1]$, and $f(t)=[a,a]$, (being any fixed constant). Then, we may observe from Lemma 2.2, that

$$\tilde{\mathcal{A}}_{m'}^{\nu_1,\nu_2}(f_\alpha^\pm;\lambda x+(1-\lambda)y)=\tilde{\mathcal{A}}_{n}^{\nu_1,\nu_2}(\bar{f}_\alpha^\pm;y)=\sum_{k=0}^{n}\binom{n}{k}\frac{(\sigma_n+\nu_1)^{n-k}(\rho_n y)^k}{(\sigma_n+\rho_n y+\nu_1)^n}\Gamma(\beta+1)\int_0^1\frac{(1-t)^{\beta-1}}{\Gamma(\beta)}a\,dt=a.$$

However, since the operator being linear and the results from Lemma 2.2 concludes that

$$\tilde{\mathcal{A}}_{m'}^{\nu_1,\nu_2}(f_\alpha^\pm;\lambda x)+\tilde{\mathcal{A}}_{m'}^{\nu_1,\nu_2}(f_\alpha^\pm;(1-\lambda)y)=a+a=2a.$$

Hence, we get $\tilde{\mathcal{A}}_{m'}^{\nu_1,\nu_2}(f_\alpha^\pm;\lambda x+(1-\lambda)y)\leq\tilde{\mathcal{A}}_{m'}^{\nu_1,\nu_2}(f_\alpha^\pm;\lambda x)+\tilde{\mathcal{A}}_{m'}^{\nu_1,\nu_2}(f_\alpha^\pm;(1-\lambda)y)$, and makes the operator convex. This completes the proof.

Theorem 2.4. *Suppose that $e_{\hat{j}}^*=(t-y)^{\hat{j}}=f(t)$, for all $\hat{j}\in\mathbb{N}_0$. Then, we have the following identity*

$$\tilde{\mathcal{A}}_{m'}^{\nu_1,\nu_2}(e_{\hat{j}}^*;y)=\sum_{\kappa=0}^{m'}\binom{m'}{\kappa}\frac{(\sigma_{m'}+\nu_1)^{m'-\kappa}(\rho_{m'}y)^\kappa}{(\sigma_{m'}+\nu_1+\rho_{m'}y)^{m'}}\frac{\Gamma(\bar{\beta}+1)}{(\tau_{m'}+\nu_2)^{\hat{j}}}\sum_{i=0}^{\hat{j}}\binom{\hat{j}}{i}\frac{\Gamma(\hat{j}-i+1)}{\Gamma(\bar{\beta}+\hat{j}-i+1)}\sum_{m=0}^{i}\kappa^m\left(-y(\tau_{m'}+\nu_2)\right)^{i-m}\binom{m}{i}.$$

Proof. The proof follows from Theorem 2.1. □

Next to the above, we have the following Lemma.

Lemma 2.5.

$$\tilde{\mathcal{A}}_{m'}^{\nu_1,\nu_2}(t-y;y)=\frac{\tau_{m'}y}{(\sigma_{m'}+\nu_1+\rho_{m'}y)(\tau_n+\nu_2)}+\frac{1}{(\bar{\beta}+1)(\tau_{m'}+\nu_2)}-y$$

$$\tilde{\mathcal{A}}_{m'}^{\nu_1,\nu_2}((t-y)^2;y)=\frac{2!}{(\tau_{m'}+\nu_2)^2(\bar{\beta}+1)(\bar{\beta}+2)}+y\left\{\frac{-2}{(\bar{\beta}+1)(\tau_{m'}+\nu_2)}+\frac{2\tau_{m'}}{(\bar{\beta}+1)(\tau_{m'}+\nu_2)^2}+\frac{\tau_{m'}}{(\tau_{m'}+\nu_2)^2}\right\}$$
$$+y^2\left\{1+\frac{\tau_{m'}^2(1-\frac{1}{m'})}{(\tau_{m'}+\nu_2)^2(\sigma_{m'}+\nu_1+\rho_{m'}y)^2}+\frac{-2\tau_{m'}}{(\tau_{m'}+\nu_2)(\sigma_{m'}+\rho_{m'}y+\nu_1)}\right\}$$

$$\tilde{\mathcal{A}}_{m'}^{\nu_1,\nu_2}((t-y)^3;y) = \frac{3!}{(\tau_{m'}+\nu_2)^3(\bar{\beta}+3)(\bar{\beta}+2)(\bar{\beta}+1)} + y\Big\{\frac{-6}{(\bar{\beta}+2)(\bar{\beta}+1)(\tau_{m'}+\nu_2)^2}$$
$$+ \frac{-6\tau_{m'}}{(\bar{\beta}+2)(\bar{\beta}+1)(\tau_{m'}+\nu_2)^3(\sigma_{m'}+\nu_1+\rho_{m'}y)} + \frac{-\tau_{m'}}{(\tau_{m'}+\nu_2)^3(\sigma_{m'}+\nu_1+\rho_{m'}y)}\Big\}$$
$$+ y^2\Big\{\frac{-6\tau_{m'}}{(\bar{\beta}+1)(\tau_{m'}+\nu_2)^2(\sigma_{m'}+\nu_1+\rho_{m'}y)} + \frac{-6\tau_{m'}}{(\bar{\beta}+1)(\tau_{m'}+\nu_2)^3(\sigma_{m'}+\nu_1+\rho_{m'}y)}$$
$$+ \frac{-\tau_{m'}}{(\tau_{m'}+\nu_2)^2(\sigma_{m'}+\nu_1+\rho_{m'}y)} + \frac{3\tau_{m'}}{(\tau_{m'}+\nu_2)^3(\sigma_{m'}+\nu_1+\rho_{m'}y)} + \frac{3}{(\bar{\beta}+1)(\tau_{m'}+\nu_2)}\Big\}$$
$$+ y^3\Big\{\frac{\tau_{m'}}{(\tau_{m'}+\nu_2)} - 1 + \Big\{\frac{-3}{(\bar{\beta}+1)(\tau_{m'}+\nu_2)^3} + \frac{-1}{(\tau_{m'}+\nu_2)^2}$$
$$+ \frac{3}{(\tau_{m'}+\nu_2)^3}\Big\} \times \frac{m'(m'-1)\rho_n^2}{(\sigma_{m'}+\nu_1+\rho_{m'}y)^2} + \frac{m'(m'-1)(m'-2)\rho_{m'}^3}{(\tau_{m'}+\nu_2)^3(\sigma_{m'}+\nu_1+\rho_{m'}y)^3}\Big\}$$

$$\tilde{\mathcal{A}}_{m'}^{\nu_1,\nu_2}((t-y)^4;y) = \frac{4!}{(\bar{\beta}+4)(\bar{\beta}+3)(\bar{\beta}+2)(\bar{\beta}+1)(\tau_{m'}+\nu_2)^4} + y\Big\{\Big\{\frac{-24}{(\tau_{m'}+\nu_2)^3(\bar{\beta}+3)(\bar{\beta}+2)(\bar{\beta}+1)}$$
$$+ \frac{24}{(\bar{\beta}+3)(\bar{\beta}+2)(\tau_{m'}+\nu_2)^4} + \frac{12\tau_{m'}}{(\bar{\beta}+2)(\bar{\beta}+1)(\tau_{m'}+\nu_2)^4} + \frac{-4}{(\bar{\beta}+1)(\tau_{m'}+\nu_2)^4} + \frac{14}{(\tau_{m'}+\nu_2)^4}\Big\}$$
$$\times \frac{\tau_{m'}}{(\sigma+\nu_1+\rho_{m'}y)}\Big\} + y^2\Big\{\Big\{\frac{-24}{(\bar{\beta}+2)(\bar{\beta}+1)(\tau_{m'}+\nu_2)^3} + \frac{-12}{(\bar{\beta}+1)(\tau_{m'}+\nu_2)^3} + \frac{4}{(\tau_{m'}+\nu_2)^3}\Big\}$$
$$\times \frac{\tau_{m'}}{(\sigma_{m'}+\nu_1+\rho_y)} + \frac{12}{(\bar{\beta}+2)(\bar{\beta}+1)(\tau_{m'}+\nu_2)^2} + \frac{12m'(m'-1)\rho_{m'}^2}{(\bar{\beta}+2)(\bar{\beta}+1)(\tau_{m'}+\nu_2)^4(\sigma_{m'}+\nu_1+\rho_{m'}y)^2}\Big\}$$
$$+ y^3\Big\{\Big\{\frac{12}{(\bar{\beta}+1)(\tau_{m'}+\nu_2)^2 + \frac{6}{(\tau_{m'}+\nu_2)^2}}\Big\}\frac{\tau_{m'}}{(\sigma_{m'}+\nu_1+\rho_{m'}y)} + \frac{-4}{(\bar{\beta}+1)(\tau_{m'}+\nu_2)}$$
$$+ \frac{-12m'(m'-1)\rho_{m'}^2}{(\sigma_{m'}+\nu_1+\rho_{m'}y)^2} + \frac{4m'(m'-1)(m'-2)\rho_{m'}^3}{(\bar{\beta}+1)(\tau_{m'}+\nu_2)^4(\sigma_{m'}+\nu_1+\rho_{m'}y)^3}\Big\}$$
$$+ y^4\Big\{\frac{-4\tau_{m'}}{(\tau_{m'}+\nu_2)} + 1 + \frac{6m'(m'-1)\rho_{m'}^2}{(\tau_{m'}+\nu_2)^2(\sigma_{m'}+\nu_1+\rho_{m'}y)^2} + \frac{-4m'(m'-1)(m'-2)\rho_{m'}^3}{(\tau_{m'}+\nu_2)^3(\sigma_{m'}+\nu_1+\rho_{m'}y)^3}$$
$$+ \frac{m'(m'-1)(m'-2)(m'-3)\rho_{m'}^4}{(\rho_{m'}y+\sigma_{m'}+\nu_1)^4(\tau_{m'}+\nu_2)^4}\Big\}$$

3. Convergence analysis of $\mathcal{A}_n^{\nu_1,\nu_2}$

Theorem 3.1. *Suppose that $f \in C_{\mathcal{F}}[0,1] = \varsigma$ (say). Then,*

$$\lim_{m\to\infty} \frac{1}{(q_m-p_m)^\gamma}\left|\left\{p_m < k < q_m : d\Big(\sum_{\xi=p_m+1}^{q_m} \tilde{a}_{m\xi}\mathcal{A}_\xi^{\nu_1,\nu_2}(f;y), f(y)\Big) > \epsilon\right\}\right| = 0,$$

uniformly on $[0,1]$.

Proof. The proof is obvious from Lemma 2.2 that $St_A^D - \lim_{n\to\infty} \tilde{\mathcal{A}}_n^{\nu_1,\nu_2}(e_j;y) = y^j$, for $j = 0,1,2$. Hence, from Bohman and Korovkin theorem [21], we reach our required result, $St_A^D - \lim_{m\to\infty} \tilde{\mathcal{A}}_m^{\nu_1,\nu_2}(f_\alpha^\pm;y) = f_\alpha^\pm(y)$. This implies that

$$\lim_{m\to\infty} \frac{1}{(q_m-p_m)^\gamma}\left|\left\{p_m < k < q_m : d\Big(\sum_{\xi=p_m+1}^{q_m} \tilde{a}_{m\xi}\mathcal{A}_\xi^{\nu_1,\nu_2}(f;y), f(y)\Big) > \epsilon\right\}\right| = 0,$$

□

Moving towards further investigation, we have to recall a few basic terminologies from [10]. By modulus of continuity in a fuzzy sense for order one ($w_1^{\mathcal{F}}(\mathfrak{f},\delta)$), we mean

$$w_1^{\mathcal{F}}(\mathfrak{f},\delta)=\sup_{y_1,y_2\in[a,b]\&|y_1-y_2|\leq\delta} d(\mathfrak{f}(y_1),\mathfrak{f}(y_2)),$$

where $\mathfrak{f}\in C_{\mathcal{F}}[a,b]$. Equivalently, it can be stated that $\mathfrak{f}_\alpha^\pm\in[a,b]$. By definition, moduli of smoothness of order $\mathfrak{r}$.

$$\omega_\gamma^{\mathfrak{r}}(\mathfrak{f}_\alpha^\pm;\delta)\equiv\sup_{0<h\leq\delta}\left\|\Delta_{h\gamma(y)}^{\mathfrak{r}}\mathfrak{f}_\alpha^\pm\right\|_\varsigma,$$

however $\delta>0$ and $\gamma(y)=\sqrt{y(1+y)}$ stands for the step weight function on $[0,1]$ and

$$\Delta_{h\gamma(y)}^{\mathfrak{r}}\mathfrak{f}_\alpha^\pm(y)=\sum_{i=0}^{\mathfrak{r}}(-1)^i\binom{\mathfrak{r}}{i}\mathfrak{f}_\alpha^\pm\left(y+\left(\frac{\mathfrak{r}}{2}-i\right)h\gamma(y)\right).$$

In fuzzy sense we may represent the modulus of smoothness for the order $\mathfrak{r}$, in the following way.

$$\omega_{\gamma,\mathfrak{r}}^{\mathcal{F}}(\mathfrak{f};\delta)\equiv\sup_{0<h\leq\delta} d\left(\Delta_{h\gamma(y)}^{\mathfrak{r},\mathcal{F}}\mathfrak{f},\bar{0}\right),$$

whereas $\bar{0}$, stands for fuzzy zero number. Moreover, Peetre's K- functional on ς, related to then step weight function γ, is defined as in [10]

$$K_{\mathfrak{r},\gamma}(\mathfrak{f}_\alpha^\pm;\delta^{\mathfrak{r}})=\inf_{g_\alpha^\pm\in\varsigma}\{\|\mathfrak{f}_\alpha^\pm-\mathfrak{g}_\alpha^\pm\|_\varsigma+\delta^{\mathfrak{r}}\|\gamma^{\mathfrak{r}}(\mathfrak{g}_\alpha^\pm)^{\mathfrak{r}}\|_\varsigma\},$$

meanwhile $\mathfrak{g}_\alpha^\pm$ is differentiable $\mathfrak{r}-1$ times and continuous absolutely in the interval $[a,b]\subset[0,1]$. It is pertinent to note the above relation in fuzzy contrast that:

$$K_{\mathfrak{r},\gamma}^{\mathcal{F}}(\mathfrak{f};\delta^{\mathfrak{r}})=\inf_{g\in C_{\mathcal{F}}[0,1]}\{d(\mathfrak{f},\mathfrak{g})+\delta^{\mathfrak{r}}d(\gamma^{\mathfrak{r}}(\mathfrak{g}'',\bar{0})\},$$

Theorem 3.2. *([10], Theorem 2.1.1) It is pertinent to note here that we can find some positive constant M_0, such that*

$$M_0^{-1}\omega_\gamma^r(g;\delta)\leq K_{r,\gamma}(g;\delta^r)\leq M_0\omega_\gamma^r(g;\delta).$$

Theorem 3.3. *Suppose that γ is any step weight function and $\mathfrak{l}\in\varsigma$. Then, we have the following estimation for $\mathcal{A}_n^{\nu_1,\nu_2}$ i.e.,*

$$d\left(\mathcal{A}_n^{\nu_1,\nu_2}(\mathfrak{l};y),l(y)\right)\leq M\omega_{\gamma,r+1}^{\mathcal{F}}\left(\mathfrak{l};\sqrt{\frac{\phi_{r+1}(y)^{r+1}+\psi_{r+1}(y)}{4\gamma^{r+1}(y)}}\right)+\omega_{\gamma,1}^{\mathcal{F}}\left(\mathfrak{l};\frac{\phi_r(y)}{\gamma^{r+1}(y)}\right).$$

Proof. Suppose that $\mathfrak{l}\in\varsigma$, which implies that $\mathfrak{l}_\alpha^\pm\in C[0,1]$. Let us consider the auxiliary operator

$$\aleph(l_\alpha^\pm;y)=\mathfrak{l}_\alpha^\pm(y)+\tilde{\mathcal{A}}_n^{\nu_1,\nu_2}(\mathfrak{l}_\alpha^\pm(t);y)-\left(\mathfrak{l}_\alpha^\pm\left(\tilde{\mathcal{A}}_n^{\nu_1,\nu_2}((t-y)^r;y)+y\right)\right)^{\frac{1}{r}},\tag{5}$$

where $\mathfrak{l}_\alpha^\pm\in C_{r+1}(\mathbb{R}^+)$ and if $\mathfrak{l}_\alpha^\pm(t)$ is any polynomial, then $r=$ Degree of $\mathfrak{l}_\alpha^\pm(t)$.
Obviously, $\aleph(1;y)=1,\aleph(t;y)=y,\aleph(t-y;y)=0,\cdots,\aleph((t-y)^r;y)=0$. Suppose that $y\in\mathbb{R}_0^+$ and $\mathfrak{l}_\alpha^\pm\in C_{r+1}(\mathbb{R}_0^+)$. Then, applying Taylor's series expansion for $\mathfrak{l}_\alpha^\pm$, up to $(r+1)^{th}$ term, and we obtain

$$\mathfrak{l}_\alpha^\pm(t)=\mathfrak{l}_\alpha^\pm(y)+(t-y)\{\mathfrak{l}_\alpha^\pm\}'(y)+(t-y)^2\{\mathfrak{l}_\alpha^\pm\}''(y)+\cdots+(t-y)^r\{\mathfrak{l}_\alpha^\pm\}^r(y)+\int_y^t\frac{(t-y)^r}{r!}\{\mathfrak{l}_\alpha^\pm\}^{r+1}(z)dz.$$

Now applying the operator $A_n^{\nu_1,\nu_2}$, on both sides

$$\aleph(\text{ɩ}_\alpha^\pm; y) = \text{ɩ}_\alpha^\pm(y) + \aleph\left(\int_y^t \frac{(t-z)^r}{r!}\{\text{ɩ}_\alpha^\pm\}^{r+1}(z); y\right). \tag{6}$$

Moreover, $\aleph(\varrho_\alpha^\pm; y) = \text{ɩ}_\alpha^\pm(y) + \tilde{\mathcal{A}}_n^{\nu_1,\nu_2}\left(\int_y^t \frac{(t-z)^r}{r!}\{\text{ɩ}_\alpha^\pm\}^{r+1}(z); y\right) - \left(\int_y^{\Phi_r(y)+y} \frac{(\Phi_r(y)+y)-z)^r}{r!}\{\text{ɩ}_\alpha^\pm\}^{r+1}(z)dz; y\right)$. For simplicity let us take $\Phi_r(y) = A_n^{\nu_1,\nu_2}((t-y)^r; y)$ and $\psi_r(y) = \{A_n^{\nu_1,\nu_2}((t-y); y)\}^r$ .This implies

$$\begin{aligned}
&\le \tilde{\mathcal{A}}_n^{\nu_1,\nu_2}\left(\int_y^t \frac{(t-z)^r}{r!}\{\varrho_\alpha^\pm\}^{r+1}(z)dz; y\right) - \left(\int_y^{\Phi_r(y)+y} \frac{(\Phi_r(y)+y)-z)^r}{r!}\{\text{ɩ}_\alpha^\pm\}^{r+1}(z)dz; y\right).\\
&\le \|\gamma^{r+1}\{\varrho_\alpha^\pm\}^{r+1}\|_w \tilde{\mathcal{A}}_{n,a}^{\nu_1,\nu_2}\left(\int_y^t \frac{(t-z)^r}{\gamma^{r+1}(y)r!}\{\varrho_\alpha^\pm\}^{r+1}(z)dz; y\right)\\
&\quad + \|\gamma^{r+1}\{\varrho_\alpha^\pm\}^{r+1}\|\left(\int_y^{\Phi_r(y)+y} \frac{(\Phi_r(y)+y)-z)^r}{r!}\text{ɩ}^{r+1}(z)\frac{dz}{\gamma^{r+1}(y)}; y\right).\\
&\le \gamma(y)^{r+1}\|\gamma^{r+1}\{\varrho_\alpha^\pm\}^{r+1}\|_w\left(\tilde{\mathcal{A}}_n^{\nu_1,\nu_2}(t-y)^{r+1}; y\right) + \gamma^{r+1}(y)\|\gamma^{r+1}\{\varrho_\alpha^\pm\}^{r+1}\|_w(\Phi_r(y))^{r+1}.
\end{aligned}$$

In other way we can re write the above expression as follows:

$$d\Big(\aleph(\varrho; y), \varrho(y)\Big) \le \gamma(y)^{r+1} d\Big(\gamma^{r+1}\{\varrho\}^{r+1}, \bar{0}\Big)\left(\tilde{\mathcal{A}}_n^{\nu_1,\nu_2}(t-y)^{r+1}; y\right) + \gamma^{r+1}(y) d\left(\gamma^{r+1}\{\varrho\}^{r+1}\right)(\Phi_r(y))^{r+1}.$$

Moreover, $|\aleph(\text{ɩ}; y)| \le 3\|\text{ɩ}\|$. Now,

$$\begin{aligned}
|\aleph(\text{ɩ}_\alpha^\pm; y) - \text{ɩ}_\alpha^\pm(y)| &\le |\aleph(\text{ɩ}_\alpha^\pm; y) - \text{ɩ}_\alpha^\pm(y) + \text{ɩ}_\alpha^\pm(\Phi_r(y)+y) - \text{ɩ}_\alpha^\pm(y)|\\
&\le \aleph(\text{ɩ}_\alpha^\pm - \varrho_\alpha^\pm; y) + |\aleph(\varrho; y) - \varrho_\alpha^\pm(y)| + |\varrho_\alpha^\pm(y) - \text{ɩ}_\alpha^\pm(y)| + |\text{ɩ}_\alpha^\pm(\Phi_r(y) + y - \text{ɩ}_\alpha^\pm(y)|\\
&\le 4\|\text{ɩ}_\alpha^\pm - \varrho_\alpha^\pm\| + \gamma(y)^{-(r+1)}(y)\|\gamma^{r+1}\{\varrho_\alpha^\pm\}''\|_w(\Phi_{r+1}(y) + \psi_{r+1}(y))\\
&\le 4K_{r+1,\gamma}(\text{ɩ}_\alpha^\pm; \frac{\Phi_{r+1}(y)^{r+1} + \psi(y)}{4\gamma(y)^{r+1}})
\end{aligned}$$

Alternatively, we obtain that $d\left(\aleph(\text{ɩ}; y), \text{ɩ}(y)\right) \le 4K_{r+1,\gamma}^{\mathcal{F}}(\text{ɩ}; \frac{\Phi_{r+1}(y)^{r+1}+\psi(y)}{4\gamma(y)^{r+1}})$.

Therefore we can find some constant $M > 0$, such that $|\aleph(\text{ɩ}_\alpha^\pm; y) - \text{ɩ}_\alpha^\pm(y)| \le Mw_\gamma^{r+1}\left(\text{ɩ}_\alpha^\pm; \sqrt{\frac{\Phi_{r+1}(y)^{r+1}+\psi_{r+1}(y)}{4\gamma(y)^{r+1}}}\right)$. This indicates that $d\left(\aleph(\text{ɩ}; y), \text{ɩ}(y)\right) \le M\omega_{\gamma,r+1}^{\mathcal{F}}\left(\text{ɩ}; \sqrt{\frac{\Phi_{r+1}(y)^{r+1}+\psi_{r+1}(y)}{4\gamma(y)^{r+1}}}\right)$.

Again from modulus of smoothness of first order we get,

$$|(\Phi_r(y) + y) - \text{ɩ}_\alpha^\pm(y)| = |\text{ɩ}_\alpha^\pm(\Phi_r(y) + y) - \text{ɩ}_\alpha^\pm(y)| \le w_1\left((\text{ɩ}_\alpha^\pm; \frac{\Phi_r(y)}{\gamma^{r+1}(y)}\right).$$

This imprecates us,

$$d\left((\Phi_r(y) + y), \text{ɩ}(y)\right) \le \omega_{\gamma,1}^{\mathcal{F}}\left((\text{ɩ}_\alpha^\pm; \frac{\Phi_r(y)}{\gamma^{r+1}(y)}\right).$$

On combining all the above inequality we get

$$\begin{aligned}
|\tilde{\mathcal{A}}_n^{\nu_1,\nu_2}(\text{ɩ}_\alpha^\pm; y) - \text{ɩ}_\alpha^\pm(y)| &\le |\aleph(\text{ɩ}_\alpha^\pm; y) - \text{ɩ}_\alpha^\pm(y)| + |\text{ɩ}_\alpha^\pm(\Phi_r(y) + y) - \text{ɩ}_\alpha^\pm(y)|\\
&\le |\aleph(\text{ɩ}_\alpha^\pm - \varrho; y)| + |\aleph(\varrho_\alpha^\pm; y) - \varrho_\alpha^\pm(y)| + |l_\alpha^\pm(y) - \varrho_\alpha^\pm(y)| + |\text{ɩ}_\alpha^\pm(\Phi_r(y) + y) - \text{ɩ}_\alpha^\pm(y)|.\\
&\le Mw_{r+1}^\gamma\left(\text{ɩ}_\alpha^\pm; \sqrt{\frac{\Phi_{r+1}(y)^{r+1} + \psi_{r+1}(y)}{4\gamma^{r+1}(y)}}\right) + w\left(\text{ɩ}_\alpha^\pm; \frac{\Phi_r(y)}{\gamma^{r+1}(y)}\right)
\end{aligned}$$

Equivalently, we can write it as follows:

$$d\left(\mathcal{A}_n^{\nu_1,\nu_2}(\mathfrak{l};y),l(y)\right) \leq M\omega_{\gamma,r+1}^{\mathcal{F}}\left(\mathfrak{l};\sqrt{\frac{\phi_{r+1}(y)^{r+1}+\psi_{r+1}(y)}{4\gamma^{r+1}(y)}}\right)+\omega_{\gamma,1}^{\mathcal{F}}\left(\mathfrak{l};\frac{\phi_r(y)}{\gamma^{r+1}(y)}\right).$$

□

Now, some local direct approximation results have been estimated with the help of the Lipschitz-type maximal function. Let us recall few basic definitions from [3] and [34].

$$Lip_M(\kappa)=\{f\in\varsigma:\ |f(t)-f(y)|\leq M\frac{|t-y|^{\kappa}}{(\alpha_1^2y^2+\alpha_2y+t)^{\frac{\kappa}{2}}},\ \text{for } \alpha_1>0,\ \alpha_2>0,\ \kappa\in(0,1] \text{ and } M>0.\},$$

is called Lipschitz-type maximal function.

Theorem 3.4. *Let us take $f\in Lip_M(\kappa)$. Then for every $\tau\in(0,1]$, we have the following inequality.*

$$d\left(\mathcal{A}_n^{\nu_1,\nu_2}(f;y),f(y)\right)\leq M\sqrt[6]{\frac{[\Phi_{n,a}^3(y)]^{\tau}}{[\alpha_1y^2+\alpha_2y]^{3\tau}}}.$$

Proof. Without loss of generality, we obtain

$$d\left(\mathcal{A}_n^{\nu_1,\nu_2}(f;y),f(y)\right)\leq d\left(\mathcal{A}_n^{\nu_1,\nu_2}(f(t)-f(y);y),\bar{0}\right)+\left(d(f(y),\bar{0}\right)d\left(\mathcal{A}_n^{\nu_1,\nu_2}(1;y),1\right)$$

Since, $f\in C_{\mathcal{F}}[0,1]$, implies $f_\alpha^{\pm}\in C[0,1]$, above inequality can be rewritten as

$$\left|\tilde{\mathcal{A}}_n^{\nu_1,\nu_2}(f_\alpha^{\pm};y)-f_\alpha^{\pm}(y)\right|\leq\left|\tilde{\mathcal{A}}_n^{\nu_1,\nu_2}(f_\alpha^{\pm}(t)-f_\alpha^{\pm}(y);y)\right|+\left|f_\alpha^{\pm}(y)\right|\left|\tilde{\mathcal{A}}_n^{\nu_1,\nu_2}(1;y)-1\right|$$

Again, from the Lipschitz maximal function, we get

$$\begin{aligned}\left|\tilde{\mathcal{A}}_n^{\nu_1,\nu_2}(f_\alpha^{\pm};y)-f_\alpha^{\pm}(y)\right| &\leq \tilde{\mathcal{A}}_n^{\nu_1,\nu_2}\left(\frac{M|t-y|^{\tau}}{\alpha_1y^2+\alpha_2y+t^{\frac{\tau}{2}}};y\right)\\ &\leq M\tilde{\mathcal{A}}_n^{\nu_1,\nu_2}\left(\frac{|t-y|^{\tau}}{(\alpha_1y^2+\alpha_2y)^{\frac{\tau}{2}}};;y\right)\end{aligned}$$

Applying Holder's inequality in the above inequality,

$$\begin{aligned}d\left(\mathcal{A}_n^{\nu_1,\nu_2}(f;y),f(y)\right) &\leq M\tilde{\mathcal{A}}_n^{\nu_1,\nu_2}\left(\frac{|t-y|^6}{(\alpha_1y^2+\alpha_2y)^3}\right)^{\frac{\tau}{6}}.\left(\tilde{\mathcal{A}}_n^{\nu_1,\nu_2}(1;y)\right)^{\frac{6-\tau}{6}}\\ &\leq M\left(\alpha_1y^2+\alpha_2y\right)^{-\frac{\tau}{2}}\left\{\tilde{\mathcal{A}}_n^{\nu_1,\nu_2}\left(|t-y|^6;y\right)\right\}^{\frac{\tau}{6}}\\ &\leq M\sqrt{\frac{[\Phi_{n,a}^6(y)]^{\tau}}{(\alpha_1y^2+\alpha_2y)^{3\tau}}},\end{aligned}$$

where $\Phi_{n,a}^6(y)=\tilde{\mathcal{A}}_n^{\nu_1,\nu_2}\left(|t-y|^6;y\right)$. □

4. Voronovskaya and Grüss Voronovskaya type estimates

In this section, we observe certain remarks and theorems for the newly positive linear operator $\tilde{\mathcal{A}}_n^{\nu_1,\nu_2}$.

Theorem 4.1. *We deduce*

$$St_A^D \lim_m m \times d(A\mathcal{A}_m^{\nu_1,\nu_2}(\hbar; y), \hbar(y)) = St_A^D \lim_m m \times d(A\mathcal{A}_m^{\nu_1,\nu_2}((\bar{\eta} - \bar{y}); \bar{y})\hbar(\bar{y}, \bar{0}) + \cdots + St_A^D \lim_m m \times d(A\mathcal{A}_m^{\nu_1,\nu_2}((\bar{\eta} - \bar{y})^r; \bar{y})\frac{\hbar^r(\bar{y})}{r!}, \bar{0})$$

Proof. Suppose that $h \in \varsigma$, this implies that $\hbar_\alpha^\pm \in [0, 1]$. Then, from Taylor's series expansion up to the r^{th} term and achieve the following,

$$\hbar_\alpha^\pm(\eta) = \hbar_\alpha^\pm(\bar{y}) + (\bar{\eta} - \bar{y})\hbar_\alpha^{\pm\prime}(\bar{y}) + (\bar{\eta} - \bar{y})^2 \frac{\hbar_\alpha^{\pm\prime\prime}(\bar{y})}{2} + \cdots + (\bar{\eta} - \bar{y})^r \frac{(\bar{\eta} - \bar{y})^r}{r!} + (\bar{\eta} - \bar{y})^r \Im(\eta, \bar{y}), \tag{7}$$

where $\Im(\eta, \bar{y}) \to 0$, as $\bar{\eta} \to \bar{y}$. Applying the operator $\tilde{\mathcal{A}}_m^{\nu_1,\nu_2}$, we have

$$\tilde{\mathcal{A}}_m^{\nu_1,\nu_2}(\hbar_\alpha^\pm(\eta); \bar{y}) = \hbar_\alpha^\pm(\bar{y}) + \tilde{\mathcal{A}}_m^{\nu_1,\nu_2}((\bar{\eta} - \bar{y})\hbar_\alpha^{\pm\prime}(\bar{y}); \bar{y}) + \tilde{\mathcal{A}}_m^{\nu_1,\nu_2}((\bar{\eta} - \bar{y})^2; \bar{y})\frac{\tilde{\hbar}_\alpha^{\pm\prime\prime}(\bar{y})}{2} + \cdots + \frac{\tilde{\mathcal{A}}_m^{\nu_1,\nu_2}((\bar{\eta} - \bar{y})^r; \bar{y})}{r!} f^r(\bar{y}) + \tilde{\mathcal{A}}_m^{\nu_1,\nu_2}((\bar{\eta} - \bar{y})^r \Im(\eta, \bar{y}); \bar{y}),$$

Again from the Cauchy-Schwartz inequality we have

$$\tilde{\mathcal{A}}_m^{\nu_1,\nu_2}((\bar{\eta} - \bar{y})^r \Im(\bar{\eta}, \bar{y}); \bar{y}) \leq \sqrt{\tilde{\mathcal{A}}_m^{\nu_1,\nu_2}(\Im(\bar{\eta}, \bar{y})^2); \bar{y})} \times \sqrt{\tilde{\mathcal{A}}_m^{\nu_1,\nu_2}(\bar{\eta} - \bar{y})^{2r}; \bar{y})}.$$

It is obvious that $\lim_{m\to\infty}(\tilde{\mathcal{A}}_m^{\nu_1,\nu_2}(\Im(\bar{\eta}, \bar{y})^2); \bar{y}) = 0$. This implies that

$$St_A^D - \lim_{m\to\infty} m\tilde{\mathcal{A}}_m^{\nu_1,\nu_2}((\bar{\eta} - \bar{y})^r \Im(\eta, \bar{y}); \bar{y}) = 0.$$

Therefore, we estimate the following result,

$$St_A^D - \lim_{m\to\infty} m \left\{ \tilde{\mathcal{A}}_m^{\nu_1,\nu_2}(\hbar_\alpha^\pm; \bar{y}) - \hbar_\alpha^\pm(\bar{y}) \right\} = St_A^D - \lim_{m\to\infty} m \Big\{ \tilde{\mathcal{A}}_m^{\nu_1,\nu_2}((\bar{\eta} - \bar{y}); \bar{y})\hbar_\alpha^{\pm\prime}(\bar{y}) + \tilde{\mathcal{A}}_m^{\nu_1,\nu_2}((\bar{\eta} - \bar{y})^2; \bar{y})\frac{\hbar_\alpha^{\pm\prime\prime}(\bar{y})}{2} + \cdots + \frac{\tilde{\mathcal{A}}_m^{\nu_1,\nu_2}((\bar{\eta} - \bar{y})^r; \bar{y})}{r!} f^r(\bar{y}) \Big\}.$$

In other way we may demonstrate that for each $\bar{\epsilon} > 0$, we can find $0 < \tilde{\epsilon} < \bar{\epsilon}$, in such a way that

$$\mathcal{B}_{p_m,q_m} = \Big\{ p_m < k < q_m : m \times d(\sum_{j=p_m+1}^{q_m} a_{mj}\mathcal{A}_j^{\nu_1,\nu_2}(\hbar; \bar{y}), \hbar(\bar{y})) \geq \epsilon \Big\},$$

and $\mathcal{B}_{p_m,q_m} = \bigcup_{i=1}^r \mathcal{B}_{p_m,q_m}^i$, while

$$\mathcal{B}_{p_m,q_m} = \Big\{ p_m < k < q_m : m \times d(\sum_{j=p_m+1}^{q_m} a_{mj}\mathcal{A}_j^{\nu_1,\nu_2}((\bar{\eta} - \bar{y})^i; \bar{y})\hbar^i(\bar{y}, \bar{0}) \geq \frac{\bar{\epsilon} - \tilde{\epsilon}}{r} \Big\}.$$

Consequently, we conclude that

$$St_A^D - \lim_m m \times d(A\mathcal{A}_m^{\nu_1,\nu-2}(\hbar; y), \hbar(y)) = St_A^D - \lim_m m \times d(A\mathcal{A}_m^{\nu_1,\nu_2}((\bar{\eta} - \bar{y}); \bar{y})\hbar((\bar{y}), \bar{0}) + \cdots + St_A^D - \lim_m m \times d(A\mathcal{A}_m^{\nu_1,\nu_2}((\bar{\eta} - \bar{y})^r; \bar{y})\frac{\hbar^r(\bar{y})}{r!}, \bar{0})$$

□

Remark 4.2. *It can be easily observed that*

$$\tilde{\mathcal{A}}_n^{\nu_1,\nu_2}((t-y)^2;y) \geq \frac{1}{3(\tau_n+\nu_2)^2}.$$

Remark 4.3. *Now, we have the following results*

$$\tilde{\mathcal{A}}_n^{\nu_1,\nu_2}((t-y)^3;y) \leq \frac{5\sigma_n^3\tau_n^3}{2(\tau_n+\nu_2)^3(\sigma_n+\rho_n y+\nu_1)^3};$$

$$\tilde{\mathcal{A}}_n^{\nu_1,\nu_2}((t-y)^4;\bar{y}) \leq \frac{\tau_n^4\sigma_n^4}{(\tau_n+\nu_2)^4}\Phi_1(\tau_n,\sigma_n).$$

Theorem 4.4. *Let us assume $f \in C_{\mathcal{F}}[0,1]$, which is impled by $f_\alpha^\pm \in C[0,1]$. Suppose that f', f'' exist in $C_{\mathcal{F}}[0,1]$. Then, we establish the following estimation:*

$$\begin{aligned} d\Big(n\times d\Big(\mathcal{A}_n^{\nu_1,\nu_2}(f;y),f(y)\Big),\Big\{\tilde{\mathcal{A}}_n^{\nu_1,\nu_2}(z-y;y)f'(y)+\frac{1}{2}\tilde{\mathcal{A}}_n^{\nu_1,\nu_2}((z-y)^2;y)f''(y)\Big\}\Big) \\ \leq K^*\Big\{w_{1,\mathcal{F}}(f',\frac{1}{\sqrt{n}})+w_{2,\mathcal{F}}(f'',\frac{1}{\sqrt{n}})\Big\} \\ +\left(\frac{1}{\zeta+1}+\nu_1+1-\nu_2\right)yf'(y) \\ \leq K^*\left\{w_{1,\mathcal{F}}(f',\frac{1}{\sqrt{n+1}})+w_{2,\mathcal{F}}(f'',\frac{1}{\sqrt{n+1}})\right\} \\ +K_1\left\{d\left(f',\bar{0}\right)+d\left(f'',\bar{0}\right)\right\}, \end{aligned}$$

where, $K_1=\sup_y\left(\frac{1}{\zeta+1}+\nu_1+1-\nu_2\right)y.$

Proof. From Gonska and Rasa [16], we get for $f_\alpha^\pm \in C[0,1]$ i.e.,

$$\begin{aligned} \Big|\bar{L}_n(f_\alpha^\pm;y)-f_\alpha^\pm(y)-\frac{1}{2}\bar{L}_n((e_1-y)^2;y){f_\alpha^\pm}''(y)-\bar{L}_n((e_1-y);y){f_\alpha^\pm}'(y)\Big| \leq \bar{L}_n((e_1-y)^2;y)\times \\ \Big\{\frac{|\bar{L}_n((e_1-y)^3;y)|}{\bar{L}_n((e_1-y)^2;y)}.\frac{5}{6h}w_1({f_\alpha^\pm}',h)+\left(\frac{3}{4}+\frac{\bar{L}_n((e_1-y)^4;y)}{\bar{L}_n((e_1-y)^2;y)}.\frac{1}{16h^2}\right)w_2({f_\alpha^\pm}'';h)\Big\}, \end{aligned}$$

where $L_n: C_{\mathcal{F}}[0,1]\to C_{\mathcal{F}}[0,1] \longleftrightarrow \bar{L}_n: C[0,1]\to C[0,1]$, and $0<h\leq\frac{1}{2}$.
From Remark 4.3, it can be observed that

$$\frac{|\tilde{\mathcal{A}}_n^{\nu_1,\nu_2}((z-y)^3;y)|}{\tilde{\mathcal{A}}_n^{\nu_1,\nu_2}((z-y)^2;y)} \leq O(1)$$

and

$$\frac{|\tilde{\mathcal{A}}_n^{\nu_1,\nu_2}((z-y)^4;y)|}{\tilde{\mathcal{A}}_n^{\nu_1,\nu_2}((z-y)^2;y)} \leq O(1).$$

Now, applying the operator $\tilde{\mathcal{A}}_n^{\nu_1,\nu_2}$,

$$\begin{aligned} \Big|\tilde{\mathcal{A}}_n^{\nu_1,\nu_2}(f_\alpha^\pm;y)-f_\alpha^\pm(y)-\tilde{\mathcal{A}}_n^{\nu_1,\nu_2}((z-y);y){f_\alpha^\pm}'(y)-\frac{1}{2}\tilde{\mathcal{A}}_n^{\nu_1,\nu_2}((z-y)^2;y).{f_\alpha^\pm}''(y)\Big| \leq \tilde{\mathcal{A}}_n^{\nu_1,\nu_2}((z-y)^2;y)\times \\ \Big\{\frac{5}{6h}O(1)w_1({f_\alpha^\pm}',h)+\left(\frac{3}{4}+O(1).\frac{1}{16h^2}\right)w_2({f_\alpha^\pm}'';y)\Big\}. \end{aligned}$$

Now, putting $h = \frac{1}{\sqrt{n}}$ and multiplying n on both sides we get,

$$\left| n\Big(\tilde{\mathcal{A}}_n^{\nu_1,\nu_2}(f_\alpha^\pm; y) - f_\alpha^\pm(y)\Big) - \tilde{\mathcal{A}}_n^{\nu_1,\nu_2}(z-y;y) f_\alpha^{\pm\prime}(y) - \frac{1}{2}\tilde{\mathcal{A}}_n^{\nu_1,\nu_2}((z-y)^2;y) f_\alpha^{\pm\prime\prime}(y)\right|$$
$$\le \frac{a}{a-1}\Big(\frac{5}{6}O(n^{-1}) w_1(f_\alpha^{\pm\prime}; \frac{1}{\sqrt{n}}) + \left(\frac{3}{4} + \frac{O(1)}{16}\right) w_2(f_\alpha^{\pm\prime\prime}; \frac{1}{\sqrt{n}})\Big) \le K^* \left\{ w_1(f_\alpha^{\pm\prime}, \frac{1}{\sqrt{n}}) + w_2(f_\alpha^{\pm\prime\prime}, \frac{1}{\sqrt{n}})\right\}$$

Furthermore,

$$\left| n\Big(\tilde{\mathcal{A}}_n^{\nu_1,\nu_2}(f_\alpha^\pm; y) - f_\alpha^\pm(y)\Big) - \tilde{\mathcal{A}}_n^{\nu_1,\nu_2}(z-y;y) f_\alpha^{\pm\prime}(y) - \frac{1}{2}\tilde{\mathcal{A}}_n^{\nu_1,\nu_2}((z-y)^2;y) f_\alpha^{\pm\prime\prime}(y)\right|$$
$$\le K^*\Big\{ w_1(f_\alpha^{\pm\prime}, \frac{1}{\sqrt{n}}) + w_2(f_\alpha^{\pm\prime\prime}, \frac{1}{\sqrt{n}})\Big\}$$
$$+ \left(\frac{1}{\zeta+1} + v_1 + 1 - v_2\right) y f_\alpha^{\pm\prime}(y)$$
$$\le K^* \left\{ w_1(f_\alpha^{\pm\prime}, \frac{1}{\sqrt{n+1}}) + w_2(f_\alpha^{\pm\prime\prime}, \frac{1}{\sqrt{n+1}})\right\}$$
$$+ K_1 \left\{ \|f_\alpha^{\pm\prime}\|_\infty + \|f_\alpha^{\pm\prime\prime}\|_\infty \right\},$$

where, $K_1 = \sup_y \left(\frac{1}{\zeta+1} + v_1 + 1 - v_2\right) y$. Alternatively, we can express the above inequality in following manner,

$$d\Big(nd\Big(\mathcal{A}_n^{\nu_1,\nu_2}(f;y), f(y)\Big), \Big\{\tilde{\mathcal{A}}_n^{\nu_1,\nu_2}(z-y;y) f'(y) + \frac{1}{2}\tilde{\mathcal{A}}_n^{\nu_1,\nu_2}((z-y)^2;y) f''(y)\Big\}\Big)$$
$$\le K^*\Big\{ w_{1,\mathcal{F}}(f', \frac{1}{\sqrt{n}}) + w_{2,\mathcal{F}}(f'', \frac{1}{\sqrt{n}})\Big\} + y f'(y)$$
$$\le K^* \left\{ w_{1,\mathcal{F}}(f', \frac{1}{\sqrt{n+1}}) + w_{2,\mathcal{F}}(f'', \frac{1}{\sqrt{n+1}})\right\}$$
$$+ K_1 \{ d(f', \bar{0}) + d(f'', \bar{0})\}$$

□

Theorem 4.5. *Suppose that* $\mathfrak{f}_\alpha^\pm \in C[0,1]$ *and* $\mathfrak{f}_\alpha^{\pm\prime}$ *exists in* $[0,1]$. *Then, we have*

$$d\Big(\mathcal{A}_n^{\nu_1,\nu_2}(\mathfrak{f}g;y), \Big\{\mathcal{A}_n^{\nu_1,\nu_2}(\mathfrak{f};\tilde{y}).\mathcal{A}_n^{\nu_1,\nu_2}(g;\tilde{y}) + \frac{\tilde{y}(1-\tilde{y})}{n}\mathfrak{f}'(\tilde{y})g'(\tilde{y})\Big\}\Big) \le \Big\{\omega^{\mathcal{F}}(\mathfrak{f}';h).\omega^{\mathcal{F}}(\mathfrak{g}';h)$$
$$+ \frac{1}{h}\omega^{\mathcal{F}}(\mathfrak{f};h)\omega^{\mathcal{F}}_{\gamma,}(\mathfrak{g}';h) + \frac{1}{h}\omega^{\mathcal{F}}_{\gamma,}(\mathfrak{f}';h)\omega^{\mathcal{F}}_{\gamma,}(\mathfrak{g};h)$$
$$+ \frac{1}{h}\max\{d(\mathfrak{f},\bar{0}), \frac{1}{h^3}\omega^{\mathcal{F}}_{\gamma,}(\mathfrak{f}';h)\}.$$
$$\max\{d(\mathfrak{g},\bar{0}), \frac{1}{h}p^3\omega^{\mathcal{F}}_{\gamma,}(\mathfrak{g}';h)\}\Big\}.$$

Proof. Let $L_n(\mathfrak{f}_\alpha^\pm, \mathfrak{g}_\alpha^\pm; \tilde{y}) = \tilde{\mathcal{A}}_n^{\nu_1,\nu_2}(\mathfrak{f}_\alpha^\pm\mathfrak{g}_\alpha^\pm; \tilde{y}) - \tilde{\mathcal{A}}_n^{\nu_1,\nu_2}(\mathfrak{f}_\alpha^\pm; \tilde{y}).\tilde{\mathcal{A}}_n^{\nu_1,\nu_2}(\mathfrak{g}_\alpha^\pm; \tilde{y}) - \frac{\tilde{y}(1-\tilde{y})}{n}\mathfrak{f}_\alpha^{\pm\prime}(\tilde{y})\mathfrak{g}_\alpha^{\pm\prime}(\tilde{y})$. Without loss of the generality we can have the following

$$L_n(\mathfrak{f}_\alpha^\pm, \mathfrak{g}_\alpha^\pm; \tilde{y}) = \left| L_n(\mathfrak{f} - u_1{}_\alpha^\pm + u_1{}_\alpha^\pm, \mathfrak{g}_\alpha^\pm + v_\alpha^\pm - v_\alpha^\pm; \tilde{y})\right|$$
$$\le |L_n(\mathfrak{f}_\alpha^\pm, \mathfrak{g}_\alpha^\pm - v_\alpha^\pm; \tilde{y})| + \left| L_n(u_1{}_\alpha^\pm, \mathfrak{g}_\alpha^\pm - v_\alpha^\pm; \tilde{y})\right| + \left| L_n(\mathfrak{f}_\alpha^\pm - u_1{}_\alpha^\pm, \mathfrak{g}_\alpha^\pm; \tilde{y})\right| + \left| L_n(u_1{}_\alpha^\pm, v_\alpha^\pm; \tilde{y})\right|,$$

where $u_\alpha^\pm, v_\alpha^\pm \in C^4[0,1]$. Now, we need to evaluate $\tilde{\mathcal{A}}_n^{\nu_1,\nu_2}(\mathfrak{f}_\alpha^\pm \mathfrak{g}_\alpha^\pm;\tilde{y}) - \tilde{\mathcal{A}}_n^{\nu_1,\nu_2}(\mathfrak{f}_\alpha^\pm;\tilde{y}).\tilde{\mathcal{A}}_n^{\nu_1,\nu_2}(\mathfrak{g}_\alpha^\pm;\tilde{y})$.
In order to do this, we consider the Taylor series expansion of function, $\mathfrak{f}_\alpha^\pm$, $\mathfrak{g}_\alpha^\pm$, and $f_\alpha^\pm \mathfrak{g}_\alpha^\pm$. Then, we get the followings

$$\tilde{\mathcal{A}}_n^{\nu_1,\nu_2}(\mathfrak{f}_\alpha^\pm \mathfrak{g}_\alpha^\pm;\tilde{y}) - \tilde{\mathcal{A}}_n^{\nu_1,\nu_2}(\mathfrak{f}_\alpha^\pm;\tilde{y}).\tilde{\mathcal{A}}_n^{\nu_1,\nu_2}(\mathfrak{g}_\alpha^\pm;y) = \Big[\tilde{\mathcal{A}}_n^{\nu_1,\nu_2}((z-\tilde{y})^2;y) - [\tilde{\mathcal{A}}_n^{\nu_1,\nu_2}(z-y;y)]^2\Big].\mathfrak{f}_\alpha^{\pm\prime}(\tilde{y})\mathfrak{g}_\alpha^{\pm\prime} - \frac{1}{2}\tilde{\mathcal{A}}_n^{\nu_1,\nu_2}(z-\tilde{y};\tilde{y})\tilde{\mathcal{A}}_n^{\nu_1,\nu_2}((z-\tilde{y})^2;\tilde{y})(\mathfrak{f}_\alpha^{\pm\prime}\mathfrak{g}_\alpha^{\pm\prime})'(\tilde{y}) + [\tilde{\mathcal{A}}_n^{\nu_1,\nu_2}((z-\tilde{y})^2;\tilde{y})]^2\frac{\mathfrak{f}_\alpha^{\pm\prime\prime}(\tilde{y})\mathfrak{g}_\alpha^{\pm\prime\prime}(\tilde{y})}{4}.$$

$$n\tilde{\mathcal{A}}_n^{\nu_1,\nu_2}(\mathfrak{f}_\alpha^\pm \mathfrak{g}_\alpha^\pm;\tilde{y}) - \tilde{\mathcal{A}}_n^{\nu_1,nu_2}(f_\alpha^p m;\tilde{y}).\tilde{\mathcal{A}}_n^{\nu_1,nu_2}(g_\alpha^p m;\tilde{y}) \le \tilde{y}\mathfrak{f}_\alpha^{\pm\prime}(\tilde{y})\mathfrak{g}_\alpha^{\pm\prime}(\tilde{y}) \le \tilde{y}\|\mathfrak{f}_\alpha^{\pm\prime}\|\|\mathfrak{g}_\alpha^{\pm\prime}\|$$

Furthermore, $|nL_n(\mathfrak{f}_\alpha^\pm \mathfrak{g}_\alpha^\pm;\tilde{y})| \le y^2\|\mathfrak{f}_\alpha^{\pm\prime}\|_\infty\|\mathfrak{g}_\alpha^{\pm\prime}\|_\infty$. But it can be observed that for $\mathfrak{f}_\alpha^\pm \in C^n[a,b]$ and $n \in \mathbb{N}$,

$$\max_{0\le k\le n}\{\|\mathfrak{f}_\alpha^{\pm k}\|\} \le C\max\{\|\mathfrak{f}_\alpha^\pm\|_\infty, \|\mathfrak{f}_\alpha^{\pm n}\|_\infty\}. \tag{8}$$

Now,

$$\begin{aligned}
&L_n(u_{1\alpha}^{\pm}, v_\alpha^\pm;\tilde{y})\\
&= \left|\tilde{\mathcal{A}}_n^{\nu_1,\nu_2}(u_{1\alpha}^{\pm}v_\alpha^\pm;\tilde{y}) - u_{1\alpha}^{\pm}v_\alpha^\pm(\tilde{y}) - \frac{\tilde{y}(1-\tilde{y})}{2n}(u_{1\alpha}^{\pm}v_\alpha^\pm)''\right| + u_{1\alpha}^{\pm}(\tilde{y})\left|\tilde{\mathcal{A}}_n^{\nu_1,\nu_2}(v_\alpha^\pm;\tilde{y}) - v_\alpha^\pm(\tilde{y}) - \frac{\tilde{y}(1-\tilde{y})}{2n}v_\alpha^{\pm\prime\prime}(\tilde{y})\right|\\
&+ v_\alpha^\pm(\tilde{y})\left|\tilde{\mathcal{A}}_n^{\nu_1,\nu_2}(u_{1\alpha}^{\pm};\tilde{y}) - u_{1\alpha}^{\pm}(\tilde{y})\frac{\tilde{y}(1-\tilde{y})}{2n}u_{1\alpha}^{\pm\prime\prime}\right| + \left|v_\alpha^\pm(\tilde{y}) - \tilde{\mathcal{A}}_n^{\nu_1,\nu_2}(v_\alpha^\pm;\tilde{y})\right|\left|\tilde{\mathcal{A}}_n^{\nu_1,\nu_2}(u_{1\alpha}^{\pm};\tilde{y}) - u_{1\alpha}^{\pm}(\tilde{y})\right|\\
&\le \frac{K}{n}\max\{\|u_{1\alpha}^{\pm\prime}\|_\infty, \|u_{1\alpha}^{\pm 4}\|_\infty\}.\max\{\|v_\alpha^{\pm\prime}\|_\infty, \|v_\alpha^{\pm 4}\|_\infty\}.
\end{aligned}$$

Now, from equation (8)

$$\begin{aligned}
|L_n(\mathfrak{f}_\alpha^\pm \mathfrak{g}_\alpha^\pm;\tilde{y})| \le \frac{K^*}{n}\Big\{&\|(\mathfrak{f}_\alpha^\pm - u_{1\alpha}^{\pm})'\|\|(\mathfrak{g}_\alpha^\pm - v_\alpha^\pm)'\| + \|(\mathfrak{f}_\alpha^\pm - u_{1\alpha}^{\pm})'\|\|v_\alpha^{\pm\prime}\|\\
&+ \|u_{1\alpha}^{\pm\prime}\|\|(\mathfrak{g}_\alpha^\pm - v_\alpha^\pm)'\|\\
&+ \frac{1}{n}\times\max\{\|u_{1\alpha}^{\pm\prime}\|_\infty, \|u_{1\alpha}^{\pm 4}\|_\infty\}.\max\{\|v_\alpha^{\pm\prime}\|_\infty, \|v_\alpha^{\pm 4}\|_\infty\}\Big\}.
\end{aligned}$$

After applying the Lemma 3.1. of [17], we get.

$$\begin{aligned}
&|L_n(\mathfrak{f}_\alpha^\pm \mathfrak{g}_\alpha^\pm;\tilde{y})| \le \Big\{w_3(\mathfrak{f}_\alpha^{\pm\prime};h).w_3(\mathfrak{g}_\alpha^{\pm\prime};h) + \frac{1}{h}w_3(\mathfrak{f}_\alpha^\pm;h)w_3(\mathfrak{g}_\alpha^{\pm\prime};h) + \frac{1}{h}w_3(\mathfrak{f}_\alpha^{\pm\prime};h)w_3(\mathfrak{g}_\alpha^\pm;h)\\
&+ \frac{1}{h}\max\{\|\mathfrak{f}_\alpha^\pm\|, \frac{1}{h^3}w_3(\mathfrak{f}_\alpha^{\pm\prime};h)\}.\max\{\|\mathfrak{g}_\alpha^\pm\|, \frac{1}{h}p^3w_3(\mathfrak{g}_\alpha^{\pm\prime};h)\}\Big\}.
\end{aligned}$$

Equivalently, we may represent this as in following fuzzy form, via the notation $\omega^{\mathcal{F}}(\mathfrak{f}';h) = w_3(\mathfrak{f}_\alpha^{\pm\prime};h)$ and the constant p^3 as in Lemma 3.1. of [17]. Hence, we get

$$\begin{aligned}
d\left(L_n(\mathfrak{f}\mathfrak{g};\tilde{y}), \bar{0}\right) \le \Big\{&\omega^{\mathcal{F}}(\mathfrak{f}';h).\omega^{\mathcal{F}}(\mathfrak{g}';h) + \frac{1}{h}\omega^{\mathcal{F}}(\mathfrak{f};h)\omega_{\gamma,}^{\mathcal{F}}(\mathfrak{g}';h) + \frac{1}{h}\omega_{\gamma,}^{\mathcal{F}}(\mathfrak{f}';h)\omega_{\gamma,}^{\mathcal{F}}(\mathfrak{g};h)\\
&+ \frac{1}{h}\max\{d(\mathfrak{f},\bar{0}), \frac{1}{h^3}\omega_{\gamma,}^{\mathcal{F}}(\mathfrak{f}';h)\}.\max\{d(\mathfrak{g},\bar{0}), \frac{1}{h}p^3\omega_{\gamma,}^{\mathcal{F}}(\mathfrak{g}';h)\}\Big\}.
\end{aligned}$$

This completes the proof. □

In last, we illustrate the following example, in order to support the Theorem 3.1.

Example

Suppose that $f(t) = cos(-(\frac{\pi}{4})t)\left(\frac{t^2}{(1+t.^2)}\right)$; for $t \in [0,1]$. Let us assume $p_n = n,\ q_n = 0$, and $\gamma = 0$ and A being the Cesàro matrix $A = (\tilde{a}_{nk})$, such that $\tilde{a}_{nk} = \begin{cases} \frac{1}{n+1}, & \text{if } 0 \le k \le n \\ 0, & \text{otherwise.} \end{cases}$.
We get from Lemma 2.2, $St_A^D - \lim_{n\to\infty} \tilde{\mathcal{A}}_n^{\nu_1,\nu_2}(e_j; y) = y^j$. It is noteworthy to conclude from Theorem 3.1 is that

$$St_A^D - \lim_{m\to\infty} \tilde{\mathcal{A}}_m^{\nu_1,\nu_2}(f; y) = f(y).$$

Similar arguments can be justified by taking $f(t) = e^{(-4t)}$, by considering the same $A = (\tilde{a}_{nk})$, as above. Graphically, this can demonstrated by taking n values (i.e., 10, 100, 1024), $\nu_1 = 0,\ \nu_2 = 0.01$ and $\beta = 0.24$, for different functions i.e., $f(t) = e^{(-4t)}$ and $f(t) = cos(-(\frac{\pi}{4})t)\left(\frac{t^2}{(1+t^2)}\right)$, in Figure 1 and Figure 2, respectively.

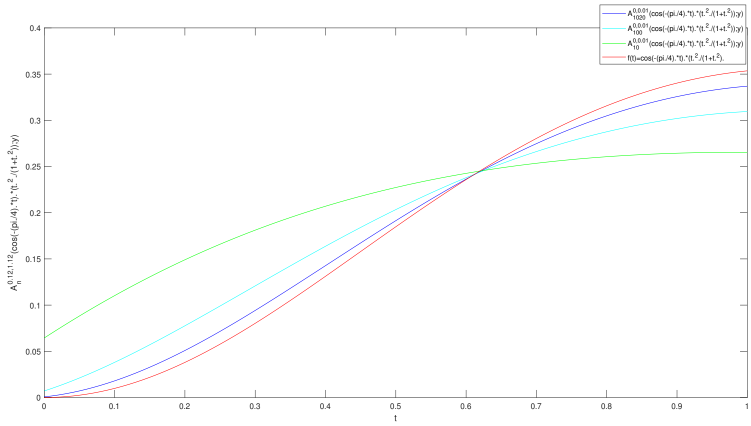

Figure 1: Plot of $\bar{\mathcal{A}}_n^{\nu_1,\nu_2}(cos(-(\frac{\pi}{4})t)\left(\frac{t^2}{(1+t.^2)}\right); y)$

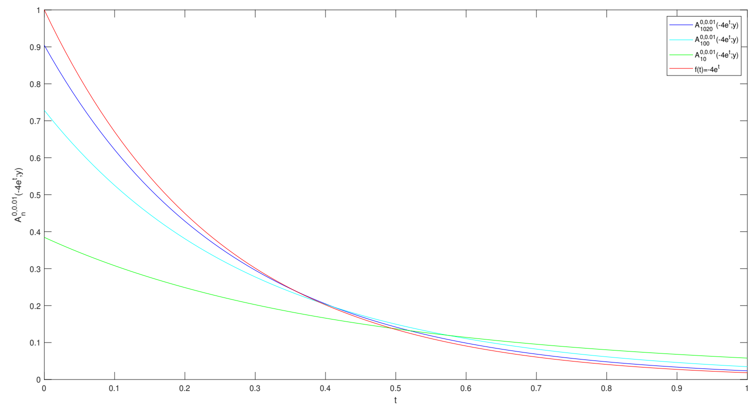

Figure 2: Plot of $\bar{\mathcal{A}}_n^{\nu_1,\nu_2}(e^{(-4t)}; y)$

Parameter discussion:

In this subsection, for validating obtained approximation results for the proposed fuzzy operator, we have performed a detail parameter analysis and represent it in the tabular form as (Table 1-5). This interpretation involves the maximum point wise error and mean squared error estimations of the approximated function by taking into account of fixed $n-$values i.e., n=1024, along with various parameters, such as $\nu_1 = \nu_2 = \{0,\ 0.5,\ 1\}$, $(\rho_n) = (\frac{1}{n^{\frac{1}{3}}})$, and $(\sigma_n) = e$. Based on the some numerical experiment for the function $\bar{\mathcal{A}}_n^{\nu_1,\nu_2}(e^{(-4t)}; y)$ in following 5 tables, for each $\beta = 0,\ 0.25,\ 0.5,\ 0.75$ and 1, respectively, have been constructed.

ν_1	ν_2	Maximum point wise error	Mean square error
0	0	0.0183	$1.2285e-04$
0	0.5	0.0183	$1.2285e-04$
0	1	0.0183	$1.2285e-04$
0.5	0	0.0183	$1.2285e-04$
0.5	0.5	0.0183	$1.2285e-04$
0.5	1	0.0183	$1.2285e-04$
1	0	0.0183	$1.2285e-04$
1	0.5	0.0183	$1.2285e-04$
1	1	0.0183	$1.2285e-04$

Table 1: Error estimations for the function $e^{(-4t)}$ and $\beta = 0$.

ν_1	ν_2	Maximum point wise error	Mean square error
0	**0**	**0.0012**	**5.2604e-07**
0	0.5	0.0013	$6.8908e-07$
0	1	0.0014	$6.0406e-07$
0.5	0	0.0078	$2.6998e-05$
0.5	0.5	0.0079	$2.7557e-05$
0.5	1	0.0080	$2.8117e-05$
1	0	0.0183	$1.2285e-04$
1	0.5	0.0183	$1.2285e-04$
1	1	0.0183	$1.2285e-04$

Table 2: Error estimations for the function $e^{(-4t)}$ and $\beta = 0.25$.

ν_1	ν_2	Maximum point wise error	Mean square error
0	0	0.0012	$5.2604e-07$
0	0.5	0.0013	$6.8908e-07$
0	1	0.0014	$6.0406e-07$
0.5	0	0.0078	$2.6998e-05$
0.5	0.5	0.0079	$2.7557e-05$
0.5	1	0.0080	$2.8117e-05$
1	0	0.0183	$1.2285e-04$
1	0.5	0.0183	$1.2285e-04$
1	1	0.0183	$1.2285e-04$

Table 3: Error estimations for the function $e^{(-4t)}$ and $\beta = 0.5$.

ν_1	ν_2	Maximum point wise error	Mean square error
0	0	0.0013	$5.7986e-07$
0	0.5	0.0014	$6.6372e-07$
0	1	0.0014	$7.5453e-07$
0.5	0	0.0079	$2.7543e-05$
0.5	0.5	0.0080	$2.8105e-05$
0.5	1	0.0080	$2.8667e-05$
1	0	0.0183	$1.2285e-04$
1	0.5	0.0183	$1.2285e-04$
1	1	0.0183	$1.2285e-04$

Table 4: Error estimations for the function $e^{(-4t)}$ and $\beta = 0.75$.

ν_1	ν_2	Maximum point wise error	Mean square error
0	0	0.0013	$5.9805e-07$
0	0.5	0.0014	$6.8374e-07$
0	1	0.0015	$7.7638e-07$
0.5	0	0.0079	$2.7716e-05$
0.5	0.5	0.0080	$2.8278e-05$
0.5	1	0.0081	$2.8841e-05$
1	0	0.0183	$1.2285e-04$
1	0.5	0.0183	$1.2285e-04$
1	1	0.0183	$1.2285e-04$

Table 5: Error estimations for the function $e^{(-4t)}$ and $\beta = 1$.

By analyzing data in the mentioned tables, we can conclude that for $\nu_1 = 0,\ \nu_2 = 0,\ \beta = 0.25,\ (\rho_n) = (\frac{1}{n^{\frac{1}{3}}})$ and $(\sigma_n) = (1 - \frac{1}{n})$, we get maximum point wise error and mean squared error , i.e., 0.0012, and $5.2604e-07$, respectively. In addition, we have pointed less error than the original operator (for the functional value lies in $[0, 1]$) defined by Özkan [35].

Conclusion

Summarising the overall theme of this present paper, we efficiently develop a fuzzy positive linear operator on combining the Riemann Liouville fractional-type operator with rational operator. Our emphasised work involves certain estimations related to the rate of convergence by surging the modulus of smoothness of higher order, Peetre's K functional, and Lipschitz-type maximal function. Further investigations regarding the Korovkin-type theorem, Voronovskaya theorem, and Grüss Voronovskaya estimates have also been characterized and strengthened by suitable graphical examples. Looking further, we intend to work further on the application prospectives of certain fuzzy neural network operators. In brief, we intend to introduce a fractional type fuzzy neural network operator encompasses with generalized activation function and its applications in the domain of image processing technique, and some fuzzy inference systems.

Disclosure statement

No potential conflict of interest was reported by the author(s).

References

[1] A.-M. Acu, I. Raşa, and A. E. Şteopoaie. Bernstein–Kantorovich operators, approximation and shape preserving properties. *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas*, 118(3):107, 2024.

[2] R. P. Agnew. On deferred Cesàro means. *Annals of Mathematics*, 33(3):413–421, 1932.

[3] A. Alamer and M. Nasiruzzaman. Approximation by Stancu variant of λ-Bernstein shifted knots operators associated by Bézier basis function. *Journal of King Saud University-Science*, 36(9):103333, 2024.
[4] G. A. Anastassiou. On basic fuzzy korovkin theory. *Studia University Babes-Bolyai Math*, 50:3–10, 2005.
[5] G. A. Anastassiou and O. Duman. Statistical fuzzy approximation by fuzzy positive linear operators. *Computers & Mathematics with Applications*, 55(3):573–580, 2008.
[6] M. Ayman-Mursaleen, M. Nasiruzzaman, N. Rao, M. Dilshad, and K. S. Nisar. Approximation by the modified λ-Bernstein-polynomial in terms of basis function. *AIMS math*, 9(2):4409–4426, 2024.
[7] F. Başar. *Summability theory and its applications*. Chapman and Hall/CRC, 2022.
[8] S. Berwal, S. Mohiuddine, A. Kajla, and A. Alotaibi. Approximation by Riemann–Liouville type fractional α-Bernstein–Kantorovich operators. *Mathematical Methods in the Applied Sciences*, 47(11):8275–8288, 2024.
[9] H. Bohman. On approximation of continuous and of analytic functions. *Arkiv för Matematik*, 2(1):43–56, 1952.
[10] Z. Ditzian and V. Totik. *Moduli of smoothness*, volume 9. Springer Science & Business Media, 2012.
[11] O. Duman. Statistical fuzzy approximation to fuzzy differentiable functions by fuzzy linear operators. *Hacettepe Journal of Mathematics and Statistics*, 39(4):497–514, 2010.
[12] A. M. El-Sayed. Fractional-order evolutionary integral equations. *Applied Mathematics and Computation*, 98(2-3):139–146, 1999.
[13] H. Fast. Sur la convergence statistique. In *Colloquium mathematicae*, volume 2, pages 241–244, 1951.
[14] J. A. Fridy. On statistical convergence. *Analysis*, 5(4):301–314, 1985.
[15] S. Gal and H. Gonska. Gr\" uss and Gr\" uss-Voronovskaya-type estimates for some Bernstein-type polynomials of real and complex variables. *arXiv preprint arXiv:1401.6824*, 2014.
[16] H. Gonska and I. Rasa. A Voronovskaya estimate with second order modulus of smoothness. *Proc. Math. Inequal.(Sibiu/Romania*, 2008.
[17] H. H. Gonska. Degree of approximation by lacunary interpolators:(0,..., r-2, r) interpolation. *The Rocky Mountain Journal of Mathematics*, pages 157–171, 1989.
[18] L. Hahn. A note on stochastic methods in connection with approximation theorems for positive linear operators. *Pacific Journal of Mathematics*, 101(2):307–319, 1982.
[19] U. Kadak. Weighted statistical convergence based on generalized difference operator involving (p, q)-gamma function and its applications to approximation theorems. *Journal of Mathematical Analysis and Applications*, 448(2):1633–1650, 2017.
[20] U. Kadak. Fractional sampling operators of multivariate fuzzy functions and applications to image processing. *Applied Soft Computing*, 132:109901, 2023.
[21] P. Korovkin. On convergence of linear positive operators in the space of continuous functions (russian). In *Doklady Akademii Nauk SSSR (NS)*, volume 90, page 961, 1953.
[22] A. Kumar, A. Verma, L. Rathour, L. N. Mishra, and V. N. Mishra. Convergence analysis of modified Szász operators associated with Hermite polynomials. *Rendiconti del Circolo Matematico di Palermo Series 2*, 73(2):563–577, 2024.
[23] A. D. Meleşteu and M. Dimitriu. A stancu type generalization of the Balázs operator. *Dolomites Research Notes on Approximation*, 17(DRNA Volume 17.2):44–51, 2024.
[24] H. Miller and C. Orhan. On almost convergent and statistically convergent subsequences. *Acta Mathematica Hungarica*, 93:135–151, 2001.
[25] S. A. Mohiuddine, A. Asiri, and B. Hazarika. Weighted statistical convergence through difference operator of sequences of fuzzy numbers with application to fuzzy approximation theorems. *International Journal of General Systems*, 48(5):492–506, 2019.
[26] M. Mursaleen and K. J. Ansari. Approximation by generalized Szász operators involving Sheffer polynomials. *arXiv preprint arXiv:1601.00675*, 2015.
[27] M. Nasiruzzaman. Approximation by gbs associated properties of Szász-Mirakjan-Jakimovski-Leviatan-Kantorovich operators. *Filomat*, 38(18):6621–6637, 2024.
[28] M. Nasiruzzaman. Approximation by Stancu-type α-Bernstein-Schurer-Kantorovich operators. *Journal of Inequalities and Applications*, 2025(1):48, 2025.
[29] M. Nasiruzzaman, M. Mursaleen, E. Alshaban, A. Alatawi, A. Alamer, N. O. Al-Atawi, and R. Kumar. Approximation by Bézier type associated shifted knots of (λ, q)-Bernstein operators. *Filomat*, 39(3):997–1018, 2025.
[30] L. Nayak, G. Das, and B. Ray. Degree of approximation of fourier series of functions in Besov space by defreed Ces 'aro mean. *J. Indian Math. Soc*, 83:161–179, 2016.
[31] L. Nayak, M. Mursaleen, and P. Baliarsingh. On deferred statistical A-convergence of fuzzy sequence and applications. *Iranian Journal of Fuzzy Systems*, 19(2):119–131, 2022.
[32] L. Nayak, B. C. Tripathy, and P. Baliarsingh. On deferred-statistical convergence of uncertain fuzzy sequences. *International Journal of General Systems*, 51(6):631–647, 2022.
[33] F. Nuray and E. Savaş. Statistical convergence of sequences of fuzzy numbers. *Mathematica Slovaca*, 45(3):269–273, 1995.
[34] M. A. Özarslan and H. Aktuğlu. Local approximation properties for certain King type operators. *Filomat*, 27(1):173–181, 2013.
[35] E. Y. Özkan. A new Kantorovich-type rational operator and inequalities for its approximation. *Mathematics*, 10(12):1982, 2022.
[36] T. Popoviciu. Asupra demonstratiei teoremei lui weierstrass cu ajutorul polinoamelor de interpolare. *Lucraarile Sesiunii Generale Stiintifice din*, pages 2–12, 1950.
[37] S. A. Rahaman and M. Mursaleen. On rough deferred statistical convergence of difference sequences in l-fuzzy normed spaces. *Journal of Mathematical Analysis and Applications*, 530(2):127684, 2024.
[38] K. Saini, K. Raj, and M. Mursaleen. Deferred Cesàro and deferred Euler equi-statistical convergence and its applications to Korovkin-type approximation theorem. *International Journal of General Systems*, 50(5):567–579, 2021.
[39] E. Savaş and M. Mursaleen. Bézier type Kantorovich q-Baskakov operators via wavelets and some approximation properties. *Bulletin of the Iranian Mathematical Society*, 49(5):68, 2023.
[40] I. J. Schoenberg. The integrability of certain functions and related summability methods. *The American mathematical monthly*,

66(5):361–775, 1959.
[41] H. Steinhaus. Sur la convergence ordinaire et la convergence asymptotique. In *Colloq. math*, volume 2, pages 73–74, 1951.
[42] H. Sun, Y. Zhang, D. Baleanu, W. Chen, and Y. Chen. A new collection of real world applications of fractional calculus in science and engineering. *Communications in Nonlinear Science and Numerical Simulation*, 64:213–231, 2018.
[43] O. Szasz. Generalization of s. Bernstein's polynomials to the infinite interval. *Journal of Research of the National Bureau of Standards*, 45(3):239–245, 1950.
[44] S. Varma and F. Taşdelen. Szász type operators involving Charlier polynomials. *Mathematical and Computer Modelling*, 56(5-6):118–122, 2012.
[45] G. Wang, D. Yu, and L. Guan. Neural network interpolation operators of multivariate functions. *Journal of Computational and Applied Mathematics*, 431:115266, 2023.
[46] K. Weierstrass. Über die analytische darstellbarkeit sogenannter willkürlicher functionen einer reellen veränderlichen. *Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften zu Berlin*, 2(633-639):364, 1885.
[47] L. A. Zadeh. Fuzzy sets. *Information and control*, 8:338–353, 1965.