Filomat 40:3 (2026), 975–993
https://doi.org/10.2298/FIL2603975S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Sequence spaces and operator ideals induced by the q-Bronze Leonardo-Lucas matrix

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Abstract. This paper introduces the q-Bronze Leonardo-Lucas matrix $\aleph(q) = (\breve{\zeta}^{(q)}_{nk})_{n,k\in\mathbb{N}}$, defined by

$$\breve{\zeta}^{(q)}_{nk} = \begin{cases} \dfrac{3\, q^{k-1}\, \breve{\zeta}_k(q)}{4\breve{\zeta}_n(q) + \breve{\zeta}_{n-1}(q) + 3n - 10}, & 1 \leq k \leq n, \\ 0, & k > n, \end{cases}$$

with $\{\breve{\zeta}_n(q)\}$ representing the q-Bronze Leonardo-Lucas sequence. Using $\breve{\zeta}_n(q)$ is defined by

$$\breve{\zeta}_n(q) = (2 + q^{k-1})\breve{\zeta}_{n-1}(q) + q^{n-1}\breve{\zeta}_{n-2}(q) - 3$$

for $n \geq 2$, $\breve{\zeta}_0(q) = 3, \breve{\zeta}_1(q) = 4$. We introduce the matrix domains $\ell_p(\aleph(q)) = (\ell_p)_{\aleph(q)}$ for $1 \leq p < \infty$, along with $\ell_\infty(\aleph(q)) = (\ell_\infty)_{\aleph(q)}$, $c_0(\aleph(q)) = (c_0)_{\aleph(q)}$, and $c(\aleph(q)) = (c)_{\aleph(q)}$, which denotes the q-Bronze Leonardo-Lucas sequence spaces. In this context, we derive the Schauder basis for the space $\ell_p(\aleph(q))$ for $1 \leq p < \infty$. We establish several results regarding the operator ideals associated with these newly defined sequence spaces. Further, we explore various geometric properties of $\ell_p(\aleph(q))$ and $\ell_\infty(\aleph(q))$. Finally, we analyze the solidity property of these sequence spaces.

1. Introduction

A Banach space V is known as a BK-space when the coordinate projections $\pi_i : V \to \mathbb{C}$ maintain continuity. These projections are defined by $\pi_i(\mathbf{v}) = v_i$ for any $\mathbf{v} = (v_i) \in V$ and for each $i \in \mathbb{N}$. The sequence spaces ℓ_p $(1 \leq p < \infty)$ and ℓ_∞, equipped with their standard norms

$$||\mathfrak{v}||_{\ell_p} = \left(\sum_{\mathfrak{i}} |\mathfrak{v}_{\mathfrak{i}}|^p\right)^{\frac{1}{p}}, \quad ||\mathfrak{v}||_{\ell_\infty} = \sup_{\mathfrak{i}\in\mathbb{N}} |\mathfrak{v}_{\mathfrak{i}}|$$

2020 *Mathematics Subject Classification*. Primary 11B37; Secondary 47L20, 46B50, 46A35, 46A45.
Keywords. q-Bronze Leonardo-Lucas numbers, Schauder basis, Operator ideal, sequence spaces.
Received: 27 June 2025; Revised: 14 October 2025; Accepted: 19 November 2025
Communicated by Dragan S. Djordjević
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are Banach spaces. Consider $\mathfrak{V}$ and $\mathfrak{W}$ as sequence spaces, and $\mathfrak{N} = (n_{ij})$ denotes an infinite real matrix. We denote the matrix simply as $\mathfrak{N} = (n_{ij})$, with the understanding that indices run to infinity.

The matrix $\mathfrak{N}$ defines a linear operator $\mathfrak{V} \to \mathfrak{W}$ if for each $\mathfrak{v} = (\mathfrak{v}_j) \in \mathfrak{V}$, its matrix transform $\mathfrak{N}\mathfrak{v} = \left(\sum_j n_{ij}\mathfrak{v}_j\right)_i$ belongs to $\mathfrak{W}$. Mathematically, the $\mathfrak{N}$-transform is defined as

$$(\mathfrak{N}\mathfrak{v})_{\mathfrak{i}} = \sum_{j=1}^{\infty} n_{\mathfrak{i}j}\mathfrak{v}_j, \quad \mathfrak{i} \in \mathbb{N}.$$

The sequence space $\mathfrak{V}_{\mathfrak{N}}$ is defined as

$$\mathfrak{V}_{\mathfrak{n}} = \{\mathfrak{v} = (\mathfrak{v}_{\mathfrak{i}}) \in \mathfrak{w} : \mathfrak{N}\mathfrak{v} \in \mathfrak{V}\},$$

is known as the domain of the matrix $\mathfrak{N}$ in the space $\mathfrak{V}$. The space ω represents the set of all real-valued sequences. Within this framework, several specialized sequence spaces are defined as follows:

- c_0: The set of sequences that converge to zero (null sequences).
- c: The set of all convergent sequences.
- ℓ_∞: The set of all bounded sequences.
- ℓ_p $(1 \leq p < \infty)$: The set of all sequences that are absolutely p-summable.
- cs: The set of sequences whose series are convergent.
- bs: The set of sequences with bounded series.

The concept of a q-analogue extends classical mathematical ideas by incorporating a parameter q. When q approaches 1, the q-analogue converges to the original formulation. Although Euler introduced the concept, Jackson later applied q-analogue methods to establish q-differentiation and q-integration [6]. For additional details on q-calculus, we refer the reader to [7]. Over time, q-analogues have been widely utilized in various mathematical fields, including algebra, combinatorics, approximation theory, and special functions. Moreover, several researchers have applied summability theory and sequence spaces. For exapmles, researchers have developed q-analogues of Cesàro sequence spaces and examined q-statistical convergence in summability methods (see [5, 8, 17]).

Recent research has increasingly focused on the q-analogues of classical sequence spaces. For example, Demiriz and Şahin [5] and Yaying et al. [17] investigated the domain $X(C(q)) = X_{C(q)}$, where X represents spaces such as ℓ_p, c_0, c, and ℓ_∞. Based on this work, Yílmaz and Akdemir [15] analyzed the topological and geometric properties of the spaces $(\ell_p)_{C(q)}$ and $(\ell_\infty)_{C(q)}$.

In a separate contribution, Alotaibi et al. [1] introduced the spaces $\ell_p(\nabla_q^2) = (\ell_p)_{\nabla_q^2}$ and $\ell_\infty(\nabla_q^2) = (\ell_\infty)_{\nabla_q^2}$, defined via the operator ∇_q^2 acting on ℓ_p and ℓ_∞. A sequence space X is termed symmetric [14] if every sequence (y_n) in X satisfies $(y_{\pi(n)}) \in X$ for any permutation π of $\mathbb{N}_0$. Interestingly, Alotaibi et al. [1] demonstrated that $\ell_\infty(\nabla_q^2)$ does not exhibit symmetry. Yaying et al. [17] introduced the q-Cesàro matrix $\mathfrak{C}(q) = (\mathfrak{c}_{n\mathfrak{v}}^q)_{n,\mathfrak{v}\in\mathbb{N}_0}$, where $c_{n\mathfrak{v}}^q$ is defined as

$$c_{n\mathfrak{v}}^q = \begin{cases} \dfrac{q^{\mathfrak{v}}}{[n+1]_q}, & 0 \leq \mathfrak{v} \leq n, \\ 0, & \mathfrak{v} > n, \end{cases}$$

where $[n+1]_q$ represents the q-analogue of $n+1$. Further contributions to the study of q-sequence spaces were made by Yaying et al. [19] introduced q-Euler sequence spaces, denoted as $K(E(q)) = K_{E(q)}$. The q-Euler matrix is defined as

$$e_{n\mathfrak{v}}^q(a,b) = \begin{cases} \dfrac{1}{(a+b)_q^n}\binom{n}{\mathfrak{v}}_q q^{\binom{\mathfrak{v}}{2}} a^{\mathfrak{v}} b^{n-\mathfrak{v}}, & 0 \leq \mathfrak{v} \leq n, \\ 0, & \mathfrak{v} > n. \end{cases}$$

In addition, they examined the domains $c_0(\nabla_q^2) := (c_0)_{\nabla_q^2}$ and $c(\nabla_q^2) := c_{\nabla_q^2}$, which incorporate the q-difference operator ∇_q^2 in the spaces c_0 and c. The second-order q-difference operator is defined by

$$(\nabla_q^2 s)_n = s_n - (1+q)s_{n-1} + qs_{n-2}.$$

Further advances in q-sequence spaces were made with the introduction of the q-Catalan sequence space $\lambda(C(q)) = \lambda_{C(q)}$ by Yaying et al. [20], where $\lambda \in \{c, c_0\}$. The q-Catalan matrix $C(q)$ is defined as

$$c_q^{n\mathfrak{v}}(q) = \begin{cases} q^{\mathfrak{v}} \dfrac{c_{\mathfrak{v}}(q)c_{n-\mathfrak{v}}(q)}{c_{n+1}(q)}, & 0 \le \mathfrak{v} \le n, \\ 0, & \mathfrak{v} > n, \end{cases}$$

where $c(q) = (c_{\mathfrak{v}}(q))_{\mathfrak{v}\in\mathbb{N}_0}$ denotes the sequence of q-Catalan numbers. Moreover, Yaying et al. [21] expanded the theory by formulating q-Pascal sequence spaces $c_0(P(q)) := (c_0)_{P(q)}$ and $c(P(q)) := c_{P(q)}$ within c_0 and c, respectively. These spaces are generated using the q-Pascal matrix $P(q) = (p^q_{n\mathfrak{v}})_{n,\mathfrak{v}\in\mathbb{N}_0}$, which is given by

$$p^q_{n\mathfrak{v}} = \begin{cases} \dbinom{n}{\mathfrak{v}} q^{\mathfrak{v}}, & \text{for } 0 \le \mathfrak{v} \le n, \\ 0, & \text{for } \mathfrak{v} > n. \end{cases}$$

A significant advance in this field was presented in Yaying et al. [18]. In this work, the authors introduced a novel framework for q-Fibonacci sequence spaces by defining a distinctive q-Fibonacci matrix and investigating its structural properties. They explored fundamental aspects of matrix domains in the spaces $\ell_p(F(q))$ and $\ell_\infty(F(q))$, examining key properties such as Schauder bases, dual spaces, and matrix transformations. Additionally, the study provided a thorough analysis of essential geometric properties, including the approximation property, the Dunford-Pettis property, the Hahn-Banach extension property, and rotundity. These findings substantially improve the theoretical foundation of q-Fibonacci sequence spaces and underscore their significance in modern functional analysis. Further advances in this area have been recently explored by Yaying et al. [22]. Yílmaz [16] investigated the structural and geometric properties of Schröder sequence spaces such as rotundity and uniform smoothness.

2. Bronze Leonardo-Lucas Sequence Spaces

The sequence of Bronze Leonardo-Lucas numbers is determined by the recurrence relation

$$\breve{\zeta}_k = 3\breve{\zeta}_{k-1} + \breve{\zeta}_{k-2} - 3, \quad [10]$$

with initial values $\breve{\zeta}_0 = 3, \breve{\zeta}_1 = 4$. According to this formula, the first few Bronze Leonardo-Lucas numbers are 3, 4, 12, 37, 120, 394, 1299, 4288, 14160, 46765.
We can easily derive the relation

$$\sum_{s=1}^{k} \breve{\zeta}_s = \frac{4\breve{\zeta}_k + \breve{\zeta}_{k-1} + 3n - 10}{3}.$$

For a nonnegative integer k, $\breve{\zeta}_k$ represent the k-th Bronze Leonardo-Lucas number. Consider the matrix $\aleph(q) = (\breve{\zeta}_{nk})$, defined by

$$\breve{\zeta}_{nk} = \begin{cases} \dfrac{3\breve{\zeta}_k}{4\breve{\zeta}_n + \breve{\zeta}_{n-1} + 3n - 10} & 1 \le k \le n, \\ 0, & k > n, \end{cases}$$

where $n, k = 1, 2, \ldots$.

$$\mathfrak{B} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \frac{12}{48} & \frac{36}{48} & 0 & 0 & 0 & 0 & \cdots \\ \frac{12}{159} & \frac{36}{159} & \frac{111}{159} & 0 & 0 & 0 & \cdots \\ \frac{12}{519} & \frac{36}{519} & \frac{111}{519} & \frac{360}{519} & 0 & 0 & \cdots \\ \frac{12}{1701} & \frac{36}{1701} & \frac{111}{1701} & \frac{360}{1701} & \frac{1182}{1701} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$\Omega'_n = (\mathfrak{B}b')_n = \frac{3}{4\breve{\zeta}_n + \breve{\zeta}_{n-1} + 3n - 10} \sum_{k=1}^{n} \breve{\zeta}_k b'_k.$$

A sequence belongs to the Bronze Leonardo-Lucas spaces $\ell_p(\mathfrak{B})$, $\ell_\infty(\mathfrak{B})$, $c(\mathfrak{B})$, or $c_0(\mathfrak{B})$ if and only if its $\mathfrak{B}$-transform lies in ℓ_p, ℓ_∞, c, or c_0, respectively.

$$\ell_p(\mathfrak{B}) = \left\{ h = (h_k) \in \omega : \sum_{n=1}^{\infty} \left| \frac{3}{4\breve{\zeta}_n + \breve{\zeta}_{n-1} + 3n - 10} \sum_{k=1}^{n} \breve{\zeta}_k h_k \right|^p < \infty \right\} (1 \le p < \infty).$$

$$\ell_\infty(\mathfrak{B}) = \left\{ h = (h_k) \in \omega : \sup_{n \in \mathbf{N}} \left| \frac{3}{4\breve{\zeta}_n + \breve{\zeta}_{n-1} + 3n - 10} \sum_{k=1}^{n} \breve{\zeta}_k h_k \right| < \infty \right\}.$$

$$c_0(\mathfrak{B}) = \left\{ h = (h_k) \in \omega : \lim_{n \to \infty} \sum_{n=1}^{\infty} \frac{3}{4\breve{\zeta}_n + \breve{\zeta}_{n-1} + 3n - 10} \sum_{k=1}^{n} \breve{\zeta}_k h_k = 0 \right\}.$$

$$c(\mathfrak{B}) = \left\{ h = (h_k) \in \omega : \lim_{n \to \infty} \sum_{n=1}^{\infty} \frac{3}{4\breve{\zeta}_n + \breve{\zeta}_{n-1} + 3n - 10} \sum_{k=1}^{n} \breve{\zeta}_k h_k = l \right\}.$$

We can express $\mathcal{G}(\aleph(q))$ as $\mathcal{G}_{(\aleph(q))}$, where $\mathcal{G}$ denotes any of the spaces $\ell_p, \ell_\infty, c_0,$ or $c,$ where $p \in [1, \infty)$.

3. q-Bronze Leonardo-Lucas Matrix

Define the matrix $\aleph(q) = \left(\breve{\zeta}^{(q)}_{nk}\right)_{n,k \in \mathbb{N}}$ by

$$\aleph(q) = \begin{cases} \dfrac{3\, q^{k-1}\, \breve{\zeta}_k(q)}{4\breve{\zeta}_n(q) + \breve{\zeta}_{n-1}(q) + 3n - 10}, & \text{if } 1 \le k \le n, \\ 0, & \text{if } k > n. \end{cases}$$

Alternatively, it can be expressed as follows

$$\aleph(q) = \begin{bmatrix} \dfrac{3\breve{\zeta}_1(q)}{4\breve{\zeta}_1(q) + \breve{\zeta}_0(q) - 7} & 0 & 0 & \cdots \\ \dfrac{3\breve{\zeta}_1(q)}{4\breve{\zeta}_2(q) + \breve{\zeta}_1(q) - 4} & \dfrac{3\, q\, \breve{\zeta}_2(q)}{4\breve{\zeta}_2(q) + \breve{\zeta}_1(q) - 4} & 0 & \cdots \\ \dfrac{3\breve{\zeta}_1(q)}{4\breve{\zeta}_3(q) + \breve{\zeta}_2(q) - 1} & \dfrac{3\, q\, \breve{\zeta}_2(q)}{4\breve{\zeta}_3(q) + \breve{\zeta}_2(q) - 1} & \dfrac{3\, q^2\, \breve{\zeta}_3(q)}{4\breve{\zeta}_3(q) + \breve{\zeta}_2(q) - 1} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

The sequence $s=(s_k)_{k\in\mathbb{N}}$ is derived from $\upsilon=(\upsilon_k)_{k\in\mathbb{N}}$ through the matrix transformation $\aleph(q)$

$$s_k=(\aleph(q)\upsilon)_k=\sum_{j=1}^{k}\frac{3\,q^{j-1}\,\breve{\zeta}_j(q)}{4\breve{\zeta}_k(q)+\breve{\zeta}_{k-1}(q)+3k-10}\upsilon_j. \tag{1}$$

Hence $s=(s_k)$ is obtained as the $\aleph(q)$-transform of $\upsilon=(\upsilon_k)$. The sequence spaces $\ell_p(\aleph(q))$, $\ell_\infty(\aleph(q))$, $c(\aleph(q))$, and $c_0(\aleph(q))$ are now introduced as follows

$$c_0(\aleph(q)):=\left\{\left\{\upsilon=(\upsilon_k)\in\omega:s=\aleph(q)\upsilon=\sum_{j=1}^{k}\frac{3\,q^{j-1}\,\breve{\zeta}_j(q)}{4\breve{\zeta}_k(q)+\breve{\zeta}_{k-1}(q)+3k-10}\upsilon_j\right\}\in c_0\right\};$$

$$c(\aleph(q)):=\left\{\left\{\upsilon=(\upsilon_k)\in\omega:s=\aleph(q)\upsilon=\sum_{j=1}^{k}\frac{3\,q^{j-1}\,\breve{\zeta}_j(q)}{4\breve{\zeta}_k(q)+\breve{\zeta}_{k-1}(q)+3k-10}\upsilon_j\right\}\in c\right\};$$

$$\ell_\infty(\aleph(q)):=\left\{\left\{\upsilon=(\upsilon_k)\in\omega:s=\aleph(q)\upsilon=\sum_{j=1}^{k}\frac{3\,q^{j-1}\,\breve{\zeta}_j(q)}{4\breve{\zeta}_k(q)+\breve{\zeta}_{k-1}(q)+3k-10}\upsilon_j\right\}\in \ell_\infty\right\};$$

$$\ell_p(\aleph(q)):=\left\{\left\{\upsilon=(\upsilon_k)\in\omega:s=\aleph(q)\upsilon=\sum_{j=1}^{k}\frac{3\,q^{j-1}\,\breve{\zeta}_j(q)}{4\breve{\zeta}_k(q)+\breve{\zeta}_{k-1}(q)+3k-10}\upsilon_j\right\}\in \ell_p\right\}.$$

The spaces $\ell_p(\aleph(q))$, $\ell_\infty(\aleph(q))$, $c(\aleph(q))$ and $c_0(\aleph(q))$ can also be understood as the domains of the q-Bronze Leonardo-Lucas matrix $\aleph(q)$ within the sequence spaces ℓ_p and ℓ_∞, c and c_0 respectively. Specifically, we have:

$$\ell_p(\aleph(q))=(\ell_p)_{\aleph(q)}\quad\text{and}\quad \ell_\infty(\aleph(q))=(\ell_\infty)_{\aleph(q)}.$$

$$c(\aleph(q))=c_{\aleph(q)}\quad\text{and}\quad c_0(\aleph(q))=(c_0)_{\aleph(q)}.$$

Clearly, when $q\to 1^-$, the sequence spaces $\ell_p(\aleph(q))$, $\ell_\infty(\aleph(q))$, $c(\aleph(q))$, and $c_0(\aleph(q))$ become $(\ell_p)_{\mathfrak{B}}$, $(\ell_\infty)_{\mathfrak{B}}$, $(c)_{\mathfrak{B}}$, and $(c_0)_{\mathfrak{B}}$, respectively.

Lemma 3.1. *The inverse of the q-Bronze Leonardo-Lucas matrix $\aleph(q)$ is denoted by $\mathcal{G}(q)=(\mathrm{g}_{nk}^{(q)})_{n,k\in\mathbb{N}_0}=\{\aleph(q)\}^{-1}$ is given by*

$$(\mathrm{g}_{nk}^{(q)})=\begin{cases}(-1)^{n-k}\dfrac{4\breve{\zeta}_k(q)+\breve{\zeta}_{k-1}(q)+3k-10}{3\,q^{n-1}\breve{\zeta}_n(q)}, & \text{if } 1\le k\le n,\\ 0, & \text{if } k>n.\end{cases}$$

The inverse $\aleph(q)$-transform or $\{\aleph(q)\}^{-1}$-transform of the sequence $s=(s_k)$ is defined as

$$\upsilon_n=\sum_{n=1}^{k}(-1)^{n-k}\frac{4\breve{\zeta}_k(q)+\breve{\zeta}_{k-1}(q)+3k-10}{3\,q^{n-1}\breve{\zeta}_n(q)}s_k. \tag{2}$$

Therefore, equations (1) and (2) are equivalent.

Theorem 3.2. *(i) If $0<p\le 1$, then $\ell_p(\aleph(q))$ is a complete p-normed space with the p-norm defined by*

$$\|\upsilon\|_{\ell_p(\aleph(q))}=\|s\|_{\ell_p}=\sum_{n=1}^{\infty}\left|\sum_{j=1}^{k}\frac{3q^{j-1}\,\breve{\zeta}_j(q)}{4\breve{\zeta}_k(q)+\breve{\zeta}_{k-1}(q)+3k-10}\upsilon_j\right|^p.$$

(ii) If $1 < p < \infty$, then $\ell_p(\aleph(q))$ is a BK-space with the norm

$$\|v\|_{\ell_p(\aleph(q))} = \|s\|_{\ell_p} = \left(\sum_{n=1}^{\infty}\left|\sum_{j=1}^{k}\frac{3q^{j-1}\breve{\zeta}_j(q)}{4\breve{\zeta}_k(q)+\breve{\zeta}_{k-1}(q)+3k-10}v_j\right|^p\right)^{1/p}.$$

(iii) The space $\ell_\infty(\aleph(q))$ is a BK-space with the norm

$$\|v\|_{\ell_\infty(\aleph(q))} = \|s\|_{\ell_\infty} = \sup_{k\in\mathbb{N}}\left|\sum_{j=1}^{k}\frac{3q^{j-1}\breve{\zeta}_j(q)}{4\breve{\zeta}_k(q)+\breve{\zeta}_{k-1}(q)+3k-10}v_j\right|.$$

Proof. This result can be confirmed through a straightforward verification. □

Theorem 3.3. *$\ell_p(\aleph(q))$ and $\ell_\infty(\aleph(q))$ are linearly isomorphic to the space ℓ_p and ℓ_∞, respectively.*

Proof. Here we proof the result only for the space $\ell_p(\aleph(q)) \cong \ell_p$ as the other one can be done in a similar way. Consider a mapping

$$\mathcal{H} : \ell_p(\aleph(q)) \to \ell_p \text{ s.t. } \mathcal{H}(v) = \aleph(q)v.$$

From the result $\mathcal{H}(v) = 0 \implies v = 0$, it implies the injection property of $\mathcal{H}$.
Let $s = (s_k) \in \ell_p$ for $1 \le p \le \infty$, and define the sequence $v = (v_k)$ as follows

$$v_k = \sum_{l=1}^{k}(-1)^{k-l}\,\frac{4\breve{\zeta}_l(q)+\breve{\zeta}_{l-1}(q)+3l-10}{3\,q^{k-1}\breve{\zeta}_k(q)}s_l, \quad (k\in\mathbb{N}). \tag{3}$$

Then, for $1 \le p \le \infty$ we have

$$\|v\|_{\ell_p(\aleph(q))} = \left(\sum_{k=1}^{\infty}\left|\aleph(q)v\right|^p\right)^{\frac{1}{p}} = \left(\sum_{k=1}^{\infty}\left(\sum_{l=1}^{k}\frac{3q^{l-1}\breve{\zeta}_l(q)}{4\breve{\zeta}_k(q)+\breve{\zeta}_{k-1}(q)+3k-10}v_l\right)^p\right)^{\frac{1}{p}}$$

$$= \left(\sum_{k=1}^{\infty}\left(\sum_{l=1}^{k}\frac{3q^{l-1}\breve{\zeta}_l(q)}{4\breve{\zeta}_k(q)+\breve{\zeta}_{k-1}(q)+3k-10}\sum_{j=l-1}^{l}(-1)^{l-j}\frac{4\breve{\zeta}_k(q)+\breve{\zeta}_{k-1}(q)+3k-10}{3q^{l-1}\breve{\zeta}_l(q)}s_j\right)^p\right)^{\frac{1}{p}}$$

$$= \left(\sum_{k=1}^{\infty}|s_k|^p\right)^{\frac{1}{p}} = \|s\|_{\ell_p} < \infty.$$

Now,

$$\|v\|_{\ell_\infty(\aleph(q))} = \sup_{k\in\mathbb{N}}|\aleph(q)_k v| = \sup_{k\in\mathbb{N}}|s_k| = \|s\|_\infty < \infty.$$

This implies that $v \in \ell_p(\aleph(q))$ (for $1 \le p \le \infty$). Therefore, $\mathcal{H}$ is both surjective and norm-preserving. Consequently, $\ell_p(\aleph(q)) \cong \ell_p$ for $1 \le p \le \infty$. □

Theorem 3.4. *$c_0(\aleph(q)) \cong c_0$ and $c(\aleph(q)) \cong c$.*

Proof. We define the mapping

$$\mathcal{T} : c_0(\aleph(q)) \to c_0 \text{ s.t. } \mathcal{T}(v) = \aleph(q)v.$$

From the result $\mathcal{T}(v) = 0 \implies v = 0$, it implies the injection property of $\mathcal{T}$.
Furthermore, let $s = (s_k) \in \ell_\infty$ and define the sequence $v = (v_k)$ by

$$v_k = \sum_{l=1}^{k}(-1)^{k-l}\,\frac{4\breve{\zeta}_l(q)+\breve{\zeta}_{l-1}(q)+3l-10}{3q^{k-1}\breve{\zeta}_k(q)}s_l, \quad (k\in\mathbb{N}). \tag{4}$$

Then

$$\begin{aligned}\lim_{k\to\infty}(\aleph(q)v)_k &= \lim_{k\to\infty}\left(\sum_{l=1}^{k}\frac{3q^{l-1}\breve{\zeta}_l(q)}{4\breve{\zeta}_k(q)+\breve{\zeta}_{k-1}(q)+3k-10}v_l\right)\\ &= \lim_{k\to\infty}\left(\sum_{l=1}^{k}\frac{2\,q^{l-1}\breve{\zeta}_l(q)}{4\breve{\zeta}_k(q)+\breve{\zeta}_{k-1}(q)+3k-10}\sum_{j=l-1}^{l}(-1)^{l-j}\frac{4\breve{\zeta}_k(q)+\breve{\zeta}_{k-1}(q)+3k-10}{2\,q^{l-1}\breve{\zeta}_l(q)}s_l\right)\\ &= \lim_{k\to\infty}s_k\\ &= 0.\end{aligned}$$

Therefore, $v\in c_0(\aleph(q))$. Thus, $\mathcal{T}$ is surjective and preserves the norm. Consequently, $c_0(\aleph(q))\cong c_0$. Other can be done in a similar way. □

Theorem 3.5. *$\ell_p\subset\ell_p(\aleph(q))$ holds.*

Proof. Let $v=(v_k)\in\ell_p$, for $1\le p<\infty$. Applying Hölder's inequality, for $n\in\mathbb{N}$, we get

$$\begin{aligned}\sum_{n=1}^{\infty}|\aleph(q)_n v|^p &\le \sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\frac{3q^{k-1}\breve{\zeta}_k(q)}{4\breve{\zeta}_n(q)+\breve{\zeta}_{n-1}(q)+3n-10}|v_k|\right)^p\\ &\le \sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\frac{2q^{k-1}p_k(q)}{4\breve{\zeta}_n(q)+\breve{\zeta}_{n-1}(q)+3n-10}|v_k|^p\right)\left(\sum_{k=1}^{n}\frac{3q^{k-1}\breve{\zeta}_k(q)}{\breve{\zeta}_{n+1}(q)+p_n(q)-1}\right)^{p-1}\\ &= \sum_{n=1}^{\infty}\frac{3}{4\breve{\zeta}_n(q)+\breve{\zeta}_{n-1}(q)+3n-10}\sum_{k=1}^{n}q^{k-1}\breve{\zeta}_k(q)|v_k|^p\\ &= \sum_{k=1}^{\infty}|v_k|^p q^{k-1}\breve{\zeta}_k(q)\sum_{n=k}^{\infty}\frac{2}{4\breve{\zeta}_n(q)+\breve{\zeta}_{n-1}(q)+3n-10}.\end{aligned}$$

Hence, $\|v\|^p_{\ell_p(\aleph(q))}\le D\|v\|^p_{\ell_p}\le\infty$, where $D=\sup_k\left(q^{k-1}\breve{\zeta}_k(q)\sum_{n=k}^{\infty}\frac{3}{4\breve{\zeta}_n(q)+\breve{\zeta}_{n-1}(q)+3n-10}\right)$. This indicates that $v\in\ell_p(\aleph(q))$. Therefore, $\ell_p\subset\ell_p(\aleph(q))$.
Similarly, it can be demonstrated that $\ell_1\subset\ell_1(\aleph(q))$, so we will omit the details. □

Theorem 3.6. *The inclusion $\ell_\infty\subset\ell_\infty(\aleph(q))$ holds.*

Proof. Consider the sequence $v=(v_k)\in\ell_\infty$. Then $\exists$ a constant $M>0$ such that $|v_k|\le M$, $\forall\,k\in\mathbb{N}$. Therefore, for $n\in\mathbb{N}$, we get

$$\begin{aligned}|\aleph(q)_n v| &\le \frac{3}{4\breve{\zeta}_n(q)+\breve{\zeta}_{n-1}(q)+3n-10}\sum_{k=1}^{n}q^{k-1}\breve{\zeta}_k(q)|v'_k|\\ &\le \frac{3M}{4\breve{\zeta}_n(q)+\breve{\zeta}_{n-1}(q)+3n-10}\sum_{k=1}^{n}q^{k-1}\breve{\zeta}_k(q)\\ &= M.\end{aligned}$$

Thus, $(\aleph(q)_n v)\in\ell_\infty$ for $n\in\mathbb{N}$, which means $v\in\ell_\infty(\aleph(q))$. Consequently, we have $\ell_\infty\subset\ell_\infty(\aleph(q))$. □

Theorem 3.7. *The inclusions $c_0\subset c_0(\aleph(q))$ and $c\subset c(\aleph(q))$ are strict.*

Proof. Since the matrix $\aleph(q)$ is regular, the inclusions are automatically valid. To demonstrate the strictness of this inclusion, consider the sequence $v = (1, 0, 1, 0, \dots)$. We now calculate

$$\begin{aligned}(\aleph(q)v)_n &= \sum_{k=1}^{n} \frac{3q^{k-1}\breve{\zeta}_k(q)}{4\breve{\zeta}_n(q) + \breve{\zeta}_{n-1}(q) + 3n - 10} \\ &= \frac{3}{4\breve{\zeta}_n(q) + \breve{\zeta}_{n-1}(q) + 3n - 10}\left(\breve{\zeta}_1(q) + q^2\breve{\zeta}_1(q) + \cdots + q^{n-1}\breve{\zeta}_n(q)\right),\end{aligned}$$

where $n \in \mathbb{N}$. This is a convergent sequence, which implies $v \in c(\aleph(q)) \setminus c$. In a similar approach, it can be applied to prove the other case. □

Theorem 3.8. *The inclusion $c_0(\aleph(q)) \subset c(\aleph(q))$ is strict.*

Proof. To demonstrate the inclusion $c_0(\aleph(q)) \subset c(\aleph(q))$, consider the sequence $v = (v_k)$, where $v_k = 1$ for all $k \geq 1$. In this case, we have the following

$$(\aleph(q)v)_n = \sum_{k=1}^{n} \frac{3q^{k-1}\breve{\zeta}_k(q)}{4\breve{\zeta}_n(q) + \breve{\zeta}_{n-1}(q) + 3n - 10} = 1, \ \ \forall n$$

Since $(\aleph(q)v) \in c$ but it is not in c_0. So, $v \in c(\aleph(q)) \setminus c_0(\aleph(q))$, this proves the result. □

Theorem 3.9. *The space $\ell_p(\aleph(q))$ is not a Hilbert space, except when $p = 2$.*

Proof. Consider the sequences $a = (a_k)$ and $b = (b_k)$ defined as

$$a = \left(1, 1, -\frac{\breve{\zeta}_1(q) + q\,\breve{\zeta}_1(q)}{q^2\breve{\zeta}_1(q)}, 0, 0, \dots\right)$$

and

$$b = \left(1, \frac{4 - 4\breve{\zeta}_1(q) - 4\breve{\zeta}_2(q)}{3\,q\breve{\zeta}_2(q)}, \frac{\breve{\zeta}_1(q) + 4\breve{\zeta}_2(q) - 4}{3\,q^2\,\breve{\zeta}_3(q)}, 0, 0, \dots\right).$$

Therefore $\aleph(q)a = (1, 1, 0, 0, \dots)$ and $\aleph(q)b = (1, -1, 0, 0, \dots)$. However, we find that

$$\|a + b\|^2_{\ell_p(\aleph(q))} = 8 \neq 4 \cdot 2^{2/p} = \|a\|^2_{\ell_p(\aleph(q))} + \|b\|^2_{\ell_p(F(q))}.$$

which clearly violates the parallelogram identity, unless $p = 2$. This completes the proof. □

Theorem 3.10. *For $1 \leq p \leq \infty$, the sequence space $\ell_p(\aleph(q))$ is not of absolute type.*

Proof. Take $a = (1, -1, 0, 0, \dots)$. Then

$$\aleph(q)a = \left(1, \frac{\breve{\zeta}_1(q) - q\breve{\zeta}_2(q)}{\breve{\zeta}_2(q) + \breve{\zeta}_1(q) - 4}, \frac{\breve{\zeta}_1(q) - q\breve{\zeta}_2(q)}{\breve{\zeta}_3(q) + \breve{\zeta}_2(q) - 1}, \dots\right),$$

while

$$\aleph(q)|a| = \left(1, \frac{\breve{\zeta}_1(q) + q\breve{\zeta}_2(q)}{\breve{\zeta}_2(q) + \breve{\zeta}_1(q) - 4}, \frac{\breve{\zeta}_1(q) + q\breve{\zeta}_2(q)}{\breve{\zeta}_3(q) + \breve{\zeta}_2(q) - 1}, \dots\right).$$

Since $\|a\|_{\ell_p\aleph(q)} \neq \||a|\|_{\ell_p\aleph(q)}$ for $0 < q < 1$, $\ell_p(\aleph(q))$ is non-absolute. □

Definition 3.11. *A normed linear space $\mathfrak{K}$, with norm $\|\cdot\|$, is defined to have a Schauder basis $u = (u_j)$ if for every element $w = (w_k) \in \mathfrak{K}$, there is a unique sequence of scalars $a = (a_j)$ such that*

$$\lim_{k\to\infty}\left\|w - \sum_{k=0}^{k} a_k u_k\right\| = 0.$$

Theorem 3.12. *Define the sequence $b^{(k)}(q) = \{b_n^{(k)}(q)\}_{k\in\mathbb{N}}$, consisting of elements of the space $\ell_p(\aleph(q))$, as follows*

$$b_n^{(k)}(q) = \begin{cases} (-1)^{n-k}\dfrac{4\breve{\zeta}_n + \breve{\zeta}_{n-1} + 3n - 10}{2\,q^{n-1}\breve{\zeta}_n(q)}, & \text{if } 1 \le k \le n, \\ 0, & \text{if } k > n. \end{cases}$$

Then, the following statements hold

(a) *The set $\left\{b^{(0)}(q), b^{(1)}(q), b^{(2)}(q), \dots\right\}$ forms a basis for $\ell_p(\aleph(q))$. Moreover, each $s \in \ell_p(\aleph(q))$ can be uniquely represented as*

$$v = \sum_{k=0}^{\infty} s_k b^{(k)}(q),$$

where $s_n = (\aleph(q)v)_n$ for each $n \in \mathbb{N}$.

(b) *The space $\ell_\infty(\aleph(q))$ does not admit a Schauder basis.*

(c) *The set $\{b^{(0)}(q), b^{(1)}(q), b^{(2)}(q), \dots\}$ forms a basis for the space $c_0(\aleph(q))$, and every $v \in c_0(\aleph(q))$ can be uniquely written as*

$$v = \sum_{k=0}^{\infty} s_k b^{(s)}(q).$$

(d) *The set $\{e, b^{(0)}(q), b^{(1)}(q), b^{(2)}(q), \dots\}$ forms a basis for the space $c(\aleph(q))$. Every $v \in c(\aleph(q))$ can be uniquely expressed as*

$$v = ze + \sum_{k=0}^{\infty} (s_k - z) b^{(k)}(q),$$

where $z = \lim_{k\to\infty} s_k = \lim_{k\to\infty} (\aleph(q)v)_k$.

4. Operator ideals

This section explores key properties of the s-type $\ell_p(\aleph(q))$ operators in the context of the q-Bronze Leonardo-Lucas sequence space. Let $\mathcal{L}(A, B)$ denote the space of bounded linear operators from A to B, and let $\mathcal{L}$ be the class of all such operators between Banach spaces. The dual of A, written A', consists of continuous linear functionals a'. For $a' \in A'$ and $b \in B$, the tensor product $a' \otimes b$ is the operator defined by $(a' \otimes b)(a) = a'(a)b$ for all $a \in A$.

Definition 4.1 ([3, 4]). *A mapping $s\colon \mathcal{L} \to \mathbb{R}^+$, where $\mathbb{R}^+$ denotes the class of positive reals, is called an s-number sequence if it satisfies the following conditions*

(a) ***Monotonicity:*** $\|s\| = s_1(\mathfrak{K}) \ge s_2(\mathfrak{K}) \ge \cdots \ge 0,$ *for $\mathfrak{K} \in \mathcal{L}(A, B)$.*

(b) ***Additivity:*** $s_{n+k}(\mathfrak{K} + \mathfrak{R}) \le s_n(\mathfrak{K}) + s_k(\mathfrak{R})$ *for $\mathfrak{K}, \mathfrak{R} \in \mathcal{L}(A, B)$ and $n, k \in \mathbb{N}$.*

(c) ***Ideal property:*** $s_n(\mathfrak{K}\mathfrak{H}\mathfrak{R}) \le \|\mathfrak{K}\| |s_n(\mathfrak{H})| \|\mathfrak{R}\|$ *for $\mathfrak{K} \in \mathcal{L}(A_0, A), \mathfrak{H} \in \mathcal{L}(A, B), \mathfrak{R} \in \mathcal{L}(B, B_0)$, and $n \in \mathbb{N}$.*

(d) ***Rank property:*** *If rank $(\mathfrak{K}) < n$, then $s_n(\mathfrak{K}) = 0$.*

(e) ***Norming property:*** $s_n\left(I_2 : \ell_2^{(n)} \to \ell_2^{(n)}\right) = 1$, *where I_2 denotes the identity operator on the n-dimensional Hilbert space.*

Definition 4.2 ([12]). *For Banach spaces A and B, let $P(A, B) = P \cap \mathcal{L}(A, B)$, where $P \subseteq \mathcal{L}$. The collection P is termed an "operator ideal" if the following hold*

(i) For every $a' \in A'$ and $b \in B$, the operator $a' \odot b$ belongs to $P(A, B)$.

(ii) If $\mathfrak{K}, \mathfrak{R} \in P(A, B)$, then $\mathfrak{K} + \mathfrak{R} \in P(A, B)$.

(iii) For any $\mathfrak{H} \in P(A, B)$, $\mathfrak{R} \in \mathcal{L}(A_0, A)$, and $\mathfrak{K} \in \mathcal{L}(B, B_0)$, the composition $\mathfrak{K}\mathfrak{H}\mathfrak{R}$ in $P(A_0, B_0)$.

Each $P(A, B)$ is referred to as a "component" of the ideal P.

Definition 4.3 ([12, 13]). *The ideal quasi-norm is a real-valued function $\mathfrak{X} : A \to \mathbb{R}^+$ that satisfy the following conditions*

(i) *If $a' \in A', b \in B$, then $\mathfrak{X}(a' \odot b) = \|a'\|\|b\|$.*

(ii) *There exists a constant $M \geq 1$ such that*

$$\mathfrak{X}(\mathfrak{K} + \mathfrak{R}) \leq M[\mathfrak{X}(\mathfrak{K}) + \mathfrak{X}(\mathfrak{R})] \quad \text{for } \mathfrak{K}, \mathfrak{R} \in P(A, B).$$

(iii) *If $\mathfrak{H} \in P(A, B)$, $\mathfrak{R} \in \mathcal{L}(A_0, A)$, and $\mathfrak{K} \in \mathcal{L}(B, B_0)$, then $\mathfrak{K}\mathfrak{H}\mathfrak{R} \in P(A_0, B_0)$.*

Lemma 4.4 ([11]). *For operators $\mathfrak{K}, \mathfrak{R} \in \mathcal{L}(A, B)$, then*

$$|s_n(\mathfrak{K}) - s_n(\mathfrak{R})| \leq \|\mathfrak{K} - \mathfrak{R}\|, \forall n \in \mathbb{N}.$$

An operator $\mathfrak{K} \in \mathcal{L}(A, B)$ is called an s-type $\ell_p(\aleph(q))$ operator if its singular values satisfy

$$\sum_{n=1}^{\infty} \left| \frac{3\sum_{k=1}^{n} q^{k-1}\breve{\zeta}_k(q) s_k(\mathfrak{K})}{4\breve{\zeta}_n(q) + \breve{\zeta}_{n-1}(q) + 3n - 10} \right|^p < \infty(1 < p < \infty).$$

The class of all such operators is denoted by $\ell_p^{(s)}(\aleph(q))$.

Theorem 4.5. *Let $1 < p < \infty$. Then the class $\ell_p^{(s)}(\aleph(q))$ is an operator ideal.*

Proof. Let A and B be any two Banach spaces. Let $a' \in A'$ and $b \in B$. Then $a' \odot b$ is a rank-one operator, and so $s_n(a' \odot b) = 0$ for all $n \geq 1$. Thus, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \left| \frac{3\sum\limits_{k=1}^{n} q^{k-1}\breve{\zeta}_k(q) s_k(a' \odot b)}{4\breve{\zeta}_n(q) + \breve{\zeta}_{n-1}(q) + 3n - 10} \right|^p &= \sum_{n=1}^{\infty} \left| \frac{3s_1(a' \odot b)}{4\breve{\zeta}_n(q) + \breve{\zeta}_{n-1}(q) + 3n - 10} \right|^p \\
&= \left[s_1(a' \odot b)\right]^p \sum_{n=1}^{\infty} \left| \frac{3}{4\breve{\zeta}_n(q) + \breve{\zeta}_{n-1}(q) + 3n - 10} \right|^p \\
&= \|a' \odot b\|^p \sum_{n=1}^{\infty} \left| \frac{3}{4\breve{\zeta}_n(q) + \breve{\zeta}_{n-1}(q) + 3n - 10} \right|^p . \\
&< \infty.
\end{aligned}$$

Thus $a' \odot b \in \ell_p^{(s)}(\aleph(q))$.

Let $\mathfrak{K}, \mathfrak{R} \in \ell_p^{(s)}(\aleph(q))$ and due to the non-negativity and non-increasing properties of s-numbers, apply

Minkowski's inequality, we get

$$\left(\sum_{n=1}^{\infty}\left(\frac{3\sum_{k=1}^{n} q^{k-1}\breve{\zeta}_k(q)s_k(\mathfrak{R}+\Re)}{4\breve{\zeta}_n(q)+\breve{\zeta}_{n-1}(q)+3n-10}\right)^p\right)^{1/p}$$

$$\leq\left(\sum_{n=1}^{\infty}\left(\frac{3\sum_{k=1}^{n} q^{2k-2}\breve{\zeta}_{k-1}(q)\, s_{2k-1}(\mathfrak{R}+\Re)+\sum_{k=1}^{n}2\, q^{2k-1}\breve{\zeta}_{2k}\, s_{2k}(\mathfrak{R}+\Re)}{4\breve{\zeta}_n(q)+\breve{\zeta}_{n-1}(q)+3n-10}\right)^p\right)^{1/p}$$

$$\leq\left(\sum_{n=1}^{\infty}\left(\frac{3\sum_{k=1}^{n}(q^{2k-2}+q^{2k-1})\breve{\zeta}_{2k-1}(q)s_{2k-1}(\mathfrak{R}+\Re)}{4\breve{\zeta}_n(q)+\breve{\zeta}_{n-1}(q)+3n-10}\right)^p\right)^{1/p}$$

$$\leq M\left(\sum_{n=1}^{\infty}\left(\frac{3\sum_{k=1}^{n} q^{k-1}\breve{\zeta}_k(q)s_k(\mathfrak{R})}{4\breve{\zeta}_n(q)+\breve{\zeta}_{n-1}(q)+3n-10}+\frac{3\sum_{k=1}^{n} q^{k-1}\breve{\zeta}_k(q)s_k(\Re)}{4\breve{\zeta}_n(q)+\breve{\zeta}_{n-1}(q)+3n-10}\right)^p\right)^{1/p}$$

$$\leq M\left[\left(\sum_{n=1}^{\infty}\left(\frac{3\sum_{k=1}^{n} q^{k-1}\breve{\zeta}_k(q)s_k(\mathfrak{R})}{4\breve{\zeta}_n(q)+\breve{\zeta}_{n-1}(q)+3n-10}\right)^p\right)^{1/p}+\left(\sum_{n=1}^{\infty}\left(\frac{3\sum_{k=1}^{n} q^{k-1}\breve{\zeta}_k(q)s_k(\Re)}{4\breve{\zeta}_n(q)+\breve{\zeta}_{n-1}(q)+3n-10}\right)^p\right)^{1/p}\right]$$

$$<\infty.$$

Thus, $\mathfrak{R}+\Re\in\ell_p^{(s)}(\aleph(q))$.

Let $\mathfrak{R}\in\mathcal{L}(A_0,A)$, $\Re\in\mathcal{L}(B,B_0)$ and $\mathfrak{H}\in\ell_p^{(s)}(\aleph(q))$. Using the property (3) in Definition 4.1, we get

$$\left(\sum_{n=1}^{\infty}\left(\frac{2\sum_{k=1}^{n} q^{k-1}\breve{\zeta}_k(q)s_k(\Re\mathfrak{H}\mathfrak{R})}{4\breve{\zeta}_n(q)+\breve{\zeta}_{n-1}(q)+3n-10}\right)^p\right)^{\frac{1}{p}}\leq\|\Re\|\|\mathfrak{R}\|\left(\sum_{n=1}^{\infty}\left(\frac{2\sum_{k=1}^{n} q^{k-1}\breve{\zeta}_k(q)s_k(\mathfrak{H})}{4\breve{\zeta}_n(q)+\breve{\zeta}_{n-1}(q)+3n-10}\right)^p\right)^{\frac{1}{p}}$$

$$\leq\infty.$$

Thus, $\Re\mathfrak{H}\mathfrak{R}\in\ell_p^{(s)}(\aleph(q))$. Hence $\ell_p^{(s)}(\aleph(q))$ is an operator ideal. □

Theorem 4.6. *Let* $1<p\leq r<\infty$. *Then* $\ell_p^{(s)}(\aleph(q))\subseteq\ell_r^{(s)}(\aleph(q))$.

Proof. This result is directly derived from the fact that $\ell_p(\aleph(q))\subseteq\ell_r(\aleph(q))$ for $1<p\leq r<\infty$. □

For the operator ideal $\ell_p^{(s)}(\aleph(q))$ with $1<p<\infty$, we define the mapping $Q^{(s)}:\ell_p^{(s)}(\aleph(q))\to\mathbb{R}^+$ as follows

$$Q^{(s)}(\mathfrak{R})=\left(\sum_{n=1}^{\infty}\left(\frac{3\sum_{k=1}^{n} q^{k-1}\breve{\zeta}_k(q)s_k(\mathfrak{R})}{4\breve{\zeta}_n(q)+\breve{\zeta}_{n-1}(q)+3n-10}\right)^p\right)^{\frac{1}{p}},$$

where $\mathfrak{R}\in\ell_p^{(s)}(\aleph(q))$.

Theorem 4.7. *For $1 < p < \infty$, the mapping $\widetilde{Q}^{(s)}$ is a quasi-norm on the operator ideal $\ell_p^{(s)}(\aleph(q))$, where*

$$\widetilde{Q}^{(s)}(\mathfrak{K}) = \frac{Q^{(s)}(\mathfrak{K})}{\left(\left(\sum_{n=1}^{\infty} \frac{3\, q^{k-1}\, \breve{\zeta}_k(q)}{4\breve{\zeta}_n(q) + \breve{\zeta}_{n-1}(q) + 3n - 10}\right)^p\right)^{\frac{1}{p}}}.$$

Proof. For arbitrary Banach spaces A and B, the rank-one operator $a' \odot b \colon A \to B$ satisfies $s_n(a' \odot b) = 0$ whenever $n \geq 2$. This leads to the following

$$\begin{aligned} Q^{(s)}(a' \odot b) &= \left(\sum_{n=1}^{\infty}\left(\frac{3\sum_{k=1}^{n} q^{k-1}\breve{\zeta}_k(q)s_k(a' \odot b)}{4\breve{\zeta}_n(q) + \breve{\zeta}_{n-1}(q) + 3n - 10}\right)^p\right)^{\frac{1}{p}} \\ &= \left(\sum_{n=1}^{\infty}\left(\frac{3\, \breve{\zeta}_1(q)s_1(a' \odot b)}{4\breve{\zeta}_n(q) + \breve{\zeta}_{n-1}(q) + 3n - 10}\right)^p\right)^{\frac{1}{p}} \\ &= \|a' \odot b\|\left(\sum_{n=1}^{\infty}\left(\frac{3}{4\breve{\zeta}_n(q) + \breve{\zeta}_{n-1}(q) + 3n - 10}\right)^p\right)^{\frac{1}{p}}. \end{aligned}$$

Since $\|a' \odot b\| = \|a'\|\|b\|$, we have

$$\widetilde{Q}^{(s)}(a' \odot b) = \|a'\|\|b\|.$$

Let $\mathfrak{K}, \mathfrak{R} \in \ell_p^{(s)}(\aleph(q))$. Then

$$\begin{aligned} Q^{(s)}(\mathfrak{K} + \mathfrak{R}) &= \left(\sum_{n=1}^{\infty}\left(\frac{3\sum_{k=1}^{n} q^{k-1}\breve{\zeta}_k(q)s_k(\mathfrak{K} + \mathfrak{R})}{4\breve{\zeta}_n(q) + \breve{\zeta}_{n-1}(q) + 3n - 10}\right)^p\right)^{\frac{1}{p}} \\ &\leq M\left(\sum_{n=1}^{\infty}\left(\frac{3\sum_{k=1}^{n} q^{k-1}\breve{\zeta}_k(q)s_k(\mathfrak{K})}{4\breve{\zeta}_n(q) + \breve{\zeta}_{n-1}(q) + 3n - 10}\right)^p\right)^{\frac{1}{p}} + \left(\sum_{n=1}^{\infty}\left(\frac{3\sum_{k=1}^{n} q^{k-1}\breve{\zeta}_k(q)s_k(\mathfrak{R})}{4\breve{\zeta}_n(q) + \breve{\zeta}_{n-1}(q) + 3n - 10}\right)^p\right)^{\frac{1}{p}} \\ &\leq M\Big(Q^{(s)}(\mathfrak{K}) + Q^{(s)}(\mathfrak{R})\Big). \end{aligned}$$

Thus,

$$\widetilde{Q}^{(s)}(\mathfrak{K} + \mathfrak{R}) \leq M\left(\widetilde{Q}^{(s)}(\mathfrak{K}) + \widetilde{Q}^{(s)}(\mathfrak{R})\right).$$

Finally, let $\mathfrak{H} \in \ell_p^{(s)}(\aleph(q))(A \to B)$, $\mathfrak{K} \in \mathcal{L}(B, B_0)$, and $\mathfrak{R} \in \mathcal{L}(A_0, A)$. Then

$$\begin{aligned}
Q^{(s)}(\mathfrak{K}\mathfrak{H}\mathfrak{R}) &= \left(\sum_{n=1}^{\infty}\left(\frac{3}{4\breve{\zeta}_n(q)+\breve{\zeta}_{n-1}(q)+3n-10}\sum_{k=1}^{n} q^{k-1}\breve{\zeta}_k(q)s_k(\mathfrak{K}\mathfrak{H}\mathfrak{R})\right)^p\right)^{\frac{1}{p}} \\
&\leq \|\mathfrak{K}\|\|\mathfrak{R}\|\left(\sum_{n=1}^{\infty}\left(\frac{3}{4\breve{\zeta}_n(q)+\breve{\zeta}_{n-1}(q)+3n-10}\sum_{k=1}^{n} q^{k-1}\breve{\zeta}_k(q)s_k(\mathfrak{H})\right)^p\right)^{\frac{1}{p}} \\
&= \|\mathfrak{K}\|\|\mathfrak{R}\|\left(\sum_{n=1}^{\infty}\left(\frac{3}{4\breve{\zeta}_n(q)+\breve{\zeta}_{n-1}(q)+3n-10}\sum_{k=1}^{n} q^{k-1}\breve{\zeta}_k(q)s_k(\mathfrak{H})\right)^p\right)^{\frac{1}{p}}.
\end{aligned}$$

Thus

$$\widetilde{Q}^{(s)}(\mathfrak{K}\mathfrak{H}\mathfrak{R}) \leq \|\mathfrak{K}\|\widetilde{Q}^{(s)}(\mathfrak{H})\|\mathfrak{R}\|.$$

Consequently, the operator ideal $\ell_p^{(s)}(\aleph(q))$ admits $\widetilde{Q}^{(s)}$ as its quasi-norm. □

Theorem 4.8. *For $1 < p < \infty$, the operator ideal $\ell_p^{(s)}(\aleph(q))$ forms a complete space under the quasi-norm $\tilde{Q}(s)$.*

Proof. For $1 < p < \infty$, we obtain

$$\begin{aligned}
\tilde{Q}(s)(\mathfrak{K}) &= \left[\sum_{n=1}^{\infty}\left(\frac{3\sum\limits_{k=1}^{n} q^{k-1}\breve{\zeta}_k(q)s_k(\mathfrak{K})}{4\breve{\zeta}_n(q)+\breve{\zeta}_{n-1}(q)+3n-10}\right)^p\right]^{\frac{1}{p}} \\
&= \left[\left(\sum_{n=1}^{\infty}\frac{3\breve{\zeta}_1(q)s_1(\mathfrak{K})}{4\breve{\zeta}_n(q)+\breve{\zeta}_{n-1}(q)+3n-10}\right)^p\right]^{\frac{1}{p}} \\
&\leq \|\mathfrak{K}\|\left[\left(\sum_{n=1}^{\infty}\frac{3}{4\breve{\zeta}_n(q)+\breve{\zeta}_{n-1}(q)+3n-10}\right)^p\right]^{\frac{1}{p}}.
\end{aligned}$$

From this we can conclude that

$$\|\mathfrak{K}\| \leq \tilde{Q}(s)(\mathfrak{K}) \quad \text{for all } \mathfrak{K} \in \ell_p^{(s)}(\aleph(q))(A \to B). \tag{5}$$

Let $(\mathfrak{K}_n)$ be a Cauchy sequence in $\ell_p^{(s)}(\aleph(q))(A \to B)$. For every $\epsilon > 0$, there exists $\kappa \in \mathbb{N}$ such that

$$\tilde{Q}(s)(\mathfrak{K}_n - \mathfrak{K}_k) < \epsilon \text{ for all } n, k \geq \kappa. \tag{6}$$

From (5), we deduce that

$$\|\mathfrak{K}_n - \mathfrak{K}_m\| \leq Q(s)(\mathfrak{K}_n - \mathfrak{K}_m).$$

Applying (6), we obtain

$$\|\mathfrak{K}_n - \mathfrak{K}_m\| \leq Q(s)(\mathfrak{K}_n - \mathfrak{K}_m) \text{ for all } n, m \geq \kappa.$$

Therefore, the sequence $(\mathfrak{K}_n)$ is Cauchy in the space $\mathcal{L}(A, B)$. Since $\mathcal{L}(A, B)$ is a Banach space, we can conclude that $\mathfrak{K}_n \to \mathfrak{K}$ as $n \to \infty$ in $\mathcal{L}(A, B)$.
Utilizing Lemma 4.4, we have

$$|s_n(\mathfrak{K}_n - \mathfrak{K}_m) - s_n(\mathfrak{K} - \mathfrak{K}_m)| \leq \|\mathfrak{K}_n - \mathfrak{K}\|.$$

Taking the limit as $n \to \infty$ gives us

$$s_n(\Re_n - \Re_m) \to s_n(\Re - \Re_m). \tag{7}$$

Now from (6), we get

$$\left(\sum_{n=1}^{\infty}\left(\frac{3\sum_{k=1}^{n} q^{k-1}\breve{\zeta}_k(q)s_k(\Re_n - \Re_m)}{4\breve{\zeta}_n(q) + \breve{\zeta}_{n-1}(q) + 3n - 10}\right)^p\right)^{\frac{1}{p}} < \epsilon\left(\sum_{n=1}^{\infty}\left(\frac{3\breve{\zeta}_k(q)}{4\breve{\zeta}_n(q) + \breve{\zeta}_{n-1}(q) + 3n - 10}\right)^p\right)^{\frac{1}{p}}.$$

For a fix $m \geq \kappa$ and taking limit as $n \to \infty$ (for $n \geq \kappa$), from (7) we derive the inequality

$$\left(\sum_{n=1}^{\infty}\left(\frac{3\sum_{k=1}^{n} q^{k-1}\breve{\zeta}_k(q)s_k(\Re - \Re_m)}{4\breve{\zeta}_n(q) + \breve{\zeta}_{n-1}(q) + 3n - 10}\right)^p\right)^{\frac{1}{p}} < \epsilon\left(\sum_{n=1}^{\infty}\left(\frac{3\breve{\zeta}_k(q)}{4\breve{\zeta}_n(q) + \breve{\zeta}_{n-1}(q) + 3n - 10}\right)^p\right)^{\frac{1}{p}},$$

which leads to

$$\tilde{Q}(s)(\Re - \Re_m) < \epsilon \text{ for all } m \geq \kappa.$$

Consequently, the sequence $(\Re_m)$ converges to $\Re$ with respect to the quasi-norm $\tilde{Q}(s)$.
We must establish the inclusion $\Re \in \ell_p^{(s)}(\aleph(q))(A \to B)$. We have

$$\begin{aligned}
\sum_{k=1}^{n} s_k(\Re) &\leq \sum_{k=1}^{n} q^{2k-2}\breve{\zeta}_{k-1}(q)s_{2k-1}(\Re) + \sum_{k=1}^{n} q^{2k-1}\breve{\zeta}_{2k}s_{2k}(\Re)\\
&\leq \sum_{k=1}^{n}(q^{2k-2} + q^{2k-1})\breve{\zeta}_{2k-1}(q)s_{2k-1}(\Re).\\
&\leq M\left(\sum_{k=1}^{n} q^{k-1}\breve{\zeta}_k(q)s_k(\Re - \Re_m) + \sum_{k=1}^{n} q^{k-1}\breve{\zeta}_k(q)s_k(\Re_m)\right).
\end{aligned}$$

Consequently

$$\begin{aligned}
&\left(\sum_{n=1}^{\infty}\left(\frac{3\sum_{k=1}^{n} q^{k-1}\breve{\zeta}_k(q)s_k(\Re)}{4\breve{\zeta}_n(q) + \breve{\zeta}_{n-1}(q) + 3n - 10}\right)^p\right)^{\frac{1}{p}}\\
&\leq M\left[\left(\sum_{n=1}^{\infty}\left(\frac{3\sum_{k=1}^{n} q^{k-1}\breve{\zeta}_k(q)s_k(\Re - \Re_m)}{4\breve{\zeta}_n(q) + \breve{\zeta}_{n-1}(q) + 3n - 10}\right)^p\right)^{\frac{1}{p}} + \left(\sum_{n=1}^{\infty}\left(\frac{3\sum_{k=1}^{n} q^{k-1}\breve{\zeta}_k(q)s_k(\Re_m)}{4\breve{\zeta}_n(q) + \breve{\zeta}_{n-1}(q) + 3n - 10}\right)^p\right)^{\frac{1}{p}}\right]
\end{aligned}$$

which is finite. since $\tilde{Q}(s)(\Re - \Re_m) \to 0$ as $m \to \infty$ and $\Re_m \in \ell_p^{(s)}(\aleph(q))(A \to B)$. Hence, we conclude that $\Re \in \ell_p^{(s)}(\aleph(q))(A \to B)$. $\square$

5. Geometric Properties

In this section, we explore various geometric properties of spaces $\ell_p(\aleph(q))$ $(1 \leq p < \infty)$ and $\ell_\infty(\aleph(q))$. A linear operator $L : X \to Y$ between Banach spaces is called compact if it maps bounded subsets of X to relatively compact subsets of Y (see [9, Definition 3.4.1]).

Definition 5.1 ([9]). *Let A be a normed space. We define a function $\rho_A : (0,\infty) \to [0,\infty)$ as follows:*

(a) *If A is not the zero space (i.e., $A \neq \{0\}$), then*

$$\rho_A(t) = \sup\left\{\frac{1}{2}\left(\|p+tq\| + \|p-tq\|\right) - 1 : p, q \in S_A\right\}$$

(b) *If A is the zero space (i.e., $A = \{0\}$), then*

$$\rho_A(t) = \begin{cases} 0 & \text{if } 0 < t < 1 \\ t-1 & \text{if } t \geq 1 \end{cases}$$

In this context, $\rho_A(t)$ represents the modulus of smoothness of the space A. The space A is uniformly smooth if:

$$\lim_{t\to 0^+} \frac{\rho_A(t)}{t} = 0.$$

Theorem 5.2. *The space $\ell_p(\aleph(q))$ is uniformly smooth for $1 < p < \infty$.*

Proof. Let $u, v \in \ell_p(\aleph(q))$, and suppose that

$$\|u+tv\| \geq 1 \quad \text{and} \quad \|u-tv\| \geq 1 \quad \text{for all } t > 0.$$

Recall that

$$\|u+tv\|_{\ell_p(\aleph(q))} = \|\aleph(q)(u+tv)\|_{\ell_p},$$

Our aim is to compute the limit $\lim_{t\to 0^+} \frac{\rho_A(t)}{t}$.
By using L'Hospital's rule, we have

$$\lim_{t\to 0^+} \frac{\rho_A(t)}{t} = \lim_{t\to 0^+} \frac{d}{dt}\rho_A(t).$$

Let us now compute $\frac{d}{dt}\rho_A(t)$. By the definition of the modulus of smoothness, we get

$$\frac{d}{dt}\rho_A(t) = \sup\left\{\frac{1}{2}\left(\frac{d}{dt}\|u+tv\| + \frac{d}{dt}\|u-tv\|\right) : x, y \in S_{\ell_p(\aleph(q))}\right\}.$$

Now,

$$\frac{d}{dt}\|u+tv\| = \frac{d}{dt}\left(\|\aleph(q)(u+tv)\|_{\ell_p}\right) = \frac{d}{dt}\left(\sum_{n=1}^{\infty} |\aleph(q)(u+tv)_n|^p\right)^{1/p}.$$

Differentiating, we have

$$\frac{d}{dt}\|u+tv\| = \frac{1}{p}\left(\sum_{n=1}^{\infty} |\aleph(q)(u+tv)_n|^p\right)^{1-\frac{1}{p}} \sum_{n=1}^{\infty} \frac{d}{dt}|\aleph(q)(u+tv)_n|^p.$$

Similarly,

$$\frac{d}{dt}\|u-tv\| = \frac{1}{p}\left(\sum_{n=1}^{\infty} |\aleph(q)(u-tv)_n|^p\right)^{1-\frac{1}{p}} \sum_{n=1}^{\infty} \frac{d}{dt}|\aleph(q)(u-tv)_n|^p.$$

In particular,

$$\frac{d}{dt}|\aleph(q)(u+tv)_n|^p = p\,|\aleph(q)(u+tv)_n|^{p-1}\frac{d}{dt}|\aleph(q)(u+tv)_n|.$$

But

$$\frac{d}{dt}|\aleph(q)(u+tv)_n| = \begin{cases}(\aleph(q)v)_n, & \text{if } \aleph(q)(u+tv)_n \geq 0,\\ -(\aleph(q)v)_n, & \text{if } \aleph(q)(u+tv)_n < 0,\end{cases}$$

because $\aleph(q)$ is a linear operator. Eventually

$$\frac{d}{dt}|\aleph(q)(u+tv)_n|^p = p\,|\aleph(q)(u+tv)_n|^{p-1}\begin{cases}(\aleph(q)v)_n, & \text{if } \aleph(q)(u+tv)_n \geq 0,\\ -(\aleph(q)v)_n, & \text{if } \aleph(q)(u+tv)_n < 0.\end{cases}$$

Similarly,

$$\frac{d}{dt}|\aleph(q)(u-tv)_n|^p = p\,|\aleph(q)(u-tv)_n|^{p-1}\begin{cases}-(\aleph(q)v)_n, & \text{if } \aleph(q)(u-tv)_n \geq 0,\\ (\aleph(q)v)_n, & \text{if } \aleph(q)(u-tv)_n < 0.\end{cases}$$

Taking limit $t \to 0^+$, we get

$$\lim_{t\to 0^+}\frac{d}{dt}\|u+tv\| = \begin{cases}\frac{1}{p}\left(\sum_{n=1}^{\infty}|\aleph(q)(u)_n|^p\right)^{1-\frac{1}{p}}\sum_{n=1}^{\infty}p(\aleph(q)(u))_n^{p-1}\,(\aleph(q)v)_n, & (\aleph(q)u)_n \geq 0,\\ -\frac{1}{p}\left(\sum_{n=1}^{\infty}|\aleph(q)(u)_n|^p\right)^{1-\dfrac{1}{p}}\sum_{n=1}^{\infty}p|\aleph(q)(u)_n|^{p-1}\,(\aleph(q)v)_n, & (\aleph(q)u)_n < 0.\end{cases}$$

and

$$\lim_{t\to 0^+}\frac{d}{dt}\|u-tv\| = \begin{cases}-\frac{1}{p}\left(\sum_{n=1}^{\infty}|\aleph(q)(u)_n|^p\right)^{1-\frac{1}{p}}\sum_{n=1}^{\infty}p(\aleph(q)(u))_n^{p-1}(\aleph(q)v)_n, & (\aleph(q)u)_n \geq 0,\\ \frac{1}{p}\left(\sum_{n=1}^{\infty}|\aleph(q)(u)_n|^p\right)^{1-\frac{1}{p}}\sum_{n=1}^{\infty}p|\aleph(q)(u)_n|^{p-1}(\aleph(q)v)_n, & (\aleph(q)u)_n < 0.\end{cases}$$

We see that

$$\lim_{t\to 0^+}\frac{d}{dt}\|u+tv\| + \lim_{t\to 0^+}\frac{d}{dt}\|u-tv\| = 0.$$

Hence, we conclude that

$$\lim_{t\to 0^+}\frac{d}{dt}\rho_A(t) = 0,$$

this completes the proof. □

Theorem 5.3 ([9]). *A normed space A is rotund, strictly convex, or strictly normed if for any two distinct points p_1 and p_2 on the unit sphere S_A and for any value $t \in (0,1)$ the following condition holds*

$$\|tp_1 + (1-t)p_2\| \leq 1.$$

Theorem 5.4 ([9]). *Let A be a normed space. Then, A is rotund (or strictly convex) if and only if the following condition holds*

$$\left\|\frac{1}{2}(p_1+p_2)\right\| < 1,$$

for any two distinct points p_1 and p_2 in the unit sphere S_A.

Theorem 5.5 ([9], Proposition 5.1.21). *A normed space is rotund if and only if every two-dimensional subspace within it is also rotund.*

Proposition 5.6 ([15]). *A linear operator H from a Banach space A to another Banach space B is considered weakly compact if for any bounded sequence (a_n) in A, there exists a subsequence (a_{n_j}) such that (Ha_{n_j}) converges weakly.*

Theorem 5.7. *For $1 < p < \infty$, the space $\ell_p(\aleph(q))$ is rotund.*

Proof. Now it is sufficient to show that by Theorem 5.5, the space $\ell_p(\mathfrak{D})$ is spanned by $\{e, r\} = \mathfrak{D}$, where e and r are elements of the unit vector basis of ℓ_p. Therefore, we construct a two dimensional subspace

$$\mathfrak{D} = \left\{(a_0, a_1, 0, 0, \dots) : (a_0, a_1, 0, 0, \dots) \in \ell_p(\aleph(q))\right\}.$$

Let u and v be arbitrary elements of $S_{\mathfrak{D}}$, and $e + r = (a_0 + b_0, a_1 + b_1, 0, 0, \dots)$. Then

$$\begin{aligned}\left\|\frac{1}{2}(u+v)\right\|_{\ell_p(\aleph(q))} &= \left\|\frac{1}{2}\left(\frac{3\breve{\zeta}_1(q)(a_0+b_0)}{4\breve{\zeta}_1(q)+\breve{\zeta}_0(q)-7}, \frac{3\breve{\zeta}_1(q)(a_0+b_0)}{4\breve{\zeta}_2(q)+\breve{\zeta}_1(q)-4} + \frac{3\,q\,\breve{\zeta}_2(q)(a_1+b_1)}{4\breve{\zeta}_2(q)+\breve{\zeta}_1(q)-4}, \dots\right)\right\|_{\ell_p}^p \\ &= \frac{1}{2^p}\left[\left|\frac{3\breve{\zeta}_1(q)(a_0+b_0)}{4\breve{\zeta}_1(q)+\breve{\zeta}_0(q)-7}\right|^p + \left|\frac{3\breve{\zeta}_1(q)(a_0+b_0)}{4\breve{\zeta}_2(q)+\breve{\zeta}_1(q)-4} + \frac{3\,q\,\breve{\zeta}_2(q)(a_1+b_1)}{4\breve{\zeta}_2(q)+\breve{\zeta}_1(q)-4}\right|^p + \dots\right]\end{aligned}$$

Recall,

$$\|a\|_{\ell_p(\aleph(q))} = \left|\frac{3\breve{\zeta}_1(q)}{4\breve{\zeta}_2(q)+\breve{\zeta}_1(q)-7}a_0\right|^p + \left|\frac{3\breve{\zeta}_1(q)}{4\breve{\zeta}_1(q)+\breve{\zeta}_0(q)-4}a_0 + \frac{3q\breve{\zeta}_2(q)}{4\breve{\zeta}_2(q)+\breve{\zeta}_1(q)-4}a_1\right|^p + \dots = 1,$$

and

$$\|b\|_{\ell_p(\aleph(q))} = \left|\frac{3\breve{\zeta}_1(q)}{4\breve{\zeta}_2(q)+\breve{\zeta}_1(q)-7}b_0\right|^p + \left|\frac{3\breve{\zeta}_1(q)}{4\breve{\zeta}_1(q)+\breve{\zeta}_0(q)-4}b_0 + \frac{3q\breve{\zeta}_2(q)}{4\breve{\zeta}_2(q)+\breve{\zeta}_1(q)-4}b_1\right|^p + \dots = 1.$$

Take

$$f_0 = \frac{3\breve{\zeta}_1(q)}{4\breve{\zeta}_2(q)+\breve{\zeta}_1(q)-7}a_0,$$

$$d_0 = \frac{3\breve{\zeta}_1(q)}{4\breve{\zeta}_2(q)+\breve{\zeta}_1(q)-7}b_0.$$

Let us consider

$$x_1 = \frac{3\breve{\zeta}_1(q)}{4\breve{\zeta}_1(q)+\breve{\zeta}_0(q)-4}a_0 + \frac{3q\breve{\zeta}_2(q)}{4\breve{\zeta}_2(q)+\breve{\zeta}_1(q)-4}a_1 + \dots,$$

and

$$y_1 = \frac{3\breve{\zeta}_1(q)}{4\breve{\zeta}_1(q)+\breve{\zeta}_0(q)-4}b_0 + \frac{3q\breve{\zeta}_2(q)}{4\breve{\zeta}_2(q)+\breve{\zeta}_1(q)-4}b_1 + \dots.$$

Hence

$$\|a\|^p_{\ell_p(\aleph(q))} = |f_0|^p + |x_1|^p = 1, \quad \|b\|^p_{\ell_p(\aleph(q))} = |d_0|^p + |y_1|^p = 1.$$

By the rotundity of the two-dimensional Banach space ℓ_p^2, where (f_0, x_1) and (d_0, y_1) are elements of ℓ_p^2, we get

$$\left(\frac{f_0 + d_0}{2}\right)^p + \left(\frac{x_1 + y_1}{2}\right)^p < 1.$$

Again, recall that

$$\left(\frac{x_1+y_1}{2}\right)^p = \frac{1}{2^p}\left(\frac{c_0+d_0}{2}+\frac{a_1+b_1}{2}\right)^p.$$

Thus, we get

$$\left\|\frac{1}{2}(a+b)\right\|^p_{\ell_p(\aleph(q))} < 1.$$

This proves that the space $\ell_p(\aleph(q))$ is rotund. □

Theorem 5.8. *$\ell_\infty(\aleph(q))$ and $\ell_1(\aleph(q))$ are not rotund.*

Proof. Now, let us consider two special elements $a, b \in \ell_\infty(\aleph(q))$.

$$a = e_1 + e_2 = (1, 1, 0, 0, \dots)$$

and

$$b = e_1 - e_2 = (1, -1, 0, 0, \dots)$$

Verify that $a, b \in S_{\ell_\infty(\aleph(q))}$. In fact,

$$\begin{aligned}\|b\|_{\ell_\infty(\aleph(q))} &= \|(1,-1,0,0,\dots)\|_{\ell_\infty(\aleph(q))}\\ &= \left\|\left(\frac{3\breve{\zeta}_1(q)}{4\breve{\zeta}_1(q)+\breve{\zeta}_0(q)-7}, \frac{3\breve{\zeta}_1(q)-3q\breve{\zeta}_2(q)}{4\breve{\zeta}_2(q)+\breve{\zeta}_1(q)-4}, \frac{3\breve{\zeta}_1(q)-3q\breve{\zeta}_2(q)}{4\breve{\zeta}_3(q)+\breve{\zeta}_2(q)-1}, \dots\right)\right\|_{\ell_\infty}\\ &= 1\end{aligned}$$

and

$$\begin{aligned}\|a\|_{\ell_\infty(\aleph(q))} &= \|(1,1,0,0,\dots)\|_{\ell_\infty(\aleph(q))}\\ &= \left\|\left(\frac{3\breve{\zeta}_1(q)}{4\breve{\zeta}_1(q)+\breve{\zeta}_0(q)-7}, \frac{3\breve{\zeta}_1(q)+3q\breve{\zeta}_2(q)}{4\breve{\zeta}_2(q)+\breve{\zeta}_1(q)-4}, \frac{3\breve{\zeta}_1(q)+3q\breve{\zeta}_2(q)}{4\breve{\zeta}_3(q)+\breve{\zeta}_2(q)-1}, \dots\right)\right\|_{\ell_\infty}\\ &= 1\end{aligned}$$

Now

$$\begin{aligned}\left\|\frac{1}{2}(a+b)\right\|_{\ell_\infty(\aleph(q))} &= \left\|\frac{1}{2}(2,0,0,\dots)\right\|_{\ell_\infty(\aleph(q))}\\ &= \|(1,0,0,\dots)\|_{\ell_\infty(\aleph(q))}\\ &= \left\|\left(\frac{3\breve{\zeta}_1(q)}{4\breve{\zeta}_1(q)+\breve{\zeta}_0(q)-7}, \frac{3\breve{\zeta}_1(q)}{4\breve{\zeta}_1(q)+\breve{\zeta}_0(q)-4}, \frac{3\breve{\zeta}_1(q)}{4\breve{\zeta}_1(q)+\breve{\zeta}_0(q)-1}, \dots\right)\right\|_{\ell_\infty}\\ &= 1.\end{aligned}$$

In this case, $\ell_\infty(\aleph(q))$ is not rotund. Similarly, the claim for $\ell_1(\aleph(q))$ can be shown. □

Definition 5.9 ([2]). *Let $\mathfrak{K}$ be a sequence space. Then $\mathfrak{K}$ is called solid if*

$$\{(b_k) \in \omega : \exists (a_k) \in \mathfrak{K}, \forall k \in \mathbb{N} : |b_k| < |a_k|)\} \subset \mathfrak{K}.$$

Theorem 5.10. *$c_0(\aleph(q))$ is a solid space.*

Proof. The solidity of $c_0(\aleph(q))$ directly follows from Definition 5.9. □

Lemma 5.11 ([2]). *Let X be a linear subspace of ω. The space X is solid if and only if $\ell_\infty X \subseteq X$, where*

$$\ell_\infty X = \{(a_k b_k) : (a_k) \in \ell_\infty,\ (b_k) \in X\}$$

Theorem 5.12. *$c(\aleph(q))$ is not a solid space.*

Proof. Consider the sequence $v = \{(-1)^k\} \in \ell_\infty$ and $u = \{1, 1, 1, 1, 1, 1, ...\} \in c(\aleph(q))$. Clearly, $uv \notin c(\aleph(q))$. Thus, by Lemma 5.11, $c(\aleph(q))$ is not a solid space. □

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