Filomat 40:3 (2026), 995–1003
https://doi.org/10.2298/FIL2603995A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Perturbations not necessarily commutative

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Abstract. This paper treatises the preservation of some spectra under perturbations not necessarily commutative and generalizes several results which have been proved in the case of commuting operators.

1. Introduction and preliminaries

According to [6], for an element a of a ring $\mathcal{A}$, denote by

$$\text{comm}_l(a) = \{b \in \mathcal{A} : ab \in \text{comm}(a) \text{ and } ba \in \text{comm}(b)\},$$

$$\text{by comm}_r(a) = \{b \in \mathcal{A} : ab \in \text{comm}(b) \text{ and } ba \in \text{comm}(a)\},$$

$$\text{and by comm}_w(a) = \text{comm}_l(a) \cap \text{comm}_r(a),$$

where comm(a) is the set of all elements that commute with a. Denote also by $\text{comm}^2(a) = \text{comm}(\text{comm}(a))$ and by Nil$(\mathcal{A})$ the nilradical of $\mathcal{A}$. If $\mathcal{A}$ is a unital complex Banach algebra, we means by $\sigma(a)$, acc $\sigma(a)$, $r(a)$ and $\exp(a)$, the spectrum of a, the accumulation points of $\sigma(a)$, the spectral radius of a and the exponential of a, respectively. We say that a is quasi-nilpotent if $r(a) = 0$. If $T \in L(X)$ the algebra of all bounded linear operators acting on an infinite dimensional complex Banach space X, then T^*, $\alpha(T)$ and $\beta(T)$ means respectively, the dual of T, the dimension of the kernel $\mathcal{N}(T)$ and the codimension of the range $\mathcal{R}(T)$, and denote by $\mathcal{R}(T^\infty) = \bigcap_{n\geq 0}\mathcal{R}(T^n)$ and $\mathcal{N}(T^\infty) = \bigcup_{n\geq 0}\mathcal{N}(T^n)$. Moreover, the ascent and the descent of T are defined by $p(T) = \inf\{n \in \mathbb{N} : \mathcal{N}(T^n) = \mathcal{N}(T^{n+1})\}$ and $q(T) = \inf\{n \in \mathbb{N} : \mathcal{R}(T^n) = \mathcal{R}(T^{n+1})\}$ (with $\inf\emptyset = \infty$). We say that a subspace M of X is T-invariant if $T(M) \subset M$ and the restriction of T on M is denoted by T_M, and we say that $(M, N) \in \text{Red}(T)$ if M, N are closed T-invariant subspaces and $X = M \oplus N$. For $n \in \mathbb{N}$, denote by $T_{[n]} = T_{\mathcal{R}(T^n)}$ and by $m_T = \inf\{n \in \mathbb{N} : \min\{\alpha(T_{[n]}), \beta(T_{[n]})\} < \infty\}$ the *essential degree* of T. An operator T is called upper semi-B-Fredholm (resp., lower semi-B-Fredholm) if the *essential ascent* $p_e(T) := \inf\{n \in \mathbb{N} : \alpha(T_{[n]}) < \infty\} < \infty$ and $\mathcal{R}(T^{p_e(T)+1})$ is closed (resp., the *essential descent* $q_e(T) := \inf\{n \in \mathbb{N} : \beta(T_{[n]}) < \infty\} < \infty$ and $\mathcal{R}(T^{q_e(T)})$

2020 *Mathematics Subject Classification*. Primary: 13AXX; Secondary 47A10, 47A11, 45A53, 47A55.
Keywords. Perturbations, spectra.
Received: 10 December 2023; Accepted: 30 September 2025
Communicated by Dragan S. Djordjević
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is closed). If T is an upper or a lower (resp., upper and lower) semi-B-Fredholm, then T it is called *semi-B-Fredholm* (resp., *B-Fredholm*) and its index is defined by $\text{ind}(T) = \alpha(T_{[m_T]}) - \beta(T_{[m_T]})$. T is said to be an upper semi-B-Weyl (resp., lower semi-B-Weyl, B-Weyl, left Drazin invertible, right Drazin invertible, Drazin invertible) if T is an upper semi-B-Fredholm with $\text{ind}(T) \leq 0$ (resp., T is a lower semi-B-Fredholm with $\text{ind}(T) \geq 0$, T is a B-Fredholm with $\text{ind}(T) = 0$, T is an upper semi-B-Fredholm and $p(T_{[m_T]}) < \infty$, T is a lower semi-B-Fredholm and $q(T_{[m_T]}) < \infty$, $p(T_{[m_T]}) = q(T_{[m_T]}) < \infty$). If T is upper semi-B-Fredholm (resp., lower semi-B-Fredholm, semi-B-Fredholm, B-Fredholm, upper semi-B-Weyl, lower semi-B-Weyl, B-Weyl, left Drazin invertible, right Drazin invertible, Drazin invertible) with essential degree $m_T = 0$, then T is said to be an upper semi-Fredholm (resp., lower semi-Fredholm, semi-Fredholm, Fredholm, upper semi-Weyl, lower semi-Weyl, Weyl, upper semi-Browder, lower semi-Browder, Browder). T is said to be bounded below if T is upper semi-Fredholm with $\alpha(T) = 0$, and is said to be Riesz if $T - \lambda I$ is Fredholm for all non-zero complex λ or equivalently $\pi(T) := T + K(X)$ is quasi-nilpotent in the Calkin algebra $L(X)/K(X)$ ($K(X)$ is the ideal of compact operators). Following [21], T is said to be generalized Drazin-Riesz invertible if there exists $(M, N) \in \text{Red}(T)$ such that T_M is invertible and T_N is Riesz. T is said to be semi-regular (resp., essentially semi-regular) if $\mathcal{R}(T)$ is closed and $\mathcal{N}(T) \subseteq \mathcal{R}(T^\infty)$ (resp., $\mathcal{R}(T)$ is closed and there exists a finite-dimensional subspace F such that $\mathcal{N}(T) \subseteq \mathcal{R}(T^\infty) + F$).

In this paper, we study the stability of the spectra summarized in the next list under the algebraic conditions considered in [6] that are weaker than the commutativity.

$\sigma_a(T)$: approximative spectrum of T	$\sigma_{bw}(T)$: B-Weyl spectrum of T
$\sigma_e(T)$: essential spectrum of T	$\sigma_{ubw}(T)$: upper semi-B-Weyl spectrum of T
$\sigma_{uf}(T)$: upper semi-Fredholm spectrum of T	$\sigma_{lbw}(T)$: lower semi-B-Weyl spectrum of T
$\sigma_{lf}(T)$: lower semi-Fredholm spectrum of T	$\sigma_{ld}(T)$: left Drazin spectrum of T
$\sigma_w(T)$: Weyl spectrum of T	$\sigma_{rd}(T)$: right Drazin spectrum of T
$\sigma_{uw}(T)$: upper semi-Weyl spectrum of T	$\sigma_d(T)$: Drazin spectrum of T
$\sigma_{lw}(T)$: lower semi-Weyl spectrum of T	$\sigma_{se}(T)$: semi-regular spectrum of T
$\sigma_b(T)$: Browder spectrum of T	$\sigma_{gd}(T)$: generalized Drazin spectrum of T
$\sigma_{ub}(T)$: upper semi-Browder spectrum of T	$\sigma_{g_zd}(T)$: g_z-invertible spectrum of T [5]
$\sigma_{lb}(T)$: lower semi-Browder spectrum of T	$\sigma_{bf}(T)$: B-Fredholm spectrum of T
$\sigma_{ubf}(T)$: upper semi-B-Fredholm spectrum of T	$\sigma_{lbf}(T)$: lower semi-B-Fredholm spectrum of T

As an extension of [10, Proposition 2.6], we prove that if $a, b \in \mathcal{A}$ are Drazin invertible such that $a \in \text{comm}_w(b)$, then $a^D \in \text{comm}(b^D)$ and ab is Drazin invertible with $(ab)^D = a^D b^D$. Moreover, we prove that if $T \in L(X)$ is generalized Drazin-Riesz invertible and R is a Riesz operator such that $T \in [\text{comm}_l(R) \cap \text{comm}(RT)] \cup [\text{comm}_r(R) \cap \text{comm}(TR)]$, then $T+R$ is generalized Drazin-Riesz invertible and $\sigma_*(T) = \sigma_*(T+R)$, where $\sigma_* \in \{\sigma_e, \sigma_{uf}, \sigma_{lf}, \sigma_w, \sigma_{uw}, \sigma_{lw}, \sigma_b, \sigma_{ub}, \sigma_{lb}\}$. If in addition T is generalized Drazin invertible and R is quasi-nilpotent, then $T+R$ is generalized Drazin invertible and $\sigma(T) = \sigma(T+R)$. This gives a generalization of some known commutative perturbation results. Among other results, we give a new characterization of power finite rank operators, by proving that if $\sigma_* \in \{\sigma_{bf}, \sigma_{ubf}, \sigma_{lbf}, \sigma_d, \sigma_{ld}, \sigma_{rd}\}$, then F is a power finite rank operator if and only if $\sigma_*(T) = \sigma_*(T + F)$ for every generalized Drazin-Riesz invertible operator $T \in \text{comm}_w(F)$.

2. Pseudo invertible elements of a ring

Definition 2.1. *Let $\mathcal{A}$ be a ring and let $a \in \mathcal{A}$. We say that a is pseudo invertible if there exists $c \in comm^2(a)$ such that $c = c^2 a$. In this case we say that c is a pseudo inverse of a.*

According to [13], an element a of a unital complex Banach algebra $\mathcal{A}$, is said to be generalized Drazin invertible if there exists $b \in \text{comm}\,(a)$ such that $b^2 a = b$ and $a - a^2 b$ is quasi-nilpotent. If so then $b \in \text{comm}^2\,(a)$ and is denoted by a^D and called the generalized Drazin inverse of a. It is proved also that a is generalized Drazin invertible if and only if $0 \notin \text{acc}\,\sigma(a)$. So every generalized Drazin invertible element a is pseudo invertible and its Drazin inverse a^D is a pseudo inverse of a.

Proposition 2.2. *Let $\mathcal{A}$ be a ring and let $a, b \in \mathcal{A}$ such that a is pseudo invertible. Then for every pseudo inverse c of a, we have*
(i) If $ba \in comm(a)$, then $bc \in comm(c)$, and if in addition $a \in comm_r(b)$, then $c \in comm_r(b)$.
(ii) If $ab \in comm(a)$, then $cb \in comm(c)$, and if in addition $a \in comm_l(b)$, then $c \in comm_l(b)$.
(iii) If $a \in comm_w(b)$, then $c \in comm_w(b)$.

Proof. By hypotheses we get $c^2 = c^2ca = acc^2$. Thus
(i) If $ba \in \mathrm{comm}(a)$, then $ba \in \mathrm{comm}(c)$ and so $bc^2 = ((ba)c)c^2 = (c(ba))c^2 = cbc$. If in addition $a \in \mathrm{comm}_r(b)$, then $bcb = ((ba)c^2)b = c^2(bab) = cb^2$. So $c \in \mathrm{comm}_r(b)$.
(ii) If $ab \in \mathrm{comm}(a)$, then $ab \in \mathrm{comm}(c)$ and thus $c^2b = c^2(c(ab)) = c^2((ab)c) = cbc$. If in addition $a \in \mathrm{comm}_l(b)$, then $bcb = b(c^2(ab)) = (bab)c^2 = b^2ac^2 = b^2c$. So $c \in \mathrm{comm}_l(b)$.
(iii) Follows directly from the previous points. □

From Proposition 2.2, we immedialtely deduce the next corollary.

Corollary 2.3. *Let $\mathcal{A}$ be a unital complex Banach algebra and let $a, b \in \mathcal{A}$ such that a is generalized Drazin invertible. Then the following assertions hold:*

(i) If $ba \in comm(a)$, then $ba^D \in comm(a^D)$, and if in addition $a \in comm_r(b)$, then $a^D \in comm_r(b)$.
(ii) If $ab \in comm(a)$, then $a^Db \in comm(a^D)$, and if in addition $a \in comm_l(b)$, then $a^D \in comm_l(b)$.
(iii) If $a \in comm_w(b)$, then $a^D \in comm_w(b)$.

Lemma 2.4. *Let $\mathcal{A}$ be a ring and let $a, b \in \mathcal{A}$ pseudo invertible. If c and d are respectively pseudo inverses of a and b, we then have*
(i) If $a \in comm_r(b)$, then $c \in comm_r(b)$ and $cb, ad, cd \in comm(b)$.
(ii) If $a \in comm_l(b)$, then $c \in comm_l(b)$ and $cb, ad, cd \in comm(a)$.
(iii) If $a \in comm_w(b)$, then $c \in comm_w(b)$ and $cb, ad, cd \in comm(a) \cap comm(b)$.

Proof. (i) Assume that $a \in \mathrm{comm}_r(b)$. Since $ba \in \mathrm{comm}(a)$ then $cb^2 = c^2ab^2 = c^2bab = (c^2(ba))b = bcb$. Thus $cb \in \mathrm{comm}(b)$. On the other hand, we have $adb = abd = ab^2d^2 = babd^2 = bad$ and $bcd = ((ba)c^2)d = c^2(b(ad)) = c^2adb = cdb$. Then $ad, cd \in \mathrm{comm}(b)$. The point (ii) goes similarly with the first point and the third point is clear. □

Let $\mathcal{A}$ be a ring. An element $a \in \mathcal{A}$ is said to be Drazin invertible of degree n if there exists $c \in \mathrm{comm}(a)$ such that $c = c^2a$ and $a^{n+1}c = a^n$. In this case $c = a^D$, $c \in \mathrm{comm}^2(a)$ and if a is of degree n and is not of degree $n-1$, then n is called the index of a and is denoted by $i(a) = n$. For more details about this definition, we refer the reader to [1]. Our next proposition gives an extension of [10, Proposition 2.6].

Proposition 2.5. *Let a, b two Drazin invertible elements of a ring $\mathcal{A}$. The following assertions hold:*
(i) If $a \in comm_r(b)$ and $ab \in comm(a)$, then ab is Drazin invertible and $(ab)^D = b^Da^D$.
(ii) If $a \in comm_l(b)$, then ab is Drazin invertible and $(ab)^D = a^Db^D$.
(iii) If $a \in comm_w(b)$, then ab is Drazin invertible, $(ab)^D = a^Db^D$ and $a^D \in comm(b^D)$.

Proof. (i) Assume that $a \in \mathrm{comm}_r(b)$ and let $n = \max\{i(a), i(b)\}$. From [6, Lemma 3.1] we deduce that $b^Da^D(ab)^{n+1} = b^Da^Da^{n+1}b^{n+1} = b^Da^nb^{n+1} = (b^D(a^nb))b^n = a^nbb^Db^n = a^nb^n = (ab)^n$. On the other hand, since $ab \in \mathrm{comm}(a)$ then $((ab)b^D)a^D) = b^D((ab)a^D) = (b^Da^D)ab$. It follows from Lemma 2.4 that $b^Da^D \in \mathrm{comm}(a)$. Hence $(b^Da^D)(ab)(b^Da^D) = (a(b^Da^D))bb^Da^D = ab^D((a^Db^D)b)a^D = ab^Db(a^D(b^Da^D)) = ab^Dbb^D(a^D)^2 = ab^D(a^D)^2 = b^Da^D$. This implies that ab is Drazin invertible, $(ab)^D = b^Da^D$ and $i(ab) \leq \max\{i(a), i(b)\}$. Also note that (again from Lemma 2.4) that $(b^Da^D)ab = b(b^D)^2a^Dab = bb^D((b^Da^D)a)b = b(b^D(ab^D))a^Db = ba((b^D)^2(a^Db)) = baa^Db(b^D)^2 = ba(a^Db^D)$. The point (ii) is done in [20, Theorem 3.1] and the point (iii) is clear. □

Proposition 2.6. *Let $\mathcal{A}$ be a ring and let $a \in \mathcal{A}$. If there exists $b \in \mathcal{A}$ and $n \in \mathbb{N}$ such that $a \in comm(ab) \cap comm(ba)$, $bab = b$ and $ba^{n+1} = a^n$, then a is Drazin invertible and $a^D = b$.*

Proof. As $a \in \mathrm{comm}(ab) \cap \mathrm{comm}(ba)$ then by [6, Lemma 3.1, Remark 3.3], we deduce that $ba = (ba)^{n+2} = b^{n+2}a^{n+2} = a^{n+2}b^{n+2} = (ab)^{n+2} = ab$. Thus $b \in \mathrm{comm}(a)$. □

3. Perturbations of pseudo invertible operators

According to [18], the analytic core $\mathcal{K}(T)$ and the quasi-nilpotent part $\mathcal{H}_0(T)$ of $T \in L(X)$ are defined by

$$\mathcal{K}(T) = \{x \in X : \exists (x_n)_n \subset X \text{ such that } \forall n \in \mathbb{N}\ x = x_0, Tx_{n+1} = x_n \text{ and } \sup_n \|x_n\|^{\frac{1}{n}} < \infty\}$$

$$\text{and } \mathcal{H}_0(T) = \{x \in X : \limsup_{n\to\infty} \|T^n x\|^{\frac{1}{n}} = 0\}.$$

Lemma 3.1. *Let $S, T \in L(X)$. The following assertions hold:*
(i) If $ST \in comm(T)$, then $\mathcal{R}(T^\infty)$ and $\mathcal{K}(T)$ are S-invariant.
(ii) If $TS \in comm(T)$, then $\mathcal{N}(T^\infty)$ and $\mathcal{H}_0(T)$ are S-invariant.

Proof. (i) Let $n \geq 1$ and let $x \in S(\mathcal{R}(T^{n+1}))$. Then there exists z such that $x = ST^{n+1}z$, and since $TST = ST^2$ then $x = T^n STz \in \mathcal{R}(T^n)$. So $S(\mathcal{R}(T^{n+1})) \subset \mathcal{R}(T^n)$. Hence $\mathcal{R}(T^\infty)$ is S-invariant. If $x \in \mathcal{K}(T)$, then there exists $(x_n) \subset X$ such that $x = x_0$, $x_n = Tx_{n+1}$ for every $n \in \mathbb{N}$ and $\sup_n \|x_n\|^{\frac{1}{n}} < \infty$. We put $y_n = Sx_n$ for every $n \in \mathbb{N}$. We then have $y_n = Sx_n = TSTx_{n+2} = Ty_{n+1}$ and $\sup_n \|y_n\|^{\frac{1}{n}} \leq \max\{\|S\|, 1\} \sup_n \|x_n\|^{\frac{1}{n}} < \infty$. Therefore $Sx \in \mathcal{K}(T)$ and then $\mathcal{K}(T)$ is S-invariant.
(ii) Since $S(\mathcal{N}(T^n)) \subset \mathcal{N}(T^{n+1})$ for every $n > 0$ then $\mathcal{N}(T^\infty)$ is S-invariant. Let $x \in \mathcal{H}_0(T)$, then $\|T^{n+1}(Sx)\|^{\frac{1}{n+1}} = \|TST^n x\|^{\frac{1}{n+1}} \leq \|TS\|^{\frac{1}{n+1}} \|T^n x\|^{\frac{1}{n+1}} = \|TS\|^{\frac{1}{n+1}} (\|T^n x\|^{\frac{1}{n}})^{\frac{n}{n+1}}$ for every integer $n > 0$. Thus $Sx \in \mathcal{H}_0(T)$ and then $\mathcal{H}_0(T)$ is S-invariant. □

The next corollary is a consequence of the previous lemma and Proposition 2.2.

Corollary 3.2. *Let $T \in L(X)$ be a pseudo invertible operator and if L is a pseudo inverse of T, then the following assertions hold:*
(i) If $ST \in comm(T)$, then $\mathcal{R}(L^\infty)$ and $\mathcal{K}(L)$ are S-invariant.
(ii) If $TS \in comm(T)$, then $\mathcal{N}(L^\infty)$ and $\mathcal{H}_0(L)$ are S-invariant.

Theorem 3.3. *Let $T \in L(X)$ and $N \in Nil(L(X))$ such that $T \in [comm_l(N) \cap comm(NT)] \cup [comm_r(N) \cap comm(TN)]$. If T is Drazin invertible, then $T + N$ is Drazin invertible. The converse is true if $T \in comm_w(N)$.*

Proof. Suppose that T is Drazin invertible, that is, $p := p(T) = q(T) < \infty$. Then $(A, B) := (\mathcal{R}(T^p), \mathcal{N}(T^p)) \in Red(T)$, T_A is invertible and T_B is nilpotent. From Lemma 3.1 we deduce that $(A, B) \in Red(N)$, and so $T = T_A \oplus T_B$ and $N = N_A \oplus N_B$. Therefore $T + N = (T + N)_A \oplus (T + N)_B$. By hypotheses and the fact that T_A is invertible and N_A is nilpotent, we conclude that $N_A \in \text{comm}(T_A)$. Thus $(T + N)_A$ is invertible and by [6, Lemma 3.8], it follows that $T+N$ is Drazin invertible. If in addition $T \in \text{comm}_w(N)$, then $T+N \in \text{comm}_w(-N)$, and thus T is Drazin invertible iff $T + N$ is Drazin invertible. □

For $T \in L(X)$, denote by $\text{IRed}(T) = \{(M, N) \in \text{Red}(T) : T_M \text{ is invertible and } (M, N) \in \text{Red}(U) \text{ for all } U \in \text{comm}(T)\}$. If T is pseudo invertible, denote by $\text{PI}(T)$ the set of its pseudo inverses. Note that the class of pseudo invertible operators is much broader, it contains in particular the class of g_z-invertible operators, see [5, Remark 4.19].

Proposition 3.4. *$T \in L(X)$ is pseudo invertible if and only if there exists $(M, N) \in Red(U)$ such that T_M is invertible for every $U \in comm(T)$. If this is the case, the map $\Phi : IRed(T) \longrightarrow PI(T)$ defined by $\Phi(M, N) = (T_M)^{-1} \oplus 0_N$ is onto.*

Proof. Assume that T is pseudo invertible and let $S \in \text{PI}(T)$. Then TS is a projection and $(M, N) := (\mathcal{R}(TS), \mathcal{N}(TS)) \in \text{Red}(T)$. Let $U \in \text{comm}(T)$, then $U(M) = \mathcal{R}(UTS) = \mathcal{R}(TSU) \subset M$ and $U(N) \subset N$. Thus $(M, N) \in \text{Red}(U)$. Moreover, if $x \in \mathcal{N}(T_M)$, then $x = TSy$ and $Tx = 0$. Therefore $x = (TS)^2 y = STx = 0$. If $x = TSy \in M$, then $x = T(TS(Sy)) \in T(M)$. Thus T_M is invertible. Let us show that $S = (T_M)^{-1} \oplus 0_N$. We have $S_N = 0_N$, since $S = STS$. Let $x = TSy \in M$. As $Sy = STSy \in M$ then $Sx = Sy = (T_M)^{-1} T_M S y = (T_M)^{-1} x$. Hence $S = (T_M)^{-1} \oplus 0_N$. Conversely, if $(M, N) \in \text{IRed}(T)$, then the operator $S = (T_M)^{-1} \oplus 0_N$ gives the desired result. Indeed, it is clear that $S = S^2 T$, and if $A \in \text{comm}(T)$, then $(M, N) \in \text{Red}(A)$. So $A_M \in \text{comm}((T_M)^{-1})$. Therefore $A \in \text{comm}(S)$ and consequently $S \in \text{comm}^2(T)$. □

It follows from Proposition 3.4 that every pseudo inverse S of a pseudo invertible operator T is Drazin invertible with $p(S) = q(S) \leq 1$. Hence $\mathrm{IRed}(T) \subset \{(\mathcal{R}(S), \mathcal{N}(S)) : S \in \mathrm{PI}(T)\}$. Moreover, if T is generalized Drazin invertible, then $(\mathcal{K}(T), \mathcal{H}_0(T)) \in \mathrm{IRed}(T)$.

Corollary 3.5. *Let $T \in L(X)$. Then $IRed(T) = \{(M, N) \in Red(T) : T_M$ is invertible and $(M, N) \in Red(U)$ for all $U \in L(X)$ such that $T \in comm(TU) \cap comm(UT)\}$.*

Proof. Let $C_T = \{(M, N) \in \mathrm{Red}(T) : T_M$ is invertible and $(M, N) \in \mathrm{Red}(U)$ for all $U \in L(X)$ such that $T \in \mathrm{comm}(TU) \cap \mathrm{comm}(UT)\}$. It is clear that $C_T \subset \mathrm{IRed}(T)$. For the opposite inclusion, assume that T is pseudo invertible (in the other case $\mathrm{IRed}(T) = \emptyset$). Let $(M, N) \in \mathrm{IRed}(T)$ and let $U \in L(X)$ such that $T \in \mathrm{comm}(TU) \cap \mathrm{comm}(UT)$ and we consider the pseudo inverse operator $S = \Phi(M, N)$ of T associated to (M, N). It is clear that S is Drazin invertible, and hence $(M, N) = (\mathcal{R}(S), \mathcal{N}(S)) = (\mathcal{R}(S^\infty), \mathcal{N}(S^\infty))$. As $T \in \mathrm{comm}(TU) \cap \mathrm{comm}(UT)$, $S \in \mathrm{comm}^2(T)$ and $S = S^2T$, it follows from Corollary 3.2 that $(M, N) \in \mathrm{Red}(U)$. □

Proposition 3.6. *Let $S, T \in L(X)$. The following assertions hold:*
(i) If T or S is Riesz and $TS \in comm(T) \cup comm(S)$, then TS is Riesz.
(ii) If T and S are Riesz and $S \in comm_r(T) \cup comm_l(T)$, then $T + S$ is Riesz.

Proof. As mentioned above, $R \in L(X)$ is Riesz if and only if $\pi(R)$ is quasi-nilpotent in the Calkin algebra $L(X)/K(X)$. The proof is then a consequence of [6, Corollary 3.10]. □

Our next theorem generalizes some known commutative perturbation results.

Theorem 3.7. *Let $R, T \in L(X)$ such that R is Riesz and $T \in [comm_l(R) \cap comm(RT)] \cup [comm_r(R) \cap comm(TR)]$. If T is generalized Drazin-Riesz invertible, then $T + R$ is generalized Drazin-Riesz invertible and*
(i) $\sigma_*(T) = \sigma_*(T + R)$, where $\sigma_* \in \{\sigma_e, \sigma_{uf}, \sigma_{lf}, \sigma_w, \sigma_{uw}, \sigma_{lw}, \sigma_b, \sigma_{ub}, \sigma_{lb}\}$.
(ii) If $R \in Nil(L(X))$, then $\sigma_+(T) \setminus \{0\} = \sigma_+(T+R) \setminus \{0\}$ and $\sigma_{++}(T) = \sigma_{++}(T+R)$, where $\sigma_+ \in \{\sigma_{bf}, \sigma_{ubf}, \sigma_{lbf}, \sigma_{bw}, \sigma_{ubw}, \sigma_{lbw}, \sigma_d, \sigma_{ld}, \sigma_{rd}\}$ and $\sigma_{++} \in \{\sigma_{bf}, \sigma_{bw}, \sigma_d\}$. If in addition X is a Hilbert space, then $\sigma_+(T) = \sigma_+(T + R)$.
(iii) If R is quasi-nilpotent, then $acc\, \sigma_-(T) \setminus \{0\} = acc\, \sigma_-(T + R) \setminus \{0\}$, where $\sigma_- \in \{\sigma, \sigma_a, \sigma_s\}$. If in addition T is generalized Drazin invertible, then $T + R$ is generalized Drazin invertible, $\sigma(T) = \sigma(T + R)$, $\sigma_\times(T) \setminus \{0\} = \sigma_\times(T + R) \setminus \{0\}$ and $acc\, \sigma_\times(T) = acc\, \sigma_\times(T + R)$, where $\sigma_\times \in \{\sigma_a, \sigma_s\}$.

Proof. Assume that T is generalized Drazin-Riesz invertible and let $(M, N) \in \mathrm{IRed}(T)$ such that T_N is Riesz.
(i) From Corollary 3.5 we have $(M, N) \in \mathrm{Red}(R)$. Thus $T_M R_M = R_M T_M$ and $T_N \in \mathrm{comm}_l(R_N) \cap \mathrm{comm}(T_N R_N)$. Hence $(T + R)_M$ is Browder, and by Proposition 3.6, $(T + R)_N$ is Riesz. From [21, Theorem 2.3], we deduce that $T + R$ is generalized Drazin-Riesz invertible. On the other hand, as $\sigma_*(T_N) \subset \sigma_b(T_N) \subset \{0\}$ then $\sigma_*(T) \setminus \{0\} = (\sigma_*(T_M) \cup \sigma_*(T_N)) \setminus \{0\} = \sigma_*(T_M) \setminus \{0\} = \sigma_*((T + R)_M) \setminus \{0\} = \sigma_*(T + R) \setminus \{0\}$. If $0 \notin \sigma_*(T)$, then T is semi-Fredholm. From [4, Corollary 3.7], there exists $(M', N') \in \mathrm{Red}(T)$ such that $T_{M'}$ is semi-regular, $T_{N'}$ is nilpotent and $\dim N' < \infty$. This entails by [5, Proposition 2.10] that $T_{M'}$ is invertible. Hence T is Browder, $(M', N') = (\mathcal{R}(T^\infty), \mathcal{N}(T^\infty))$ and $0 \notin \sigma_*(T + R)$. Thus $\sigma_*(T) = \sigma_*(T + R)$. Assume now that $0 \notin \sigma_*(T + R)$. As $T + R$ is generalized Drazin-Riesz invertible then $T + R$ is Browder. Since $T + R \in \mathrm{comm}_l(-R) \cup \mathrm{comm}_r(-R)$, by Proposition [6, Proposition 4.18], we deduce that $0 \notin \sigma_e(T)$, and hence $\sigma_*(T) = \sigma_*(T+R)$. If $0 \notin \sigma_*(T) \cup \sigma_*(T+R)$, then $\sigma_*(T) = \sigma_*(T) \setminus \{0\} = \sigma_*(T + R) \setminus \{0\} = \sigma_*(T + R)$.
(ii) Assume that $R \in \mathrm{Nil}(L(X))$. We have $\sigma_+(T) \setminus \{0\} = (\sigma_+(T_M) \cup \sigma_+(T_N)) \setminus \{0\} = \sigma_+(T_M) \setminus \{0\} = \sigma_+((T+R)_M) \setminus \{0\} = \sigma_+(T + R) \setminus \{0\}$, since $\sigma_+(T_N) \subset \sigma_b(T_N) \subset \{0\}$ and σ_+ is stable under commuting nilpotent perturbations. If $0 \notin \sigma_{++}(T)$, then T is B-Fredholm, which implies from [3, Theorem 2.21] and [5, Proposition 2.10] that T is Drazin invertible. We conclude from Theorem 3.3 that $\sigma_{++}(T) = \sigma_{++}(T+R)$. Assume now that $0 \notin \sigma_{++}(T+R)$, then $T+R$ is B-Fredholm, and thus $T+R$ is Drazin invertible. Hence $\sigma((T+R)_N)$ is a finite set. We have from Proposition [6, Proposition 4.14], $\sigma(T_N) \setminus \{0\} \subset \sigma((T + R)_N) \setminus \{0\}$. Thus $0 \notin \mathrm{acc}\, \sigma(T)$, and then T is generalized Drazin invertible. So $(M', N') := (\mathcal{K}(T), \mathcal{H}_0(T)) \in \mathrm{IRed}(T)$ and $T_{N'}$ is quasi-nilpotent. As $T + R$ is Drazin invertible then $(T + R)_{N'}$ is nilpotent. Since $T + R \in \mathrm{comm}_l(-R) \cup \mathrm{comm}_r(-R)$, by [6, Lemma 3.8], we deduce that $T_{N'}$ is nilpotent. Thus T is Drazin invertible and $\sigma_{++}(T) = \sigma_{++}(T + R)$. If in addition X is a Hilbert space, using [9, Theorem 2.6] and the same argument as above, we deduce that $\sigma_+(T) = \sigma_+(T + R)$. The point (iii) goes similarly and is left to the reader. □

Let $T \in L(X)$ and let $Q \in L(X)$ be a quasi-nilpotent operator which commutes with T. It is well known that $\sigma(T) = \sigma(T + Q)$. The proof of Theorem 3.7 suggests the following question.
Question: This equality $\sigma(T) = \sigma(T + Q)$ remains true if we only have $Q \in \text{comm}_l(T)$ and $T \in \text{comm}(TQ)$ [or $Q \in \text{comm}_r(T)$ and $T \in \text{comm}(QT)$]? Note that Theorem 3.7 gives a partial answer to this question.

We give in the next result an extension of [1, Lemma 3.81]. Recall that $T \in L(X)$ is said to be algebraic if there exists a non-null polynomial P such that $P(T) = 0$.

Corollary 3.8. *If $T \in L(X)$ is algebraic and $N \in Nil(L(X))$ such that $T \in [comm_l(N) \cap comm(NT)] \cup [comm_r(N) \cap comm(TN)]$, then $T + N$ is algebraic.*

Proof. The proof follows from Theorem 3.7 and the fact that T is algebraic if and only if it has empty Drazin spectrum. □

Theorem 3.9. *Let $R \in L(X)$ and $T \in comm(TR) \cap comm(RT)$. The following assertions hold:*
(i) If T is essentially semi-regular and R is Riesz, then $T + R$ is essentially semi-regular.
(ii) If T is semi-regular and $R = Q$ is quasi-nilpotent, then $T + Q$ is semi-regular.

Proof. Let $M = \mathcal{R}(T^\infty)$.
(i) Assume that T is essentially semi-regular. From [16, Proposition 13] and Lemma 3.1, we conclude that M is closed R-invariant. Consider the operators $\overline{T}, \overline{R} \in L(X/M)$ induced by T and R, respectively. As R is Riesz then [16, Lemma 15] implies that R_M and $\overline{R}$ are Riesz, and since T_M is onto, it follows from [6, Corollary 4.2] that $T_M \in \text{comm}(R_M)$. Hence $(T + R)_M$ is lower semi-Browder. Since $\overline{T}$ is upper semi-Browder, from [6, Proposition 4.18] we conclude that $\overline{T + R} = \overline{T} + \overline{R}$ is upper semi-Fredholm. We deduce then from [16, Theorem 14] that $T + R$ is essentially semi-regular.
(ii) If T is semi-regular then M is closed Q-invariant and T_M is onto. Consider the operators $\overline{T}, \overline{Q} \in L(X/M)$ induced by T and Q, respectively. As Q is quasi-nilpotent then Q_M and $\overline{Q}$ are quasi-nilpotent. Moreover, by Lemma [14, Lemma 1] we have T_M is onto and $\overline{T}$ is bounded below. As $T \in \text{comm}(TQ) \cap \text{comm}(QT)$ then $\overline{Q} \in \text{comm}(\overline{T})$ and $Q_M \in \text{comm}(T_M)$. Hence $(T + Q)_M$ is onto and $\overline{T} + \overline{Q}$ is bounded below. Again by [14, Lemma 1], we deduce that $T + Q$ is semi-regular. □

4. Perturbations by finite rank operators

We begain this part by the next lemma which gives an extension of [12, Lemma 2.1] proved in the case of commuting operators.

Lemma 4.1. *Let $S, T \in L(X)$ such that $S \in comm_w(T)$. Then for every integers $m \geq 1$ and $n \geq 3$, we have*

(i) $$max\left\{dim\frac{\mathcal{N}(T^n)}{\mathcal{N}[(T+S)^{n+m-1}] \cap \mathcal{N}(T^n)}, dim\frac{\mathcal{N}[(T+S)^n]}{\mathcal{N}(T^{n+m-1}) \cap \mathcal{N}[(T+S)^n]}\right\} \leq dim\,\mathcal{R}(S^m).$$

(ii) $$max\left\{dim\frac{\mathcal{R}(T^{n+m-1})}{\mathcal{R}[(T+S)^n] \cap \mathcal{R}(T^{n+m-1})}, dim\frac{\mathcal{R}[(T+S)^{n+m-1}]}{\mathcal{R}(T^n) \cap \mathcal{R}[(T+S)^{n+m-1}]}\right\} \leq dim\,\mathcal{R}(S^m).$$

Proof. (i) Since $S \in \text{comm}_w(T)$, from [6, Corollary 3.6], we have for every $x \in \mathcal{N}(T^n)$ $(T+S)^{n+m-1}x = S^m Ax$, where $A = \sum_{i=0}^{n-1} C_{n+m-1}^{i+m} S^i T^{n-i-1}$. Thus $(T+S)^{n+m-1}(\mathcal{N}(T^n)) \subset \mathcal{R}(S^m)$. Let M be a subspace such that $\mathcal{N}(T^n) = \left(\mathcal{N}[(T+S)^{n+m-1}] \cap \mathcal{N}(T^n)\right) \oplus M$. As $T^n(T+S)^{n+m-1} = (T+S)^{n+m-1}T^n$ then $(T+S)^{n+m-1}(\mathcal{N}(T^n)) \subset \mathcal{N}(T^n)$. And since $M \cap \mathcal{N}[(T+S)^{n+m-1}] = \{0\}$ it then follows that $\dim M \leq \dim \mathcal{R}(S^m)$. Since $S \in \text{comm}_w(T)$ if and only if $(-S) \in \text{comm}_w(T+S)$, the proof is complete.

(ii) Let M be a subspace such that $\mathcal{R}[(T+S)^{n+m-1}] = \Big(\mathcal{R}(T^n)\cap\mathcal{R}[(T+S)^{n+m-1}]\Big)\oplus M$ and let $(e_i := (T+S)^{n+m-1}v_i)_{i=1,\dots,k}$ be a linearly independent family of M. From [6, Corollary 3.6], we deduce that

$$\begin{aligned}(T+S)^{n+m-1} &= \sum_{i=0}^{n+m-1} C^i_{n+m-1}T^{n+m-1-i}S^i\\ &= \sum_{i=0}^{m-1} C^i_{n+m-1}T^{n+m-1-i}S^i + \sum_{i=m}^{n+m-1} C^i_{n+m-1}T^{n+m-1-i}S^i\\ &= T^nA + S^mB,\end{aligned}$$

where $A = \sum_{i=0}^{m-1} C^i_{n+m-1}T^{m-1-i}S^i$ and $B = \sum_{i=m}^{n+m-1} C^i_{n+m-1}T^{n+m-1-i}S^{i-m}$. If $k > \dim\mathcal{R}(S^m)$, then there exist $\lambda_1,\dots,\lambda_k$ not all zero such that $\sum_{i=1}^{k}\lambda_iS^mBv_i = 0$. Hence $\sum_{i=1}^{k}\lambda_i(T+S)^{n+m-1}v_i = \sum_{i=1}^{k}\lambda_iT^nAv_i$ and thus $\sum_{i=1}^{k}\lambda_ie_i \in \mathcal{R}(T^n)\cap M = \{0\}$. But this is a contradiction. Therefore $k \le \dim\mathcal{R}(S^m)$ and then $\dim M \le \dim\mathcal{R}(S^m)$. The proof is complete. □

Denote by $\mathcal{F}_0(X)$ the class of power finite rank operators acting on X. The next theorem extends [12, Theorem 2.2], [8, Proposition 3.1] and a special case of the direct implication of [7, Theorem 3.1].

Theorem 4.2. *Let $T \in L(X)$ and $F \in \mathcal{F}_0(X)$ such that $F \in comm_w(T)$. The following equivalences hold:*
(i) $q(T) < \infty$ if and only if $q(T+F) < \infty$.
(ii) $p(T) < \infty$ if and only if $p(T+F) < \infty$.
(iii) $q_e(T) < \infty$ if and only if $q_e(T+F) < \infty$.
(iv) $p_e(T) < \infty$ if and only if $p_e(T+F) < \infty$.
(v) $m_T < \infty$ if and only if $m_{T+F} < \infty$.

Proof. Let $m \ge 1$ be an integer such that $\dim\mathcal{R}(F^m) < \infty$.
(i) Assume that $q := q(T) < \infty$ and let $n \ge \max\{3, q\}$. Then

$$c_n := \dim\frac{\mathcal{R}(T^{n+m-1})}{\mathcal{R}[(T+F)^n]\cap\mathcal{R}(T^{n+m-1})} = \dim\frac{\mathcal{R}(T^q)}{\mathcal{R}[(T+F)^n]\cap\mathcal{R}(T^q)},$$

$$c_n^{'} := \dim\frac{\mathcal{R}[(T+F)^{n+m-1}]}{\mathcal{R}(T^n)\cap\mathcal{R}[(T+F)^{n+m-1}]} = \dim\frac{\mathcal{R}[(T+F)^{n+m-1}]}{\mathcal{R}(T^q)\cap\mathcal{R}[(T+F)^{n+m-1}]}.$$

From Lemma 4.1 we have $\max\{c_n, c_n^{'}\} < \infty$ for all $n \ge \max\{3, q\}$. As $(c_n)_n$ is increasing then there exists an integer $k \ge \max\{3, q\}$ such that $\mathcal{R}[(T+F)^n]\cap\mathcal{R}(T^q) = \mathcal{R}[(T+F)^k]\cap\mathcal{R}(T^q)$, for every $n \ge k$. Thus $c_n^{'} = \dim\frac{\mathcal{R}[(T+F)^{n+m-1}]}{\mathcal{R}(T^q)\cap\mathcal{R}[(T+F)^k]}$ for every $n \ge k$. Therefore $(c_n^{'})_{n\ge k}$ is a decreasing sequence. So there exists $r \ge k$ such that for every $n \ge r$ we have $c_n^{'} = c_r^{'}$. Hence $q(T+F) \le r+m-1$ and the converse is obvious. The point (ii) goes similarly.
(iii) Assume that $e := q_e(T) < \infty$. By Lemma 4.1 we obtain $\dim\frac{\mathcal{R}(T^e)}{\mathcal{R}[(T+F)^n]\cap\mathcal{R}(T^{n+m-1})} = \dim\frac{\mathcal{R}(T^e)}{\mathcal{R}(T^{n+m-1})} + \dim\frac{\mathcal{R}(T^{n+m-1})}{\mathcal{R}[(T+F)^n]\cap\mathcal{R}(T^{n+m-1})} < \infty$ for every $n \ge l = \max\{3, e\}$. Thus $\dim\frac{\mathcal{R}(T^e)}{\mathcal{R}[(T+F)^n]\cap\mathcal{R}(T^e)} < \infty$ for every $n \ge l$. On the other hand, from the proof of Lemma 4.1, we have $\mathcal{R}[(T+F)^{n+m-1}] \subset \mathcal{R}(T^e) + \mathcal{R}(F^m)$ for every $n \ge l$. Hence

$$\dim\frac{\mathcal{R}(T^e)+\mathcal{R}(F^m)}{\mathcal{R}[(T+F)^{n+m-1}]} = \dim\frac{\mathcal{R}(T^e)+\mathcal{R}(F^m)}{\mathcal{R}[(T+F)^{n+m-1}]\cap\mathcal{R}(T^e)} - \dim\frac{\mathcal{R}[(T+F)^{n+m-1}]}{\mathcal{R}[(T+F)^{n+m-1}]\cap\mathcal{R}(T^e)} < \infty,$$

and consequently $\dim \dfrac{\mathcal{R}[(T+F)^{n+m-1}]}{\mathcal{R}[(T+F)^{n+m}]} < \infty$ for every $n \geq l$. Therefore $q_e(T+F) \leq \max\{m+2, q_e(T)+m-1\} < \infty$ and the converse is obvious. The point (iv) goes similarly. For the proof of the point (v), the reader is referred to [4] in which we mentioned that $m_T = \min\{p_e(T), q_e(T)\}$. □

The following theorem extends [19, Lemma 2.1, Lemma 2.2].

Theorem 4.3. *Let $T \in L(X)$ and $F \in \mathcal{F}_0(X) \cap comm_w(T)$. Then T is upper semi-B-Fredholm (resp., lower semi-B-Fredholm, B-Fredholm, left Drazin invertible, right Drazin invertible, Drazin invertible) if and only if $T+F$ is.*

Proof. Suppose that T is upper semi-B-Fredholm. Theorem 4.2 implies that $\mathcal{R}(T^{p_e(T)+1})$ is closed and $p_e(T+F) < \infty$. Since for every $n \geq 2$, $FT^n = T^nF$ then $F(\mathcal{N}(T^n)) \subset \mathcal{N}(T^n)$. Consider $\tilde{T}$ and $\tilde{F}$ the operators induced by T and F on $\tilde{X} = X/\mathcal{N}(T^{d+3})$, where $d = \mathrm{dis}(T)$. It is easily seen that $\tilde{T}$ is upper semi-Fredholm and $\tilde{F} \in \mathcal{F}_0(\tilde{X})$. From [6, Propositon 4.18], we deduce that $\tilde{T}+\tilde{F}$ is upper semi-Fredholm. Hence $\mathcal{R}[(T+F)^l] + \mathcal{N}(T^{d+3})$ is closed for every $l \in \mathbb{N}$. Furthermore, Lemma 4.1 implies that $\dim \dfrac{\mathcal{N}(T^n)}{\mathcal{N}[(T+F)^{n+m-1}] \cap \mathcal{N}(T^n)} < \infty$, where $n \geq 3$ and $m \geq 1$ such that $\dim \mathcal{R}(F^m) < \infty$. Hence

$$\dim \frac{\mathcal{R}[(T+F)^{n+m-1}] \cap \mathcal{N}(T^n)}{\mathcal{R}[(T+F)^{n+m-1}] \cap \mathcal{N}[(T+F)^{n+m-1}] \cap \mathcal{N}(T^n)} < \infty.$$

As $\alpha((T+F)^{n+m-1}_{[n+m-1]}) < \infty$ then $\dim\left(\mathcal{R}[(T+F)^{n+m-1}] \cap \mathcal{N}(T^n)\right) < \infty$ for every integer $n \geq \max\{3, p_e(T+F)\}$. From the Neubauer Lemma [17, Proposition 2.1.1], we conclude that $\mathcal{R}[(T+F)^{n+m-1}]$ is closed. Hence $T+F$ is upper semi-B-Fredholm. If T is lower semi-B-Fredholm, then from [15], T^* is upper semi-B-Fredholm, and consequently T^*+F^* is upper semi-B-Fredholm. Thus $T+F$ is lower semi-B-Fredholm (see again [15]). If T is left Drazin invertible, then T is upper semi-B-Fredholm and $p(T) < \infty$. So $T+F$ is upper semi-B-Fredholm, and from Theorem 4.2 we have $p(T+F) < \infty$. Thus $T+F$ is left Drazin invertible. The other cases go similarly. Since $F \in \mathrm{comm}_w(T)$ if and only if $(-F) \in \mathrm{comm}_w(T+F)$, the proof is complete. □

Corollary 4.4. *If $T \in L(X)$ is generalized Drazin-Riesz invertible and $F \in \mathcal{F}_0(X)$ such that $T \in [comm_l(F) \cap comm(TF)] \cup [comm_r(F) \cap comm(FT)]$, then $\sigma_*(T)\setminus\{0\} = \sigma_*(T+F)\setminus\{0\}$, where $\sigma_* \in \{\sigma_{bf}, \sigma_{ubf}, \sigma_{lbf}, \sigma_d, \sigma_{ld}, \sigma_{rd}, \sigma_{gd}, \sigma_{g_zd}\}$. If in addition $F \in comm_w(T)$, then $\sigma_*(T) = \sigma_*(T+F)$.*

Proof. We will leave these routine arguments as exercise for the reader. □

Remark 4.5. *In [11, Proposition 3.3], the authors proved that if X is an infinite dimensional complex Banach space and $T \in L(X)$, then there exists a non-algebraic operator $S \in comm(T)$. From the proof of this result and the one of [1, Lemma 3.83], it is easy to see that if in addition T is an algebraic operator, then we can consider S as a compact operator.*

The next proposition gives a new characterization of power finite rank operators.

Proposition 4.6. *Let $F \in L(X)$ and $\sigma_* \in \{\sigma_{bf}, \sigma_{ubf}, \sigma_{lbf}, \sigma_d, \sigma_{ld}, \sigma_{rd}\}$. The following statements are equivalent:*
(i) $F \in \mathcal{F}_0(X)$;
(ii) $\sigma_*(T) = \sigma_*(T+F)$ for every generalized Drazin-Riesz invertible operator $T \in comm_w(F)$;
(iii) $\sigma_*(T) = \sigma_*(T+F)$ for every $T \in comm(F)$.

Proof. (i) $\Longrightarrow$ (ii) Is a consequence of Corollary 4.4.
(ii) $\Longrightarrow$ (i) We have $\sigma_*(F) = \emptyset$. So F is algebraic and $\sigma(F) = \{\lambda_1, \ldots, \lambda_n\}$. Thus $X = X_1 \oplus \cdots \oplus X_n$, where $X_i = \mathcal{N}((F-\lambda_i I)^{m_i})$ for some m_i. If $F \notin \mathcal{F}_0(X)$, then there exists $1 \leq i \leq n$ such that $\lambda_i \neq 0$ and $\dim X_i = \infty$. As $F_i - \lambda_i I$ is nilpotent, from Remark 4.5, there exists a non-algebraic compact operator $S_i \in \mathrm{comm}(F_i)$, where $F_i = F_{X_i}$. The operator $S = 0_1 \oplus \cdots \oplus S_i \oplus \cdots \oplus 0_n$, where $0_j = 0_{X_j}$, is non-algebraic, compact and commutes with F. By hypothesis we have $\sigma_*(S) = \sigma_*(S+F)$, this entails, from [19, Corollary 2.10] that $\sigma_*(S) = \sigma_*(S_i) = \sigma_*(S_i + F_i) = \sigma_*(S_i + \lambda_i I) = \sigma_*(S+F)$, since $F_i - \lambda_i I$ is nilpotent. Hence $\lambda_i = 0$ and this is a contradiction. Thus $F \in \mathcal{F}_0(X)$.
(i) $\Longrightarrow$ (iii) Is a consequence of Theorem 4.3, and (iii) $\Longrightarrow$ (i) is proved in [19, Theorem 2.11]. □

The next proposition extends the second assertion of [10, Corollary 3.5]. $\mathcal{F}(X)$ denotes the ideal of finite rank operator in $L(X)$.

Proposition 4.7. *Let $S, T \in L(X)$ be B-Fredholm operators, if $S \in comm_r(T) \cup comm_l(T)$, then TS is B-Fredholm.*

Proof. Assume that $S \in \text{comm}_r(T)$, then $S^* \in \text{comm}_l(T^*)$. As T and S are B-Fredholm then T^* and S^* are B-Fredholm. From [10, Theorem 3.4], $\pi_f(S^*) := S^* + \mathcal{F}(X)$ and $\pi_f(T^*) := T^* + \mathcal{F}(X)$ are Drazin invertible. Since $\pi_f(S^*) \in \text{comm}_l(\pi_f(T^*))$, from Proposition 2.5 we get $\pi_f(S^*T^*) := S^*T^* + \mathcal{F}(X)$ is Drazin invertible. We conclude again by [10, Theorem 3.4] that TS is a B-Fredholm operator. The case $S \in \text{comm}_l(T)$ is analogous. □

As a continuation to what has been done in the paper [5], we end this part by the following theorem which improves [2, Theorem 2.1].

Theorem 4.8. *Let $T \in L(X)$. Then*

$$\sigma_d(T) = \sigma_{bf}(T) \cup acc(acc\,\sigma(T)),$$

$$\sigma_b(T) = \sigma_e(T) \cup acc(acc\,\sigma(T)).$$

Proof. Let us prove that $\sigma_d(T) = \sigma_{bf}(T) \cup \text{acc}(\text{acc}\,\sigma(T))$. Let $\lambda \notin \sigma_{bf}(T) \cup \text{acc}(\text{acc}\,\sigma(T))$ and without loss of generality we can assume that $\lambda = 0$. Then T is B-Fredholm and $0 \notin \text{acc}\,(\text{acc}\,\sigma(T))$. This entails from [3, Theorem 2.21] and [5, Theorem 4.11] that T is a g_z-invertible operator and $T = T_M \oplus T_N$ for some $(M, N) \in \text{Red}(T)$ such that T_M is semi-regular and T_N is nilpotent. This implies again by [5, Theorem 4.7] that $p(T_M) = q(T_M) = \tilde{p}(T) = \tilde{q}(T) = 0$, and so T_M is invertible. Hence T is Drazin invertible. The converse is clear, since $\text{acc}\,\sigma(T) \subset \sigma_d(T)$. The second equality goes similarly. For the definition of $\tilde{p}(T)$ and $\tilde{q}(T)$ of a g_z-invertible operator T, see [5]. □

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