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Generalized quasi-Einstein Weyl manifolds

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Abstract. In this paper, we introduce the notion of generalized quasi-Einstein Weyl manifold which extends the concept of quasi-Einstein Weyl manifold. We provide an explicit example to demonstrate its existence. Furthermore, we present several results concerning generalized quasi-Einstein Weyl manifolds that admit certain special vector fields. Finally, we investigate some special conformal transformation between such manifolds.

1. Introduction

A Riemannian manifold M_n $(n > 2)$ with metric g is an Einstein manifold if its Ricci tensor Ric is of the form

$$Ric(X, Y) = \frac{r}{n} g(X, Y) \tag{1}$$

where r is the scalar curvature of M_n [2].

A non-Einstein Riemannian manifold M_n $(n > 2)$ is called a quasi-Einstein manifold if its Ricci tensor Ric of type $(0, 2)$ is not identically zero and is of the form

$$Ric(X, Y) = ag(X, Y) + bA(X)A(Y), \tag{2}$$

where a, b are functions and A is a non-zero 1-form defined by $A(X) = g(X, U)$ for all vector fields X and a unit vector field U. A is called the associated 1-form and U is called the generator of the manifold [4].

The notion of quasi-Einstein manifolds were generalized in different ways such as generalized quasi-Einstein manifolds ([3],[5], [8]), nearly quasi-Einstein manifolds [7], generalized Einstein manifolds [1], super quasi-Einstein manifolds [6], pseudo quasi-Einstein manifolds [21], extended quasi-Einstein manifolds [12], mixed quasi-Einstein manifolds [15], etc.

A non-flat Riemannian manifold M_n $(n > 2)$ is called a generalized quasi-Einstein manifold if its Ricci tensor Ric of type $(0, 2)$ is not identically zero and is of the form

$$Ric(X, Y) = ag(X, Y) + bA(X)A(Y) + cB(X)B(Y), \tag{3}$$

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where a, b, c are certain nonzero functions and A, B are two non-zero 1-forms. The unit vector fields U and V corresponding to 1-forms A and B respectively defined by

$$g(X, U) = A(X), \quad g(X, V) = B(X)$$

are orthogonal, i.e., $g(U, V) = 0$. The vector fields U and V are called the generators of the manifold [8].

The aim of this paper is to introduce the notion of generalized quasi-Einstein Weyl manifold which generalizes the concept of quasi-Einstein Weyl manifolds. A Weyl manifold is a conformal manifold equipped with a torsion free connection preserving the conformal structure. It is said to be Einstein-Weyl if the symmetric part of the Ricci tensor of the Weyl connection is proportional with its conformal metric. This condition is a generalization of the usual Einstein-Weyl manifolds.

Einstein-Weyl manifolds were studied by Folland [9], Tod [23], Pedersen and Tod [18] and many others. Quasi-Einstein Weyl manifolds were defined and studied by Gül and Canfes [13].

In the present work, we define generalized quasi-Einstein Weyl manifolds and give an example for the existence of them. Then, we examine some special vector fields on quasi-Einstein Weyl manifolds and present results. Moreover, we obtain some results about conformal, generalized concircular, and conharmonic transformations of quasi-Einstein Weyl manifolds.

2. Preliminaries

An n-dimensional differentiable manifold M having a torsion-free connection D and a conformal class $C[g]$ of metrics preserved by D is called a Weyl manifold which will be denoted by $M_n(g, \omega)$, where $g \in C[g]$ and ω is a 1-form satisfying the compatibility condition

$$Dg = 2(g \otimes \omega). \tag{4}$$

If ω is a closed form, then $M_n(g, \omega)$ is conformal to a Riemannian manifold.

Under the conformal change

$$\bar{g} = \lambda^2 g, \quad \lambda > 0 \tag{5}$$

of the representative metric tensor g, the 1-form ω changes by the law $\bar{\omega} = \omega + d \ln \lambda$.

It is easy to see that $M_n(\bar{g}, \bar{\omega})$ satisfies the compatibility condition and hence, it generates the same Weyl manifold ([11],[17], [22]).

Assume that $M_n(g, \omega)$ is a Weyl manifold of class C^∞, covered by a system of coordinate neighborhoods (U, x^h). Then, Eq. (4) can be written in local coordinates by

$$D_k g_{ij} = 2\omega_k g_{ij}. \tag{6}$$

We note that, throughout the paper, we will use Einstein summation convention over the repeated indices.

The curvature tensor, the covariant curvature tensor, the Ricci tensor, and the scalar curvature of $M_n(g, \omega)$ are defined, respectively, by:

$$v^j W^p_{jkl} = (D_k D_l - D_l D_k) v^p, \tag{7}$$

$$W_{hjkl} = g_{hp} W^p_{jkl}, \tag{8}$$

$$W_{ij} = W^p_{ijp} = g^{hk} W_{hijk}, \tag{9}$$

$$s = g^{ij} W_{ij}. \tag{10}$$

From (7) it follows that

$$W^p_{jkl} = \partial_k \Gamma^p_{jl} - \partial_l \Gamma^p_{jk} + \Gamma^p_{hk} \Gamma^h_{jl} - \Gamma^p_{hl} \Gamma^h_{jk}, \tag{11}$$

where $\partial_k = \dfrac{\partial}{\partial x^k}$ and Γ^i_{kl} are the coefficients of the Weyl connection D given by

$$\Gamma^i_{kl} = \left\{ \begin{matrix} i \\ kl \end{matrix} \right\} - g^{im}(g_{mk}\omega_l + g_{ml}\omega_k - g_{kl}\omega_m), \tag{12}$$

in which $\left\{ \begin{matrix} i \\ kl \end{matrix} \right\}$ are the coefficients of the Levi-Civita connection.

Definition 2.1. *A tensor field A is called a satellite of g with weight p if it admits a transformation of the form*

$$\bar{A} = \mu^p A,$$

under the change (5) of the metric tensor g ([11],[17], [22]).

From (9), (10), (11) and (12), it is easy to see that the Ricci tensor, the curvature tensor and Weyl connection coefficients are satellites of g with weight 0, and the scalar curvature s is a satellite of g with weight -2.

Definition 2.2. *The prolonged derivative of a satellite A of g with weight p is defined by ([11],[17], [22])*

$$\dot{\partial}_k A = \partial_k A - p\omega_k A.$$

Definition 2.3. *The prolonged covariant derivative of a satellite A of g with weight p is defined by ([11],[17], [22])*

$$\dot{D}_k A = D_k A - p\omega_k A. \tag{13}$$

We note that the prolonged covariant derivative and the prolonged derivative preserve the weights of the tensors.

From (5), (6) and (13) it follows that

$$\dot{D}_k g_{ij} = 0.$$

Moreover, since $\dot{\partial}_k g_{ij} = \partial_k g_{ij} - 2\omega_k g_{ij}$, it follows from (12) that

$$\Gamma^i_{kl} = \frac{1}{2} g^{im} \left(\dot{\partial}_k g_{lm} + \dot{\partial}_l g_{km} - \dot{\partial}_m g_{kl} \right).$$

Definition 2.4. *A satellite of g is called prolonged covariantly constant if its prolonged covariant derivative is zero.*

3. Generalized Quasi-Einstein Weyl Manifolds

In this section, we define generalized quasi-Einstein Weyl manifolds and give an example for the existence. Also, we give a result about the associated scalars of the manifold.

A Weyl manifold $M_n(g, \omega)$ is called a generalized quasi-Einstein Weyl manifold if

$$S_{ij} = a g_{ij} + b A_i A_j + c B_i B_j, \tag{14}$$

where a, b, c are functions of weight -2, S_{ij} denotes the symmetric part of the Ricci tensor W_{ij} of weight 0, and A_i, B_i are non-zero 1-forms of weight 1 satisfying

$$g^{ij} A_i A_j = 1, \quad g^{ij} B_i B_j = 1, \quad g^{ij} A_i B_j = 0. \tag{15}$$

In this case, A_i and B_i are called associated 1-forms, and a, b, c are called associated functions. If $c = 0$, then $M_n(g, \omega)$ is called quasi-Einstein Weyl manifold [13].

Multiplying (14) by g^{ij}, we get

$$s = an + b + c, \tag{16}$$

which is the scalar curvature of a generalized quasi-Einstein Weyl manifold.

Theorem 3.1. *If the associated scalar functions of a generalized quasi-Einstein Weyl manifold $M_n(g,\omega)$ are prolonged covariantly constant, then $M_n(g,\omega)$ is conformal to a generalized quasi-Einstein manifold.*

Proof. Assume that $M_n(g,\omega)$ is a generalized quasi-Einstein Weyl manifold. The prolonged covariant derivative of (16) is

$$\dot{D}_k s = \dot{D}_k(an+b+c) = n\dot{D}_k a + \dot{D}_k b + \dot{D}_k c. \tag{17}$$

If the associated scalar functions a, b, c are prolonged covariantly constant, and since the weight of s is -2, then from (17) we find

$$\dot{D}_k s = D_k s + 2\omega_k s = 0.$$

Hence, we have $\omega_k = -\frac{D_k s}{2s}$, from which it follows that ω_k is locally a gradient. Thus, $M_n(g,\omega)$ is conformal to generalized a quasi-Einstein manifold. □

Furthermore, contracting Eq. (14) with A^iA^j and then with B^iB^j, we obtain

$$A^iA^jS_{ij} = a+b, \;\; B^iB^jS_{ij} = a+c. \tag{18}$$

From these, it follows that

$$s = (n-2)a + (A^iA^j + B^iB^j)S_{ij}.$$

Similarly, multiplying Eq. (14) by A^i and then by B^i, respectively, yields

$$A^iS_{ij} = (a+b)A_j, \;\; B^iS_{ij} = (a+c)B_j,$$

from which we conclude that the dual of the associated 1-forms are eigenvectors of symmetric part of the Ricci tensor of the generalized quasi-Einstein Weyl manifold.

Example 3.2. *We consider a 3−dimensional Weyl manifold $M_3(g,\omega)$ endowed with a metric by $ds^2 = g_{ij}dx^idx^j = (dx^1)^2 + (dx^2)^2 + e^{x^1}(dx^3)^2$ and a 1-form $\omega = \omega_i dx^i = e^{x^1}dx^3$. The nonzero Weyl connection coefficients are*

$$\begin{aligned}\Gamma^1_{13} &= \Gamma^1_{31} = -e^{x^1}, \;\; \Gamma^1_{33} = -\frac{e^{x^1}}{2}, \;\; \Gamma^2_{23} = \Gamma^2_{32} = -e^{x^1}, \;\; \Gamma^3_{11} = 1,\\ \Gamma^3_{13} &= \Gamma^3_{31} = \frac{1}{2}, \;\; \Gamma^3_{22} = 1, \;\; \Gamma^3_{33} = -e^{x^1}\end{aligned}$$

A direct computation yields the following nonzero components of the Ricci tensor.

$$W_{11} = \frac{1}{4} + e^{x^1}, \quad W_{13} = -W_{31} = \frac{3}{2}e^{x^1}, \quad W_{22} = e^{x^1}, \quad W_{33} = \frac{1}{4}e^{x^1}.$$

Moreover, the nonzero components of symmetric parts of the Ricci tensor S_{ij} and the scalar curvature s are

$$S_{11} = \frac{1}{4} + e^{x^1}, \quad S_{22} = e^{x^1}, \quad S_{33} = \frac{1}{4}e^{x^1}, \qquad s = \frac{1}{2} + 2e^{x^1}.$$

If the associated scalar functions are given by

$$a = \frac{1}{4}, \; b = e^{x^1}, \; c = e^{x^1} - \frac{1}{4},$$

and the associated 1− forms are $A = A_idx^i = dx^1, B = B_idx^i = dx^2$ then, the Eqs. (14) and (15) are satisfied. Therefore, $M_3(g,\omega)$ is a generalized quasi-Einstein Weyl manifold.

4. Some Special Vector Fields On Generalized Quasi-Einstein Weyl Manifolds

In this section, we define some special vector fields on a Weyl manifold and then present the results related to a generalized quasi-Einstein Weyl manifolds.

A vector field ϕ in a Riemannian manifold M is called torse-forming if it satisfies the condition

$$\nabla_i \phi^h = \alpha \delta_i^h + \phi^h \gamma_i,$$

where α is a smooth function, ϕ^h and γ_i are the components of the vector field ϕ and 1-form γ, respectively, and δ_i^h is the Kronecker symbol [27]. For details, see [16, p. 168].

If $\alpha = 0$, then the torse-forming vector field is called recurrent vector field, that is, the vector field ϕ satisfies

$$\nabla_i \phi^h = \phi^h \gamma_i.$$

If $\gamma = 0$, then the torse-forming vector field is called concircular vector field, that is, the vector field ϕ satisfies

$$\nabla_i \phi^h = \alpha \delta_i^h.$$

A $\varphi(Ric)$-vector field is a vector field φ on a Riemannian manifold M satisfying

$$\nabla_i \varphi^h = \beta R_i^h, \tag{19}$$

where φ^h and R_i^h are the components of the vector field φ and the Ricci tensor of the Riemannian manifold M, respectively, and β is a constant [10]. Equation (19) can also be expressed in index-lowered for as

$$\nabla_i \varphi_j = \beta R_{ij},$$

where R_{ij} is the Ricci tensor of the Riemannian manifold. We note that generalized $\varphi(Ric)$-vector fields are also defined by taking β as a function [20]. A detailed analysis of these equations for (pseudo)-Riemannian spaces is carried out in [19]. We follow the methodology of that paper in our present work.

Now, we define these vector fields by using prolonged covariant derivative on a Weyl manifold.

A vector field ϕ of weight p in a Weyl manifold $M_n(g, \omega)$ is called generalized torse-forming vector field if it satisfies the condition

$$\dot{D}_i \phi^h = \alpha \delta_i^h + \phi^h \gamma_i, \tag{20}$$

where α is a smooth function of weight p, ϕ^h and γ_i are the components of the vector field ϕ and 1-form γ of weights p and 0, respectively.

If $\alpha = 0$, then the generalized torse-forming vector field is called generalized recurrent vector field, that is, the vector field ϕ satisfies

$$\dot{D}_i \phi^h = \phi^h \gamma_i. \tag{21}$$

If $\gamma = 0$, then the generalized torse-forming vector field is called generalized concircular vector field, that is, the vector field ϕ satisfies

$$\dot{D}_i \phi^h = \alpha \delta_i^h. \tag{22}$$

Since, the Ricci tensor W_{ij} of a Weyl manifold $M_n(g, \omega)$ is not symmetric, we define generalized $\varphi(Ric)$-vector fields by taking symmetric part of the Ricci tensor S_{ij}.

Definition 4.1. *A generalized $\varphi(Ric)$-vector field of weight p is a vector field φ on a Weyl manifold $M_n(g,\omega)$ satisfying*

$$\dot{D}_i\varphi^h = \beta S_i^h, \tag{23}$$

where φ^h are the components of the vector field φ, S_i^h is defined $S_i^h = g^{hj}S_{ij}$ with weight -2, and β is a function of weight $p+2$.

Equation (23) can also be written index-lowered form as

$$\dot{D}_i\varphi_j = \beta S_{ij}, \tag{24}$$

where $\varphi_j = \varphi^h g_{hj}$, which is of weight $p+2$.

Theorem 4.2. *If $M_n(g,\omega)$ is a generalized quasi-Einstein Weyl manifold, then both of the vector fields dual to the associated 1-forms cannot be generalized torse-forming vector fields.*

Proof. Assume that the vector fields dual to the associated 1-forms are both generalized torse-forming vector fields on a generalized quasi-Einstein Weyl manifold. Then, from (20) we have:

$$\dot{D}_iA^h = \alpha\delta_i^h + A^h\gamma_i, \quad \dot{D}_iB^h = \breve{\alpha}\delta_i^h + B^h\breve{\gamma}_i. \tag{25}$$

Multiplying (25) by g_{hj}, we obtain:

$$\dot{D}_iA_j = \alpha g_{ij} + A_j\gamma_i, \quad \dot{D}_iB_j = \breve{\alpha} g_{ij} + B_j\breve{\gamma}_i, \tag{26}$$

where A_j and B_j are the components of the associated 1-forms of weight 1, α and $\breve{\alpha}$ are nonzero scalar functions of weight -1, γ_i and $\breve{\gamma}_i$ are 1-forms of weight 0. Contracting Eq. (26) with A^j and B^j, respectively, we get

$$0 = \alpha A_i + \gamma_i, \quad 0 = \breve{\alpha}B_i + \breve{\gamma}_i,$$

since $A^j\dot{D}_iA_j = B^j\dot{D}_iB_j = 0$ and $A^jA_j = B^jB_j = 1$. Hence we obtain

$$\gamma_i = -\alpha A_i, \quad \breve{\gamma}_i = -\breve{\alpha}B_i. \tag{27}$$

Using Eq. (27) in (26), we get

$$\dot{D}_iA_j = \alpha\left(g_{ij} - A_iA_j\right), \quad \dot{D}_iB_j = \breve{\alpha}\left(g_{ij} - B_iB_j\right). \tag{28}$$

From (15), we have $A^iB_i = 0$. Taking prolonged covariant derivative of this equation and using (28), we obtain

$$0 = \dot{D}_k\left(A^iB_i\right) = A^i\dot{D}_kB_i + B_i\dot{D}_kA^i = A^i\dot{D}_kB_i + B^i\dot{D}_kA_i = \breve{\alpha}A_k + \alpha B_k. \tag{29}$$

Contracting (29) with A^k and B^k, respectively, we get $\alpha = \breve{\alpha} = 0$, which contradicts the assumption that α and $\breve{\alpha}$ are nonzero scalar functions. Therefore, both of the vector fields dual to the associated 1-forms cannot be generalized torse-forming vector fields in a generalized quasi-Einstein Weyl manifold. □

Theorem 4.3. *If the vector fields dual to the associated 1-forms are generalized concircular vector fields on a generalized quasi-Einstein Weyl manifold, then the associated 1-forms are prolonged covariantly constant.*

Proof. Assume that the vector fields dual to the associated 1-forms are generalized concircular vector fields on a generalized quasi-Einstein Weyl manifold. Then, by Eq. (22), we have

$$\dot{D}_iA^h = \alpha\delta_i^h, \quad \dot{D}_iB^h = \breve{\alpha}\delta_i^h,$$

from which it follows that

$$\dot{D}_iA_j = \alpha g_{ij}, \quad \dot{D}_iB_j = \breve{\alpha}g_{ij}, \tag{30}$$

where A_j and B_j are the components of the associated 1-forms of weight 1, α and $\breve{\alpha}$ are scalar functions of weight -1. Multiplying Eq. (30) by A^j and B^j, respectively, we get

$$0 = \alpha A_i, \quad 0 = \breve{\alpha}B_i, \tag{31}$$

since $A^j\dot{D}_iA_j = 0$ and $B^j\dot{D}_iB_j = 0$. Multiplying Eq. (31) by A^j and B^j, respectively, we obtain $\alpha = \breve{\alpha} = 0$. Therefore, by Eq. (30), the result follows. □

Theorem 4.4. *If the vector fields dual to the associated 1-forms are generalized recurrent vector fields on a generalized quasi-Einstein Weyl manifold, then the associated 1-forms are prolonged covariantly constant.*

Proof. Assume that the vector fields dual to the associated 1-forms are generalized recurrent vector fields on a generalized quasi-Einstein Weyl manifold. Then, by Eq. (21), we have

$$\dot{D}_iA^h = A^h\gamma_i, \quad \dot{D}_iB^h = B^h\breve{\gamma}_i,$$

from which it follows that

$$\dot{D}_iA_j = A_j\gamma_i, \quad \dot{D}_iB_j = B_j\breve{\gamma}_i, \tag{32}$$

where A_j and B_j are the components of the associated 1-forms of weight 1, γ_i and $\breve{\gamma}_i$ are components of the 1-forms γ and $\breve{\gamma}$ of weight 0.

Multiplying Eq. (32) by A^j and B^j, respectively, we get $\gamma_i = \breve{\gamma}_i = 0$, since $A^j\dot{D}_iA_j = 0$ and $B^j\dot{D}_iB_j = 0$. Hence, the theorem is proved. □

Definition 4.5. *A Weyl manifold $M_n(g,\omega)$ is said to be a nearly quasi-Einstein Weyl manifold if S_{ij}, the symmetric part of the Ricci tensor W_{ij} of the Weyl manifold, satisfies the condition*

$$S_{ij} = ag_{ij} + bE_{ij},$$

where a is a scalar function of weight -2, b is a scalar function and E_{ij} are components of the symmetric (0,2)-tensor E, such that sum of weight of b and E_{ij} is zero [14].

Theorem 4.6. *If the vector fields dual to the associated 1-forms are generalized $A(Ric)$ and $B(Ric)$ vector fields on a generalized quasi-Einstein Weyl manifold, then the manifold is nearly quasi-Einstein Weyl manifold.*

Proof. Suppose that the vector fields dual to the associated 1-forms are generalized $A(Ric)$ and $B(Ric)$ vector fields on a generalized quasi-Einstein Weyl manifold. Then, by (24), we have

$$\dot{D}_iA_j = \beta S_{ij}, \quad \dot{D}_iB_j = \breve{\beta}S_{ij}, \tag{33}$$

where A_j and B_j are the components of the associated 1-forms of weight 1, β and $\breve{\beta}$ are nonzero scalar functions of weight 1, and S_{ij} denotes the components of the symmetric part of the Ricci tensor of the Weyl manifold, which has weight 0.

Multiplying (33) by A^j and B^j, respectively, we get

$$0 = \beta A^jS_{ij}, \quad 0 = \breve{\beta}B^jS_{ij}, \tag{34}$$

where we used the facts $A^j\dot{D}_iA_j = 0$ and $B^j\dot{D}_iB_j = 0$.

Multiplying (34) again by A^j and B^j, respectively, and using Eq. (18), we get

$$0 = (a + b)\beta, \quad 0 = (a + c)\breve{\beta}. \tag{35}$$

From (35), since β and $\breve{\beta}$ are nonzero functions, we obtain $b = c = -a$ from which, by (14), we get

$$S_{ij} = ag_{ij} + bE_{ij},$$

where $E_{ij} = A_iA_j + B_iB_j$ are the components of the symmetric (0,2)-tensor E of weight 2. Therefore, generalized quasi-Einstein Weyl manifold is a nearly quasi-Einstein Weyl manifold. □

5. Some Special Conformal Transformations On Generalized Quasi-Einstein Weyl Manifolds

In this section, we examine generalized concircular transformations and conharmonic transformations of generalized quasi-Einstein Weyl manifolds.

Let $\tau : M_n(g, \omega) \longrightarrow \tilde{M}_n(\tilde{g}, \tilde{\omega})$ be a conformal transformations of the Weyl manifold $M_n(g, \omega)$ onto another Weyl manifold $\tilde{M}_n(\tilde{g}, \tilde{\omega})$. Then, at the corresponding points of $M_n(g, \omega)$ and $\tilde{M}_n(\tilde{g}, \tilde{\omega})$ one can choose [24]

$$g_{ij} = \tilde{g}_{ij}.$$

Let D, $\tilde{D}$ be the connections of $M_n(g, \omega)$, $\tilde{M}_n(\tilde{g}, \tilde{\omega})$, respectively, with connection coefficients Γ^i_{jk} and $\tilde{\Gamma}^i_{jk}$. Then, the relation between the curvature tensors of $M_n(g, \omega)$ and $\tilde{M}_n(\tilde{g}, \tilde{\omega})$ is given by

$$\tilde{W}^p_{jkl} = W^p_{jkl} + \delta^p_l P_{jk} - \delta^p_k P_{jl} + g_{jk}g^{pm}P_{ml} - g_{jl}g^{pm}P_{mk} + 2\delta^p_j \dot{D}_{[k}P_{l]}, \tag{36}$$

where $\dot{D}_{[k}P_{l]}$ denotes the antisymmetric part of $\dot{D}_kP_l$, $P = \omega - \tilde{\omega}$ is a covector field of weight 0 and

$$P_{kl} = \dot{D}_lP_k - P_kP_l + \frac{1}{2}g_{kl}P^sP_s. \tag{37}$$

Contraction on the indices p and l in (36) gives us

$$\tilde{W}_{jk} = W_{jk} + (n-2)P_{jk} + g_{jk}g^{lm}P_{ml} + 2\dot{D}_{[k}P_{j]}, \ \ n > 2. \tag{38}$$

Using (38), we obtain

$$\tilde{S}_{jk} = S_{jk} + (n-2)P_{(jk)} + g_{jk}g^{lm}P_{ml}, \tag{39}$$

where $P_{(jk)}$ is symmetric part of P_{jk}.

In [25] and [13], the following theorems are proved.

Theorem 5.1. *The conformal transformation $\psi : M_n(g, \omega) \longrightarrow \tilde{M}_n(\tilde{g}, \tilde{\omega})$ which preserves circles is called generalized concircular if and only if*

$$P_{kl} = \phi g_{kl},$$

where ϕ is a smooth scalar function of weight -2, $P_{kl} = \nabla_lP_k - P_kP_l + \frac{1}{2}g_{kl}g^{rs}P_rP_s$, and $P = \omega - \tilde{\omega}$ is the covector field of the conformal transformations of weight zero.

Theorem 5.2. *The tensor Z of type $(1,3)$ whose components are given by*

$$Z^p_{jkl} = W^p_{jkl} - \frac{s}{n(n-1)}(\delta^p_l g_{jk} - \delta^p_k g_{jl}) \tag{40}$$

is invariant under a generalized concircular mapping of $M_n(g, \omega)$. Such a tensor is called the generalized concircular curvature tensor of $M_n(g, \omega)$.

Contraction on the indices p and l in (40) gives the generalized concircularly invariant tensor

$$Z^p_{jkp} = Z_{jk} = W_{jk} - \frac{s}{n} g_{jk}. \tag{41}$$

Theorem 5.3. *If the symmetric part $P_{(kl)}$ of the tensor P_{kl} is zero, then the symmetric part of the Ricci tensor for a Weyl manifold $M_n(g,\omega)$ $(n > 2)$ is preserved by the conformal mapping*

$$\tau : M_n(g,\omega) \longrightarrow \tilde{M}_n(\tilde{g},\tilde{\omega}).$$

So, we have,

Theorem 5.4. *Let $M_n(g,\omega)$ and $\tilde{M}_n(\tilde{g},\tilde{\omega})$ be two Weyl manifolds and let τ be a conformal mapping of $M_n(g,\omega)$ into $\tilde{M}_n(\tilde{g},\tilde{\omega})$. If $M_n(g,\omega)$ is a generalized quasi-Einstein (or Einstein) Weyl manifold and the symmetric part $P_{(kl)}$ of the tensor P_{kl} is zero, then $\tilde{M}_n(\tilde{g},\tilde{\omega})$ is a generalized quasi-Einstein (or Einstein) Weyl manifold.*

Proof. The result follows directly from Theorem 5.3. □

Theorem 5.5. *A generalized quasi-Einstein Weyl manifold is transformed into a generalized quasi-Einstein Weyl manifold by a generalized concircular mapping.*

Proof. Let $\psi : M_n(g,\omega) \longrightarrow \tilde{M}_n(\tilde{g},\tilde{\omega})$, be a generalized concircular mapping between two Weyl manifolds. Assume that $M_n(g,\omega)$ is a generalized quasi-Einstein Weyl manifold. From (41), we can write

$$\tilde{W}_{jk} - \frac{\tilde{s}}{n}\tilde{g}_{jk} = W_{jk} - \frac{s}{n} g_{jk}, \tag{42}$$

since the generalized concircular tensor Z^p_{jkl} and its contracted tensor Z_{jk} are invariant under a generalized concircular mapping.

From (42), we obtain

$$\tilde{S}_{jk} - \frac{\tilde{s}}{n}\tilde{g}_{jk} = S_{jk} - \frac{s}{n} g_{jk}.$$

Since the map ψ is concircular, we may take $\tilde{g}_{jk} = g_{jk}$. Moreover, using Eq. (14), it follows that

$$\tilde{S}_{jk} = \left(a - \frac{s}{n} + \frac{\tilde{s}}{n}\right) g_{jk} + bA_jA_k + cB_jB_k.$$

Therefore, $\tilde{M}_n(\tilde{g},\tilde{\omega})$ is a generalized quasi-Einstein Weyl manifold, which completes the proof. □

Now, we consider a conharmonic transformation between two generalized quasi-Einstein Weyl manifolds. The conharmonic transformation is a conformal transformation preserving the harmonicity of a certain function. If the conformal mapping is also conharmonic, then we have

$$g^{kl}\dot{D}_kP_l + \frac{1}{2}(n-2)P^kP_k = 0, \tag{43}$$

where P_k is the components of the covector field $P = \omega - \tilde{\omega}$ [26].

Theorem 5.6. *A conformal transformation between two generalized quasi-Einstein Weyl manifolds is conharmonic if and only if the associated scalars are preserved.*

Proof. Let $\tau : M_n(g,\omega) \longrightarrow \tilde{M}_n(\tilde{g},\tilde{\omega})$ be a conformal transformation of a generalized quasi- Einstein Weyl manifold $M_n(g,\omega)$ onto another generalized quasi- Einstein Weyl manifold $\tilde{M}_n(\tilde{g},\tilde{\omega})$. Then, from (14) and (39), we obtain

$$\tilde{a}\tilde{g}_{ij} + \tilde{b}\tilde{A}_i\tilde{A}_j + \tilde{c}\tilde{B}_i\tilde{B}_j = ag_{ij} + bA_iA_j + cB_iB_j + (n-2)P_{(ij)} + g_{ij}g^{rs}P_{rs}. \tag{44}$$

Multiplying (44) by $g^{ij} = \tilde{g}^{ij}$ and summing up on the indices i and j gives

$$\tilde{a}n + \tilde{b} + \tilde{c} = an + b + c + (2n-2)g^{rs}P_{(rs)}. \tag{45}$$

Suppose that τ is also a conharmonic transformation. Then, it follows from Eqs. (37) and (43) that

$$g^{rs}P_{(rs)} = g^{rs}P_{rs} = 0. \tag{46}$$

Therefore, using Eqs. (45) and (46), we obtain $\tilde{a} = a, \tilde{b} = b, \tilde{c} = c$.

Conversely, suppose that $\tilde{a} = a, \tilde{b} = b, \tilde{c} = c$. Then, from Equation (45) it follows that

$$g^{rs}P_{(rs)} = g^{rs}P_{rs} = 0. \tag{47}$$

Therefore, Eq. (47) implies Eq. (43) which shows that conformal transformation τ is also a conharmonic transformation. □

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References

[1] C. L. Bejan and T. Q. Binh, *Generalized Einstein Manifolds*, Proceedings of the 10th international conference on differential geometry and its applications, , World Scientific, 57–64 (2008).
[2] A. L. Besse, *Einstein Manifolds*, Springer- Verlag, Berlin, 2008.
[3] G. Catino, *Generalized quasi-Einstein manifolds with harmonic Weyl tensor*, Math. Z. **271** (3–4) (2012), 751–756.
[4] M. C. Chaki and R. K. Maity, *On quasi Einstein manifolds*, Publ. Math. Debrecen **57** (2000), 297–306.
[5] M. C. Chaki, *On generalized quasi Einstein manifolds*, Publ. Math. Debrecen **58** (2001), 683–691.
[6] M. C. Chaki, *On super quasi Einstein manifolds*, Publ. Math. Debrecen **64** (2004), 481–488.
[7] U. C. De and A. K. Gazi, *On nearly quasi Einstein manifolds*, Novi Sad J. Math. **38** (2008), 115–121.
[8] U. C. De and G. C. Ghosh, *On generalized quasi Einstein manifolds*, Kyungpook Math. J. **44** (4) (2004), 607–615.
[9] G. B. Folland, *Weyl manifolds*, J. Differential Geom. **4** (1970), 145–153.
[10] I. Hinterleitner and V. A. Kiosak, *$\varphi(Ric)$-vector fields in Riemannian spaces*, Arch. Math. (Brno) **44** (5)(2008), 385–390.
[11] V. Hlavaty, *Theorie d'immersion d'une W_m dans W_n*, Ann. Soc. Polon. Math. **21** (1949), 196–206.
[12] Z. Huang, F. Su and W. Lu, *The extended quasi-Einstein manifolds*, Filomat **38** 11 (2024), 3761–3775.
[13] İ. Gül and E. Ö. Canfes, *On quasi-Einstein Weyl manifolds*, Int. J. Geom. Methods Mod. Phys. **14** (2017), 1750122 (12 pages).
[14] İ. Gül, *On Generalized Recurrent and Generalized Concircularly Recurrent Weyl Manifolds*, Sakarya University Journal of Science **25** (5) (2021), 1189–1196.
[15] S. Mallick, A. Yıldız, and U. C. De, *Characterizations of mixed quasi-Einstein manifolds*, Int. J. Geom. Methods Mod. Phys. **14** 6 (2017), 1750096.
[16] J. Mikeš et al., *Differential Geometry of Special Mappings*, (2nd edition), Palacky Univ. Press, Olomouc, 2019.
[17] A. P. Norden, *Spaces with affin connection*, Moskow: Nauka, 1976.
[18] H. Pedersen and K. P. Tod, *Three-dimensional Einstein-Weyl geometry*, Adv. Math. **97** (1993), 74–109.
[19] P. Peška, L. Vítkova and J. Mikeš, *Fundamental equations of generalized $\varphi(Ric)$-vector fields*, Geom. Integrability & Quantization **30** (2024), 81–86.
[20] A. Savchenko, N. Vashpanova and N. Vasylieva, *Generalized $\varphi(Ric)$-vector fields in special pseudo-Riemannian spaces*, Proceedings of the International Geometry Center **14** 4 (2021), 231–242.
[21] A. A. Shaikh, *On pseudo quasi Einstein manifold*, Period. Math. Hungar. **59** (2009), 119–146.
[22] B. Tsareva and G. Zlatanov, *On the geometry of the nets in the n-dimensional space of Weyl*, J. Geom. **38** (1/2) (1990), 182–197.
[23] K. P. Tod, *Compact 3-dimensional Einstein-Weyl structures*, J. Lond. Math. Soc. **45** (1992), 341–351.
[24] A. Özdeğer, *Conformal and generalized concircular mappings of Einstein-Weyl manifolds*, Acta Math. Sci. Ser. B Engl. Ed. **30** (5) (2010), 1739–1745.
[25] A. Özdeğer and Z. Şentürk, *Generalized circles in a Weyl space and their conformal mapping*, Publ. Math.Debrecen **60** (1/2) (2002), 75–87.
[26] F. Özdemir and E. Ö. Canfes, *On generalized conformally recurrent kaehlerian Weyl spaces*, Iran. J. Sci. Technol. Trans. A Sci. **36** (3) (2012), 299–304.
[27] K. Yano, *On the torse-forming directions in Riemannian spaces*, Proc. Imp. Acad. **20** (6) (1944), 340–345.