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Surfaces with $H^2 = K$ along a given curve

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Abstract. In the present paper, we handle the problem of construction of surfaces with a given curve whose squared mean curvature is equal to the Gaussian curvature, that is $H^2 = K$, along the given curve. We express surfaces parametrically possessing the given curve and obtain conditions for the coefficient functions to satisfy $H^2 = K$ along the curve. The theory is validated by illustrative examples..

1. Introduction

Curves and surfaces are among the subjects studied in differential geometry for many years. These topics are covered in almost all differential geometry books [9], [17], [20], [21].

We can think of a curve as a geometric shape obtained by bending and twisting a wire from various parts. Surfaces are shapes such as planes, spheres and saddle surfaces in three-dimensional space.

There are special curves such as geodesic, asymptotic curve and curvature line on a surface. Geodesic is the shortest path connecting two different points on a surface. When creating ship and aircraft routes, care is taken to ensure that the route is a geodesic. The asymptotic curve is the curve in which the tangent vector field at each point is an asymptotic direction. If the tangent vector field at every point on the curve is a principal direction, then the curve is called a line of curvature.

A curve in three-dimensional space is completely evident by its curvature and torsion. The mean curvature and Gaussian curvature defined on surfaces provide information about the shape of the surface. Let the principal curvatures at a point of the surface be k_1 and k_2. The mean curvature is denoted by H and it is defined as $H = \frac{k_1+k_2}{2}$. The Gaussian curvature at a point is denoted by K and is defined as $K = k_1 k_2$.

In differential geometry, classification of surfaces with constant mean curvature is a fundamental problem. A surface of constant mean curvature can be thought of as a surface in which tensile forces and external pressure are balanced. There are many studies on surfaces with constant mean curvature. In 1951, Hopf proved that any immersed surface of constant mean curvature in $\mathbb{R}^3$ is a sphere [15]. Alexandrov showed that any compact embedded surface of constant mean curvature in $\mathbb{R}^3$ is a sphere [2]. Meeks and Tinaglia derived intrinsic curvature and discrete diameter estimates for $\mathbb{R}^3$-embedded compact discs with constant non-zero mean curvature and applied these estimates to study the global geometry of $\mathbb{R}^3$-embedded complete surfaces with constant non-zero curvature [19].

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A lot of work has been done related to Gaussian curvature. Bayram [3] examined surfaces with constant Gaussian curvature along a given curve. Guan and Spruck discussed surfaces with constant Gaussian curvature, accepting the curves given in the n-dimensional sphere as boundary curves [12]. Chen [10] examined minimal surfaces with constant mean curvature in his study.

Surface creation problems, specifically problems of finding a surface that passes through a given curve and accepts this curve as a special curve, attract the attention of many scientists [3–7], [11], [13], [14], [16], [22], [24]. Wang et al. [23] first posed this type of problem in 2004 and obtained surfaces passing through a given curve and accepting this curve as a common geodesic. Kasap et al. [16] generalized the functions of Wang et al. [23] and presented a more general family of surfaces with common geodesic. In 2011, Li et al. [18] presented sufficient conditions for a surface family with a common line of curvature. Ergün et al. [11] carried the work done by Li et al. [18] to 3 dimensional Minkowski space. Surface family with common asymptotic curve defined by Bayram et al. [4]. Bayram [5] constructed surfaces with constant mean curvature along a given timelike curve in Minkowski 3 space. Şaffak et al. [22] presented a different approach for designing a surface pencil through a given geodesic curve. Yaman and Kasap [24] investigated the problem of the construction of a ruled surface family passing through the striction curve of a ruled surface.

It is a well known fact that if $H^2 = K$ at every point of the surface in $\mathbb{R}^3$, then the surface is a piece of a plane or a sphere. In this study, we handle the problem of finding surfaces where $H^2 = K$ along a given curve. We express surfaces parametrically using the Frenet frame defined along the curve. The conditions that the so called marching scale functions, which are the coefficient functions of the vector fields of the frame, should meet are investigated. Examples are given to support the results obtained.

2. Preliminaries

Let I be an open interval and $\alpha : I \to \mathbb{R}^3$ be a regular curve, that is $\alpha'(u) \neq 0$, $\forall u \in I$. Then $T(u) = \frac{\alpha'(u)}{\|\alpha'(u)\|}$, $N(u) = B(u) \times T(u)$, $B(u) = \frac{\alpha'(u)\times\alpha''(u)}{\|\alpha'(u)\times\alpha''(u)\|}$ are called the tangent, the principal normal and the binormal vector fields of α, respectively. The curvature and the torsion of α are defined as $\kappa(u) = \frac{\|\alpha'(u)\times\alpha''(u)\|}{\|\alpha'(u)\|^3}$, $\tau(u) = \frac{\det(\alpha'(u),\alpha''(u),\alpha'''(u))}{\|\alpha'(u)\times\alpha''(u)\|^2}$, $\forall u \in I$ respectively. If $\kappa > 0$, then we have

$$\begin{aligned} T'(u) &= \lambda(u)\kappa(u)N(u), \\ N'(u) &= -\lambda(u)\kappa(u)T(u) + \lambda(u)\tau(u)B(u), \\ B'(u) &= -\lambda(u)\tau(u)N(u), \end{aligned}$$

where $\lambda(u) = \|\alpha'(u)\|$, $\forall u \in I$ [20]. A differentiable function $F : \mathbb{R}^n \to \mathbb{R}^m$ is called regular if its derivative map F_{*p} is one-to-one $\forall p \in \mathbb{R}^n$. Let $D \subset \mathbb{R}^2$ be an open set and $X : D \to \mathbb{R}^3$ be a differentiable function. If X is regular and one-to-one, then it is called a coordinate patch. A coordinate patch is called proper if the inverse function $X^{-1} : X(D) \to D$ is continuous. A surface in $\mathbb{R}^3$ is a subset M of $\mathbb{R}^3$ such that for each point p of M there exists a proper patch in M whose image contains a neighborhood of p in M [20].

Lemma 2.1. *Let $X : D \to \mathbb{R}^3$ be a coordinate patch. The Gauss and the mean curvature of X is given by*

$$\begin{aligned} K &= \frac{\det(X_{uu}, X_u, X_v)\det(X_{vv}, X_u, X_v) - (\det(X_{uv}, X_u, X_v))^2}{\left(\|X_u\|^2\|X_v\|^2 - \langle X_u, X_v\rangle^2\right)^2}, \\ H &= \frac{\det(X_{uu}, X_u, X_v)\det(X_{vv}, X_u, X_v)\langle X_u, X_v\rangle + \det(X_{vv}, X_u, X_v)\|X_u\|^2}{2\left(\|X_u\|^2\|X_v\|^2 - \langle X_u, X_v\rangle^2\right)^{\frac{3}{2}}}, \end{aligned}$$

respectively, where the subscript denotes the partial derivative with respect to indicated variable [1].

3. Main Results

In this section, we handle the problem of constructing surfaces possessing a given curve along which the equality $H^2 = K$ holds. For the sake of simplicity, we may assume that the given curve is a parameter curve.

Let $\alpha(u)$, $u_1 \le u \le u_2$, be a regular curve, that is $\|\alpha'(u)\| = \lambda \neq 0$, $\forall u \in [u_1, u_2]$. To have the Frenet frame defined along α, assume that the curvature does not vanish. Otherwise, the curve turns in to a straight line.

Surfaces possessing the curve $\alpha(u)$ can be represented parametrically as

$$X(u,v) = \alpha(u) + l(u,v)\,T(u) + m(u,v)\,N(u) + n(u,v)\,B(u), \tag{1}$$

$u_1 \le u \le u_2$, $v_1 \le v \le v_2$. The functions $l(u,v)$, $m(u,v)$ and $n(u,v)$ are real valued C^2 functions, which are called marching scale functions. To satisfy the parameter curve requirement for the curve α, we should have

$$l(u,v_0) = m(u,v_0) = n(u,v_0) = 0,\ u_1 \le u \le u_2, \tag{2}$$

for some fixed $v_0 \in [v_1, v_2]$. Note that choosing different marching scale functions yields different surfaces accepting the curve α as a common parameter curve.

We want to obtain conditions for surfaces $X(u,v)$ to satisfy $H^2 = K$ along the curve α. We need following calculations to find the mean and the Gauss curvature of these surfaces along the given curve. Using Eqn. 1, we have the following

$$X_u(u,v) = (\lambda + l_u - m\lambda\kappa)\,T + (\lambda l\kappa + m_u - n\lambda\tau)\,N + (m\lambda\tau + n_u)\,B,$$

$$X_v(u,v) = l_v(u,v)\,T(u) + m_v(u,v)\,N(u) + n_v(u,v)\,B(u).$$

Along the curve α, we have

$$X_u(u,v_0) = \lambda(u)\,T(u),$$

$$X_{uu}(u,v_0) = \lambda'(u)\,T(u) + \lambda^2(u)\,\kappa(u)\,N(u),$$

$$X_{uv}(u,v_0) = (l_{vu} - \lambda m_v\kappa)\,T + (\lambda l_v\kappa + m_{vu} - \lambda n_v\tau)\,N + (\lambda m_v\tau + n_{vu})\,B,$$

$$X_{vv}(u,v_0) = l_{vv}(u,v_0)\,T(u) + m_{vv}(u,v_0)\,N(u) + n_{vv}(u,v_0)\,B(u).$$

Also, we have

$$\det(X_{uu}, X_u, X_v) = -\lambda^3\kappa n_v,$$

$$\det(X_{uv}, X_u, X_v) = -\lambda^2\kappa l_v n_v - \lambda m_{vu} n_v + \lambda^2\tau n_v^2 + \lambda^2\tau m_v^2 + \lambda m_v n_{vu},$$

$$\det(X_{vv}, X_u, X_v) = -\lambda\,(m_{vv} n_v - n_{vv} m_v),$$

$$\|X_u\|^2 = \langle \lambda T, \lambda T\rangle = \lambda^2,$$

$$\|X_v\|^2 = l_v^2 + m_v^2 + n_v^2,$$

$$\langle X_u, X_v\rangle = \lambda l_v,$$

along the curve α. Plugging above calculations in Lemma 2.1, we have the mean and the Gaussian curvature of the surface $X(u,v)$ along the curve $\alpha(u)$ as

$$\left\{\begin{array}{c} H(u,v_0) = \Big[(-\lambda\kappa n_v)\left(l_v^2 + m_v^2 + n_v^2\right) \\ +2l_v\left[n_v\left(\lambda\kappa l_v + m_{vu} - \lambda\tau n_v\right) - m_v\left(\lambda\tau m_v + n_{vu}\right)\right] \\ -\lambda\left(\tau m_{vv} n_v - n_{vv} m_v\right)\Big]\,\dfrac{1}{2\lambda\left(m_v^2+n_v^2\right)^{\frac{3}{2}}}, \end{array}\right. \tag{3}$$

and

$$K(u,v_0)=\frac{\lambda^2\kappa n_v\left(m_{vv}n_v-n_{vv}m_v\right)-\left(-\lambda\kappa l_v n_v-m_{vu}n_v+\lambda\tau n_v^2+\lambda\tau m_v^2+m_v n_{vu}\right)^2}{\lambda^2\left(m_v^2+n_v^2\right)^2}, \tag{4}$$

respectively.

Theorem 3.1. *Sufficient conditions for the surface $X(u,v)$ to satisfy $H^2=K$ along the curve α are*

$$\begin{aligned} l_v(u,v_0) &= n_v(u,v_0)=\left(n_{vu}(u,v_0)+\lambda\tau m_v(u,v_0)\right)=0,\\ l(u,v_0) &= m(u,v_0)=n(u,v_0)=0\neq m_v(u,v_0), \end{aligned}$$

$\forall u\in[u_1,u_2]$ and for some fixed $v_0\in[v_1,v_2]$.

Theorem 3.2. *Sufficient conditions for the surface $X(u,v)$ to satisfy $H^2=K$ along the curve α are*

$$\begin{aligned} l(u,v_0) &= m(u,v_0)=n(u,v_0)=m_v(u,v_0)=n_{vv}(u,v_0)=0,\\ l_{vv}(u,v_0) &= l_v(u,v_0)=\tau(u)=0\neq n_v(u,v_0),\ \frac{m_{vv}(u,v_0)}{\left(n_v(u,v_0)\right)^2}=\kappa(u), \end{aligned}$$

$\forall u\in[u_1,u_2]$ and for some fixed $v_0\in[v_1,v_2]$.

Corollary 3.3. *The curve α is a geodesic on the surface $X(u,v)$ satisfying Theorem 3.2.*

Theorem 3.4. *Sufficient conditions for the surface $X(u,v)$ to satisfy $H^2=K$ along the curve α are*

$$\begin{aligned} l(u,v_0) &= m(u,v_0)=n(u,v_0)=n_v(u,v_0)=0\neq l_v(u,v_0)=m_v(u,v_0),\\ n_{vv}(u,v_0) &= n_{uv}(u,v_0)=n_{vu}(u,v_0)=\tau(u)=0, \end{aligned}$$

$\forall u\in[u_1,u_2]$ and for some fixed $v_0\in[v_1,v_2]$.

Corollary 3.5. *The curve α is an asymptotic curve on the surface $X(u,v)$ satisfying Theorem 3.4.*

Corollary 3.6. *There exists a ruled surface satisfying $H^2=K$ along a planar curve.*

Proof. Assume that α is a planar curve. Choosing marching scale functions as

$$l(u,v)=m(u,v)=v,\ n(u,v)=0,$$

we obtain the ruled surface

$$X(u,v)=\alpha(u)+v\left(T(u)+N(u)\right).$$

Since α is a planar curve, we have $\tau(u)=0$ along the curve, which satisfies Theorem 3.4 and completes the proof. □

Corollary 3.7. *There exists a developable surface satisfying $H^2=K$ along a planar curve.*

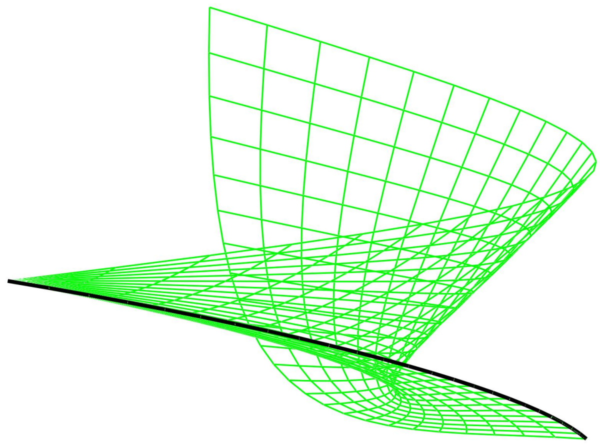

Figure 1: The curve α (black in color) and the surface $X_1(u,v)$ which satisfies $H^2 = K$ along this curve.

4. Numerical Examples

Example 4.1. *For the curve* $\alpha = (\cos u, \sin u, 0)$ *we have*

$$\begin{aligned} T(u) &= (-\sin u, \cos u, 0), \\ N(u) &= (-\cos u, -\sin u, 0), \\ B(u) &= (0, 0.1). \end{aligned}$$

Choosing marching scale functions as

$$l(u,v) = 2uv^2,\ m(u,v) = uv,\ n(u,v) = uv^3,\ 0 < u < 1,\ 0 \leq v < 1,$$

and $v_0 = 0$ *satisfies Theorem 3.1 and we obtain the surface*

$$X_1(u,v) = \left((1-uv)\cos u - 2uv^2 \sin u, (1-uv)\sin u + 2uv^2 \cos u, uv^3\right),$$

satisfying $H^2 = K$ *along the curve* α *(Figure 1).*

For the same curve, if we choose the marching scale functions as

$$l(u,v) = v^3,\ \ m(u,v) = \frac{v^2}{2},\ \ n(u,v) = v,\ \ 0 \leq u \leq 3,\ \ 0 \leq v \leq 1,$$

and $v_0 = 0$*, then Theorem 3.2 and Corollary 3.3 are satisfied and we obtain the surface*

$$X_2(u,v) = \left(\left(1 - \frac{v^2}{2}\right)\cos u - v^3 \sin u, \left(1 - \frac{v^2}{2}\right)\sin u + v^3 \cos u, v\right),$$

satisfying $H^2 = K$ *along the geodesic* α *(Figure 2).*

To obtain another surface with the same property, we choose marching scale functions as

$$l(u,v) = uv,\ \ m(u,v) = uv,\ \ n(u,v) = uv^3,\ \ 0 \leq u \leq 3,\ \ -1 \leq v \leq 1,$$

and $v_0 = 0$ *which satisfies Theorem 3.4 and Corollary 3.5. Thus, we obtain the surface*

$$X_3(u,v) = \left((1-uv)\cos u - uv \sin u, (1-uv)\sin u + uv \cos u, uv^3\right),$$

with $H^2 = K$ *along the asymptotic curve* α *(Figure 3).*

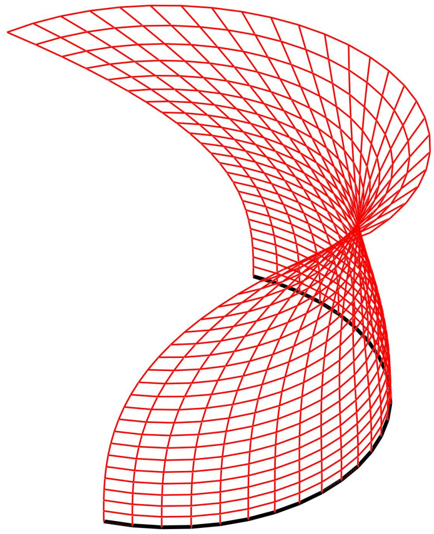

Figure 2: The geodesic α (black in color) and the surface $X_2(u,v)$ which satisfies $H^2 = K$ along this curve.

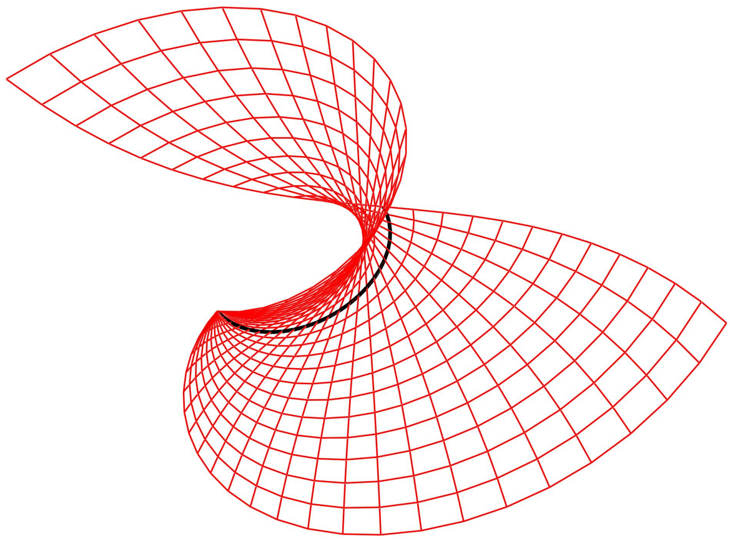

Figure 3: The asymptotic curve α (black in color) and the surface $X_3(u,v)$ which satisfies $H^2 = K$ along this curve.

Example 4.2. *Now we take the skloid curve $\alpha(u) = (u - \sin u, 1 - \cos u, 0)$. A straight calculation gives*

$$\begin{aligned} T(u) &= \left(\sin\frac{u}{2}, \cos\frac{u}{2}, 0\right), \\ N(u) &= \left(-\cos\frac{u}{2}, \sin\frac{u}{2}, 0\right), \\ B(u) &= (0,0,1). \end{aligned}$$

Choosing marching scale functions as

$$l(u,v) = uv, \;\; m(u,v) = u^2 v, \;\; n(u,v) = uv^3, \;\; 0 \le u \le 1, \;\; 0 \le v \le 1,$$

and $v_0 = 0$ *satisfies Theorem 3.4 and Corollary 3.5. Hence, we get the surface*

$$X_4(u,v) = \left(1 - \cos u + uv \sin\frac{u}{2} - u^2 v \cos\frac{u}{2}, \sin u + uv\cos\frac{u}{2} + u^2 v \sin\frac{u}{2}, uv^3\right),$$

with $H^2 = K$ *along the asymptotic curve* α *(Figure 4).*

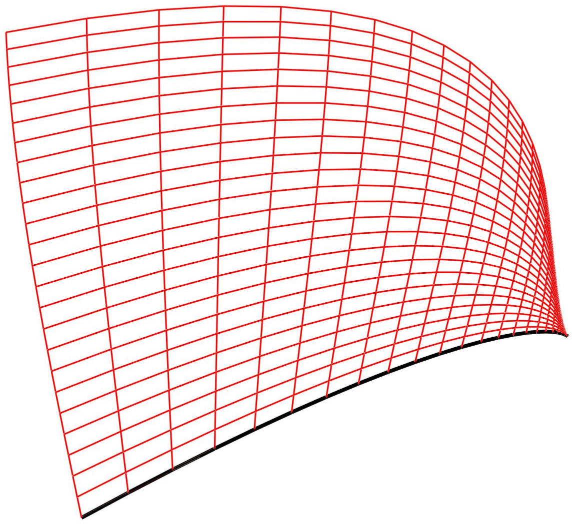

Figure 4: The asymptotic curve α (black in color) and the surface $X_4(u,v)$ which satisfies $H^2 = K$ along this curve.

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