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Sensitivity to initial condition on interval maps

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Abstract. Deterministic chaos is commonly characterized by sensitivity to initial conditions. This paper investigates the relationships between sensitivity, recurrence, and R-mixing in interval maps. We provide a characterization of R-mixing based on endpoint accessibility, establish links between sensitivity and recurrent interval cycles, and show that sensitive systems exhibit a unique recurrent structure. Our main results include new equivalence conditions for local eventual surjectivity, as well as explicit constructions of sensitive maps admitting a unique recurrent cycle.

1. Introduction

Throughout this paper, X denotes a compact metric space. An *interval map* refers to a continuous map acting on a real interval. For integers $n \leq m$, the notation $[\![n, m]\!]$ denotes the closed integer interval

$$[\![n, m]\!] := \{k \in \mathbb{Z} \mid n \leq k \leq m\}.$$

The notion of chaos has attracted considerable attention in recent decades, and various definitions have been proposed. One widely accepted approach emphasizes *sensitivity to initial conditions*. The concept was first formalized by Guckenheimer [18], who described systems in which the set of β-unstable points has positive Lebesgue measure for some $\beta > 0$. In this work, however, we adopt Devaney's definition [15].

Given a topological dynamical system (X, f) and $\beta > 0$, a point $x \in X$ is called *β-unstable* if, for every neighborhood U of x, there exist $y \in U$ and $m \in \mathbb{N}$ such that

$$d(f^m(x), f^m(y)) \geq \beta.$$

The set of all β-unstable points is denoted by $U_\beta(f)$. A map f is *β-sensitive* if $U_\beta(f) = X$, and *sensitive* if this holds for some $\beta > 0$.

Sensitivity reflects the property that initially close trajectories eventually separate by at least β. For interval maps, Ruette [28] proved that transitivity implies sensitivity, while sensitivity implies transitivity for an iterate of the map on a subinterval, demonstrating the strong connection between these notions.

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Recurrence is another classical concept in topological dynamics. A point $x \in X$ is *recurrent* (or T-recurrent for an operator T) if there exists a strictly increasing sequence (k_m) of positive integers such that

$$T^{k_m}x \longrightarrow x \quad \text{as } m \to \infty.$$

The operator T is *recurrent* if for every nonempty open set $U \subset X$, there exists $m \in \mathbb{N}$ with

$$T^{-m}(U) \cap U \neq \emptyset.$$

A map $f\colon X \to X$ is *topologically R-mixing* if for every nonempty open set $U \subset X$, there exists $M \geq 0$ such that

$$f^m(U) \cap U \neq \emptyset \quad \text{for all } m \geq M.$$

For interval maps $f\colon [s,t] \to [s,t]$, it was shown in [11] that f is R-mixing if and only if, for every $\epsilon > 0$ and every nondegenerate interval $K \subset [s,t]$, there exists M such that

$$[s, s+\epsilon] \subset f^m(K) \quad (\text{resp. } [t-\epsilon, t]) \qquad \text{for all } m \geq M.$$

For additional material on recurrent operators, R-mixing, and related notions in linear and nonlinear dynamics, we refer the reader to [1–3, 7, 8, 11–14, 16, 19–27]

Outline of the paper. In Section 2, we study R-mixing, accessible endpoints, and the property of being locally eventually onto (LEO). We prove that a map $f\colon [s,t] \to [s,t]$ is LEO if and only if both endpoints are accessible. We also show that if s is the only non-accessible endpoint, then s must be a fixed point; moreover, if s (resp. t) is non-accessible and fixed, then there exists a decreasing (resp. increasing) sequence of fixed points $(y_m)_{m\geq 0}$ converging to s (resp. t), and $f|_{[y_{m+1}, y_m]}$ is non-monotone for all $m \geq 0$. We also construct a map on $[0,1]$ which is R-mixing but not locally eventually onto.

In Section 3, we investigate sensitivity to initial conditions and its links with recurrence and R-mixing. We prove that R-mixing implies β-sensitivity under appropriate conditions on β. Furthermore, if $U_\beta(f)$ has nonempty interior, then there exists a recurrent cycle of intervals $(I_1, \ldots, I_q)$ such that

$$I_1 \cup \cdots \cup I_q \subset \overline{U_\beta(f)} \quad \text{and} \quad |I_j| \geq \beta \text{ for some } j \in [\![1, q]\!].$$

In Section 4, we show that even if all points are β-unstable for some $\beta > 0$, the system admits a *unique* recurrent cycle of intervals. To illustrate this, we construct a sensitive interval map with exactly one recurrent cycle of $q+1$ intervals.

2. Some properties of R-mixing interval maps

We begin this section by establishing a condition ensuring that an interval map admitting the R-mixing property must possess a periodic point.

Corollary 2.1. *Let $f\colon I \to I$ be an interval map. If f is topologically R-mixing, then f admits a periodic point of odd period greater than* 1.

Proof. Let $I = [s,t]$ and let

$$F = \{x \in I : f(x) = x\}$$

denote the set of fixed points of f. Since a nonempty interior of F would contradict the R-mixing property, the set F is closed and has empty interior. Consequently, one may choose a closed, nondegenerate subinterval $K \subset I$ such that K contains no fixed point.

Since f is topologically R-mixing, Proposition 2.4 of [11] ensures that there exists an integer M such that

$$K \subset f^m(K) \qquad \text{for all } m \geq M.$$

Choose an odd integer $m \geq M$. By Lemma 1.11 of [28], there exists a point $x \in K$ satisfying $f^m(x) = x$.

The minimal period of x divides m, and therefore must be odd. Moreover, since K contains no fixed point, its period cannot equal 1. Thus the period of x is an odd integer greater than 1. □

3. Accessible endpoints and R-mixing

Let $f\colon [s,t] \to [s,t]$ be an interval map. Recall that f is *topologically R-mixing* if, for every nonempty open set $K \subset [s,t]$ and every $\epsilon > 0$, there exists $M \geq 0$ such that for all $m \geq M$,

$$f^m(K) \supset [s, s+\epsilon] \quad (\text{resp. } [t-\epsilon, t]),$$

that is, the iterates of every nondegenerate interval K eventually cover arbitrarily large portions of the interval $[s,t]$.

A natural question is: under what conditions do the iterates of a nondegenerate interval eventually cover the whole interval $[s,t]$? As we shall see, this happens precisely when both endpoints s and t are *accessible* in the sense of Definition 3.4. The notions of accessible and non-accessible endpoints are discussed in [9] and further developed in the published paper [10].

Definition 3.1 (locally eventually onto). *A topological dynamical system (X, f) is termed* locally eventually onto *(or* topologically exact*) if, for every nonempty open subset $U \subseteq X$, there exists an integer $M \geq 1$ such that*

$$f^m(U) = X \quad \textit{for all } m \geq M.$$

Remark 3.2. *This property implies that the dynamical system is topologically R-mixing by definition.*

Lemma 3.3. *Let $f\colon I \to I$ be an interval map. The following statements are equivalent:*

1. *f is locally eventually onto.*
2. *For every $\epsilon > 0$, there exists $M \geq 0$ such that any subinterval $K \subset I$ with $|K| > \epsilon$ satisfies $f^m(K) = I$ for all $m \geq M$.*

Proof. **(2) $\Longrightarrow$ (1):** Assume (2) holds. Let $U \subset I$ be any nonempty open set. Then U contains a nondegenerate subinterval K. Set $\epsilon = |K|$. By (2), there exists $M \geq 0$ such that $f^m(K) = I$ for all $m \geq M$. Hence f is locally eventually onto.

(1) $\Longrightarrow$ (2): Suppose f is locally eventually onto. Fix $\epsilon > 0$. Divide $I = [s,t]$ into k equal subintervals

$$K_j = \Big(s + \tfrac{j}{k}(t-s),\ s + \tfrac{j+1}{k}(t-s)\Big), \quad j \in [\![0, k-1]\!],$$

where k is chosen such that

$$\frac{t-s}{k} < \frac{\epsilon}{2}.$$

By (1), for each K_j, there exists $M_j \geq 0$ such that $f^m(K_j) = I$ for all $m \geq M_j$. Let $M = \max\{M_j : 0 \leq j \leq k-1\}$. Now, let $K \subset I$ be any subinterval with $|K| > \epsilon$. Then K contains at least one of the K_j, so for all $m \geq M$ we have

$$f^m(K) \supset f^m(K_j) = I,$$

which proves (2). □

Definition 3.4 (accessible endpoint). *Let $f\colon [s,t] \to [s,t]$ be an interval map. We say that the endpoint s (resp. t) is* accessible *if there exist $y \in (s,t)$ and $m \geq 1$ such that $f^m(y) = s$ (resp. $f^m(y) = t$).*

Proposition 3.5. *Let $f\colon [s,t] \to [s,t]$ be a topologically R-mixing interval map. The following statements are equivalent:*

1. *f is locally eventually onto.*
2. *Both endpoints s and t are accessible.*

More precisely, for every $\epsilon > 0$ and any nontrivial subinterval $K \subset (s,t)$:

- $[s, t-\epsilon] \subset f^m(K)$ *for all* $m \geq M$ *if and only if* s *is accessible.*
- $[s+\epsilon, t] \subset f^m(K)$ *for all* $m \geq M$ *if and only if* t *is accessible.*

Proof. The equivalence (1) $\iff$ (2) follows immediately from the definition of locally eventually onto together with the definition of accessible endpoints, so we focus on the refined statements.

Assume first that s is accessible. Then there exist $y_0 \in (s, t)$ and $m_0 \geq 1$ such that $f^{m_0}(y_0) = s$. Fix $\epsilon > 0$ small enough so that $y_0 \in [s+\epsilon, t-\epsilon]$. By the R-mixing property (Proposition 3.4 in [11]), there exists $M \geq 0$ such that

$$[s, s+\epsilon] \subset f^m(K) \quad \text{for all } m \geq M.$$

Since $y_0 \in [s+\epsilon, t-\epsilon]$ and $f^{m_0}(y_0) = s$, the intermediate value theorem gives

$$[s, t-\epsilon] \subset f^{m+m_0}(K) \quad \text{for all } m \geq M.$$

Conversely, suppose that for some $\epsilon > 0$ and some $m \geq 1$ we have

$$[s, t-\epsilon] \subset f^m(K).$$

Then, in particular, $s \in f^m(K)$. As $s \notin K$, there exists $y \in K$ such that $f^m(y) = s$; hence s is accessible. The argument for the endpoint t is entirely similar. □

In the next lemma we show that, near a non-accessible endpoint, an R-mixing map exhibits infinite oscillatory behavior: if an endpoint is not accessible, its neighborhood contains infinitely many fixed points accumulating at the endpoint, and the map fails to be monotone on each interval between consecutive fixed points.

Lemma 3.6. *Let* $f: [s, t] \to [s, t]$ *be a topologically* R*-mixing interval map.*

1. *If* s *(resp.* t*) is the unique non-accessible endpoint, then it is a fixed point. If both* s *and* t *are non-accessible, then either* $f(s) = s$ *and* $f(t) = t$*, or* $f(s) = t$ *and* $f(t) = s$*.*
2. *If* s *(resp.* t*) is a fixed non-accessible endpoint, then there exists a decreasing (resp. increasing) sequence of fixed points* $(y_m)_{m\geq 0}$ *converging to* s *(resp.* t*). Furthermore, for all* $m \geq 0$*, the restriction* $f|_{[y_{m+1}, y_m]}$ *is not monotone.*

Proof.

1. Assume that s is not accessible, so that $s \notin f((s, t))$. Since f is topologically R-mixing, Lemma 3.5 of [11] implies that f is onto. Therefore, either $f(s) = s$ or $f(t) = s$. If t is accessible and $f(t) = s$, then s would be accessible, a contradiction. Thus, if s is the only non-accessible endpoint, then $f(s) = s$. By a similar argument, if t is the only non-accessible endpoint, then $f(t) = t$. If both s and t are non-accessible, then either $f(s) = s$ and $f(t) = t$, or $f(s) = t$ and $f(t) = s$.
2. Suppose that s is a fixed non-accessible endpoint, i.e. $f(s) = s$ and $s \notin f((s, t))$. Fix $\epsilon \in (0, t-s)$. By R-mixing, we cannot have $f([s, s+\epsilon]) \subset [s, s+\epsilon]$ (otherwise one obtains a forward invariant subinterval, contradicting R-mixing). Hence, there exists $z \in (s, s+\epsilon]$ such that $f(z) \geq s+\epsilon$.
If $f(x) \geq x$ for all $x \in [s, t]$, set

$$y = \min\Big\{z,\ \min f([z, t])\Big\}.$$

Then $f([y, t]) \subset [y, t]$, again contradicting the R-mixing property. Therefore, there exists $x \in [s, z]$ such that $f(x) < x$. By Lemma 1.11 in [11], there is a fixed point in $[x, z]$. Let y_0 be such a fixed point, and choose ϵ arbitrarily small to construct a decreasing sequence $(y_m)_{m\geq 0}$ of fixed points converging to s.
Finally, suppose that for some m the restriction $f|_{[y_{m+1}, y_m]}$ were monotone. Then $f([y_{m+1}, y_m])$ would be the interval $[y_{m+1}, y_m]$, giving a nontrivial forward-invariant subinterval, again contradicting R-mixing. Hence $f|_{[y_{m+1}, y_m]}$ is not monotone for all $m \geq 0$.
□

Remark 3.7. *Lemma 3.6 shows that if s is a non-accessible endpoint which is not fixed, then by (1) we have $f^2(s) = s$, so statement (2) applies to the map f^2.*

The following result states that the type of behavior described in Lemma 3.6(2) cannot occur if f is piecewise monotone or C^1. The case of piecewise monotonicity is discussed in [12]. Recall that f is *piecewise monotone* if the interval can be divided into finitely many subintervals on each of which f is monotone (see [28, p. 5] for more details).

Proposition 3.8. *Let $f\colon [s,t] \to [s,t]$ be a topologically R-mixing interval map. If f is either piecewise monotone or C^1, then the endpoints s and t are accessible. Consequently, f is locally eventually onto.*

Proof. Suppose that s is not accessible. Then, by Lemma 3.6(1), $f^2(s) = s$. Set $g := f^2$. Since f is topologically R-mixing, so is g.

If f is C^1, then g is C^1 as well. Since $g(s) = s$, the case $g'(s) < 0$ is impossible (it would force nontrivial intervals to be strictly repelled from s in one direction), and if $g'(s) = 0$ then, as g is R-mixing, one obtains a contradiction from the existence of a small neighborhood on which $g(x) \le x$ or $g(x) \ge x$. Thus we must have $g'(s) > 0$, so g is increasing in a neighborhood of s.

If instead f is piecewise monotone, then g is also piecewise monotone, and there exists $r \in (s,t)$ such that $g|_{[s,r]}$ is increasing.

In both cases, there exists $r \in (s,t)$ such that g is increasing on $[s,r]$. However, Lemma 3.6(2) applied to g ensures the existence of two distinct points $y < z$ in (s,r) such that $g|_{[y,z]}$ is not monotone, which is impossible since g is increasing on $[s,r]$. This contradiction shows that s must be accessible. A similar argument applies at t, hence both endpoints are accessible. By Proposition 3.5, f is locally eventually onto. □

Remark 3.9. *Proposition 3.8 remains valid if the R-mixing map f is either monotone or C^1 only in a neighborhood of the two endpoints.*

Example 3.10. *We build a continuous map $f\colon [0,1] \to [0,1]$ which is topologically R-mixing but not locally eventually onto. This example, adapted from [4], illustrates the existence of R-mixing maps with non-accessible endpoints.*

Consider a bi-infinite sequence $(t_n)_{n\in\mathbb{Z}}$ in $(0,1)$ with $t_n < t_{n+1}$ for all $n \in \mathbb{Z}$ and

$$\lim_{n\to-\infty} t_n = 0, \qquad \lim_{n\to+\infty} t_n = 1.$$

Define intervals $I_n := [t_n, t_{n+1}]$ for each $n \in \mathbb{Z}$. For each n, construct a piecewise linear map

$$f_n\colon I_n \to I_{n-1} \cup I_n \cup I_{n+1}$$

such that

$$f_n(t_n) := t_n, \qquad f_n(t_{n+1}) := t_{n+1},$$

$$f_n\left(\frac{2t_n + t_{n+1}}{3}\right) = t_{n+2}, \qquad f_n\left(\frac{t_n + 2t_{n+1}}{3}\right) = t_{n-1},$$

and f_n is linear between these points.

Define the global map f by

$$f(0) = 0, \quad f(1) = 1, \quad \text{and} \quad f(y) = f_n(y) \quad \text{for all } y \in I_n.$$

It is straightforward to check that f is continuous. Moreover, there is no $y \in (0,1)$ and $m \ge 1$ such that $f^m(y) \in \{0,1\}$, so 0 and 1 are non-accessible endpoints. Hence f is not locally eventually onto by Proposition 3.5.

We now show that f is R-mixing. Let $K \subset [0,1]$ be any nontrivial interval. By the 3-expanding property (Lemma 2.10 in [28]), there exists $n \ge 0$ such that $f^n(K)$ contains at least two critical points of f, and therefore

$$f^{n+1}(K) \supset I_j \quad \text{for some } j \in \mathbb{Z}.$$

Iterating further, for all $m \geq 0$ we have

$$f^m(I_j) \supset [t_{j-m},\ t_{j+m+1}].$$

As $m \to \infty$, these intervals expand and converge to $[0,1]$ in the sense that for every $\epsilon > 0$, there exists M such that

$$f^m(K) \supset [0,\epsilon] \cup [1-\epsilon, 1] \quad \text{for all } m \geq M.$$

By definition, this implies that f is topologically R-mixing.

4. Links between recurrence, R-mixing and sensitivity

In simple terms, sensitivity to initial conditions means that there exist points arbitrarily close to each other whose trajectories eventually separate by at least a fixed amount. We shall show that, for interval maps, recurrence implies sensitivity. This demonstrates that the notions of recurrence and sensitivity are closely related on the interval.

Definition 4.1 (unstable point, sensitivity to initial conditions). *Let (X, f) be a topological dynamical system and $\epsilon > 0$. A point $x \in X$ is* ϵ-unstable *(in the sense of Lyapunov) if, for every neighborhood U of x, there exists $y \in U$ and $n \geq 0$ such that $d(f^n(x), f^n(y)) \geq \epsilon$. The set of all ϵ-unstable points is denoted by $U_\epsilon(f)$. A point is* unstable *if it is ϵ-unstable for some $\epsilon > 0$.*

Definition 4.2. *Let (X, f) be a topological dynamical system and $\epsilon > 0$. The map f is called* ϵ-sensitive to initial conditions *(or simply* ϵ-sensitive*) if $U_\epsilon(f) = X$. The map is* sensitive to initial conditions *if it is ϵ-sensitive for some $\epsilon > 0$.*

4.1. Sensitivity and recurrence

Proposition 4.3. *Let $f : I \to I$ be an interval map. Then:*

- *If f is topologically R-mixing, then f is β-sensitive for all $\beta \in (0, \frac{|I|}{2})$.*

Proof. Let $I = [s,t]$ and assume that f is topologically R-mixing. Fix $\epsilon \in (0, \frac{|I|}{2})$, and let $y \in [s,t]$ and V be a neighborhood of y. By Proposition 3.3 of [11], there exists $m \geq 0$ such that

$$[s, s+\epsilon] \subset f^m(V) \quad \text{or} \quad [t-\epsilon, t] \subset f^m(V).$$

Thus there exist $x, z \in V$ with $f^m(x) = s + \epsilon$ and $f^m(z) = t - \epsilon$. Hence

$$\max\Big\{|f^m(y) - f^m(x)|,\ |f^m(y) - f^m(z)|\Big\} \geq \frac{(t-s) - 2\epsilon}{2} = \frac{|I|}{2} - \epsilon.$$

Thus y is β-unstable with $\beta := \frac{|I|}{2} - \epsilon$. Since ϵ is arbitrary, the result follows. □

The converse of Proposition 4.3 is false in general. However, somewhat surprisingly, a partial converse holds: instability on a subinterval guarantees the existence of a recurrent cycle of intervals.

Proposition 4.4. *Let f be an interval map. Suppose that, for some $\epsilon > 0$, the set $U_\epsilon(f)$ of ϵ-unstable points has nonempty interior. Then there exists a cycle of intervals $(K_1, \ldots, K_q)$ such that $f|_{K_1 \cup \cdots \cup K_q}$ is recurrent. Moreover,*

$$K_1 \cup \cdots \cup K_q \subset \overline{U_\epsilon(f)} \qquad \text{and} \qquad \exists j \in [\![1, q]\!] \text{ such that } |K_j| \geq \epsilon.$$

Proof. Define the family

$$F := \left\{X \subset \overline{U_\epsilon(f)} :\ X \text{ closed},\ f(X) \subset X,\ \mathrm{Int}(X) \neq \emptyset\right\}.$$

Since $U_\epsilon(f)$ contains a nonempty open interval J, Lemma (iii) of [28] implies $f^m(J) \subset U_\epsilon(f)$ for all $m \geq 0$. Define

$$\tilde{X} := \overline{\bigcup_{m\geq 0} f^m(J)}.$$

Then $\tilde{X} \in F$, so $F \neq \emptyset$.

Let $X \in F$ and let $K \subset X$ be a nondegenerate interval. Since $\mathrm{Int}(X) \cap U_\epsilon(f) \neq \emptyset$, Lemma (v) of [28] ensures that there exists $m \geq 0$ such that $|f^m(K)| \geq \epsilon$. Thus:

$$\textbf{Every } X \in F \textbf{ has a connected component } C \textbf{ with } |C| \geq \epsilon. \tag{1}$$

Equip F with the partial order $\subset$. Given a totally ordered subfamily $\{X_\alpha\}$, put

$$X = \bigcap_\alpha X_\alpha.$$

Then X is closed, forward invariant ($f(X) \subset X$), contains a connected component of length $\geq \epsilon$ by Lemma (v) of [28], and satisfies $\mathrm{Int}(X) \neq \emptyset$. Thus $X \in F$. By Zorn's lemma, F has a minimal element Y.

Now Y has at least one connected component J_j of length $\geq \epsilon$. By Lemma (v) of [11], repeated iteration yields an $m_j \geq 1$ such that $f^{m_j}(J_j) \subset J_j$. More generally, the image of J_j must fall into some J_{τ_j}. Since the total number of components is finite, there exist i, n with $f^n(J_j) \subset J_i$. Set:

$$Y' = \bigcup_{m=0}^{n} f^m(J_j).$$

Then $Y' \in F$ and $Y' \subset Y$. By minimality of Y, we get $Y' = Y$.

Hence Y has finitely many connected components $I_1, \ldots, I_q$, cyclically permuted by f:

$$f(I_j) = I_{j+1}, \quad j \in [\![1, q-1]\!], \qquad f(I_q) = I_1.$$

To show recurrence, suppose for contradiction that $f|_Y$ is not recurrent. Then there exists an open set $V \subset Y$ such that

$$f^m(V \cap Y) \cap (V \cap Y) = \emptyset \quad \forall m \geq 0.$$

Let $I \subset V \cap Y$ be a nonempty open interval and define

$$Z := \overline{\bigcup_{m\geq 0} f^m(I)}.$$

Then $Z \in F$ and $Z \subset Y$, yet $Z \cap V = \emptyset$, contradicting the minimality of Y. Hence $f|_Y$ is recurrent. □

Example 4.5. *We construct a sensitive interval map with a unique recurrent cycle of intervals. Even when a map has $\epsilon > 0$ such that every point is ϵ-unstable, the union of its recurrent cycles of intervals need not be dense. We illustrate this by constructing a sensitive interval map with exactly one recurrent cycle of $q+1$ intervals.*

Fix $q \geq 1$. Define

$$y_j := \frac{j}{2q+1}, \quad j \in [\![0, 2q+1]\!], \qquad I_j := [y_j, y_{j+1}],\ j \in [\![0, 2q]\!].$$

Define a continuous map $f\colon [0,1] \to [0,1]$ *by*

$$f(y_0) := y_3, \qquad f\left(\frac{y_0+y_1}{2}\right) := y_2, \qquad f(y_1) := y_3,$$
$$f(y_{2q-1}) := y_{2q+1}, \qquad f(y_{2q}) := y_0, \qquad f(y_{2q+1}) := y_1,$$

and extend linearly between these points. On $[y_1, y_{2q-1}]$, f *has slope* 1 *and satisfies* $f(y_j) = y_{j+2}$ *for all* $j \in [\![1, 2q-1]\!]$.

All points are ϵ*-unstable for* $\epsilon = \frac{1}{8(2q+1)}$, *but the system has exactly one recurrent cycle of intervals:*

$$(I_0, I_2, \dots, I_{2q}).$$

The cycle $C = I_0 \cup I_2 \cup \cdots \cup I_{2q}$ *is recurrent because* $f^{q+1}|_{I_0}$ *is conjugate to an inverted rescaled tent map* T_2 *([28]), and by Proposition 4.3* $f^{q+1}|_{I_0}$ *is* β*-sensitive with* $\beta := \frac{1}{2(2q+1)}$. *Hence* $C \subset U_\beta(f)$.

We show f *is* $\frac{\beta}{4}$*-sensitive.*

Case 1: the orbit of y ***meets the cycle*** C. *Assume* $f^m(y) \in C$ *for some* m. *The recurrence of* C *implies the existence of* $\epsilon_0 > 0$ *such that either*

$$f^m([y, y+\epsilon_0]) \subset C \quad \text{or} \quad f^m([y-\epsilon_0, y]) \subset C.$$

For any $\epsilon \in (0, \epsilon_0]$, $f^m([y-\epsilon, y+\epsilon])$ *contains a nondegenerate subinterval of* C. *By* β*-sensitivity on* C *there exist* $k \geq m$ *and* $x, z \in [y-\epsilon, y+\epsilon]$ *with*

$$|f^k(x) - f^k(z)| \geq \beta.$$

Consequently,

$$\max\{|f^k(y) - f^k(x)|, |f^k(y) - f^k(z)|\} \geq \frac{\beta}{2}.$$

Thus

$$\bigcup_{n\geq 0} f^{-n}(C) \subset U_{\beta/2}(f). \tag{1}$$

Case 2: the orbit of y ***never meets*** C. *Define*

$$Y := \bigcap_{m\geq 0} f^{-m}\Big(\bigcup_{j=0}^{q-1} I_{2j+1}\Big).$$

We claim that Y *is a Cantor set (see [28] for the definition). On each odd interval* I_{2j+1}, f *is a linear homeomorphism:*

$$f|_{I_{2j+1}} : I_{2j+1} \xrightarrow{\sim} I_{2j+3}\ (j < q-1), \qquad f|_{I_{2q-1}} : I_{2q-1} \xrightarrow{\sim} [0,1].$$

Hence the set of $y \in I_j$ *such that* $f^{q-j+1}(y)$ *lies in* $I_1 \cup I_3 \cup \cdots \cup I_{2q-1}$ *is a union of* q *disjoint closed intervals of equal length. Iterating this argument constructs* Y *as a Cantor set: closed, nowhere dense, perfect, with empty interior.*

Thus $\overline{[0,1] \setminus Y} = [0,1]$. *By* (1) *and the density lemma (Lemma (iv) of [28]),* f *is* $\frac{\beta}{4}$*-sensitive.*

Finally, suppose there existed another recurrent cycle C'. *Since* C *and* C' *are disjoint, there would exist a nondegenerate interval* I *such that* $f^m(I) \cap C = \emptyset$ *for all* m, *so* $I \subset Y$. *But* Y *is a Cantor set (no intervals), a contradiction. Thus* C *is the unique recurrent cycle of* f.

Conflict of Interest Statement.
The authors declare that they have no conflict of interest.
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