Filomat 40:3 (2026), 1045–1054
https://doi.org/10.2298/FIL2603045L

Published by Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

The complete w-core inverse in semigroups

Qi Lu[a,*], Shibo Shi[b], Long Wang[b]

[a]*School of Mathematics and Physics, Bengbu University, Bengbu, 233030, P. R. China*
[b]*School of Mathematical Sciences, Yangzhou University, Yangzhou 225002, P. R. China*

Abstract. Let S be a $*$-semigroup and let $a, w \in S$. The initial goal of this work is to consider commuting properties of a new class of generalized inverses, called the complete w-core inverse of a, extending the EP invertibility of generalized inverses. It is shown that a is completely w-core invertible if and only if a is w-core invertible and $a_w^{\#}$ commutes with aw if and only if a is w-EP invertible. Moreover, the representations of complete w-core inverses and w-EP inverses are both presented in rings. Also, the connections between the complete w-core inverse and other generalized inverses are given. More criterions for the complete w-core inverse is derived by units and one-sided ideals in rings.

1. Introduction

The idea of generalized inverses was first proposed by Ivar Fredholm in the process of studying integral equations [14]. Over the past hundred years, it appears in numerous applications that include areas such as networks [2, 4], linear estimation [13, 18], Markov chains [15, 19], coding theory and cryptography [6, 25, 30]. Many different types of generalized inverses have been introduced and studied whether at the level of matrices and operators, or at the level of elements in rings and semigroups. In this article, we focus on commuting properties of some given generalized inverses, as the generalization of the EP invertibility of generalized inverses theory.

Throughout, unless otherwise indicated, S denotes a semigroup. Following Drazin, an element $a \in S$ is Drazin invertible [8] if there exists some $x \in S$ such that (2) $xax = x$, (5) $xa = ax$ and (1′) $a^k = a^{k+1}x$ for some nonnegative integer k. Such an x is called a Drazin inverse of a. It is unique if it exists, and is denoted by a^D. The smallest nonnegative integer k in the condition (1′) is called the Drazin index of a, and is denoted by $ind(a)$. The element a is called group invertible if $ind(a) = 1$, and the group inverse of a is denoted by $a^{\sharp}$. The set of all group invertible elements of S is denoted by $S^{\sharp}$. In order to force its uniqueness, further conditions have to be imposed:

2020 *Mathematics Subject Classification*. Primary 15A09; Secondary 16E50, 16U80, 16W10.
Keywords. Commuting property, core inverse, complete w-core inverse, EP inverse, w-EP inverse, the inverse along an element.
Received: 23 February 2025; Revised: 18 November 2025; Accepted: 01 December 2025
Communicated by Dijana Mosić
The research is supported by the National Natural Science Foundation of China (12471133), Natural Science Foundation of Jiangsu Province (BK20200944), Natural Science Foundation of Anhui Higher Education Institutions of China (KJ2021A1131), the Key Scientific Research Foundation of the Education Department of Province Anhui (2022AH051907, 2025AHGXZK30855).
* Corresponding author: Qi Lu
Email addresses: luqi_bbu@163.com (Qi Lu), lwangmath@yzu.edu.cn (Long Wang)
ORCID iDs: https://orcid.org/0009-0004-6941-2369 (Qi Lu), https://orcid.org/0009-0004-3935-0718 (Shibo Shi), https://orcid.org/0009-0009-1690-987X (Long Wang)

$$(1)\ a = axa, \quad (3)\ (ax)^* = ax, \quad (4)\ (xa)^* = xa,$$
$$(6)\ xa^2 = a, \quad (7)\ ax^2 = x, \quad (8)\ a^2x = a, \quad (9)\ x^2a = x,$$
$$(\text{i})\ a \in aa^*R, \quad (\text{ii})\ a \in Ra^*a, \quad (\text{iii})\ a \in a^*R, \quad (\text{iv})\ a \in Ra^*.$$

If the equation (1) is solvable, then we say a is regular, and x is called an inner inverse (also called $\{1\}$-inverse) of a. In Eqs. (3, 4), the semigroup S is assumed to carry an involution $*$ (unary operation that satisfies $(x^*)^* = x$ and $(xy)^* = y^*x^*$ for all $x, y \in S$). A semigroup with a specific involution $*$ is called a $*$-semigroup. In a $*$-semigroup, a solution to the system (1, 2, 3, 4) is called the Moore-Penrose inverse (MP-inverse) of a, denoted by $x = a^\dagger$. If $x \in S$ satisfies both Eqs. (1, 3), then x is called an $\{1, 3\}$-inverse of a and denoted by $a^{(1,3)}$. The set of all $\{1, 3\}$-invertible elements of R is denoted by $S^{\{1,3\}}$. Similarly, $a^{(1,4)}$ and $S^{\{1,4\}}$ are defined, respectly. Specially, if $aa^\dagger = a^\dagger a$, then we say a is EP invertible. As we known, there are a variety ways to characterize elements with EP properties. It is proven that a is EP invertible if and only if $(aa^\sharp)^* = aa^\sharp$ if and only if there exists x satisfies Eqs. (3, 8, 9) [31] if and only if there exists x satisfies Eqs. (4, 6, 7) [31] if and only if a satisfies Eqs. (i, ii, iii, iv) [23].

Suppose that R is a $*$-ring, that is an associative ring with an involution $*$ satisfying $(x^*)^* = x$, $(xy)^* = y^*x^*$ and $(x + y)^* = x^* + y^*$ for all $x, y \in R$. The notion of core inverses was first introduced by Baksalary and Trenkler [1] for square complex matrices of index one. Later, Rakić et al. [24] extended this concept from $\mathbb{C}_{n\times n}$ to arbitrary $*$-rings. For any $*$-ring element a, the core inverse $a^{\#}$ of a is the unique solution to the system (1, 2, 3, 6, 7). In 2017, Xu et al. [32] found that the equations (1) and (2) above can be dropped, more precisely, they characterized the core inverse by the solution of three equations (3, 6, 7). Moreover, it was pointed out that a is core invertible if and only if $a \in R^\sharp \cap R^{\{1,3\}}$, in which case, $a^{\#} = a^\sharp aa^{(1,3)}$. For more relevant literature on the core inverse, the reader is referred to the references [1, 16, 20, 21, 24, 32]. As weaker versions of the core invertibility, one-sided core invertibility is introduced [26, 27, 29], the ideal of one-sided generalized invertibility is mainly origin from one-sided (b, c)-invertibility proposed by Drazin [12]. More details on (one-sided) core invertibility and (b, c)-invertibility can be found in [5, 7–11, 16, 22, 26, 34]. As the generalization of the core invertibility, the one-sided core invertibility is one of the generalization, the other one is weighted core invertibility, such as w-core and (b, c)-core invertibility introduced by Zhu et. al [34–36].

Recently, the authors consider commuting properties of the generalized inverse such as the complete inverse of a along d (abbr. completely Mary inverse) and the complete (b, c)-inverse [28]. As we known, the core inverse with the commutative property is EP invertible. Hence, the study on commutative property of w-core inverse can be considered as the deep research of generalized EP invertibility. Along this research thread, the complete w-core invertibility and the w-EP invertibility are introduced and investigated in this article. Specifically, it is proved that an element a is completely w-core invertible if and only if a is w-core invertible and $a_w^{\#}$ commutes with aw. The paper is organized as follows. In Section 2, w-EP inverses and complete w-core inverses were introduced and the relationship between them was investigated. It is proven that a is w-EP invertible if and only if aw is EP invertible and $a \in awS$ if and only if a is completely w-core invertible. The different between w-EP inverses and complete w-core inverses is the different expression. It is proven that if a is completely w-core invertible, then aw is EP and the complete w-core inverse of a is $(aw)^\sharp$. Moreover, the w-EP inverse of a is not unique and it is equal to $z + awzR(1 - awz)$, where $z = (aw)^\sharp$. In Section 3, more characterizations of complete w-core inverses and the related generalized inverses are provided.

2. w-EP inverses and complete w-inverses of a

At the begin of this section, we firstly provide some definitions and characterizations of well known generalized inverses in a semigroup.

Definition 2.1. *(1). [32, Theorem 3.1] Let S be a semigroup and $a \in S$. It is said that a is core invertible if there exists some $x \in S$ such that*

$$ax^2 = x,\ xa^2 = a \text{ and } ax = (ax)^*.$$

Such an x is called a core inverse of a and it is denoted by $a^{\text{\textcircled{\#}}}$.

(2). [36, Definition 2.1] Let $a, w \in S$. An element a is said to be w-core invertible if there exists some $x \in S$ such that

$$awx^2 = x,\ xawa = a \text{ and } awx = (awx)^*.$$

Such an x is called a w-core inverse of a and it is denoted by $a_w^{\text{\textcircled{\#}}}$.

(3). [31, Theorem 2.2, 2.7] We say that a is EP invertible if there exists $y \in S$ such that

$$y^2a = y,\ a^2y = a \text{ and } ay = (ay)^*,$$

or equivalently if there exists $z \in S$ such that

$$az^2 = z,\ za^2 = a \text{ and } za = (za)^*.$$

Compare with the several definitions in Definition 2.1, we extend the EP invertibility to the w-EP invertibility, and provide the definition of w-EP inverse as follows. It is worth noting that in Reference [33], the authors also proposed the corresponding definition of the w-EP inverse by means of the Mary inverse, and [33, Theorem 16] confirms that their definition is consistent with that of the w-EP inverse presented in this paper. For related results, the reader is referred to Reference [33]. To ensure consistency in the exposition of this paper, we present the required proof processes herein.

Definition 2.2. *Let $a, w \in S$. We say that a is w-EP invertible if there exists $x \in S$ such that*

$$awx^2 = x,\ xawa = a \text{ and } (xaw)^* = xaw.$$

In this case, x is called a w-EP inverse of a, denote it by a_w^{EP}.

Proposition 2.3. *Let $a, w \in S$. If a is w-EP invertible, then aw is EP invertible.*

Proof. Set $y = x^2aw$, where $x = a_w^{EP}$. Then we can obtain $awy = awx^2aw = xaw = x(xawa)w = yaw$. Moreover, it is easy to check that $awyaw = xawaw = aw$ and $yawy = (xaw)x^2aw = x^2aw = y$. This implies that $aw \in R^\sharp$ and $y = (aw)^\sharp$. Note that $awy = xaw = (awy)^*$. Hence, it shows that aw is EP invertible. □

Remark 2.4. *If aw is EP invertible, a is not w-EP invertible, in general. We only have to set $a = w = e_{12}$ in $M_2(\mathbb{C})$. It is clear that $aw = O_2$ is EP invertible. However, a is not w-EP invertible. Indeed, if a is w-EP invertible, then there exsits some x such that $xawa = a$. It leads to $a = O_2$, it is a contradiction.*

Lemma 2.5. *For any $a, w \in S$, if $x \in S$ is a w-EP inverse of a, then $x = xawx$ and $a = awxa$.*

Proof. As x is a w-EP inverse of a, then $awx^2 = x$ and $xawa = a$, which give that $xawx = xawawx^2 = awx^2 = x$ and $a = xawa = awx^2awa = awxa$. □

In the following, we will provide the relationship between the w-EP invertibility and the EP invertibility.

Theorem 2.6. *For any $a, w \in S$, a is w-EP invertible if and only if aw is EP invertible and $a \in awS$.*

Proof. Firstly, suppose that a is w-EP invertible. By Proposition 2.3 and Lemma 2.5, we known that aw is EP invertible and $a \in awS$. Conversely, since aw is EP invertible, we have $(aw)^\sharp$ and $(aw)^\dagger$ both exist and $(aw)^\sharp = (aw)^\dagger$. Set $x = (aw)^\sharp$. Then it is clear that $(xaw)^* = xaw$, $awx^2 = x$ and $xawaw = aw$. Hence, $xawa = xawaws = aws = a$ since $a = aws$ for some $s \in S$. □

Remark 2.7. *Let $a, w \in S$ and a be w-EP invertible. We claim that the w-EP inverse of a is not unique. Moreover, there exists some w-EP inverse x of a, such that $awx \neq xaw$. It means that there exists at least one w-EP inverse of a, which is not commutative with aw. Indeed, set $S = M_2(\mathbb{C})$, $a = e_{11}$ and $w = E_2$. Then, one can see $aw = a$ is EP invertible and $(aw)^\dagger = (aw)^\sharp = a = e_{11}$. Clearly, $x_1 = a$ is one w-EP inverse of a. Moreover, by a computation, we can obtain $x_2 = e_{11}M_2(1)$ is also one w-EP inverse of a. Further, $awx_2 \neq x_2aw$. We denote by $\{a_w^{EP}\}$ the set of all w-EP inverse of a.*

Assume that a is core invertible and x is the core inverse of a. It is well known that if a and x are commutative each other, then a is EP invertible. In [36], the authors introduced the w-core inverse, as the generalization of core inverse. In the following, we will consider the commutativity of the w-core inverse as the generalization of EP-invertibility. We called the w-core inverse with the commutative property as the complete w-core inverse.

Definition 2.8. *Let $a, w \in S$. We say that a is completely w-core invertible if there exists $z \in S$ such that*

$$awz^2 = z,\ a^* = a^*zaw \text{ and } awz = (awz)^*.$$

In this case, z is called a complete w-core inverse of a.

Remark 2.9. *Let $a, w \in S$. If a is completely w-core invertible and z is a complete w-core inverse of a, then we have $zawz = z$. Indeed, $awza = (wz)^*a^*a = (wz)^*a^*zawa = awz^2awa = zawa$. Post-multiplying by wz^2, we have $awza(wz^2) = zawa(wz^2)$, and consequently, $z = zawz$ since $awz^2 = z$.*

Theorem 2.10. *Let $a, w \in S$. Then a is completely w-core invertible if and only if a is w-core invertible and $a_w^{\textcircled{\#}}$ commutes with aw.*

Proof. If a is completely w-core invertible and z is a complete w-core inverse of a, then by Remark 2.9, one can see that $z = zawz = z(awz)^* = z(wz)^*a^*$. It gives that $z^2aw = z(wz)^*a^*zaw = z(wz)^*a^* = z$. Combine $z^2aw = z$ and $awz^2 = z$, it follows that $awz = awz^2aw = zaw$. Furthermore, this means that a is w-core invertible and $a_w^{\textcircled{\#}}$ commutes with aw.

Conversely, we only have to prove $a^* = a^*zaw$ where $z = a_w^{\textcircled{\#}}$. Indeed, by $zawa = a$ and $awz = zaw$, we can obtain $a = awza$, it follows from $(awz)^* = awz$ that $a^* = a^*awz = a^*zaw$. □

Remark 2.11. *In view of Theorem 2.10, it is known that the complete w-core inverse is unique if it exists. In fact, the complete w-core inverse of a is the w-core inverse of a which is commutative with aw. The complete w-core inverse of a is denoted by $a_w^{\circledast}$. We denote by $S_w^{\circledast}$ the set of all completely w-core invertible elements in S.*

In the following, we will provide the relationship between the complete w-core invertibility and the w-EP invertibility,

Theorem 2.12. *Let $a, w \in S$. Then a is completely w-core invertible if and only if a is w-EP invertible.*

Proof. As a is completely w-core invertible, by Theorem 2.10, we know that $a_w^{\circledast}$ commutes with aw. And then $a_w^{\circledast}$ is also a w-EP inverse of a.

Conversely, as we know that a is w-EP invertible, it follows from Theorem 2.6 that aw is EP invertible and $a \in awS$. Set $z = (aw)^\dagger = (aw)^\sharp$. We claim that z is the complete w-core inverse of a. Indeed, it is not difficult to check that z is a w-EP inverse of a. Herein, we only have to prove $a^*zaw = a^*$ By Lemma 2.5, one can see that $a^*zaw = a^*(aw)^\dagger aw = [aw(aw)^\dagger a]^* = [aw(aw)^\dagger awza]^* = (awza)^* = a^*$. □

Remark 2.13. *According to Remark 2.11 and Theorem 2.12, we can obtain if a is completely w-core invertible, then the complete w-core inverse is unique, and furthermore $a_w^{\circledast} = a_w^{\textcircled{\#}} = (aw)^\dagger = (aw)^\sharp$.*

In fact, we know that if a is completely w-core invertible, then a is w-core invertible and aw is EP invertible. Next, we will consider the converse and provide the following result.

Corollary 2.14. *Let $a, w \in S$. If a is w-core invertible and $x = a_w^{\textcircled{\#}}$, then the followings are equivalent:*

(i) a is completely w-core invertible.

(ii) $xaw = awx$.

(iii) $(xaw)^* = xaw$.

(iv) aw is EP invertible.

Proof. $(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (iii)$ By Definition 2.8 and Theorem 2.10, they are clear.

$(iii) \Rightarrow (vi)$ By [31, Theorem 2.2], it is clear.

$(vi) \Rightarrow (i)$ By Theorem 2.6 and Theorem 2.12, we only have to prove $a \in awS$. In fact, as a is w-core invertible, it follows from [36, Theorem 2.10] that $a \in awS$. □

By Theorem 2.12, we know that the complete w-core invertibility and the w-EP invertibility are coincide. However, by Remark 2.7 and Remark 2.11, one can see that the expressions of the w-EP inverse and the complete w-core inverse are different. Indeed, the complete w-core inverse is unique, and the w-EP inverse is not unique. Moreover, it is easy to find that the complete w-core inverse is a w-EP inverse of a. In the following, we will consider the expression of the w-EP inverse of a in a $*$-ring R.

Proposition 2.15. *Let $a, w \in R$. If a is w-EP invertible, then $\{a_w^{EP}\} = z + awzR(1 - awz)$, where $z = a_w^{\circledast}$.*

Proof. Since a is w-EP invertible, it follows from Theorem 2.12 and Remark 2.13 that a is completely w-core invertible, and aw is EP. Write $z = a_w^{\circledast} = (aw)^{\dagger}$. By a computation, we can obtain $x = z + awzr(1 - awz)$ is a w-EP inverse of a for any $r \in R$. That is, $z + awzR(1 - awz) \subseteq \{a_w^{EP}\}$. Indeed, by Theorem 2.10, we know that $a = zawa = awza$, and then $xaw = zaw = (xaw)^*$ and $xawa = zawa = a$. Moreover, by Remark 2.9, it is easy to check that $x^2 = z^2 + zr(1 - awz)$ and $awx^2 = z + awzr(1 - awz) = x$.

Conversely, set x be a w-EP inverse of a and $z = (aw)^{\dagger}$. Then we can check that $awz(x-z) = awzx - awz^2 = awzawx^2 - z = x - z$, and $(x - z)awz = xawz - z = (xawa)wz^2 - z = 0$. Hence, $awz(x - z)(1 - awz) = x - z$, and consequently, $x = z + awz(x - z)(1 - awz)$. So, one can see that $\{a_w^{EP}\} \subseteq z + awzR(1 - awz)$. □

As the special case of complete w-core inverses and w-EP inverses, when $w = 1$, we can prove that a is completely 1-core invertible if and only if a is 1-EP invertible if and only if a is EP invertible. And in this case, $a_1^{\circledast} = a^{\#} = a^{\dagger}$ and $\{a_1^{EP}\} = \{a^{\#} + aa^{\#}r(1 - aa^{\#})\}$ for any $r \in R$. In fact, we also can provide a equivalent definition of w-EP invertibility as follows. And moreover, it is necessary to repeat that the ideal of this equivalent definition is similar to the Definition 2.1 (3).

Proposition 2.16. *Let $a, w \in S$. Then the followings are equivalent:*

(i) a is w-EP invertible.

(ii) There exists $y \in S$ such that $(awy)^* = awy$, $y^2aw = y$ and $awawy^2a = a$.

Proof. We claim that the condition (ii) is also equivalent to aw is EP invertible and $a \in awS$. Indeed, by the condition (ii) $y^2aw = y$ and $awawy^2a = a$, one can see that $a \in awS$ and $awawy^2aw = aw$, consequently, we have $awawy = aw$. It follows from [31, Theorem 2.7] that aw is EP invertible. Conversely, if aw is EP invertible and $a \in awS$, then, by [31, Theorem 2.7], there exists some $y \in S$ such that $(awy)^* = awy$, $y^2aw = y$ and $awawy = aw$. Furthermore, we can obtain $awawy^2aw = aw$. Then, it follows from $awawy^2aw = aw$ that $awawy^2a = a$ since $a = aws$ for some $s \in S$. □

Remark 2.17. *In a $*$-ring R, we claim that $\{y\} = z + (1 - awz)Rawz$, where $z = a_w^{\circledast}$, under the condition of Proposition 2.16. Since a is w-EP invertible, we have aw is EP invertible and $z = (aw)^{\dagger}$. Set $y = z + (1 - awz)rawz$ for any $r \in R$. By a computation, we can obtain y satisfies the condition (ii) of Proposition 2.16. Indeed, we can check that $awy = awz = (awy)^*$ and $y^2 = z^2 + (1 - awz)rz$. It gives that $y^2aw = z^2aw + (1 - awz)rzaw = z + (1 - awz)rawz = y$ and $awy^2 = awz^2 = z$. Furthermore, since a is w-EP invertible, by Theorem 2.6, we have $a \in awR$, and then $awawy^2a = awza = awzaws = aws = a$ for some $s \in R$. This implies that $z + (1 - awz)Rawz \subseteq \{y\}$. Conversely, for any element y satisfies the condition (ii) of Proposition 2.16, we can check that $(y - z)awz = (1 - awz)(y - z) = y - z$. Indeed, by a computation, we can obtain $(y - z)awz = yawz - zawz = y^2awawz - z = y^2aw - z = y - z$ and $awz(y - z) = awzy - awz^2 = awzy^2aw - z = z^2(aw)^2y^2aw - z = z^2aw - z = 0$. Hence, we have that $y - z = (1 - awz)(y - z)awz$, furthermore, $y = z + (1 - awz)(y - z)awz \in z + (1 - awz)Rawz$.*

Remark 2.18. *Note that in Proposition 2.16 it is not difficult to find that $awya = a$. Indeed, in the proof of Proposition 2.16, we know that aw is EP invertible, and then $awya = awy^2awa = z(aw)^2y^2awa = zawa = awz(awawy^2a) = awawy^2a = a$, where z is the EP inverse of aw.*

(1). Maybe, we can guess that a is w-EP invertible if and only if there exists $y \in S$ such that $(awy)^* = awy$, $y^2aw = y$ and $awya = a$. In fact, it is not true, in generally. This is a new one-sided EP inverse, the authors will discuss this topic in a new article. Herein, we can provide a counter-example, by checking the following Example 2.19.

(2). Moreover, similar to the definition of Definition 2.1 (3), maybe we guess that a is w-EP invertible if and only if there exists $y \in S$ such that $(awy)^* = awy$, $y^2aw = y$ and $away = a$. In fact, it is also not true, in generally. By checking the following Remark 2.20, herein, we only have to set $a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $w = \begin{pmatrix} 3 & 6 \\ 1 & 0 \end{pmatrix}$. Then a is w-EP invertible. However, it is easy to check that $awa = O_2$. And it is a contradiction that there exists some $y \in S$ such that $away = a$ if a is w-EP invertible. Of course, by this counter example, we can see that it is also not true that a is w-EP invertible if and only if there exists $y \in S$ such that $(awy)^* = awy$, $y^2aw = y$ and $a^2wy = a$.

Example 2.19. *Let S be the semigroup of infinite matrices over the real number field $\mathbb{R}$, which are row and column-finite. The involution $*$ of S is the transpose of real matrices. Let*

$$a = \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & 1 & 0 & \\ & & \ddots & \ddots \end{pmatrix}, \; y = a^* = \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & \ddots \\ & & & \ddots \end{pmatrix} \text{ and } w = E.$$

Then $awy = \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & 1 & \\ & & & \ddots \end{pmatrix} = (awy)^*$, $ya = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & \ddots \end{pmatrix}$. *It is clear that $awya = a$ and $y^2aw = y$. Moreover, it is not difficult to find that $awawy^2a = a^2y \neq a$. In fact, we know that when a is w-EP invertible, then $a \in (aw)^2S$. However, in this case, a is not w-EP invertible, since $a \notin (aw)^2S$.*

Remark 2.20. *For any $a, w \in S$, if a is completely w-core invertible, then a is also w-EP invertible. Set $z = a^{\circledast}_w$. Then, we know z is also a w-EP inverse of a. By Lemma 2.5, we know that $z = zawz$ and $a = awza$. However, we claim that $azaw \neq a$. Indeed, we only have to set $a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $w = \begin{pmatrix} 3 & 6 \\ 1 & 0 \end{pmatrix}$. Then a is completely w-core invertible and $z = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Clearly, $azaw = O_2$.*

According to the above discuss, we can obtain the following corollary, when $w = 1$.

Corollary 2.21. *Let $a \in R$. Then the following conditions are equivalent:*
(i) a is EP invertible.
(ii) There exists $x \in S$ such that $(xa)^* = xa$, $ax^2 = x$ and $xa^2 = a$.
(iii) There exists $y \in S$ such that $(ay)^* = ay$, $y^2a = y$ and $a^2y = a$.
(iv) There exists $z \in S$ such that $(az)^* = az$, $az^2 = z$ and $a^*za = a^*$.
In this case, $z = a^\dagger = a^\sharp$, $\{y\} = a^\sharp + (1 - aa^\sharp)Raa^\sharp$ and $\{x\} = a^\sharp + aa^\sharp R(1 - aa^\sharp)$.

In what follows, we will continue to consider the special case of the complete w-core inverse, when we choose different elements for w.

Proposition 2.22. *Let $a \in S$. Then the following conditions are equivalent:*
(i) $a \in S^{\circledast}_a$.
(ii) $a \in S^{EP}$.

Proof. From Theorem 2.10, we know that if a is completely a-core invertible, then a is a-core invertible and $a^{\textcircled{\#}}_a a^2 = a^2 a^{\textcircled{\#}}_a$. By [36, Proposition 2.15], we can find that, in this case, a is core invertible and $a^{\textcircled{\#}}_a = a^\sharp a^{\textcircled{\#}}$. As $a^{\textcircled{\#}}_a a^2 = a^2 a^{\textcircled{\#}}_a$, it gives that $a^\sharp a^{\textcircled{\#}} a^2 = aa^{\textcircled{\#}}$, and consequently, $aa^\sharp = aa^{(1,3)}$ since $a^{\textcircled{\#}} = a^\sharp aa^{(1,3)}$, hence we have $a \in S^{EP}$. Conversely, we only have to set $x = (a^\sharp)^2$. Then it is easy to check that x is the complete a-core inverse of a. □

Proposition 2.23. *Let $a \in S$. Then the following conditions are equivalent:*
(i) $a \in S^{\circledast}_{a^*}$.
(ii) $a \in S^{\dagger}$.

Proof. From Theorem 2.10, we know that if a is completely a^*-core invertible, then a is a^*-core invertible, and hence $a \in S^{\dagger}$ by [36, Proposition 2.22]. Conversely, assume that $a \in S^{\dagger}$, and set $x = (a^{\dagger})^* a^{\dagger}$. Then it is easy to check that x is the complete a^*-core inverse of a. □

In fact, we can extend the result of Proposition 2.22 as the following form.

Proposition 2.24. *Let $a \in S$ and let $n \geq 2$ be an integer. Then the following conditions are equivalent:*
(i) $a \in S^{\circledast}_{a^n}$.
(ii) $a \in S^{EP}$.

Proof. Since a is completely a^n-core invertible, we have a^{n+1} is EP invertible and $a \in a^{n+1}S$. Set $z = a^n(a^{n+1})^{\#} = a^n(a^{n+1})^{\dagger}$. By the double commutativity property of the group inverse, it leads to $za = [a^n(a^{n+1})^{\#}]a = a^{n+1}(a^{n+1})^{\#} = az$. This implies $az = (az)^* = za = (za)^*$. Furthermore, since $a \in a^{n+1}S$, we can see $aza = a^{n+1}(a^{n+1})^{\#}a = a^{n+1}(a^{n+1})^{\#}a^{n+1}t = a$ for some $t \in S$. Additionally, $zaz = (a^{n+1})^{\#}a^{n+1}a^n(a^{n+1})^{\#} = z$. Thus, a is EP invertible with $a^{\dagger} = a^{\#} = z$.

Conversely, if a is EP invertible, then we only have to set $x = (a^{\#})^n = (a^{\dagger})^n$. Then it is easy to check that x is the complete a^n-core inverse of a. □

According to Proposition 2.23 and Proposition 2.24, we will guess that, for any integer $n \geq 2$, a is completely $(a^n)^*$-core invertible if and only if a is Moore-Penrose invertible. In the following result, we will explain that if a is completely $(a^n)^*$-core invertible then a is Moore-Penrose invertible. However, unfortunately, we have the following counter-example to show that the converse is not true, in generally.

Remark 2.25. *As we known, a is completely $(a^n)^*$-core invertible, then there exists $x \in S$ such that $a(a^n)^*x^2 = x$, $a^* = a^*xa(a^n)^*$ and $[a(a^n)^*x]^* = a(a^n)^*x$. Moreover, we know that $a(a^n)^*x = xa(a^n)^*$, and consequently, we can obtain that $a = x^*a^na^*a$. It gives that $a \in Saa^*a$ and then $a \in S^{\dagger}$.*

Example 2.26. *Set $S = M_2(\mathbb{C})$ and $a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then we can obtain that a is Moore-Penrose invertible and $a^{\dagger} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. However, it is easy to find that $a(a^n)^* = O_2$, where $n \geq 2$. And it implies that $a \notin a(a^2)^*S$. By Theorem 2.6 and Theorem 2.12, we know that a is not completely $(a^n)^*$-core invertible.*

3. More characterizations of complete w-inverses of a

In [36], the authors point out one important characterization of the w-core invertibility. That is, a is w-core invertible if and only if $w^{\|a}$, $a^{(1,3)}$ both exist. And in this case, $w^{\|a} = a^{\circledast}_{w}a$ and $a^{\circledast}_{w} = w^{\|a}a^{(1,3)}$. In the following, we will consider more characterizations of the completely w-core invertibility.

Proposition 3.1. *Let $a, w \in S$. Then a is completely w-core invertible if and only if $w^{\|a}$, $a^{(1,3)}$ both exist and $w^{\|a}w = aa^{(1,3)}$.*

Proof. ⇒: Since a is completely w-core invertible, by [36, Theorem 2.6], we have that $w^{\|a}$, $a^{(1,3)}$ both exist and $w^{\|a} = a^{\circledast}_{w}a$, which guarantee $w^{\|a}w = a^{\circledast}_{w}aw$. As $aw(a^{\circledast}_{w})^2 = a^{\circledast}_{w}$, then $aa^{(1,3)}a^{\circledast}_{w} = a^{\circledast}_{w}$, and consequently, we can obtain

$$w^{\|a}w = aa^{(1,3)}a^{\circledast}_{w}aw = (a^{(1,3)})^*a^*a^{\circledast}_{w}aw = (a^{(1,3)})^*a^* = aa^{(1,3)}.$$

⇐: By the hypothesis, $w^{\|a}$ and $a^{(1,3)}$ both exist, then a is w-core invertible. It follows from [36, Theorem 2.10] that aw is core invertible and $a \in awS$. Next, we only have to prove aw is EP invertible and then, by Theorem 2.6 and Theorem 2.12, we can obtain a is completely w-core invertible. Indeed, in terms of [17, Theorem 7], we know that if $w^{\|a}$ exists, then $(aw)^{\#}$ exists and $w^{\|a} = (aw)^{\#}a$. As $w^{\|a}w = aa^{(1,3)}$, then $aa^{(1,3)} = w^{\|a}w = (aw)^{\#}aw$, which implies $[(aw)^{\#}aw]^* = (aw)^{\#}aw$ and consequently, aw is EP invertible. □

Remark 3.2. *By checking the proof of Proposition 3.1, one can see that a is completely w-core invertible if and only if* $w^{\|a}$, $a^{(1,3)}$ *both exist and* $(w^{\|a}w)^* = w^{\|a}w$.

Next, we want to consider what will happened when we replace $\{1,3\}$-invertibility into EP invertibility in Remark 3.2.

Remark 3.3. *In fact, by Remark 2.20, we know a is completely w-core invertible, but not be EP invertible, in generally. We only have to set* $a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ *and* $w = \begin{pmatrix} 3 & 6 \\ 1 & 0 \end{pmatrix}$. *Then a is completely w-core invertible and* $a^{\circledast}_w = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. *It is clear that a is not group invertible and consequently a is not EP, since* $a^2 = 0$ *and* $a \notin a^2S$.

Moreover, in this case, we know that $awa = a$ *and* $w^{\|a}$ *exists. It means that if* $w^{\|a}$ *exists, then a is completely w-core invertible can not imply that a is EP.*

In [28], the authors consider commuting properties of the generalized inverse along one element, and introduce one kind of generalized inverse, the complete inverse of a along d. Let $a, d \in S$. It is said that a is completely invertible along d if there exists $x \in S$ such that $axd = d = dxa$ and $x \leq_{\mathcal{H}} d$. In this case, x is called a complete inverse of a along d, and use the notation $a^{\sharp d}$ to denote.

Theorem 3.4. *Let* $a, w \in R$. *If* $w^{\sharp a}$ *exists, then a is EP invertible if and only if a is completely w-core invertible.*

Proof. $\Leftarrow$ Since a is completely w-core invertible, by Remark 2.13, we have that aw is EP invertible and $a^{\circledast}_w = (aw)^{\sharp} = (aw)^{\dagger}$. Set $z = wa^{\circledast}_w$. Then, we can obtain $az = aw(aw)^{\dagger}$ and $(az)^* = az$. And by $awa^{\circledast}_w a = a$ and $a^{\circledast}_w awa^{\circledast}_w = a^{\circledast}_w$, it is easy to find $aza = aw(aw)^{\dagger}a = aw(aw)^{\dagger}(awa^{\circledast}_w a) = a$ and $zaz = wa^{\circledast}_w awa^{\circledast}_w = z$. Moreover, by hypothesis, we know that $w^{\|a} = (aw)^{\sharp}a$ and $w^{\|a}w = ww^{\|a}$. Then, it means that $w(aw)^{\sharp}a = (aw)^{\sharp}aw$, and consequently, we have $za = w(aw)^{\sharp}a = (aw)^{\sharp}aw = az$. Hence, one can see that a is EP invertible, and $a^{\dagger} = a^{\sharp} = z$.

$\Rightarrow$ By [36, Theorem 2.6], if a is EP invertible and $w^{\|a}$ exists, then a is w-core invertible and $a^{\circledast}_w = w^{\|a}a^{\sharp}$. Herein, we only have to prove that $a^{\circledast}_w$ commutes with aw. That is to say, we need to prove this equality $w^{\|a}a^{\sharp}aw = aww^{\|a}a^{\sharp}$. Herein, note that when $w^{\sharp a}$ exists, then $aa^{\sharp}w = waa^{\sharp}$ [28, Theorem 3.12]. Hence, we can obtain that the left side of the equality is $w^{\|a}a^{\sharp}aw = w^{\|a}waa^{\sharp} = aa^{\sharp}$. And the right side of the equality is $aww^{\|a}a^{\sharp} = aa^{\sharp}$. It is proven that $a^{\circledast}_w$ commutes with aw. $\square$

Lemma 3.5. *Let* $a, w \in R$. *If* $w^{\|a}$ *exists and* $aw = wa$, *then* $w^{\sharp a}$ *exists.*

Proof. By hypothesis, $w^{\|a}$ exists and by [17, Theorem 7], one can see that $(aw)^{\sharp}$ exists and $w^{\|a} = (aw)^{\sharp}a$. From [10, Theorem 2.2] and $aw = wa$, then we have $w(aw)^{\sharp} = (aw)^{\sharp}w$. This implies that $w^{\|a}w = (aw)^{\sharp}aw = w(aw)^{\sharp}a = ww^{\|a}$, and consequently, $w^{\sharp a}$ exists. $\square$

Corollary 3.6. *Let* $a, w \in R$ *and* $aw = wa$. *The following conditions are equivalent:*

(i) a is completely w-core invertible.

(ii) a is EP invertible and $w^{\|a}$ *exists.*

In this case, $a^{\circledast}_w = w^{\|a}a^{\sharp}$.

We need to say, at last part of this section, more criterions for the complete w-core inverse is derived by units and one-sided ideals in a $*$-ring R. In [3, Theorem 2.1], the author gave the characterizations and expressions of EP invertibility by a projection and units in a ring. It is proven that a is EP invertible if and only if there exists a projection p such that $pa = ap = 0$ and $a + p \in R^{-1}$. In this section, we present some equivalent conditions for the existence of completely w-core inverses. Before we start, look at the following results.

Theorem 3.7. *Let* $a, w \in R$. *The following conditions are equivalent:*

(i) a is completely w-core invertible.

(ii) There exists a unique projection $p \in R$ *such that* $awp = pa = 0$ *and* $u = p + aw \in R^{-1}$.

(iii) There exists a projection $p \in R$ *such that* $awp = pa = 0$ *and* $u = p + aw \in R^{-1}$.

In this case, $a^{\circledast}_w = u^{-1}(1 - p)$.

Proof. $(i) \Rightarrow (ii)$ Suppose that $a \in R$ is completely w-core invertible, then, by Theorem 2.10 and Theorem 2.12, aw is EP invertible and $a \in awR$. So, there exists a unique projection $p \in R$ such that $awp = paw = 0$ and $u = p + aw \in R^{-1}$. Since $a \in awR$, the equality $paw = 0$ implies $pa = 0$.

$(ii) \Rightarrow (iii)$ It is clear.

$(iii) \Rightarrow (i)$ Set $z = u^{-1}(1-p)$. By $(1-p)u = aw$, it gives that $awz = awu^{-1}(1-p) = 1-p$. As $pu = p$ and $u(1-p) = aw$, then $p = pu^{-1}$ and $1-p = u^{-1}aw$, and hence, we can obtain that $awz^2 = (1-p)u^{-1}(1-p) = u^{-1}(1-p) - pu^{-1}(1-p) = z$. Moreover, $zaw = u^{-1}(1-p)aw = u^{-1}aw = 1-p$, it shows that $a^*zaw = a^*(1-p) = a^*$. This means that a is completely w-core invertible and $a_w^{\circledast} = u^{-1}(1-p)$. □

Following Drazin, given any $a \in R$, the left annihilator of a is defined by $l(a) = \{x \in R : xa = 0\}$, and the right annihilator of a is defined by $r(a) = \{x \in R : ax = 0\}$.

Theorem 3.8. *Let $a, w \in R$. Then the following conditions are equivalent:*

(i) a is completely w-core invertible.

(ii) There exists some $x \in R$ such that $a^*xaw = a^*$, $xR = aR$ and $Rx = Ra^*$.

(iii) There exists some $x \in R$ such that $a^*xaw = a^*$, $l(x) = l(a)$ and $r(x) = r(a^*)$.

(iv) There exists some $x \in R$ such that $a^*xaw = a^*$, $l(x) = l(a)$ and $r(x) \subseteq r(a^*)$.

(v) There exists some $x \in R$ such that $a^*xaw = a^*$, $l(a) \subseteq l(x)$ and $r(x) \subseteq r(a^*)$.

Proof. $(i) \Rightarrow (ii)$ by [36, Theorem 2.13].

$(ii) \Rightarrow (iii)$, $(iii) \Rightarrow (iv)$ and $(iv) \Rightarrow (v)$ are clear.

$(v) \Rightarrow (i)$ As $a^*xaw = a^*$, we have $a = (xaw)^*a$, and it gives that $1-(xaw)^* \in l(a) \subseteq l(x)$. Then we can obtain $x = (xaw)^*x$. It follows from $xaw = (xaw)^*xaw$ that $xaw = (xaw)^*$ and $x = xawx$. By $x = xawx$, it gives that $1-awx \in r(x) \subseteq r(a^*)$, and then $a^* = a^*awx$. It implies $a = (awx)^*a$ and $awx = (awx)^*awx$. Hence, we can obtain $(awx)^* = awx$ and $a = awxa$. It follows from $a = awxa$ that $1-awx \in l(a) \subseteq l(x)$, and then $x = awx^2$. □

Set $w = 1$ in Theorem 3.8, we get the characterization for EP inverses in $*$-rings.

Corollary 3.9. *Let $a \in R$. Then the following conditions are equivalent:*

(i) a is EP invertible.

(ii) There exists some $x \in R$ such that $a^*xa = a^*$, $xR = aR$ and $Rx = Ra^*$.

(iii) There exists some $x \in R$ such that $a^*xa = a^*$, $l(x) = l(a)$ and $r(x) = r(a^*)$.

(iv) There exists some $x \in R$ such that $a^*xa = a^*$, $l(x) = l(a)$ and $r(x) \subseteq r(a^*)$.

(v) There exists some $x \in R$ such that $a^*xa = a^*$, $l(a) \subseteq l(x)$ and $r(x) \subseteq r(a^*)$.

References

[1] O.M. Baksalary, G. Trenkler, *Core inverse of matrices*, Linear Multilinear Algebra **58** (2010) 681-697.

[2] A. Ben-Israel, *Applications of generalized inverses to programming, games and networks, in: Generalized Inverses and Applications*, Academic Press 1976.

[3] J. Benítez, *Moore–Penrose inverses and commuting elements of C^*-algebras*, J. Math. Anal. Appl. **345** (2008) 766-770.

[4] R. Bott, R.J. Duffin, *On the algebra of networks*, Trans. Am. Math. Soc. **74** (1953) 99-109.

[5] X.F. Chen, J.L. Chen, *Right core inverses of a product and a companion matrix*, Linear Multilinear Algebra **69** (2021) 2245-2263.

[6] E. Dawson, C.K. Wu, *Key agreement scheme based on generalised inverses of matrices*, Electron. Lett. **33** (1997) 1210-1211.

[7] M.P. Drazin, *A class of outer generalized inverses*, Linear Algebra Appl. **436** (2012) 1909-1923.

[8] M.P. Drazin, *Pseudo-inverses in associative rings and semigroups*, Amer. Math. Monthly **65** (1958) 506-514.

[9] M.P. Drazin, *EP properties of (b,c)-invertible matrices*, Linear and Multilinear Algebra **12** (2020) 1675-1681.

[10] M.P. Drazin, *Commuting properties of generalized inverses*, Linear and Multilinear Algebra **61** (2013) 1675-1681.

[11] M.P. Drazin, *Weighted (b,c)-inverses in categories and semigroups*, Commun. Algebra **48** (2020) 1423-1438.

[12] M.P. Drazin, *Left and right generalized inverses*, Linear Algebra Appl. **510** (2016) 64-78.

[13] R.J. Elliott, J. Van der Hoek, *Optimal linear estimation and data fusion*, IEEE Trans. Autom. Control **51** (2006) 686-689.

[14] I. Fredholm, *Sur une classe d'équations fonctionnelles*, Acta Math. **27** (1903) 365–390.

[15] J.J. Hunter, *Generalized inverses of Markovian kernels in terms of properties of the Markov chain*, Linear Algebra Appl. **447** (2014) 38-55.

[16] T.T. Li, J.L. Chen, *Characterizations of core and dual core inverses in rings with involution*, Linear Multilinear Algebra **66** (2018) 717-730.

[17] X. Mary, *On generalized inverses and Green's relations*, Linear Algebra Appl. **434** (2011) 1836-1844.

[18] D.W. Marquaridt, *Generalized inverses, ridge regression, biased linear estimation, and nonlinear estimation*, Technometrics **12** (1970) 591-612.

[19] C.D. Meyer, *The role of the group generalised inverse in the theory of finite Markov chains*, SIAM Rev. **17** (1975) 443-464.
[20] D. Mosić, *New generalizations of the core and dual core inverses*, Publicationes Mathematicae **105** (2024) 119-140.
[21] D. Mosić, C.Y. Deng, H.F. Ma, *On a weighted core inverse in a ring with involution*, Commun. Algebra **6** (2018) 2332-2345.
[22] D. Mosić, P.S. Stanimirović, J.K. Sahoo, R. Behera, V.N. Katsikis, *One-sided weighted outer inverses of tensors*, J. Comput. Appl. Math. **388** (2021) 113293.
[23] P. Patrićio, R. Puystjens, *Drazin-Moore-Penrose invertibility in rings*, Linear Algebra Appl. **389** (2004) 159-173.
[24] D.S. Rakić, N.S. Dinčić, D.S. Djordjević, *Group, Moore-Penrose, core and dual core inverse in rings with involution*, Linear Algebra Appl. **463** (2014) 115-133.
[25] H.M. Sun, *Cryptanalysis of a public-key cryptosystem based on generalized inverses of matrices*, IEEE Commun. Lett. **5** (2001) 61-63.
[26] L. Wang, D. Mosić, *The one-sided inverse along two elements in rings*, Linear Multilinear Algebra **69** (2021) 2410-2422.
[27] L. Wang, D. Mosić, Y.F. Gao, *Right core inverse and the related generalized inverses*, Commun. Algebra **47** (2019) 4749-4762.
[28] L. Wang, D. Mosić, H. Yao, *The commuting inverse along an element in semigroups*, Commun. Algebra **50** (2022) 433-443.
[29] C. Wu, J.L. Chen, *Left core inverses in rings with involution*, Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matematicas **67** (2022) 116.
[30] C.K. Wu, E. Dawson, *Generalised inverses in public key cryptosystem design*, IEE Proc., Comput. Digit. Tech. **145** (1998) 321-326.
[31] S.Z. Xu, J. L.Chen, J. Benítez, *EP Elements in Rings with Involution*, Bull. Malays. Math. Sci. Soc. **6** (2016) Page 18.
[32] S.Z. Xu, J.L.Chen, X.X. Zhang, *New characterizations for core inverses in rings with involution*, Front. Math. China **12** (2017) 231-246.
[33] Y.X. Yang, H.H. Zhu, O.M. Baksalary, *On a generalization of an EP element*, Commun. Algebra, **1** (2025) 1-14.
[34] H.H. Zhu, *The (b, c)-core inverse and its dual in semigroups with involution*, J. Pure Appl. Algebra **228** (2024) 107526.
[35] H. H. Zhu, *Further results on several types of generalized inverses*, Comm. Algebra **46** (2018) 3388-3396.
[36] H.H. Zhu, L.Y. Wu, J.L. Chen, *A new class of generalized inverses in semigroups and rings with involution*, Commun. Algebra **51** (2023) 2098-2113.