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The WGGD-inverse for square matrices

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Abstract. This paper will introduce a new class of generalized inverses for square matrices: the weak group G-Drazin (WGGD) inverse. The WGGD inverse is not unique and defined as a proper composition of the weak group inverse and the G-Drazin inverse. We investigate the characterizations, representations, and properties of the inverse. A variant of the successive matrix squaring computational iterative scheme is given for calculating the WGGD inverse. Moreover, the Cramer's rule for the solution of a singular equation $Ax = b$ is presented. In addition, we consider some additional properties of the WGGD inverse through an induced binary relation. In the final, the WGGD inverse being used in solving appropriate systems of linear equations is established.

1. Introduction

Throughout this paper, we denote the set of all $m \times n$ complex matrices by $\mathbb{C}^{m\times n}$. For $A \in \mathbb{C}^{m\times n}$, the symbols A^*, $\mathrm{rank}(A)$, $N(A)$ and $R(A)$ stand for the conjugate transpose, the rank, the null space and the range space of A, respectively. Moreover, I_n will refer to the $n \times n$ identity matrix.

Let $A \in \mathbb{C}^{n\times n}$, the smallest positive integer k for which $\mathrm{rank}(A^k)$ $=\mathrm{rank}(A^{k+1})$ is called the index of A and is denoted by $\mathrm{Ind}(A) = k$. Then $\mathbb{C}_k^{n\times n}$ represents all $n \times n$ complex matrices with index k. $P_{E,F}$ represents the projector on the subspace E along the subspace F. For $A \in \mathbb{C}^{n\times n}$, P_A stands for the orthogonal projection onto $R(A)$. The symbol $\mathbb{C}_n^{CM}$ represents the subset of all $n \times n$ complex matrices with index 1.

Next, let's review the definitions of some generalized inverses. For $A \in \mathbb{C}^{m\times n}$, the Moore-Penrose inverse $A^\dagger$ of A is the unique matrix $X \in \mathbb{C}^{n\times m}$ satisfying the following four Penrose equations [1]:

$$AXA = A, \quad XAX = X, \quad (AX)^* = AX, \quad (XA)^* = XA.$$

The Moore-Penrose inverse can be used to represent orthogonal projectors $P_A := AA^\dagger$ onto $R(A)$ and $Q_A := A^\dagger A$ onto $R(A^*)$, respectively. A matrix $X \in \mathbb{C}^{n\times m}$ that satisfies the equality $AXA = A$ is called an

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inner inverse or $\{1\}$-inverse of A, and a matrix $X \in \mathbb{C}^{n\times m}$ that satisfies the equality $XAX = X$ is called an outer inverse or $\{2\}$-inverse of A.

The Drazin inverse is a kind of outer inverse defined for square matrices. For $A \in \mathbb{C}^{n\times n}$ and $\mathrm{Ind}(A) = k$, the Drazin inverse A^D of A is the unique matrix $X \in \mathbb{C}^{n\times n}$ satisfying the following three equations [24]:

$$A^{k+1}X = A^k, \qquad XAX = X, \qquad AX = XA.$$

In particular, if $\mathrm{Ind}(A) = 1$, $A^D = A^\#$ is the group inverse of A.

For $A \in \mathbb{C}^{n\times n}$ with $\mathrm{Ind}(A) = k$, the core-EP inverse $A^{\textcircled{\dagger}}$ of A is the unique matrix $X \in \mathbb{C}^{n\times n}$ satisfying the following conditions [16]:

$$XAX = X, \qquad R(A^k) = R(X) = R(X^*).$$

Obviously, the core-EP inverse is an outer inverse of A. Recall that, by [7], the core-EP inverse can be expressed as $A^{\textcircled{\dagger}} = A^D A^k (A^k)^\dagger$.

The weak group inverse is proposed by Wang and Chen [22] for square matrices of an arbitrary index as an extension of the group inverse. For $A \in \mathbb{C}^{n\times n}$, the weak group inverse $A^{\textcircled{w}}$ of A is the uniquely determined matrix that satisfying:

$$AX^2 = X, \qquad AX = A^{\textcircled{\dagger}}A.$$

Notice that, by [22], we have $A^{\textcircled{w}} = (A^{\textcircled{\dagger}})^2 A$. Two new generalized inverses have emerged by combining Moore-Penrose inverse and the weak group inverse, which are the weak core inverse (WCI) $A^{\textcircled{w},\dagger}$ and the dual weak core inverse (d-WCI) $A^{\dagger,\textcircled{w}}$ [3]. Precisely, the weak core inverse of $A \in \mathbb{C}^{n\times n}$ presents a unique solution to the matrix system [3]:

$$XAX = X, \quad AX = CA^\dagger, \quad XA = A^D C,$$

where C is the weak core part of A with $C = AA^{\textcircled{w}}A$. Notice that $A^{\textcircled{w},\dagger} = A^{\textcircled{w}}AA^\dagger$ and $A^{\dagger,\textcircled{w}} = A^\dagger A A^{\textcircled{w}}$.

In [3], for $A\in\mathbb{C}^{n\times n}$ and $\mathrm{Ind}(A) = k$, the weak core part C of A satisfies the following equations:

$$CA^k = A^{k+1}, \qquad C = A^{\textcircled{\dagger}}A^2, \qquad (I - AA^D)C = 0, \tag{1}$$

$$(I - AA^{\textcircled{\dagger}})C = (I - AA^{\textcircled{w}})C = 0, \qquad C(I - Q_A) = 0. \tag{2}$$

The DMP-inverse of $A \in \mathbb{C}_k^{n\times n}$, written by $A^{D,\dagger}$, was defined in [14] as the unique matrix $X \in \mathbb{C}_k^{n\times n}$ satisfying

$$XAX = X, \qquad XA = A^D A, \qquad A^k X = A^k A^\dagger.$$

Moreover, it was proved that $A^{D,\dagger} = A^D A A^\dagger$. Also, the dual DMP-inverse of A was introduced in [14], namely $A^{\dagger,D} = A^\dagger A A^D$.

The MPCEP inverse for bounded linear Hilbert space operators was proposed in [13] as a combination of the Moore-Penrose inverse with the core-EP inverse. More precisely, the MPCEP (or MP-Core-EP) inverse of $A\in\mathbb{C}^{n\times n}$ is presented as a composed generalized inverse defined by the matrix expression

$$A^{\dagger,\textcircled{\dagger}} = A^\dagger A A^{\textcircled{\dagger}},$$

and it presents the unique solution to the matrix system

$$XAX = X, \;\; XA = A^\dagger A A^{\textcircled{\dagger}} A \text{ and } AX = AA^{\textcircled{\dagger}}.$$

Notice that, for $A\in\mathbb{C}^{n\times n}$, the *CEPMP inverse of A, defined in [13], by $A_{\textcircled{\dagger},\dagger} = A_{\textcircled{\dagger}} A A^\dagger$ is unique solution to the system

$$XAX = X, \;\; AX = AA_{\textcircled{\dagger}} AA^\dagger \text{ and } XA = A_{\textcircled{\dagger}} A,$$

where $A_{\textcircled{\dagger}}$ is the dual core-EP inverse of A.

In [23], let $A\in\mathbb{C}^{n\times n}$ and $\mathrm{Ind}(A) = k$. Then, $X\in\mathbb{C}^{n\times n}$ is called the G-Drazin inverse of A if it serves as a solution for the following equations:

$$AXA = A,\ (^k1)XA^{k+1} = A^k,\ (1^k)A^{k+1}X = A^k. \tag{3}$$

The G-Drazin inverse of A is denoted by A^{GD}. It is known that this type inverse is not unique and can be represented by the set in the general case. See [18] for details.

The ICEP inverse was introduced in [19]. In the following, we will present the definition of the ICEP inverse. Firstly, let's use $A\{1\}$ to denote the set of all $\{1\}$-inverse of A. For $A\in\mathbb{C}^{n\times n}$, $\mathrm{Ind}(A) = k$ and an arbitrary $G\in A\{1\}$, the matrix X is the unique solution to the matrix equations

$$XAX = X,\ \ XA = GAA^{\textcircled{\dagger}}A \text{ and } AX = AA^{\textcircled{\dagger}}.$$

We call X the ICEP inverse of A and is denoted by $X = A^{-,\textcircled{\dagger}} = GAA^{\textcircled{\dagger}}$.

In [12], for $A\in\mathbb{C}^{n\times n}$ with $\mathrm{Ind}(A) = k$, and each $A^{GD}\in\mathcal{A}\{GD\}$, where $\mathcal{A}\{GD\}$ stands for the set of all G-Drazin inverse of A, the GDMP inverse of A, denoted by $A^{GD,\dagger}$, is the $n\times n$ matrix

$$A^{GD,\dagger} = A^{GD}AA^{\dagger}.$$

Moreover, the dual GDMP inverse of A is also introduced in [8, 10, 12], namely $A^{\dagger,GD} = A^{\dagger}AA^{GD}$.

Two new generalized inverses have emerged by combining core-EP inverse and the G-Drazin inverse, which is the CEPGD inverse [17]. For a fixed G-Drazin inverse $A^{GD}\in\mathcal{A}\{GD\}$, the CEPGD inverse of A, denote by $A^{\textcircled{\dagger},GD}$, is the unique matrix X satisfying the following equations:

$$XAX = X,\ \ XA = A^{\textcircled{\dagger}}A \text{ and } AX = AA^{\textcircled{\dagger}}AA^{GD}.$$

Notice that $A^{\textcircled{\dagger},GD} = A^{\textcircled{\dagger}}AA^{GD}$.

Next, let us review the core-EP decomposition. Wang gave the core-EP decomposition in the document [21]. Let $A\in\mathbb{C}^{n\times n}$ with $\mathrm{Ind}(A) = k$, $\mathrm{rank}(A^k) = p$. Then, one has $A = A_1 + A_2$, where $A_1\in\mathbb{C}_n^{CM}$, $A_2^k = 0$, $A_1^*A_2 = A_2A_1 = 0$. Furthermore, there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that

$$A = U\begin{pmatrix} T & S \\ 0 & N \end{pmatrix}U^*,\ A_1 = U\begin{pmatrix} T & S \\ 0 & 0 \end{pmatrix}U^*,\ A_2 = U\begin{pmatrix} 0 & 0 \\ 0 & N \end{pmatrix}U^*, \tag{4}$$

where $T\in\mathbb{C}^{p\times p}$ is nonsingular and $S\in\mathbb{C}^{p\times(n-p)}$, $N\in\mathbb{C}^{(n-p)\times(n-p)}$ is nilpotent of index k, i.e., $N^k = 0$.

Lemma 1.1. *[4, 6, 21, 25] Let $A\in\mathbb{C}_k^{n\times n}$ as in (4). Then*

(i) $A^{\dagger} = U\begin{pmatrix} T^*\triangle & -T^*\triangle SN^{\dagger} \\ (I_{n-p} - N^{\dagger}N)S^*\triangle & N^{\dagger} - (I_{n-p} - N^{\dagger}N)S^*\triangle SN^{\dagger} \end{pmatrix}U^*,$

(ii) $A^{\textcircled{\dagger}} = U\begin{pmatrix} T^{-1} & 0 \\ 0 & 0 \end{pmatrix}U^*,$

(iii) $A^{\textcircled{w}} = (A^{\textcircled{\dagger}})^2A = U\begin{pmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{pmatrix}U^*,$

(iv) $A^{D} = U\begin{pmatrix} T^{-1} & T^{-(k+1)}\widetilde{S} \\ 0 & 0 \end{pmatrix}U^*,$

where $\triangle = [TT^* + S(I_{n-p} - N^{\dagger}N)S^*]^{-1}$.

Lemma 1.2. *[9] Let $A\in\mathbb{C}^{n\times n}$ with rank $r > 0$. Then there exists a unitary matrix $U\in\mathbb{C}^{n\times n}$ such that*

$$A = U \begin{pmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{pmatrix} U^*, \tag{5}$$

where $\Sigma = diag(\sigma_1 I_{r1}, \sigma_2 I_{r2}, \ldots, \sigma_t I_{rt})$ is the diagonal matrix of singular values of A, $\sigma_1 > \sigma_2 > \ldots > \sigma_t > 0$, $r_1 + r_2 + \ldots + r_t = r$, and $K\in\mathbb{C}^{r\times r}$, $L\in\mathbb{C}^{r\times(n-r)}$ satisfy $KK^* + LL^* = I_r$.

Lemma 1.3. *Let $A\in\mathbb{C}^{n\times n}$ be a matrix written as in (5). Then*

(i)[5] the core-EP inverse of A is

$$A^{\textcircled{\dagger}} = U \begin{pmatrix} (\Sigma K)^{\textcircled{\dagger}} & 0 \\ 0 & 0 \end{pmatrix} U^*.$$

(ii)[3] the weak group inverse of A is

$$A^{\textcircled{w}} = U \begin{pmatrix} ((\Sigma K)^{\textcircled{\dagger}})^2 \Sigma K & ((\Sigma K)^{\textcircled{\dagger}})^2 \Sigma L \\ 0 & 0 \end{pmatrix} U^* = U \begin{pmatrix} (\Sigma K)^{\textcircled{w}} & ((\Sigma K)^{\textcircled{\dagger}})^2 \Sigma L \\ 0 & 0 \end{pmatrix} U^*.$$

Lemma 1.4. *[3, 22] The following statements concerning $A^{\textcircled{w}}$ are true.*

(i) $A^{\textcircled{w}}$ is an outer inverse of A,

(ii) $R(A^{\textcircled{w}}) = R(A^k)$,

(iii) $A^{\textcircled{w}} A^{k+1} = A^k$,

(iv) $AA^{\textcircled{w}} = A^k B$ for some matrix B,

(v) $A^{\textcircled{w}} = A^k Z$ for some matrix Z.

The main structure of this paper is as follows. In Sect. 2, we introduce the WGGD inverse. Then, we give some representations and characterizations of this type inverse. In Sect. 3, we develop the SMS method for finding the WGGD inverse. In Sect. 4, the Cramer's rule for the solution of a singular equation $Ax = b$ is presented. In Sect. 5, a binary relation for this inverse is introduced along with some derived properties. In Sect. 6, we give the application of the WGGD inverse in solving linear equations.

2. The WGGD-Inverse

In this part, we establish the weak group G-Drazin (WGGD) inverse on the set of square matrices. Furthermore, we discuss a few characterizations of the WGGD inverse and their relation with the main classes of generalized inverses. From here onward, we will consider a matrix $A\in\mathbb{C}^{n\times n}$ with $\mathrm{Ind}(A) = k$. Theorem 2.2 provides the motivation to investigate the WGGD inverse.

Definition 2.1. *(a) Let $A\in\mathbb{C}^{n\times n}$ with $\mathrm{Ind}(A) = k$ and $A^{GD}\in\mathcal{A}\{GD\}$ be a fixed G-Drazin inverse. Then, the WGGD inverse of A is termed as $A^{\textcircled{w},GD}$ and defined by the expression*

$$A^{\textcircled{w},GD} = A^{\textcircled{w}} A A^{GD}.$$

(b) The WGGD family of A is marked with $\mathcal{A}\{\textcircled{w}, GD\}$ and defined as the set

$$\mathcal{A}\{\textcircled{w}, GD\} = A^{\textcircled{w}} A A^{GD} = \left\{ A^{\textcircled{w}} A A^{GD} : A^{GD}\in\mathcal{A}\{GD\} \right\}.$$

Theorem 2.2. *Let $A\in\mathbb{C}^{n\times n}$ with $\mathrm{Ind}(A) = k$ and $A^{GD}\in\mathcal{A}\{GD\}$ be a fixed G-Drazin inverse. C is the weak core part of A. The matrix expression $X = A^{ⓦ}AA^{GD}$ is the solution of the subsequent matrix equations*

$$XAX = X,\quad XA = A^{ⓦ}A,\quad AX = CA^{GD}. \tag{6}$$

The symbol $\mathcal{A}\{ⓦ, GD\}$ stands for the set of all WGGD-inverses of A; clearly $\mathcal{A}\{ⓦ, GD\} \neq \emptyset$ because $\mathcal{A}\{GD\}$ is a nonempty set. Hence, $\mathcal{A}\{ⓦ, GD\} = \{A^{ⓦ}AA^{GD} : A^{GD}\in\mathcal{A}\{GD\}\}$. Therefore, WGGD-inverses of A always exist and, in general, they are not unique.

Proof. Let $X = A^{ⓦ}AA^{GD}$. Then

$$XA = A^{ⓦ}AA^{GD}A = A^{ⓦ}A,$$
$$AX = AA^{ⓦ}AA^{GD} = CA^{GD}$$

and

$$XAX = A^{ⓦ}AA^{GD}AA^{ⓦ}AA^{GD} = A^{ⓦ}AA^{ⓦ}AA^{GD} = A^{ⓦ}AA^{GD} = X.$$

□

Theorem 2.3. *Let $A\in\mathbb{C}^{n\times n}$ with $\mathrm{Ind}(A) = k$. Suppose that C is the weak core part of A and $A^{GD}\in\mathcal{A}\{GD\}$ is a fixed G-Drazin inverse. Then*

$$A^{ⓦ,GD} = A^DCA^{GD}.$$

Proof. From [22] we have $R(A^{ⓦ}) = R(A^k)$, and so $A^{ⓦ} = A^kZ$ from some matrix Z. Then

$$A^{ⓦ,GD} = A^{ⓦ}AA^{GD} = A^kZAA^{GD} = A^DAA^kZAA^{GD} = A^DAA^{ⓦ}AA^{GD} = A^DCA^{GD}.$$

□

Theorem 2.4. *Let $A\in\mathbb{C}^{n\times n}$ with $\mathrm{Ind}(A) = k$ and $A^{GD}\in\mathcal{A}\{GD\}$ be a fixed G-Drazin inverse. Then $R(A^{ⓦ,GD}) = R(A^k)$.*

Proof. In fact, by (6) we obtain

$$R(A^{ⓦ,GD}) \subseteq R(A^{ⓦ,GD}A) = R(A^DC) \subseteq R(A^D) = R(A^k).$$

On the other hand, $R(A^k) \subseteq R(A^{ⓦ,GD})$ because applying (6) and using Lemma 1.4 we have

$$A^{ⓦ,GD}A^{k+1} = (A^{ⓦ}AA^{GD})A^{k+1} = A^{ⓦ}AA^k = A^k. \tag{7}$$

Hence, $R(A^k) = R(A^{ⓦ,GD})$. □

Example 2.5. *Consider a matrix* $A = \begin{pmatrix} 2 & 2 & 1 & 0 \\ 3 & 4 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. *Clearly, $rank(A) = 3$, $rank(A^2) = rank(A^3) = 2$, so that $k = \mathrm{Ind}(A) = 2$. Then*

$$A^\dagger = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -6/5 & 4/5 & 0 & 0 \\ -3/5 & 2/5 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},\quad A^D = A^2(A^5)^\dagger A^2 = \begin{pmatrix} 2 & -1 & -1/2 & -3/2 \\ -3/2 & 1 & 1/2 & 5/4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A^{\dagger,D} = A^\dagger AA^D = \begin{pmatrix} 2 & -1 & -1/2 & -3/2 \\ -6/5 & 4/5 & 2/5 & 1 \\ -3/5 & 2/5 & 1/5 & 1/2 \\ 0 & 0 & 0 & 0 \end{pmatrix},\quad A^{D,\dagger} = A^DAA^\dagger = \begin{pmatrix} 2 & -1 & -1/2 & 0 \\ -3/2 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A^{\textcircled{\dagger}} = A^2(A^3)^{\dagger} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -3/2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ A^{\textcircled{w}} = (A^{\textcircled{\dagger}})^2 A = \begin{pmatrix} 2 & -1 & -1/2 & 0 \\ -3/2 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A^{\dagger,\textcircled{\dagger}} = A^{\dagger}AA^{\textcircled{\dagger}} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -6/5 & 4/5 & 0 & 0 \\ -3/5 & 2/5 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ A_{\textcircled{\dagger},\dagger} = A_{\textcircled{\dagger}}AA^{\dagger} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -3/2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A^{\textcircled{w},\dagger} = A^{\textcircled{w}}AA^{\dagger} = \begin{pmatrix} 2 & -1 & -1/2 & 0 \\ -3/2 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ A^{\dagger,\textcircled{w}} = A^{\dagger}AA^{\textcircled{w}} = \begin{pmatrix} 2 & -1 & -1/2 & 0 \\ -6/5 & 4/5 & 2/5 & 0 \\ -3/5 & 2/5 & 4/5 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Furthermore, for a fixed $A^{-} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 1 & 1/2 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$, *and* $A^{GD} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -2 & 1 & -1/2 & 3 \\ 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & 8 \end{pmatrix}$,

we have

$$A^{-,D} = A^{-}AA^{D} = \begin{pmatrix} 2 & -1 & -1/2 & -3/2 \\ -1 & 1 & 1/2 & 1 \\ -1 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ A^{D,-} = A^{D}AA^{-} = \begin{pmatrix} 2 & -1 & -1/2 & 0 \\ -3/2 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A^{-,\textcircled{\dagger}} = A^{-}AA^{\textcircled{\dagger}} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ A^{\textcircled{\dagger},-} = A^{\textcircled{\dagger}}AA^{-} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -3/2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A^{-,\textcircled{w}} = A^{-}AA^{\textcircled{w}} = \begin{pmatrix} 2 & -1 & -1/2 & 0 \\ -1 & 1 & 1/2 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ A^{\textcircled{w},-} = A^{\textcircled{w}}AA^{-} = \begin{pmatrix} 2 & -1 & -1/2 & 0 \\ -3/2 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A^{GD,\dagger} = A^{GD}AA^{\dagger} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -2 & 1 & -1/2 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \ A^{\dagger,GD} = A^{\dagger}AA^{GD} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -6/5 & 8/5 & 0 & 16/5 \\ -3/5 & 4/5 & 1 & 8/5 \\ 0 & 0 & 1 & 8 \end{pmatrix},$$

$$A^{GD,\textcircled{\dagger}} = A^{GD}AA^{\textcircled{\dagger}} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ A^{\textcircled{\dagger},GD} = A^{\textcircled{\dagger}}AA^{GD} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -3/2 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A^{\textcircled{w},GD} = A^{\textcircled{w}}AA^{GD} = \begin{pmatrix} 2 & -1 & -1/2 & -4 \\ -3/2 & 2 & 1/2 & 8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ A^{GD,\textcircled{w}} = A^{GD}AA^{\textcircled{w}} = \begin{pmatrix} 2 & -1 & -1/2 & 0 \\ -2 & 1 & 1/2 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

It is observable that the WGGD inverse of A differs from the selected inner inverse, core-EP inverse, Drazin inverse, Moore-Penrose inverse, weak group inverse, G-Drazin inverse, DMP inverse, MPCEP inverse, *CEPMP inverse, MPD inverse, inner Drazin inverse, Drazin inner inverse, inner core-EP inverse, core-EP-inner inverse, inner weak group inverse, weak group inner inverse, ICEP inverse, GDMP-inverse, MPGD-inverse, CEPGD inverse, and GDCEP inverse. □

Theorem 2.6. *Let $A\in\mathbb{C}^{n\times n}$ be written as in (4) with $\mathrm{Ind}(A) = k$ and $A^{GD}\in\mathcal{A}\{GD\}$ be a fixed G-Drazin inverse. Then*

$$A^{Ⓦ,GD} = U\begin{pmatrix} T^{-1} & X_2 + T^{-1}SN^{-} + T^{-2}SNN^{-} \\ 0 & 0 \end{pmatrix} U^*,$$

where $(TX_2 + SN^-)N = 0,\ X_2 = T^{-(k+1)}\widetilde{S} - T^{-k}\widetilde{S}N^-$ and $\widetilde{S} = \sum_{i=0}^{k-1} T^i SN^{k-1-i}$.

Proof. Let $A = U\begin{pmatrix} T & S \\ 0 & N \end{pmatrix} U^*$. From $AXA = A$, we obtain

$$\begin{aligned} U\begin{pmatrix} T & S \\ 0 & N \end{pmatrix} U^* &= U\begin{pmatrix} T & S \\ 0 & N \end{pmatrix}\begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}\begin{pmatrix} T & S \\ 0 & N \end{pmatrix} U^* \\ &= U\begin{pmatrix} TX_1T + SX_3T & TX_1S + TX_2N + SX_3S + SX_4N \\ NX_3T & NX_3S + NX_4N \end{pmatrix} U^*. \end{aligned}$$

Thus, $NX_3 = 0,\ X_4 = N^-,\ TX_1 + SX_3 = I,\ (TX_2 + SX_4)N = 0$. Next, we evaluate

$$A^k = U\begin{pmatrix} T^k & \widetilde{S} \\ 0 & 0 \end{pmatrix} U^*,$$

where $\widetilde{S} = \sum_{i=0}^{k-1} T^i SN^{k-1-i}$. Hence,

$$XA^k = U\begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}\begin{pmatrix} T^k & \widetilde{S} \\ 0 & 0 \end{pmatrix} U^* = \begin{pmatrix} X_1T^k & X_1\widetilde{S} \\ X_3T^k & X_3\widetilde{S} \end{pmatrix} U^*,$$

and

$$A^kX = U\begin{pmatrix} T^k & \widetilde{S} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} U^* = U\begin{pmatrix} T^kX_1 + \widetilde{S}X_3 & T^kX_2 + \widetilde{S}X_4 \\ 0 & 0 \end{pmatrix} U^*.$$

Using the condition $A^kX = XA^k$, we obtain $X_1T^k = T^kX_1,\ X_3 = 0,\ X_1\widetilde{S} = T^kX_2 + \widetilde{S}X_4$. Hence, the GD inverses of A are of the form

$$A^{GD} = U\begin{pmatrix} T^{-1} & X_2 \\ 0 & N^- \end{pmatrix} U^*, \tag{8}$$

where $(TX_2 + SN^-)N = 0,\ X_2 = T^{-(k+1)}\widetilde{S} - T^{-k}\widetilde{S}N^-$. By Lemma 1.1 and (8), we can see the GD inverses are of the form

$$A^{Ⓦ,GD} = U\begin{pmatrix} T^{-1} & X_2 + T^{-1}SN^{-} + T^{-2}SNN^{-} \\ 0 & 0 \end{pmatrix} U^*.$$

□

We discuss the general form of the WGGD inverse via the Hartwig and Spindelböck decomposition (in short, HS decomposition) [9].

Lemma 2.7. *[17] Consider A as defined in (5). Then, the GD inverses of A are of the form*

$$A^{GD} = U\begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} U^*,$$

where $\Sigma KX_1 + \Sigma LX_3 = I_r,\ X_1(\Sigma K)^k = (\Sigma K)^{k-1},\ X_3(\Sigma K)^{k-1} = 0$ and

$$(\Sigma K)^{k+1}X_2 + (\Sigma K)^k\Sigma LX_4 = (\Sigma K)^{k-1}\Sigma L.$$

Theorem 2.8. *Consider the A as defined in* (5) *and* $A^{Ⓦ}$ *be written as in Lemma* 1.3. *Then, the WGGD inverses of A are of the form*

$$A^{Ⓦ,GD} = U\begin{pmatrix} (\Sigma K)^{Ⓦ}X_1 + ((\Sigma K)^{Ⓣ})^2\Sigma L X_3 & (\Sigma K)^{Ⓦ}X_2 + ((\Sigma K)^{Ⓣ})^2\Sigma L X_4 \\ 0 & 0 \end{pmatrix} U^*,$$

where

$$\Sigma K X_1 + \Sigma L X_3 = I_r,\ X_1(\Sigma K)^k = (\Sigma K)^{k-1},\ X_3(\Sigma K)^{k-1} = 0,$$

and

$$(\Sigma K)^{k+1}X_2 + (\Sigma K)^k\Sigma L X_4 = (\Sigma K)^{k-1}\Sigma L.$$

Proof. The proof can be demonstrated by Lemma 1.3 and Lemma 2.7. □

Theorem 2.9. *Let* $A\in\mathbb{C}^{n\times n}$ *with* $\mathrm{Ind}(A) = k$ *and* $A^{GD}\in\mathcal{A}\{GD\}$ *be a fixed G-Drazin inverse. C is the weak core part of A. The system of conditions*

$$AX = CA^{GD} \ \text{ and } \ R(X) \subseteq R(A^k), \tag{9}$$

is consistent and it has the unique solution $X = A^{Ⓦ,GD}$.

Proof. Let $X = A^{Ⓦ,GD}$. Clearly, from (6) we obtain $AX = CA^{GD}$. On the other hand, according to Theorem 2.4, we have $R(X) \subseteq R(A^k)$. So, we deduce that $A^{Ⓦ,GD}$ satisfies the two conditions in (9).

In order to show that system (9) has a unique solution, assume that both X_1 and X_2 satisfy (9), that is, $AX_1 = CA^{GD} = AX_2$, $R(X_1) \subseteq R(A^k)$, and $R(X_2) \subseteq R(A^k)$. Since $A(X_1 - X_2) = 0$, we obtain $R(X_1 - X_2) \subseteq N(A) \subseteq N(A^k)$. We also get $R(X_1 - X_2) \subseteq R(A^k)$. Therefore, $R(X_1 - X_2) \subseteq N(A^k) \cap R(A^k) = \{0\}$ because $\mathrm{Ind}(A) = k$. Thus, $X_1 = X_2$. □

Lemma 2.10. *Let* $A\in\mathbb{C}^{n\times n}$ *with* $\mathrm{Ind}(A) = k$ *and* $A^{GD}\in\mathcal{A}\{GD\}$ *be a fixed G-Drazin inverse. Then the weak core part C of A satisfies* $CA^{GD}C = C$.

Proof. It is clear that

$$CA^{GD}C = AA^{Ⓦ}AA^{GD}AA^{Ⓦ}A = AA^{Ⓦ}AA^{Ⓦ}A = AA^{Ⓦ}A = C.$$

□

Theorem 2.11. *Let* $A\in\mathbb{C}^{n\times n}$ *with* $\mathrm{Ind}(A) = k$ *and* $A^{GD}\in\mathcal{A}\{GD\}$ *be a fixed G-Drazin inverse. C is the weak core part of A. Then, X is a WGGD inverse of A if and only if* $XCX = X$, $CX = CA^{GD}$, *and* $XC = A^DC$.

Proof. Let $X = A^{Ⓦ,GD}$. According to Lemma 2.10 and A^D is an inner inverse of C in [3], we have

$$XCX = A^D(CA^{GD}C)A^DCA^{GD} = A^D(CA^{GD}C)A^{GD} = A^DCA^{GD} = X,$$

and

$$CX = (CA^DC)A^{GD} = CA^{GD},\ XC = A^D(CA^{GD}C) = A^DC.$$

Conversely, if X satisfies $XCX = X$, $CX = CA^{GD}$, $XC = A^DC$, then we have $XCX = A^DCX = X$. Applying Theorem 2.3 and $CX = CA^{GD}$, we can get

$$X - A^{Ⓦ,GD} = A^DCX - A^{Ⓦ,GD} = 0.$$

Thus, $X = A^{Ⓦ,GD}$, i.e. X is a *WGGD* inverse of A. □

Theorem 2.12. *Let* $A\in\mathbb{C}^{n\times n}$ *with* $\mathrm{Ind}(A) = k$ *and* $A^{GD}\in\mathcal{A}\{GD\}$ *be a fixed G-Drazin inverse. Then*

$$A^{Ⓦ,GD} = A^{(2)}_{R(A^k),N((A^k)^*A^2A^{GD})}.$$

Proof. By Theorem 2.3 and Theorem 2.14 we have that $A^{Ⓦ,GD}$ is an outer inverse of A with $R(A^{Ⓦ,GD}) = R(A^k)$. On the other hand, we are going to prove that $N(A^{Ⓦ,GD}) = N((A^k)^*A^2A^{GD})$ holds. In fact, by Theorem 2.3 and (1) we get

$$N(A^{Ⓦ,GD}) = N(AA^{Ⓦ,GD}) = N(CA^{GD}) = N(A^{\textcircled{\dagger}}A^2A^{GD}).$$

Then $x \in N(A^{Ⓦ,GD})$ if and only if $A^2A^{GD}x \in N(A^{\textcircled{\dagger}}) = N((A^k)^*)$, since in [6], $A^{\textcircled{\dagger}} = A^{(2)}_{R(A^k),N((A^k)^*)}$. Therefore, $x \in N(A^{Ⓦ,GD})$ if and only if $x \in N((A^k)^*A^2A^{GD})$. □

Theorem 2.13. *Let $A \in \mathbb{C}^{n\times n}$ with $\mathrm{Ind}(A) = k$ and $A^{GD} \in \mathcal{A}\{GD\}$ be a fixed G-Drazin inverse. The WGGD inverse $A^{Ⓦ,GD}$ is the unique solution to the following constrained matrix equations:*

(1) *$AX = P_{R(A^k),N((A^k)^*A^2A^{GD})}$ and $R(X) \subseteq R(A^k)$, where $P_{R(A^k),N((A^k)^*A^2A^{GD})}$ is a projection onto $R(A^k)$ along $N((A^k)^*A^2A^{GD})$ satisfying $AA^{Ⓦ,GD} = CA^{GD}$;*

(2) *$XA = P_{R(A^k),N((A^k)^*A^2)}$ and $R(X^*) \subseteq R((AA^{GD})^*)$, where $P_{R(A^k),N((A^k)^*A^2)}$ is a projection onto $R(A^k)$ along $N((A^k)^*A^2)$ satisfying $A^{Ⓦ}A = A^{Ⓦ,GD}C$.*

Proof. Since, by definition, $A^{Ⓦ,GD}$ is an outer inverse of A, we obtain that $A^{Ⓦ,GD}A$ idempotent and $N(AA^{Ⓦ,GD}) = N(A^{Ⓦ,GD})$ and $R(A^{Ⓦ,GD}A) = R(A^{Ⓦ,GD})$. Therefore, Theorem 2.13 implies $N(A^{Ⓦ,GD}) = N((A^k)^*A^2A^{GD})$ and $R(A^{Ⓦ,GD}A) = R(A^k)$.

(1) According to Theorem 2.3, we have

$$R(AA^{Ⓦ,GD}) = AR(A^{Ⓦ,GD}) = AR(A^k) = R(A^{k+1}) = R(A^k).$$

On the other hand, by the definition of the WGGD inverse and A^D is an inner inverse of C in [3] we obtain

$$AA^{Ⓦ,GD} = AA^{Ⓦ}AA^{GD} = CA^{GD}.$$

(2) Firstly, we are going to prove that $N(A^{Ⓦ,GD}A) = N((A^k)^*A^2)$ holds. In fact, $x \in N(A^{Ⓦ,GD}A)$ if and only if $Ax \in N(A^{Ⓦ,GD}) = N((A^k)^*A^2A^{GD})$. Therefore, $x \in N(A^{Ⓦ,GD}A)$ if and only if $x \in N((A^k)^*A^2A^{GD}A) = N((A^k)^*A^2)$.

Finally, by the definition of the WGGD inverse, we get $A^{Ⓦ,GD}C = A^{Ⓦ}AA^{GD}AA^{Ⓦ}A = A^{Ⓦ}A$. □

Theorem 2.14. *Let $A \in \mathbb{C}^{n\times n}$ with $\mathrm{Ind}(A) = k$ and $A^{GD} \in \mathcal{A}\{GD\}$ be a fixed G-Drazin inverse. Then, the following statements are equivalent.*

(1) *X is a WGGD inverse inverse of the given matrix A.*

(2) *$AXA = A^{\textcircled{\dagger}}A^2,\ XAX = X,\ AX = A^{\textcircled{\dagger}}A^2A^{GD},\ XA = A^{Ⓦ}A.$*

(3) *$AXA = A^{\textcircled{\dagger}}A^2,\ XAX = X,\ XA^{\textcircled{\dagger}}A^2 = A^{Ⓦ}A,\ A^{\textcircled{\dagger}}A^2X = A^{\textcircled{\dagger}}A^2A^{GD}.$*

(4) *$XA^{\textcircled{\dagger}}A^2X = X,\ XA^{\textcircled{\dagger}}A^2 = A^{Ⓦ}A,\ AX = A^{\textcircled{\dagger}}A^2A^{GD}.$*

(5) *$A^{\textcircled{\dagger}}A^2XA^{\textcircled{\dagger}}A^2 = A^{\textcircled{\dagger}}A^2,\ A^{\textcircled{\dagger}}A^2X = A^{\textcircled{\dagger}}A^2A^{GD},\ A^{Ⓦ}AX = X.$*

(6) *$XAA^{Ⓦ}AX = X,\ A^{Ⓦ}AXAX = X,\ XA = A^{Ⓦ}A,\ XAA^{GD} = X.$*

Proof.

(1) ⇒ (2): On the basis of $AX = A^{\textcircled{\dagger}}A^2A^{GD}$. One can verify $AXA = A^{\textcircled{\dagger}}A^2A^{GD}A = A^{\textcircled{\dagger}}A^2$. The rest of the proof is evident by Theorem 2.2.

(2) ⇒ (3): Using $XA = A^{Ⓦ}A$ and $AXA = A^{\textcircled{\dagger}}A^2$, then

$$XA^{\textcircled{\dagger}}A^2 = XAXA = XA = A^{Ⓦ}A,\ A^{\textcircled{\dagger}}A^2X = AXAX = AX = A^{\textcircled{\dagger}}A^2A^{GD}.$$

(3) ⇒ (4): By $AXA = A^{\textcircled{\dagger}}A^2$ and $XAX = X$, we have

$$XA^{\textcircled{\dagger}}A^2X = XAXAX = XAX = X.$$

Using $AA^{\textcircled{W}} = A^{\textcircled{\dagger}}A$ and $A^{\textcircled{\dagger}}A^2X = A^{\textcircled{\dagger}}A^2A^{GD}$, we get

$$AA^{\textcircled{W}}AX = AX = A^{\textcircled{\dagger}}A^2A^{GD}.$$

(4) ⇒ (5): The assumptions imply $X = XA^{\textcircled{\dagger}}A^2X = A^{\textcircled{W}}AX$.
Since $AX = A^{\textcircled{\dagger}}A^2A^{GD}$, we can obtain

$$A^{\textcircled{\dagger}}A^2X = A^{\textcircled{\dagger}}AA^{\textcircled{\dagger}}A^2A^{GD} = A^{\textcircled{\dagger}}A^2A^{GD}.$$

By $XA^{\textcircled{\dagger}}A^2 = A^{\textcircled{W}}A$, we get

$$A^{\textcircled{\dagger}}A^2XA^{\textcircled{\dagger}}A^2 = A^{\textcircled{\dagger}}A^2A^{\textcircled{W}}A = A^{\textcircled{\dagger}}A^2(A^{\textcircled{\dagger}})^2A^2 = A^{\textcircled{\dagger}}A^2.$$

(5) ⇒ (1): Since $A^{\textcircled{W}} = (A^{\textcircled{\dagger}})^2A$, this implication follows obviously from

$$X = A^{\textcircled{W}}AX = (A^{\textcircled{\dagger}})^2A^2X = (A^{\textcircled{\dagger}})^2A^2A^{GD} = A^{\textcircled{W}}AA^{GD}.$$

(1) ⇒ (6): The definition $X = A^{\textcircled{W}}AA^{GD}$ yields

$$XAA^{\textcircled{W}}AX = A^{\textcircled{W}}AA^{GD}AA^{\textcircled{W}}AA^{\textcircled{W}}AA^{GD} = A^{\textcircled{W}}AA^{GD} = X.$$

And

$$A^{\textcircled{W}}AXAX = A^{\textcircled{W}}AA^{\textcircled{W}}AA^{GD}AA^{\textcircled{W}}AA^{GD} = A^{\textcircled{W}}AA^{GD} = X.$$

It is observable that

$$XA = A^{\textcircled{W}}AA^{GD}A = A^{\textcircled{W}}A, \;\; XAA^{GD} = A^{\textcircled{W}}AA^{GD}AA^{GD} = A^{\textcircled{W}}AA^{GD} = X.$$

(6) ⇒ (1): By applying $X = XAA^{GD}$ and $XA = A^{\textcircled{W}}A$, this implication follows obviously from

$$XAA^{GD} = A^{\textcircled{W}}AA^{GD} = X.$$

Corollary 2.15. *Let $A \in \mathbb{C}^{n\times n}$ and $A^{GD} \in \mathcal{A}\{GD\}$ be a fixed G-Drazin inverse.Then*

$$A^{\textcircled{W},GD} \in (A^{\textcircled{\dagger}}A^2)\{1,2\}.$$

Proof. It follows from Theorem 2.14, part (4) and (5). □

Definition 2.16. *[2, 26] The (B, C)-inverse of $A \in \mathbb{C}^{m\times n}$ denoted by $A^{(B,C)}$, is the unique matrix $X \in \mathbb{C}^{n\times m}$ satisfying $XAB = B$, $CAX = C$, $N(X) = N(C)$ and $R(X) = R(B)$, where $B, C \in \mathbb{C}^{n\times m}$.*

Theorem 2.17. *Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{Ind}(A) = k$ and $A^{GD} \in \mathcal{A}\{GD\}$ be a fixed G-Drazin inverse. Then $A^{\textcircled{W},GD}$ is a $(A^k, A^{\textcircled{\dagger}}A^2A^{GD})$ inverse of A.*

Proof. By Lemma 1.4, we can get

$$A^{\textcircled{W},GD}AA^k = A^{\textcircled{W}}AA^{GD}A^k = A^{\textcircled{W}}AA^k = A^k,$$

and

$$A^{\textcircled{\dagger}}A^2A^{GD}AA^{\textcircled{W},GD} = A^{\textcircled{\dagger}}A^2A^{GD}AA^{\textcircled{W}}AA^{GD} = AA^{\textcircled{W}}AA^{GD}AA^{\textcircled{W}}AA^{GD} = A^{\textcircled{\dagger}}A^2A^{GD}.$$

On the other hand, from Theorem 2.12, we have

$$R(A^{\textcircled{W},GD}) = R(A^k), \; N(A^{\textcircled{W},GD}) = N(A^{\textcircled{\dagger}}A^2A^{GD}).$$

□

Theorem 2.18. *Let $A \in \mathbb{C}^{n\times n}$ with $\mathrm{Ind}(A) = k$ and $A^{GD} \in \mathcal{A}\{GD\}$ be a fixed G-Drazin inverse. For $l \geqslant k$,*

$$A^{Ⓦ,GD} = A^k(A^{k+2})^\dagger A^2 A^{GD}. \tag{10}$$

Proof. According to [11], it follows $A^{Ⓦ} = A^k(A^{k+2})^\dagger A$. By the corresponding Theorem 2.2, we get the equality (10). □

Theorem 2.19. *Let $A \in \mathbb{C}^{n\times n}$ with $\mathrm{Ind}(A) = k$ and $A^{GD} \in \mathcal{A}\{GD\}$ be a fixed G-Drazin inverse. Then*

$$A^{Ⓦ,GD} = A^D A^{\textcircled{\dagger}} A^2 A^{GD}.$$

Proof. Since $AA^{Ⓦ} = A^{\textcircled{\dagger}} A$, we get $A^{\textcircled{\dagger}} A^2 = AA^{Ⓦ} A$. Then

$$A^{Ⓦ,GD} = A^D C A^{GD} = A^D AA^{Ⓦ} AA^{GD} = A^D A^{\textcircled{\dagger}} A^2 A^{GD}.$$

□

Theorem 2.20. *Let $A \in \mathbb{C}^{n\times n}$ with $\mathrm{Ind}(A) = k$ and $A^{GD} \in \mathcal{A}\{GD\}$ be a fixed G-Drazin inverse. Then*

$$A^{Ⓦ,GD} = A^k((A^k)^* A^{k+3})^\dagger (A^k)^* A^2 A^{GD}.$$

Proof. Using $A^{Ⓦ,GD} = A^{(2)}_{R(A^k),N((A^k)^* A^2 A^{GD})}$, we obtain on the basis of Urquhart formula [3, 20],

$$A^{Ⓦ,GD} = A^k((A^k)^* A^2 AA^k)^\dagger (A^k)^* A^2 A^{GD} = A^k((A^k)^* A^{k+3})^\dagger (A^k)^* A^2 A^{GD}.$$

□

In the first result, we discuss a few properties of the WGGD inverse, which can be verified easily.

Proposition 2.21. *For each $A^{GD} \in \mathcal{A}\{GD\}$, the WGGD inverse $A^{Ⓦ,GD}$ satisfies the subsequent properties:*

(1) $A^{Ⓦ,GD} A = A^{Ⓦ} A$;

(2) $A^{Ⓦ,GD} A^{k+1} = A^k, k = \mathrm{ind}(A)$;

(3) $A^{Ⓦ,GD} = A^{Ⓦ,GD} AA^{GD}$;

(4) $A^{Ⓦ,GD} AA^{Ⓦ,GD} = A^{Ⓦ,GD}$;

(5) $A^{Ⓦ,GD} = A^l(A^{l+2})^\dagger A^2 A^{GD}$, *where* $l \geqslant k = \mathrm{ind}(A)$.

Theorem 2.22. *Let $A \in \mathbb{C}^{n\times n}$ with $\mathrm{Ind}(A) = k$ and $A^{GD} \in \mathcal{A}\{GD\}$ be a fixed G-Drazin inverse. C is the weak core part of A. The following statements are equivalent:*

(1) $X = A^{Ⓦ,GD} = A^{Ⓦ} AA^{GD}$.

(2) $XCX = X,\ CX = CA^{GD},\ XC = A^D C$.

(3) $X = A^D AA^{Ⓦ,GD}$.

Proof.

That (1) implies all other items (2) − (3) can be checked directly.

(2) ⇒ (1): It is obvious that $X = XCX = A^{Ⓦ} AA^{GD}$.

(3) ⇒ (1): Since $A^{Ⓦ} = A^k Z$ for some matrix Z. It follows that

$$\begin{aligned} X &= A^D AA^{Ⓦ,GD} = A^D AA^{Ⓦ} AA^{GD} = A^k ZAA^{GD} \\ &= A^k ZAA^{GD} = A^{Ⓦ} AA^{GD}. \end{aligned}$$

□

3. The successive matrix squaring algorithm for the WGGD inverse

In this section, we give the successive matrix squaring algorithms for computing the WGGD inverse. The development of the SMS iterations start from the transformations.

Since

$$\begin{aligned}(A^{k+2})^{\dagger}A(AA^{Ⓦ,GD}) &= (A^{k+2})^{\dagger}A^2A^k(A^{k+2})^{\dagger}A^2A^{GD}\\ &= (A^{k+2})^{\dagger}A^{k+2}(A^{k+2})^{\dagger}A^2A^{GD} = (A^{k+2})^{\dagger}A^2A^{GD},\end{aligned}$$

we have

$$\begin{aligned}A^{Ⓦ,GD} &= A^{Ⓦ,GD} - \beta((A^{k+2})^{\dagger}A(AA^{Ⓦ,GD}) - (A^{k+2})^{\dagger}A^2A^{GD})\\ &= (I - \beta(A^{k+2})^{\dagger}A^2)A^{Ⓦ,GD} + \beta(A^{k+2})^{\dagger}A^2A^{GD}.\end{aligned}$$

Observe the following matrices

$$P = I - \beta(A^{k+2})^{\dagger}A^2,\ \ Q = \beta(A^{k+2})^{\dagger}A^2A^{GD},\ \ \beta > 0.$$

It is obvious that $A^{Ⓦ,GD}$ is the unique solution of $X = PX + Q$. Then an iterative procedure for computing the WGGD inverse $A^{Ⓦ,GD}$ can be defined as follows

$$X_1 = Q,\ X_{m+1} = PX_m + Q. \tag{11}$$

This algorithm can be implemented in parallel by considering the block matrix

$$T = \begin{pmatrix} P & Q\\ 0 & I\end{pmatrix},\ \ T^m = \begin{pmatrix} P^m & \Sigma_{i=0}^{m-1}P^iQ\\ 0 & I\end{pmatrix}.$$

The top right block of T^m is X^m, the mth approximation to $A^{Ⓦ,GD}$. The matrix power T^m can be computed by the successive squaring, i.e.

$$T_0 = T,\ T_{i+1} = T_i^2,\ i = 0, 1, \ldots, j,$$

where the integer j is such that $2^j \geqslant m$. The following theorem gives the sufficient condition for the convergence of the iterative process (11).

Theorem 3.1. *Let $A\in\mathbb{C}^{n\times n}$ with $\mathrm{Ind}(A) = k$ and $A^{GD}\in\mathcal{A}\{GD\}$ be a fixed G-Drazin inverse. Suppose rank$(A^k) = r$. Then the approximation*

$$X_{2^m} = \sum_{i=0}^{2^m-1}(I - \beta(A^{k+2})^{\dagger}A^2)^i\beta(A^{k+2})^{\dagger}A^2A^{GD},$$

defined by the iterative process (11) converges to the WGGD inverse $A^{Ⓦ,GD}$ if the spectral radius $\rho(I - X_1A) \leqslant 1$. Moreover, the following error estimation holds:

$$\left\|A^{Ⓦ,GD} - X_{2^m}\right\| \leqslant \left\|(I - X_1A)^{2^m}\right\|.$$

As a result,

$$\lim_{m\to\infty} sup \sqrt[2^m]{\|A^{Ⓦ,GD} - X_{2^m}\|} \leqslant (I - X_1A).$$

Proof. We know that

$$A^{Ⓦ,GD}AA^{Ⓦ,GD} = A^{Ⓦ,GD},\ \ \ X_{2^m}AA^{Ⓦ,GD} = X_{2^m}.$$

By the mathematical induction, we can get

$$I - X_{2^m}A = (I - X_1A)^{2^m}.$$

Therefore,

$$\begin{aligned}\left\|A^{Ⓦ,GD} - X_{2^m}\right\| &= \left\|A^{Ⓦ,GD} - X_{2^m}AA^{Ⓦ,GD}\right\| \\ &= \left\|(I - X_{2^m}AA^{Ⓦ,GD}\right\| \\ &\leqslant \left\|A^{Ⓦ,GD}\right\| \left\|I - X_{2^m}A\right\| \\ &= \left\|A^{Ⓦ,GD}\right\| \left\|(I - X_1A)^{2^m}\right\|,\end{aligned}$$

and

$$\begin{aligned}\lim_{m\to\infty} sup \sqrt[2^m]{\left\|A^{Ⓦ,GD} - X_{2^m}\right\|} &\leqslant \lim_{m\to\infty} sup \sqrt[2^m]{\left\|A^{Ⓦ,GD}\right\| \left\|(I - X_1A)^{2^m}\right\|} \\ &= \rho(I - X_1A).\end{aligned}$$

In the last equality, we use the fact that $\lim_{m\to\infty} \|B^n\|^{1/n} = \rho(B)$, for any square matrix B.

If β is a real parameter such that $\underset{1\leqslant i\leqslant t}{max} |1 - \beta\lambda_i| < 1$, where λ_i $(i = 1, 2, \ldots, s)$ are the nonzero eigenvalues of $(A^{k+2})^{\dagger}A^2A^{GD}$, then

$$\rho(I - X_1A) = \rho(I - \beta(A^{k+2})^{\dagger}A^2) \leqslant 1.$$

It completes the proof. □

Example 3.2. *Consider the following matrix:*

$$A = \begin{pmatrix} 0 & 4/3 & -1/3 \\ -1/3 & 1 & -1/3 \\ -2/3 & -2/3 & 0 \end{pmatrix}, \ \mathrm{ind}(A) = 2.$$

Let

$$P = I - \beta(A^4)^{\dagger}A^2, \ \ Q = \beta(A^4)^{\dagger}A^2A^{GD}, \ \beta = 0.6.$$

The eigenvalues λ_i of QA are included in the set $\{0, 0, 0.5\}$. The nonzero eigenvalues λ_i satisfy

$$\underset{i}{max} |1 - \lambda_i| = |1 - 0.5| = 0.5 < 1.$$

Then we obtain the satisfactory approximation for $A^{Ⓦ,GD}$ after the 12th iteration of the successive matrix squaring algorithm.

$$(T^2)^{12} \approx \begin{pmatrix} 0.9466 & 0.3792 & 0.9238 & 0.1617 & -0.1617 & 0 \\ 0.1256 & 0.1233 & 0.3780 & 0.3757 & 0.3757 & 0 \\ -0.0351 & 0.2512 & 0.8940 & -0.1081 & -0.1081 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The upper right corner of $(T^2)^{12}$ is an approximation of the WGGD inverse, that is

$$A^{Ⓦ,GD} = \begin{pmatrix} 0.9466 & 0.3792 & 0.9238 \\ 0.1256 & 0.1233 & 0.3780 \\ -0.0351 & 0.2512 & 0.8940 \end{pmatrix}.$$

4. The Cramer's rule for the solution of a singular equation $Ax = b$

Theorem 4.1. *Let $A \in \mathbb{C}^{n\times n}$ with $\mathrm{Ind}(A) = k$ and $A^{GD} \in \mathcal{A}\{GD\}$ be a fixed G-Drazin inverse. Suppose $U \in \mathbb{C}^{n\times r}$ and $V^* \in \mathbb{C}^{n\times r}$ having full column rank such that*

$$R(U) = N(A^{Ⓦ,GD}) \text{ and } R(A^k) \subseteq N(V).$$

Then, the bordered matrix

$$X = \begin{pmatrix} A & U \\ V & 0 \end{pmatrix}$$

is nonsingular and

$$X^{-1} = \begin{pmatrix} A^{Ⓦ,GD} & (I_n - A^{Ⓦ,GD}A)V^{\dagger} \\ U^{\dagger}(I_n - AA^{Ⓦ,GD}) & -U^{\dagger}(A - AA^{Ⓦ,GD}A)V^{\dagger} \end{pmatrix}. \quad (12)$$

Proof. Since $R(A^{Ⓦ,GD}) = R(A^k) \subseteq N(V)$, we obtain $VA^{Ⓦ,GD} = 0$. By

$$R(U) = R(UU^{\dagger}) = N(I_n - UU^{\dagger}),$$

we can obtain

$$I_n - AA^{Ⓦ,GD} = UU^{\dagger}(I_n - AA^{Ⓦ,GD}).$$

Let

$$\mathfrak{Y} = \begin{pmatrix} A^{Ⓦ,GD} & (I_n - A^{Ⓦ,GD}A)V^{\dagger} \\ U^{\dagger}(I_n - AA^{Ⓦ,GD}) & -U^{\dagger}(A - AA^{Ⓦ,GD}A)V^{\dagger} \end{pmatrix},$$

we have

$$\begin{aligned} X\mathfrak{Y} &= \begin{pmatrix} AA^{Ⓦ,GD} + UU^{\dagger}(I_n - AA^{Ⓦ,GD}) & A(I_n - A^{Ⓦ,GD}A)V^{\dagger} - UU^{\dagger}(A - AA^{Ⓦ,GD}A)V^{\dagger} \\ VA^{Ⓦ,GD} & V(I_r - A^{Ⓦ,GD}A)V^{\dagger} \end{pmatrix} \\ &= \begin{pmatrix} AA^{Ⓦ,GD} + (I_n - AA^{Ⓦ,GD}) & A(I_n - AA^{Ⓦ,GD})V^{\dagger} - UU^{\dagger}(I_n - AA^{Ⓦ,GD})AV^{\dagger} \\ VA^{Ⓦ,GD} & VV^{\dagger} - VA^{Ⓦ,GD}AV^{\dagger} \end{pmatrix} \\ &= \begin{pmatrix} I_n & A(I_n - A^{Ⓦ,GD})V^{\dagger} - (I_n - AA^{Ⓦ,GD})AV^{\dagger} \\ VA^{Ⓦ,GD} & VV^{\dagger} \end{pmatrix} \\ &= \begin{pmatrix} I_n & 0 \\ 0 & I_r \end{pmatrix} \\ &= I_{n+r}. \end{aligned}$$

In an analogous way, it is possible to verify that $\mathfrak{Y}X = I_{n+r}$. Thus, X is nonsingular and $X^{-1} = \mathfrak{Y}$. □

Theorem 4.2. *Let $A \in \mathbb{C}^{n\times n}$ with $\mathrm{Ind}(A) = k$ and $A^{GD} \in \mathcal{A}\{GD\}$ be a fixed G-Drazin inverse. If $R(B) \subseteq R(AA^{Ⓦ})$, then*

$$AX = B, \ R(X) \subseteq R(A^k) \quad (13)$$

has the unique solution $X = A^{Ⓦ,GD}B$.

Proof. Since $R(B) \subseteq R(AA^{Ⓦ})$, we have $B = AA^{Ⓦ}Z$, for some $Z \in \mathbb{C}^{n\times n}$. If $X = A^{Ⓦ,GD}B$, then we can obtain

$$AX = AA^{Ⓦ,GD}B = AA^{Ⓦ}AA^{GD}AA^{Ⓦ}Z = AA^{Ⓦ}Z = B.$$

Thus, $X = A^{Ⓦ,GD}B$ is a solution of (13). Finally, we show the uniqueness of X. Let $X_1 \in R(A^k)$ also satisfies (13). Then

$$X - X_1 \in R(A^{Ⓦ}) \cap N(A) \subseteq R(A^{Ⓦ}) \cap N(A^{Ⓦ}A) = R(A^{Ⓦ}A) \cap N(A^{Ⓦ}A) = \{0\}.$$

Hence, $X = X_1$. □

Using the relationship between the WGGD inverse of A and a nonsingular bordered matrix, we give the Cramer's rule for solving a singular linear equation $Ax = B$. $A(ij \to b_j)$ denotes the matrix obtained by replacing ith column of A with b_j, where b_j is the jth column of B.

Theorem 4.3. *Let $A \in \mathbb{C}^{n\times n}$ with $\mathrm{Ind}(A) = k$ and $A^{GD} \in \mathcal{A}\{GD\}$ be a fixed G-Drazin inverse. Suppose $U \in \mathbb{C}^{n\times r}$ and $V^* \in \mathbb{C}^{n\times r}$ having full column rank such that*

$$R(A^k) = N(V), \; R(U) = N(A^{\textcircled{W},GD}).$$

If $R(B) \subseteq R(A^k)$, then the unique solution $X = A^{\textcircled{W},GD}B$ of the singular linear equation (13) is given by

$$x_{ij} = \frac{det\begin{pmatrix} (A(i\to b_j) & U \\ V(i\to 0) & 0 \end{pmatrix}}{det\begin{pmatrix} A & U \\ V & 0 \end{pmatrix}}, \; i = 1, 2, \dots n, \; j = 1, 2, \dots n. \tag{14}$$

Proof. Since $X = A^{\textcircled{W},GD}B \in R(AA^{\textcircled{W}}) = N(V)$ and $B \in R(AA^{\textcircled{W}})$, we have

$$VX = 0, \; (I_n - AA^{\textcircled{W},GD})B = 0. \tag{15}$$

It follows from (15) that the solution of $AX = B$ satisfies

$$\begin{pmatrix} A & U \\ V & 0 \end{pmatrix}\begin{pmatrix} X \\ 0 \end{pmatrix} = \begin{pmatrix} B \\ 0 \end{pmatrix}. \tag{16}$$

By Theorem 4.1, the coefficient matrix of (16) is nonsingular. Using (12) and (15), we can obtain

$$\begin{pmatrix} X \\ 0 \end{pmatrix} = \begin{pmatrix} A^{\textcircled{W},GD} & (I_n - A^{\textcircled{W},GD}A)V^{\dagger} \\ U^{\dagger}(I_n - AA^{\textcircled{W},GD}) & -U^{\dagger}(A - AA^{\textcircled{W},GD}A)V^{\dagger} \end{pmatrix}\begin{pmatrix} B \\ 0 \end{pmatrix} = \begin{pmatrix} A^{\textcircled{W},GD}B \\ 0 \end{pmatrix}.$$

Therefore, $X = A^{\textcircled{W},GD}B$ and (14) follows from the classical Cramer's rule [24]. □

5. Binary relation on the WGGD inverse

It is well known that a reflexive and transitive binary relation on a non-empty set is a pre-order [15]. In addition, if the relation is also anti-symmetric, it is termed as a partial order.

Definition 5.1. *[15] Let A, $B \in \mathbb{C}^{n\times n}$ with $\mathrm{Ind}(A) = 1$. Then, A is below B under the sharp order $A \leqslant^{\sharp} B$ if there exists commuting g-inverses A^- and $A^=$, such that $AA^- = BA^-$ and $A^=A = A^=B$.*

Definition 5.2. *For A, $B \in \mathbb{C}^{n\times n}$, we will say that A is below B under the relation $\leqslant^{\textcircled{W},GD}$ if $A^{\textcircled{W},GD}A = A^{\textcircled{W},GD}B$ and $AA^{\textcircled{W},GD} = BA^{\textcircled{W},GD}$ for a fixed $A^{GD} \in \mathcal{A}\{GD\}$. Such a relation is termed as $A \leqslant^{\textcircled{W},GD} B$.*

Naturally, we will consider whether this binary relationship can become a partial order. The answer to this question is No. A binary relation is called a partial order if it is reflexive, transitive, and anti-symmetric on a non-empty set. Next, we give a concrete example to prove that this relationship is not satisfied anti-symmetry.

Example 5.3. *Consider the matrices*

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since

$$A^{\textcircled{W}} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; B^{\textcircled{W}} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A^{GD} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, B^{GD} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

we can get

$$AA^{Ⓦ,GD} = BA^{Ⓦ,GD} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, A^{Ⓦ,GD}A = A^{Ⓦ,GD}B = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$BB^{Ⓦ,GD} = AB^{Ⓦ,GD} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, B^{Ⓦ,GD}B = B^{Ⓦ,GD}A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus,

$$AA^{Ⓦ,GD} = BA^{Ⓦ,GD}, A^{Ⓦ,GD}A = A^{Ⓦ,GD}B,$$
$$AB^{Ⓦ,GD} = BB^{Ⓦ,GD}, B^{Ⓦ,GD}B = B^{Ⓦ,GD}A.$$

Clearly, $A \leqslant^{Ⓦ,GD} B$ and $B \leqslant^{Ⓦ,GD} A$ hold, but $A \neq B$. Hence, The relation $\leqslant^{Ⓦ,GD}$ is not anti-symmetric.

Example 5.4. *Consider the matrices*

$$A = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 2 & -2 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since

$$A^{Ⓦ} = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, B^{Ⓦ} = \begin{pmatrix} 1 & 2 & -2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A^{GD} = \begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, B^{GD} = \begin{pmatrix} 1/6 & 1/3 & 0 & 0 \\ 1/3 & 2/3 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1/6 & 1/3 & 0 & 0 \end{pmatrix},$$

then, by calculating we get

$$A^{Ⓦ,GD} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, B^{Ⓦ,GD} = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

It is easy to check $A^{Ⓦ,GD}A \neq A^{Ⓦ,GD}B$ and $B^{Ⓦ,GD}B \neq B^{Ⓦ,GD}A$, we can not get $A \leqslant^{Ⓦ,GD} B$ and $B \leqslant^{Ⓦ,GD} A$. Thus, the relation $\leqslant^{Ⓦ,GD}$ is not symmetric.

Next, we discuss the conditions that make it becoming a partial order.

Proposition 5.5. *If $A^{Ⓦ,GD}$ is the WGGD inverse of A, then subsequent statements are mutually equivalent for A, $B \in \mathbb{C}^{n\times n}$:*

(1) $A \leqslant^{Ⓦ,GD} B$.

(2) $AA^{\textcircled{W}}A = BA^{\textcircled{W}}A = AA^{\textcircled{W},GD}B$.

(3) $A^{\textcircled{W}}A = A^{\textcircled{W},GD}B$ and $AA^{\textcircled{W}} = BA^{\textcircled{W}}$.

Proof. (1) ⇒ (2): Let $A \leqslant^{\textcircled{W},GD} B$. Then

$$AA^{\textcircled{W}}A = AA^{\textcircled{W}}AA^{\textcircled{W}}A = AA^{\textcircled{W}}AA^{GD}AA^{\textcircled{W}}A = BA^{\textcircled{W}}A,$$

and

$$AA^{\textcircled{W}}A = AA^{\textcircled{W}}AA^{GD}A = AA^{\textcircled{W}}AA^{GD}B = AA^{\textcircled{W},GD}B.$$

(2) ⇒ (3): Let $AA^{\textcircled{W}}A = BA^{\textcircled{W}}A = AA^{\textcircled{W},GD}B$. Then

$$A^{\textcircled{W}}AA^{\textcircled{W}}A = A^{\textcircled{W}}BA^{\textcircled{W}}A = A^{\textcircled{W}}AA^{\textcircled{W},GD}B = A^{\textcircled{W},GD}B,$$

and

$$AA^{\textcircled{W}} = AA^{\textcircled{W}}AA^{\textcircled{W}} = AA^{\textcircled{W}}AA^{GD}AA^{\textcircled{W}} = BA^{\textcircled{W}}.$$

(3) ⇒ (1): Assume (3) holds. Then, it follows:

$$A^{\textcircled{W},GD}A = A^{\textcircled{W}}A = A^{\textcircled{W},GD}B$$

in conjunction with

$$AA^{\textcircled{W},GD} = AA^{\textcircled{W}}AA^{GD} = BA^{\textcircled{W}}AA^{GD} = BA^{\textcircled{W},GD}.$$

□

Theorem 5.6. *Assume that $A\in\mathbb{C}^{n\times n}$ is represented by (4). In addition, if $B\in\mathbb{C}^{n\times n}$, the subsequent statements are equivalent:*

(1) $A \leqslant^{\textcircled{W},GD} B$.

(2) $B = U\begin{pmatrix} T & S-(TX_2+SN^- +T^{-1}SNN^-)B_4 \\ 0 & B_4 \end{pmatrix}U^*$.

Proof. (1) ⇒ (2): Let $A \leqslant^{\textcircled{W},GD} B$ and consider $B = U\begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}U^*$, where B_i $(i = 1,2,3,4)$ are arbitrary. By comparing $AA^{\textcircled{W},GD} = BA^{\textcircled{W},GD}$, using Theorem 2.6 and the matrix form of $A^{\textcircled{W},GD}$, we obtain

$$B_1 = T,\ B_3 = 0.$$

Applying $A^{\textcircled{W},GD}A = A^{\textcircled{W},GD}B$, we get

$$T^{-1}S + (X_2 + T^{-1}SN^- + T^{-2}SNN^-)N = T^{-1}B_2 + (X_2 + T^{-1}SN^- + T^{-2}SNN^-)B_4. \tag{17}$$

Using $(TX_2 + SN^-)N = 0$ (see Theorem 2.6) and the equality (17), we have

$$B_2 = S - (TX_2 + SN^- + T^{-1}SNN^-)B_4.$$

(2) ⇒ (1): It follows by direct verification. □

Definition 5.7. *For A, $B\in\mathbb{C}^{n\times n}$, we will say that A is below B under the relation $\leqslant^{\textcircled{W},-}$ if $A^{\textcircled{W},-}A = A^{\textcircled{W},-}B$ and $AA^{\textcircled{W},-} = BA^{\textcircled{W},-}$ for a fixed $A^{GD}\in\mathcal{A}\{GD\}$. Such a relation is termed as $A \leqslant^{\textcircled{W},-} B$.*

Theorem 5.8. *Let $A, B\in\mathbb{C}_n^{CM}$. If $A^{GD} = A^-$, then $A \leqslant^{\textcircled{W},-} B$ equivalent to $A\sharp \leqslant B$, where $\sharp\leqslant$ is the left sharp partial order.*

Proof. Suppose that $A \leqslant^{\textcircled{W},-} B$ for $A, B\in\mathbb{C}_n^{CM}$. Then $A^{\textcircled{W},-}A = A^{\textcircled{W},-}B$ and $AA^{\textcircled{W},-} = BA^{\textcircled{W},-}$. Since $A\in\mathbb{C}_n^{CM}$, we have $A^{\textcircled{W}} = A^\sharp$, where $A^\sharp$ denotes the group inverse of A (that is, $AA^\sharp A = A, A^\sharp AA^\sharp$ and $AA^\sharp = A^\sharp A$). Now from $A^{\textcircled{W},-}A = A^{\textcircled{W},-}B$, we get $A^\sharp AA^-A = A^\sharp AA^-B$ and $AA^\sharp A^-A = AA^\sharp A^-B$. Multiplying $AA^\sharp A^-A = AA^\sharp A^-B$ by A^2 from the left side, we have $A^2 = AB$. On the other hand, similarly from $AA^{\textcircled{W},-} = BA^{\textcircled{W},-}$, we obtain $A^2 = BA$, and then $R(A) \subseteq R(B)$. By Definition 6.3.1 in [15], we have that $A\sharp \leqslant B$, that is, A is a predecessor of B under the left sharp partial order. □

6. Applications

In this section, we will give the application of the WGGD inverse in solving linear equations.

Theorem 6.1. *Let* $A\in\mathbb{C}^{n\times n}$ *with* $\mathrm{Ind}(A)=k$ *and* $A^{GD}\in\mathcal{A}\{GD\}$ *be a fixed G-Drazin inverse. The equation*

$$(A^{k+2})^*A^2x=(A^{k+2})^*A^2A^{GD}b, \tag{18}$$

is consistent and its general solution is

$$x=A^{\textcircled{W},GD}b+(I-A^{\textcircled{W},GD}A)y, \tag{19}$$

for arbitrary $y\in\mathbb{C}^n$.

Proof. Suppose that x has the form (19). Applying $A^{\textcircled{W},GD}=A^k(A^{k+2})^\dagger A^2A^{GD}$, we have

$$\begin{aligned}(A^{k+2})^*A^2A^{\textcircled{W},GD} &= (A^{k+2})^*A^2A^k(A^{k+2})^\dagger A^2A^{GD}\\ &= (A^{k+2})^*A^{k+2}(A^{k+2})^\dagger A^2A^{GD}\\ &= (A^{k+2})^*A^2A^{GD}.\end{aligned}$$

Therefore $(A^{k+2})^*A^2A^{\textcircled{W},GD}b=(A^{k+2})^*A^2A^{GD}b$, which implies that (18) holds for x.

For a solution x to (18), we obtain

$$\begin{aligned}A^{\textcircled{W},GD}b &= A^k(A^{k+2})^\dagger A^2A^{GD}b\\ &= A^k(A^{k+2})^\dagger((A^{k+2})^\dagger)^*(A^{k+2})^*A^2A^{GD}b\\ &= A^k(A^{k+2})^\dagger((A^{k+2})^\dagger)^*(A^{k+2})^*A^2x\\ &= A^k(A^{k+2})^\dagger((A^{k+2})^\dagger)^*(A^{k+2})^*A^2A^{GD}Ax\\ &= A^{\textcircled{W},GD}Ax.\end{aligned}$$

Now, we get

$$x=A^{\textcircled{W},GD}b+x-A^{\textcircled{W},GD}Ax=A^{\textcircled{W},GD}b+(I-A^{\textcircled{W},GD}A)x.$$

i.e., x possesses the form (19). □

Theorem 6.2. *Let* $A\in\mathbb{C}^{n\times n}$ *with* $\mathrm{Ind}(A)=k$ *and* $A^{GD}\in\mathcal{A}\{GD\}$ *be a fixed G-Drazin inverse. Then the general solution to*

$$A^{\textcircled{W}}Ax=A^{\textcircled{W},GD}b \tag{20}$$

is given by

$$x=A^{\textcircled{W},GD}b+(I-A^{\textcircled{W}}A)y, \tag{21}$$

for arbitrary $y\in\mathbb{C}^n$.

Proof. Notice that x of the form (21) is a solution to (20):

$$A^{\textcircled{W}}Ax=A^{\textcircled{W}}AA^{\textcircled{W},GD}b=A^{\textcircled{W},GD}b.$$

Let x be a solution to (20). Then, by $A^{\textcircled{W}}Ax=A^{\textcircled{W},GD}b$, we deduce that x has the form (21):

$$x=A^{\textcircled{W},GD}b+x-A^{\textcircled{W}}Ax=A^{\textcircled{W},GD}b+(I-A^{\textcircled{W}}A)x.$$

□

Theorem 6.3. *Let $A \in \mathbb{C}^{n\times n}$ with $\mathrm{Ind}(A) = k$ and $A^{GD} \in \mathcal{A}\{GD\}$ be a fixed G-Drazin inverse. Then the general solution to*

$$A^{Ⓦ}Ax = A^{Ⓦ}b, \ b \in R(A^k), \tag{22}$$

is given by

$$\begin{aligned} x &= A^{Ⓦ,GD}b + (I - A^{Ⓦ}A)y, \\ &= A^{Ⓦ}b + (I - A^{Ⓦ}A)y \end{aligned} \tag{23}$$

for arbitrary $y \in \mathbb{C}^n$.

Proof. If x is represented by (23), then

$$A^{Ⓦ}Ax = A^{Ⓦ}AA^{Ⓦ,GD}b = A^{Ⓦ}P_{R(A^k),N(I-A^{Ⓦ}A)}b = A^{Ⓦ}b.$$

Hence, x is a solution to (22).

On the other hand, assume that x is a solution to (22). Using

$$A^{Ⓦ,GD}b = A^{Ⓦ}P_{R(A^k),N(I-A^{Ⓦ}A)}b = A^{Ⓦ}b = A^{Ⓦ}Ax,$$

one can conclude that

$$x = A^{Ⓦ,GD}b + x - A^{Ⓦ}Ax = A^{Ⓦ,GD}b + (I - A^{Ⓦ}A)x.$$

Thus, the solution x to (22) possesses the form (23). Since $b \in R(A^k)$, we have $P_{R(A^k),N(I-A^{Ⓦ}A)}b = b$, then we observe the identities $A^{Ⓦ,GD}b = A^{Ⓦ}P_{R(A^k),N(I-A^{Ⓦ}A)}b = A^{Ⓦ}b = A^{Ⓦ}Ax$, which confirms the second identity in (23). □

7. Conclusion

A novel class of outer generalized inverses, termed as the WGGD inverse, is introduced as a proper composition of the weak group and the G-Drazin inverse. A few properties and computationally efficient representations of the WGGD inverse are presented and investigated. The image and nullity of the WGGD inverse are considered. The representations of the WGGD inverse based on the core-EP decomposition and the Hartwig-Spindelböck decomposition are established. A binary relation induced by this inverse is introduced along with some derived properties. Some encouraging subjects for future investigation are mentioned as follows:

- Perturbations, limit representations, and continuity of the WGGD inverse;
- Studying of the WGGD inverse for tensors;
- Investigation of the WGGD inverse for Hilbert spaces operators;
- The proposed combination of two types of generalized inverses can be an inspiration for future composite generalized inverses defined on the basis of existing ones.

References

[1] Ben-Israel A, Greville T N E. Generalized inverses: theory and applications. Springer Science and Business Media. 2003.
[2] Benítez J, Boasso E, Jin H. On one-sided (B, C)-inverses of arbitrary matrices. The Electronic Journal of Linear Algebra. 2017, 32: 391-422.
[3] Ferreyra D E, Levis F E, Priori A N, Thome N. The weak core inverse. Aequationes mathematicae. 2021, 95(2): 351-373.

[4] Ferreyra D E, Levis F E, Thome N. Maximal classes of matrices determining generalized inverses. Applied Mathematics and Computation. 2018, 333: 42-52.
[5] Ferreyra D E, Levis F E, Thome N. Revisiting the core EP inverse and its extension to rectangular matrices. Quaestiones Mathematicae. 2018, 41(2): 265-281.
[6] Ferreyra D E, Levis F E, Thome N. Characterizations of k-commutative equalities for some outer generalized inverses. Linear and Multilinear Algebra. 2020, 68(1): 177-192.
[7] Gao Y, Chen J. Pseudo core inverses in rings with involution. Communications in Algebra. 2018, 46(1): 38-50.
[8] Hernández M V, Lattanzi M B, Thome N. GDMP-inverses of a matrix and their duals. Linear and Multilinear Algebra. 2022, 70(19): 3923-3935.
[9] Hartwig R E, Spindelböck K. Matrices for which A^* and $A^\dagger$ conmmute. Linear and Multilinear Algebra. 1983, 14(3): 241-256.
[10] Mosić D. GOMP inverse of Hilbert spaces operators. TWMS Journal of Pure and Applied Mathmatics. 2023, 14(2): 195-213.
[11] Mosić D, Stanimirović P S. Representations for the weak group inverse. Applied Mathematics and Computation. 2021, 397: 125957.
[12] Mosić D, Stanimirović P S. Strong weighted GDMP inverse for operators. Bulletin of the Iranian Mathematical Society. 2024, 50(3): 43.
[13] Mosić D, Xu S, Chen J. On a new generalized inverse for Hilbert space operators. Quaestiones Mathematicae. 2020, 43(9): 1331-1348.
[14] Malik S B, Thome N. On a new generalized inverse for matrices of an arbitrary index. Applied Mathematics and Computation. 2014, 226: 575-580.
[15] Mitra S K, Bhimasankaram P, Malik S B. Matrix partial orders, shorted operators and applications. World Scientific, 2010.
[16] Prasad K M, Mohana K S. Core-EP inverse. Linear and Multilinear Algebra. 2014, 62(6): 792-802.
[17] Panda S K, Sahoo J K, Behera R. The CEPGD-Inverse for Square Matrices. Bulletin of the Iranian Mathematical Society. 2023, 49(6): 79.
[18] Romo F P. On G-Drazin inverses of finite potent endomorphisms and arbitrary square matrice. Linear and Multilinear Algebra. 2022, 70(12): 2227-2247.
[19] Sitha B, Sahoo J K, Behera R, Stanimirović P S, Stupina A A. Generalized core-EP inverse for square matrices. Computational and Applied Mathematics. 2023, 42(8): 348.
[20] Urquhart N S. Computation of generalized inverse matrices which satisfy specified conditions. SIAM Review. 1968, 10(2): 216-218.
[21] Wang H. Core-EP decomposition and its applications. Linear Algebra and its Applications. 2016, 508: 289-300.
[22] Wang H, Chen J. Weak group inverse. Open Mathematics. 2018, 16(1): 1218-1232.
[23] Wang H, Liu J. Partial orders based on core-nilpotent decomposition. Linear Algebra and its Applications. 2016, 488: 235-248.
[24] Wang G, Wei Y, Qiao S. Generalized inverses: Theory and Computations. Developments in Mathematics 53. Singapore, Springer, Beijing, 2018. Science Press.
[25] Yan H, Wang H, Zuo K, Chen Y. Further characterizations of the weak group inverse of matrices and the weak group matrix. AIMS Mathematics. 2021, 6(9): 9322-9341.
[26] Zuo K, Cvetković Ilić D S, Cheng J. Different characterizations of DMP-inverse of matrices. Linear and Multilinear Algebra. 2022, 70(3): 411-418.