Filomat 40:3 (2026), 1075–1092
https://doi.org/10.2298/FIL2603075S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On some convergence type in Octonion-valued b-metric spaces

Ekrem Savaş[a], Selim Çetin[b], Ömer Kişi[c], Mehmet Gürdal[d,*]

[a]*Department of Mathematics, Uşak University, 64100, Uşak, Turkey*
[b]*Department of Mathematics, Mehmet Akif Ersoy University, Burdur, Turkey*
[c]*Department of Mathematics, Bartın University, 74100, Bartın, Turkey*
[d]*Department of Mathematics, Süleyman Demirel University, 32260, Isparta, Turkey*

Abstract. This paper will extend some notions, such as convergence, to b-metric spaces with octonion valued b-metric spaces constructed by Qiu et al. Octonion-valued metric spaces are based on modifying the triangle inequality of a semi-metric space by multiplying one side of the inequality by a scalar b. This new generalisation of metric spaces is very interesting since octonions are not even a ring since they do not have the associative property of multiplication and the spaces do not satisfy the standard triangle inequality. Through these concepts, statistical convergence and related concepts are generalised. Properties associated with these concepts are given and the connections between them are established. Moreover, the influence of some structures of octonions on statistical convergence is analysed. Octonion-valued structures extend beyond quaternion-valued frameworks by incorporating non-associative properties, offering a richer setting for studying convergence phenomena. The present work provides new insights by establishing results that are not obtainable in associative settings. These and similar facts make the results obtained in these defined b-metric spaces of particular interest.

1. Introduction

Sequence convergence and summability theory research has long been a significant field in pure mathematics, with significant contributions to fields like computer science, topology, functional analysis, Fourier analysis, applied mathematics, mathematical modeling, and measure theory. In recent years, the idea of statistical convergence of sequences has grown in significance. The notion of statistical convergence was previously mentioned as ”almost convergence” by Zygmund in the initial version of his renowned monograph issued in 1935, (see [38]). Later on, the notion was introduced by Steinhaus [34]. This progress initiated a series of studies of statistical convergence. In 1951, for the first time, the concept of statistical convergence and its properties were given by Fast [11]. Later on, it was represented by Schoenberg [33] who investigated statistical convergence as a summability method and also outlined some fundamental characteristics of statistical convergence. Additional some interesting characteristics of statistical convergence

2020 *Mathematics Subject Classification*. Primary 11B05; Secondary 11N05, 11A99.
Keywords. natural density, prime numbers, statistical convergence, octonion valued b-metric space.
Received: 29 January 2025; Revised: 23 November 2025; Accepted: 26 November 2025
Communicated by Ljubiša D. R. Kočinac
* Corresponding author: Mehmet Gürdal
Email addresses: ekremsavas@yahoo.com (Ekrem Savaş), selimcetin@mehmetakif.edu.tr (Selim Çetin), okisi@bartin.edu.tr (Ömer Kişi), gurdalmehmet@sdu.edu.tr (Mehmet Gürdal)
ORCID iDs: https://orcid.org/0000-0003-2135-3094 (Ekrem Savaş), https://orcid.org/0000-0002-9017-1465 (Selim Çetin), https://orcid.org/0000-0001-6844-3092 (Ömer Kişi), https://orcid.org/0000-0003-0866-1869 (Mehmet Gürdal)

were studied by Fridy [14] and Salat [30]. For work in other areas related to statistical convergence, we refer the reader to [18, 19, 25, 31, 32, 37]. In mathematics, metric spaces are crucial, particularly in topology and analysis. Frechet [13] initially proposed the idea of metric space in 1906. Since then, the generalization of metric spaces has attracted the attention of several scholars, who have written numerous articles on the topic [8, 15, 24]. The idea of octonion-valued b-metric space, which was put up by Qiu et al. [28] in 2025 as a generalization of octonion metrics—a logical extension of complex and quaternion-valued metrics as well as the traditional concept of metric—is one outcome of their investigations. Examining statistical convergence and its characteristics in octonion valued b–metric spaces is the goal of this research. However, most existing studies are confined to associative structures such as real, complex, or quaternion-valued spaces. Since octonions are non-associative, extending convergence concepts to this framework requires new techniques. The present paper addresses this gap by formulating and analyzing statistical convergence in octonion-valued b-metric spaces.

Soon after quaternions were discovered in 1843, John T. Graves created octonions. Arthur Cayley later expanded and improved this idea on his own. A systematic extension in hypercomplex number theory controlled by the Cayley-Dickson structure is demonstrated by the evolution from real numbers to complex numbers, then to quaternions, and eventually to octonions. Because of their unique mathematical characteristics, octonions stand out in this sequence. Octonions are neither commutative nor associative, in contrast to real and complex numbers, which are commutative, and quaternions, which are non-commutative but nonetheless associative. In addition to their theoretical significance, octonions' special non-associative properties are useful in applications that need to manage multidimensional data relationships. Octonions have been employed in physics to create duality-invariant field equations for dyons, according to Kansu et al.[23]. These equations effectively represent electric-magnetic dualities, much like Maxwell's equations do. Eight-dimensional octonions' multicomponent character unifies the intricate interactions between electric and magnetic components. Octonions have emerged as a practical tool for processing and expressing high-dimensional data in the field of machine learning. Deep octonion networks (DONs), which incorporate multidimensional characteristics into various layers of neural networks by taking use of the compact structure of octonions, were first presented by Wu et al. Octonions facilitate effective data representation and processing in this context; activities like picture classification exhibit enhanced performance and convergence. Accordingly, octonions' associative and non-commutative characteristics have enabled creative applications in contemporary theoretical physics, artificial intelligence, and control systems where multidimensionality and flexible data representation are essential, despite initially posing difficulties for conventional algebraic applications. According to [1, 4, 7, 9, 26], octonions, their subalgebraic structure, and multidisciplinary applications are covered in detail. The theoretical foundations of non-associative algebras and their analytical extensions have been developed in several directions [4, 26]. Parallel to this, operator-theoretic investigations and Banach algebraic approaches [16, 17, 21, 22] have yielded valuable techniques for studying functional relationships within reproducing kernel and Wiener-type frameworks. These results inspire the present research, which establishes new types of convergence in octonion-valued b-metric spaces and highlights the algebraic–analytic interplay between octonion structures and generalized metric notions.

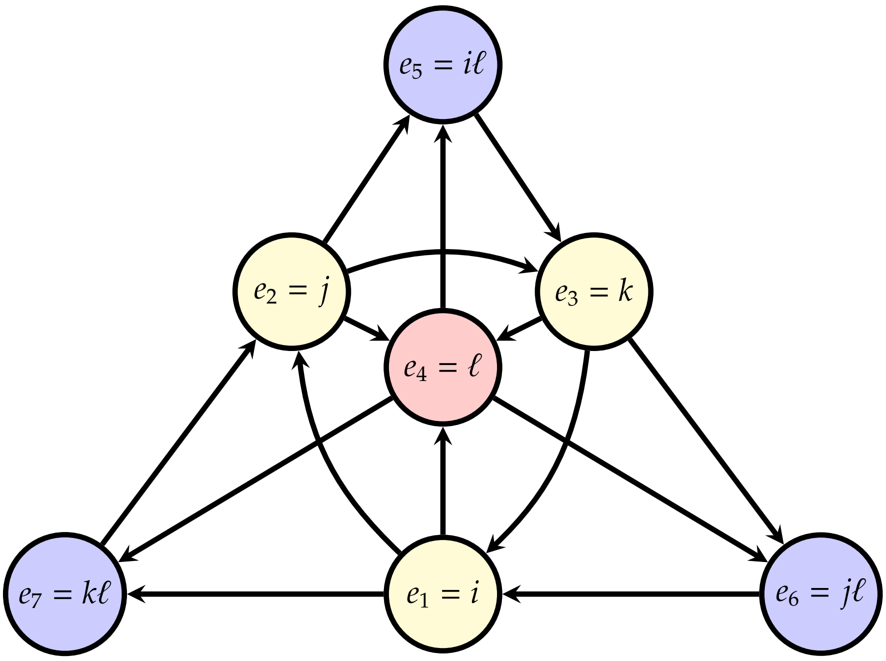
In this paper, we extend certain fundamental concepts such as convergence to octonion valued b-metric spaces, which were initially constructed by Qiu et al. [28] in 2025. By defining a partial ordering relation on octonions, we present the necessary concepts related to this unique mathematical structure, including statistical convergence, statistical Cauchy sequences, and statistically dense subsequences. These concepts are generalized within the framework of octonion valued b-metric spaces, allowing us to explore their properties and the connections between them.

The structure of the paper is as follows. Concepts and qualities that will be helpful in the future are covered in Section 2. Summability theory and some concepts of statistical convergence on b-metric spaces are covered in Section 3. Unlike classical metric spaces, octonion-valued b-metric spaces operate in a non-associative setting, leading to behaviors and structural properties not observed in real, complex, or quaternion-valued cases. Example 2.7 in this paper illustrates this distinction explicitly.

2. Preliminaries

2.1. Preliminaries on Octonions

In the follow-up we will to examine on $\mathbb{O}$, Octonions, a non-associative generalization of the division algebra of quaternions. In this section, we will begin by extending the basis elements of quaternions, represented as $\{1, i, j, k\}$, by incorporating an additional basis element ℓ. This extension enables us to construct the eight-dimensional octonion division algebra in detail, as described in [12], including its diagrammatic representation and algebraic operations.



(1)

Thus, each element $\rtimes \in \mathbb{O}$ can be expressed in the form:

$$\rtimes = o_0 + o_1e_1 + o_2e_2 + o_3e_3 + o_4e_4 + o_5e_5 + o_6e_6 + o_7e_7, \quad o_n \in \mathbb{R}, \quad \text{where } n = 0, 1, 2, 3, 4, 5, 6, 7.$$

The basis elements of $\mathbb{O}$ are $1, e_1, e_2, e_3, e_4, e_5, e_6, e_7$. The detailed multiplication of these basis elements is shown in the table below.

$\cdot$	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
1	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	-1	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	e_2	$-e_3$	-1	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_3	e_2	$-e_1$	-1	e_7	$-e_6$	e_5	$-e_4$
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	-1	e_1	e_2	e_3
e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	-1	$-e_3$	e_2
e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	-1	$-e_1$
e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	-1

The conjugate element $\bar{\rtimes}$ is given by

$$\bar{\rtimes} = o_0 - o_1e_1 - o_2e_2 - o_3e_3 - o_4e_4 - o_5e_5 - o_6e_6 - o_7e_7.$$

The norm of an arbitrary octonion is calculated as

$$\|\rtimes\| = \sqrt{\rtimes \cdot \bar{\rtimes}} = \sqrt{o_0^2 + o_1^2 + o_2^2 + o_3^2 + o_4^2 + o_5^2 + o_6^2 + o_7^2}.$$

Additionally, the inverse of an arbitrary octonion $\rtimes$ is given in the form

$$\rtimes^{-1} = \frac{\bar{\rtimes}}{\|\rtimes\|^2}.$$

Any quaternion's imaginary part can be represented as a vector in three-dimensional Euclidean space, analogous to a movement vector, with its real part indicating the time of this movement. Similarly, octonions can be redefined in a seven-dimensional Euclidean space as a pair consisting of a scalar and a vector, allowing for a different perspective. While quaternions differ from real and complex numbers in their non-commutative multiplication, octonions, as a more complex structure, lose the associative property from the group axioms in multiplication. Consequently, division algebra over octonions becomes non-associative, adding to its intriguing properties.

We can represent octonions as an ordered set of eight real numbers $(o_0, o_1, o_2, o_3, o_4, o_5, o_6, o_7)$ with coordinate-wise addition and multiplication defined by a specific table. Here, the first component, o_0, is called the real part, while the remaining seven-tuple $(o_1, o_2, o_3, o_4, o_5, o_6, o_7)$ constitutes the imaginary part. Thus, as noted above, any quaternion can be written in the form $\left(o_0, \overrightarrow{u}\right)$,where $\overrightarrow{u} = (o_1, o_2, o_3, o_4, o_5, o_6, o_7)$ and o_0 represents the real part. From here, the following properties can be easily observed:

$$\begin{aligned}
\rtimes :&= \left(o_0, \overrightarrow{u}\right), \quad \overrightarrow{u} \in \mathbb{R}^7; \quad o_0 \in \mathbb{R} \\
&= \left(o_0, (o_1, o_2, o_3, o_4, o_5, o_6, o_7)\right); \quad o_0, o_1, o_2, o_3, o_4, o_5, o_6, o_7 \in \mathbb{R} \\
&= o_0 + o_1e_1 + o_2e_2 + o_3e_3 + o_4e_4 + o_5e_5 + o_6e_6 + o_7e_7.
\end{aligned}$$

Now, let us define a partial ordering relation $\preceq$ on the non-associative and non-commutative octonion algebra $\mathbb{O}$ as follows.

$\rtimes \preceq \rtimes'$ if and only if $\mathrm{Re}(\rtimes) \leq \mathrm{Re}(\rtimes')$, $\mathrm{Im}_e(\rtimes) \leq \mathrm{Im}_e(\rtimes')$, $\rtimes, \rtimes' \in \mathbb{O}$; $e = e_1, e_2, e_3, e_4, e_5, e_6, e_7$, where $\mathrm{Im}_{e_n} = o_n; n = 1, 2, 3, 4, 5, 6, 7$. To confirm that it is $\rtimes \preceq \rtimes'$, satisfying any one of the 256 conditions derived from the sum of all possible combinations of 8, from 0 to 8 in respectively, will suffice. Obtained from the 0-combinations of 8, meaning none of its components are equal; this 1 case constitute

(1) $\mathrm{Re}(\rtimes) < \mathrm{Re}(\rtimes')$; $\mathrm{Im}_{e_n}(\rtimes) < \mathrm{Im}_{e_n}(\rtimes')$, where $n = 1, 2, 3, 4, 5, 6, 7$.

Obtained from the 1-combinations of 8, meaning only one component is equal; these 8 cases constitute

(2) $\mathrm{Re}(\rtimes) = \mathrm{Re}(\rtimes')$; $\mathrm{Im}_{e_n}(\rtimes) < \mathrm{Im}_{e_n}(\rtimes')$, where $n = 1, 2, 3, 4, 5, 6, 7$.

(3) $\mathrm{Re}(\rtimes) < \mathrm{Re}(\rtimes')$; $\mathrm{Im}_{e_n}(\rtimes) < \mathrm{Im}_{e_n}(\rtimes')$, where $n = 2, 3, 4, 5, 6, 7$; $\mathrm{Im}_{e_1}(\rtimes) = \mathrm{Im}_{e_1}(\rtimes')$.

(4) $\mathrm{Re}(\rtimes) < \mathrm{Re}(\rtimes')$; $\mathrm{Im}_{e_n}(\rtimes) < \mathrm{Im}_{e_n}(\rtimes')$, where $n = 1, 3, 4, 5, 6, 7$; $\mathrm{Im}_{e_2}(\rtimes) = \mathrm{Im}_{e_2}(\rtimes')$.

(5) $\mathrm{Re}(\rtimes) < \mathrm{Re}(\rtimes')$; $\mathrm{Im}_{e_n}(\rtimes) < \mathrm{Im}_{e_n}(\rtimes')$, where $n = 1, 2, 4, 5, 6, 7$; $\mathrm{Im}_{e_3}(\rtimes) = \mathrm{Im}_{e_3}(\rtimes')$.

(6) $\mathrm{Re}(\rtimes) < \mathrm{Re}(\rtimes')$; $\mathrm{Im}_{e_n}(\rtimes) < \mathrm{Im}_{e_n}(\rtimes')$, where $n = 1, 2, 3, 5, 6, 7$; $\mathrm{Im}_{e_4}(\rtimes) = \mathrm{Im}_{e_4}(\rtimes')$.

(7) $\mathrm{Re}(\rtimes) < \mathrm{Re}(\rtimes')$; $\mathrm{Im}_{e_n}(\rtimes) < \mathrm{Im}_{e_n}(\rtimes')$, where $n = 1, 2, 3, 4, 6, 7$; $\mathrm{Im}_{e_5}(\rtimes) = \mathrm{Im}_{e_5}(\rtimes')$.

(8) $\mathrm{Re}(\rtimes) < \mathrm{Re}(\rtimes')$; $\mathrm{Im}_{e_n}(\rtimes) < \mathrm{Im}_{e_n}(\rtimes')$, where $n = 1, 2, 3, 4, 5, 7$; $\mathrm{Im}_{e_6}(\rtimes) = \mathrm{Im}_{e_6}(\rtimes')$.

(9) $\mathrm{Re}(\rtimes) < \mathrm{Re}(\rtimes')$; $\mathrm{Im}_{e_n}(\rtimes) < \mathrm{Im}_{e_n}(\rtimes')$, where $n = 1, 2, 3, 4, 5, 6$; $\mathrm{Im}_{e_7}(\rtimes) = \mathrm{Im}_{e_7}(\rtimes')$.

Obtained from the 2-combinations of 8, meaning only two components are equal; these 27 cases constitute

(10) $\mathrm{Re}(\rtimes) = \mathrm{Re}(\rtimes')$; $\mathrm{Im}_{e_n}(\rtimes) < \mathrm{Im}_{e_n}(\rtimes')$, where $n = 2, 3, 4, 5, 6, 7$; $\mathrm{Im}_{e_1}(\rtimes) = \mathrm{Im}_{e_1}(\rtimes')$.

(11) $\mathrm{Re}(\rtimes) = \mathrm{Re}(\rtimes');\ \mathrm{Im}_{e_n}(\rtimes) < \mathrm{Im}_{e_n}(\rtimes')$, where $n = 1, 3, 4, 5, 6, 7;\ \mathrm{Im}_{e_2}(\rtimes) = \mathrm{Im}_{e_2}(\rtimes')$.

(12) $\mathrm{Re}(\rtimes) = \mathrm{Re}(\rtimes');\ \mathrm{Im}_{e_n}(\rtimes) < \mathrm{Im}_{e_n}(\rtimes')$, where $n = 1, 2, 4, 5, 6, 7;\ \mathrm{Im}_{e_3}(\rtimes) = \mathrm{Im}_{e_3}(\rtimes')$.

(13) $\mathrm{Re}(\rtimes) = \mathrm{Re}(\rtimes');\ \mathrm{Im}_{e_n}(\rtimes) < \mathrm{Im}_{e_n}(\rtimes')$, where $n = 1, 2, 3, 5, 6, 7;\ \mathrm{Im}_{e_4}(\rtimes) = \mathrm{Im}_{e_4}(\rtimes')$.

(14) $\mathrm{Re}(\rtimes) = \mathrm{Re}(\rtimes');\ \mathrm{Im}_{e_n}(\rtimes) < \mathrm{Im}_{e_n}(\rtimes')$, where $n = 1, 2, 3, 4, 6, 7;\ \mathrm{Im}_{e_5}(\rtimes) = \mathrm{Im}_{e_5}(\rtimes')$.

(15) $\mathrm{Re}(\rtimes) = \mathrm{Re}(\rtimes');\ \mathrm{Im}_{e_n}(\rtimes) < \mathrm{Im}_{e_n}(\rtimes')$, where $n = 1, 2, 3, 4, 5, 7;\ \mathrm{Im}_{e_6}(\rtimes) = \mathrm{Im}_{e_6}(\rtimes')$.

(16) $\mathrm{Re}(\rtimes) = \mathrm{Re}(\rtimes');\ \mathrm{Im}_{e_n}(\rtimes) < \mathrm{Im}_{e_n}(\rtimes')$, where $n = 1, 2, 3, 4, 5, 6;\ \mathrm{Im}_{e_7}(\rtimes) = \mathrm{Im}_{e_7}(\rtimes')$.

(17) $\mathrm{Re}(\rtimes) < \mathrm{Re}(\rtimes');\ \mathrm{Im}_{e_n}(\rtimes) < \mathrm{Im}_{e_n}(\rtimes'),\ n = 3, 4, 5, 6, 7;\ \mathrm{Im}_{e_1}(\rtimes) = \mathrm{Im}_{e_1}(\rtimes');\ \mathrm{Im}_{e_2}(\rtimes) = \mathrm{Im}_{e_2}(\rtimes')$.

(18) $\mathrm{Re}(\rtimes) < \mathrm{Re}(\rtimes');\ \mathrm{Im}_{e_n}(\rtimes) < \mathrm{Im}_{e_n}(\rtimes'),\ n = 2, 4, 5, 6, 7;\ \mathrm{Im}_{e_1}(\rtimes) = \mathrm{Im}_{e_1}(\rtimes');\ \mathrm{Im}_{e_3}(\rtimes) = \mathrm{Im}_{e_3}(\rtimes')$.

(19) $\mathrm{Re}(\rtimes) < \mathrm{Re}(\rtimes');\ \mathrm{Im}_{e_n}(\rtimes) < \mathrm{Im}_{e_n}(\rtimes'),\ n = 2, 3, 5, 6, 7;\ \mathrm{Im}_{e_1}(\rtimes) = \mathrm{Im}_{e_1}(\rtimes');\ \mathrm{Im}_{e_4}(\rtimes) = \mathrm{Im}_{e_4}(\rtimes')$.

(20) $\mathrm{Re}(\rtimes) < \mathrm{Re}(\rtimes');\ \mathrm{Im}_{e_n}(\rtimes) < \mathrm{Im}_{e_n}(\rtimes'),\ n = 2, 3, 4, 6, 7;\ \mathrm{Im}_{e_1}(\rtimes) = \mathrm{Im}_{e_1}(\rtimes');\ \mathrm{Im}_{e_5}(\rtimes) = \mathrm{Im}_{e_5}(\rtimes')$.

(21) $\mathrm{Re}(\rtimes) < \mathrm{Re}(\rtimes');\ \mathrm{Im}_{e_n}(\rtimes) < \mathrm{Im}_{e_n}(\rtimes'),\ n = 2, 3, 4, 5, 7;\ \mathrm{Im}_{e_1}(\rtimes) = \mathrm{Im}_{e_1}(\rtimes');\ \mathrm{Im}_{e_6}(\rtimes) = \mathrm{Im}_{e_6}(\rtimes')$.

(22) $\mathrm{Re}(\rtimes) < \mathrm{Re}(\rtimes');\ \mathrm{Im}_{e_n}(\rtimes) < \mathrm{Im}_{e_n}(\rtimes'),\ n = 2, 3, 4, 5, 6;\ \mathrm{Im}_{e_1}(\rtimes) = \mathrm{Im}_{e_1}(\rtimes');\ \mathrm{Im}_{e_7}(\rtimes) = \mathrm{Im}_{e_7}(\rtimes')$.

(23) $\mathrm{Re}(\rtimes) < \mathrm{Re}(\rtimes');\ \mathrm{Im}_{e_n}(\rtimes) < \mathrm{Im}_{e_n}(\rtimes'),\ n = 1, 4, 5, 6, 7;\ \mathrm{Im}_{e_2}(\rtimes) = \mathrm{Im}_{e_2}(\rtimes');\ \mathrm{Im}_{e_3}(\rtimes) = \mathrm{Im}_{e_3}(\rtimes')$.

(24) $\mathrm{Re}(\rtimes) < \mathrm{Re}(\rtimes');\ \mathrm{Im}_{e_n}(\rtimes) < \mathrm{Im}_{e_n}(\rtimes'),\ n = 1, 3, 5, 6, 7;\ \mathrm{Im}_{e_2}(\rtimes) = \mathrm{Im}_{e_2}(\rtimes');\ \mathrm{Im}_{e_4}(\rtimes) = \mathrm{Im}_{e_4}(\rtimes')$.

(25) $\mathrm{Re}(\rtimes) < \mathrm{Re}(\rtimes');\ \mathrm{Im}_{e_n}(\rtimes) < \mathrm{Im}_{e_n}(\rtimes'),\ n = 1, 3, 4, 6, 7;\ \mathrm{Im}_{e_2}(\rtimes) = \mathrm{Im}_{e_2}(\rtimes');\ \mathrm{Im}_{e_5}(\rtimes) = \mathrm{Im}_{e_5}(\rtimes')$.

(26) $\mathrm{Re}(\rtimes) < \mathrm{Re}(\rtimes');\ \mathrm{Im}_{e_n}(\rtimes) < \mathrm{Im}_{e_n}(\rtimes'),\ n = 1, 3, 4, 5, 7;\ \mathrm{Im}_{e_2}(\rtimes) = \mathrm{Im}_{e_2}(\rtimes');\ \mathrm{Im}_{e_6}(\rtimes) = \mathrm{Im}_{e_6}(\rtimes')$.

(27) $\mathrm{Re}(\rtimes) < \mathrm{Re}(\rtimes');\ \mathrm{Im}_{e_n}(\rtimes) < \mathrm{Im}_{e_n}(\rtimes'),\ n = 1, 3, 4, 5, 6;\ \mathrm{Im}_{e_2}(\rtimes) = \mathrm{Im}_{e_2}(\rtimes');\ \mathrm{Im}_{e_7}(\rtimes) = \mathrm{Im}_{e_7}(\rtimes')$.

(28) $\mathrm{Re}(\rtimes) < \mathrm{Re}(\rtimes');\ \mathrm{Im}_{e_n}(\rtimes) < \mathrm{Im}_{e_n}(\rtimes'),\ n = 1, 2, 5, 6, 7;\ \mathrm{Im}_{e_3}(\rtimes) = \mathrm{Im}_{e_3}(\rtimes');\ \mathrm{Im}_{e_4}(\rtimes) = \mathrm{Im}_{e_4}(\rtimes')$.

(29) $\mathrm{Re}(\rtimes) < \mathrm{Re}(\rtimes');\ \mathrm{Im}_{e_n}(\rtimes) < \mathrm{Im}_{e_n}(\rtimes'),\ n = 1, 2, 4, 6, 7;\ \mathrm{Im}_{e_3}(\rtimes) = \mathrm{Im}_{e_3}(\rtimes');\ \mathrm{Im}_{e_5}(\rtimes) = \mathrm{Im}_{e_5}(\rtimes')$.

(30) $\mathrm{Re}(\rtimes) < \mathrm{Re}(\rtimes');\ \mathrm{Im}_{e_n}(\rtimes) < \mathrm{Im}_{e_n}(\rtimes'),\ n = 1, 2, 4, 5, 7;\ \mathrm{Im}_{e_3}(\rtimes) = \mathrm{Im}_{e_3}(\rtimes');\ \mathrm{Im}_{e_6}(\rtimes) = \mathrm{Im}_{e_6}(\rtimes')$.

(31) $\mathrm{Re}(\rtimes) < \mathrm{Re}(\rtimes');\ \mathrm{Im}_{e_n}(\rtimes) < \mathrm{Im}_{e_n}(\rtimes'),\ n = 1, 2, 4, 5, 6;\ \mathrm{Im}_{e_3}(\rtimes) = \mathrm{Im}_{e_3}(\rtimes');\ \mathrm{Im}_{e_7}(\rtimes) = \mathrm{Im}_{e_7}(\rtimes')$.

(32) $\mathrm{Re}(\rtimes) < \mathrm{Re}(\rtimes');\ \mathrm{Im}_{e_n}(\rtimes) < \mathrm{Im}_{e_n}(\rtimes'),\ n = 1, 2, 3, 6, 7;\ \mathrm{Im}_{e_4}(\rtimes) = \mathrm{Im}_{e_4}(\rtimes');\ \mathrm{Im}_{e_5}(\rtimes) = \mathrm{Im}_{e_5}(\rtimes')$.

(33) $\mathrm{Re}(\rtimes) < \mathrm{Re}(\rtimes');\ \mathrm{Im}_{e_n}(\rtimes) < \mathrm{Im}_{e_n}(\rtimes'),\ n = 1, 2, 3, 5, 7;\ \mathrm{Im}_{e_4}(\rtimes) = \mathrm{Im}_{e_4}(\rtimes');\ \mathrm{Im}_{e_6}(\rtimes) = \mathrm{Im}_{e_6}(\rtimes')$.

(34) $\mathrm{Re}(\rtimes) < \mathrm{Re}(\rtimes');\ \mathrm{Im}_{e_n}(\rtimes) < \mathrm{Im}_{e_n}(\rtimes'),\ n = 1, 2, 3, 5, 6;\ \mathrm{Im}_{e_4}(\rtimes) = \mathrm{Im}_{e_4}(\rtimes');\ \mathrm{Im}_{e_7}(\rtimes) = \mathrm{Im}_{e_7}(\rtimes')$.

(35) $\mathrm{Re}(\rtimes) < \mathrm{Re}(\rtimes');\ \mathrm{Im}_{e_n}(\rtimes) < \mathrm{Im}_{e_n}(\rtimes'),\ n = 1, 2, 3, 4, 7;\ \mathrm{Im}_{e_5}(\rtimes) = \mathrm{Im}_{e_5}(\rtimes');\ \mathrm{Im}_{e_6}(\rtimes) = \mathrm{Im}_{e_6}(\rtimes')$.

(36) $\mathrm{Re}(\rtimes) < \mathrm{Re}(\rtimes');\ \mathrm{Im}_{e_n}(\rtimes) < \mathrm{Im}_{e_n}(\rtimes'),\ n = 1, 2, 3, 4, 6;\ \mathrm{Im}_{e_5}(\rtimes) = \mathrm{Im}_{e_5}(\rtimes');\ \mathrm{Im}_{e_7}(\rtimes) = \mathrm{Im}_{e_7}(\rtimes')$.

Following a similar approach, we can easily list the 56 cases where exactly 3 components are equal (derived from the 3-combinations of 8), 70 cases with 4 equal components, 56 cases with 5 equal components, and 27 cases with 6 equal components. However, to avoid making the article overly tedious, we will not elaborate in detail on the remaining 211 intermediate cases. For simplicity, let us focus only on the 8 cases with exactly 7 equal components, corresponding to the 7-combinations of 8 where just one component differs.

(248) $\mathrm{Re}(\rtimes) < \mathrm{Re}(\rtimes');\ \mathrm{Im}_{e_n}(\rtimes) = \mathrm{Im}_{e_n}(\rtimes')$, where $n = 1, 2, 3, 4, 5, 6, 7$.

(249) $\mathrm{Re}(\rtimes) = \mathrm{Re}(\rtimes');\ \mathrm{Im}_{e_n}(\rtimes) = \mathrm{Im}_{e_n}(\rtimes')$, where $n = 2, 3, 4, 5, 6, 7;\ \mathrm{Im}_{e_1}(\rtimes) < \mathrm{Im}_{e_1}(\rtimes')$.

(250) $\mathrm{Re}(\rtimes) = \mathrm{Re}(\rtimes')$; $\mathrm{Im}_{e_n}(\rtimes) = \mathrm{Im}_{e_n}(\rtimes')$, where $n = 1, 3, 4, 5, 6, 7$; $\mathrm{Im}_{e_2}(\rtimes) < \mathrm{Im}_{e_2}(\rtimes')$.

(251) $\mathrm{Re}(\rtimes) = \mathrm{Re}(\rtimes')$; $\mathrm{Im}_{e_n}(\rtimes) = \mathrm{Im}_{e_n}(\rtimes')$, where $n = 1, 2, 4, 5, 6, 7$; $\mathrm{Im}_{e_3}(\rtimes) < \mathrm{Im}_{e_3}(\rtimes')$.

(252) $\mathrm{Re}(\rtimes) = \mathrm{Re}(\rtimes')$; $\mathrm{Im}_{e_n}(\rtimes) = \mathrm{Im}_{e_n}(\rtimes')$, where $n = 1, 2, 3, 5, 6, 7$; $\mathrm{Im}_{e_4}(\rtimes) < \mathrm{Im}_{e_4}(\rtimes')$.

(253) $\mathrm{Re}(\rtimes) = \mathrm{Re}(\rtimes')$; $\mathrm{Im}_{e_n}(\rtimes) = \mathrm{Im}_{e_n}(\rtimes')$, where $n = 1, 2, 3, 4, 6, 7$; $\mathrm{Im}_{e_5}(\rtimes) < \mathrm{Im}_{e_5}(\rtimes')$.

(254) $\mathrm{Re}(\rtimes) = \mathrm{Re}(\rtimes')$; $\mathrm{Im}_{e_n}(\rtimes) = \mathrm{Im}_{e_n}(\rtimes')$, where $n = 1, 2, 3, 4, 5, 7$; $\mathrm{Im}_{e_6}(\rtimes) < \mathrm{Im}_{e_6}(\rtimes')$.

(255) $\mathrm{Re}(\rtimes) = \mathrm{Re}(\rtimes')$; $\mathrm{Im}_{e_n}(\rtimes) = \mathrm{Im}_{e_n}(\rtimes')$, where $n = 1, 2, 3, 4, 5, 6$; $\mathrm{Im}_{e_7}(\rtimes) < \mathrm{Im}_{e_7}(\rtimes')$.

Finally, let us consider the case derived from the 8-combinations of 8, where all corresponding components are equal, which indicates that the two octonions are identical.

(256) $\mathrm{Re}(\rtimes) = \mathrm{Re}(\rtimes')$; $\mathrm{Im}_{e_n}(\rtimes) = \mathrm{Im}_{e_n}(\rtimes')$, where $n = 1, 2, 3, 4, 5, 6, 7$.

Specifically, if $\|\rtimes\| \neq \|\rtimes'\|$ and any condition between (1) and (256) is satisfied, $\rtimes \preceq \rtimes'$ will be written. If only condition (256) is satisfied, we will denote this by $\rtimes \prec \rtimes'$. We will briefly denote this situation as

$$\rtimes \preceq \rtimes' \implies \|\rtimes\| \leq \|\rtimes'\|. \tag{2}$$

A careful examination of the 256 conditions above reveals that we can introduce octonion-valued metric spaces, which generalize the complex metric spaces defined by Azam et al. [3], by taking the codomain as the field of complex numbers. If a complex-valued metric space satisfies condition

$$\Omega_{\mathbb{C}}(s,t) \preceq b \cdot \left(\Omega_{\mathbb{C}}(s,v) + \Omega_{\mathbb{C}}(v,t)\right) \tag{3}$$

for all $s, t, v \in S$, which is a relaxed version of the triangle inequality for $b \geq 1$ derived using the partial ordering in the third property, such a space is called a complex-valued $b-$metric space. Detailed information about this space can be found in the literature, specifically in [10, 29].

These are then generalized to quaternion-valued metric spaces, as defined by Ahmed et al. [2], taking the codomain as the skew field of quaternions, which serve as a non-commutative extension of these metric spaces to Clifford algebra analysis.

2.2. *Definitions and notations*

Following, we will define octonion-valued metric spaces, an interesting generalization of metric spaces that are neither commutative nor associative.

Definition 2.1. ([6]) Given a nonempty set S. If the transformation $\Omega_{\mathbb{O}} : S \times S \to \mathbb{O}$ on this set satisfies following conditions,

(1) $0_{\mathbb{O}} \preceq \Omega_{\mathbb{O}}(s,t)$ for all $s, t \in S$ and $\Omega_{\mathbb{O}}(s,t) = 0_{\mathbb{O}}$ if and only if $s = t$,

(2) $\Omega_{\mathbb{O}}(s,t) = \Omega_{\mathbb{O}}(t,s)$ for all $s, t \in S$,

(3) $\Omega_{\mathbb{O}}(s,t) \preceq \Omega_{\mathbb{O}}(s,v) + \Omega_{\mathbb{O}}(v,t)$ for all $s, t, v \in S$.

Then $\Omega_{\mathbb{O}}$ is called an octonion valued metric on S, and the pair $(S, \Omega_{\mathbb{O}})$ is called an octonion valued metric space.

Example 2.2. Let $\Omega_{\mathbb{O}} : \mathbb{O} \times \mathbb{O} \to \mathbb{O}$ be an octonion valued function defined as $\Omega_{\mathbb{O}}(\rtimes, \rtimes') = |o_0 - o_0'| + |o_1 - o_1'|e_1 + |o_2 - o_2'|e_2 + |o_2 - o_2'|e_2 + |o_3 - o_3'|e_3 + |o_4 - o_4'|e_4 + |o_5 - o_5'|e_5 + |o_6 - o_6'|e_6 + |o_7 - o_7'|e_7$, where $\rtimes, \rtimes' \in \mathbb{O}$ with

$$\rtimes = o_0 + o_1e_1 + o_2e_2 + o_3e_3 + o_4e_4 + o_5e_5 + o_6e_6 + o_7e_7,$$

$$\rtimes' = o_0' + o_1'e_1 + o_2'e_2 + o_3'e_3 + o_4'e_4 + o_5'e_5 + o_6'e_6 + o_7'e_7;$$

$$o_i, o_i' \in \mathbb{R};\ i = 0, 1, 2, 3, 4, 5, 6, 7.$$

Then $(\mathbb{O}, \Omega_{\mathbb{O}})$ defines an octonion valued metric space.

Below, we provide an example of an octonion-valued metric that does not have a known numerical set as its domain.

Example 2.3. Let $X = \{a, b, c\}$ be an arbitrary set with three elements. Define the distances between the elements of the set by

$$\Omega_{\mathbb{O}}(a,b) = \Omega_{\mathbb{O}}(b,a) = 3 + 4e_1 - 6e_2 + 4e_3 + 3e_4 + 3e_5 - 2e_6 + e_7$$
$$\Omega_{\mathbb{O}}(b,c) = \Omega_{\mathbb{O}}(c,b) = 1 + 2e_1 + 3e_3 - 5e_4 - 3e_6 + 4e_7$$
$$\Omega_{\mathbb{O}}(a,c) = \Omega_{\mathbb{O}}(c,a) = 2 + 3e_1 + e_2 + e_3 - 2e_4 + 2e_5 - e_6 + 5e_7$$
$$\Omega_{\mathbb{O}}(a,a) = \Omega_{\mathbb{O}}(b,b) = \Omega_{\mathbb{O}}(c,c) = 0 + 0e_1 + 0e_2 + 0e_3 + 0e_4 + 0e_5 + 0e_6 + 0e_7.$$

Since they are $\|\Omega_{\mathbb{O}}(a,b)\| = 10$, $\|\Omega_{\mathbb{O}}(a,c)\| = 7$, $\|\Omega_{\mathbb{O}}(c,b)\| = 8$, $\|\Omega_{\mathbb{O}}(a,b) + \Omega_{\mathbb{O}}(a,c)\| = \sqrt{195}$, $\|\Omega_{\mathbb{O}}(a,b) + \Omega_{\mathbb{O}}(b,c)\| = \sqrt{200}$ and $\|\Omega_{\mathbb{O}}(c,b) + \Omega_{\mathbb{O}}(a,c)\| = \sqrt{169} = 13$, it can be seen through straightforward calculations that the conditions given in Definition 2.1 above are satisfied.

Definition 2.4. ([28]) Given a nonempty set S. If the transformation $\Omega_{\mathbb{O}} : S \times S \to \mathbb{O}$ on this set satisfies following conditions,

(1) $0_{\mathbb{O}} \preceq \Omega_{\mathbb{O}}(s,t)$ for all $s, t \in S$ and $\Omega_{\mathbb{O}}(s,t) = 0_{\mathbb{O}}$ if and only if $s = t$,

(2) $\Omega_{\mathbb{O}}(s,t) = \Omega_{\mathbb{O}}(t,s)$ for all $s, t \in S$,

(3) $\Omega_{\mathbb{O}}(s,t) \preceq b \cdot \Big(\Omega_{\mathbb{O}}(s,v) + \Omega_{\mathbb{O}}(v,t)\Big)$ for all $s, t, v \in S$, $1 \leq b \in \mathbb{R}$.

Then $\Omega_{\mathbb{O}}$ is called an octonion valued $b-$metric on S, and the pair $(S, \Omega_{\mathbb{O}})$ is called an octonion valued $b-$metric space.

Example 2.5. Example 2.2 and Example 2.3 are instances of octonion-valued 1-metric spaces for the real scalar $b = 1$.

Remark 2.6. It should be explicitly noted that, as seen from Definition 2.1 and Definition 2.4, every octonion-valued metric space is an octonion-valued $b-$metric space in the special case where $b = 1$.

The converse of the remark we provided above is not true, except for the special case of $b = 1$. The next example we will present is an octonion-valued $b-$metric space for $b = 2$, yet it is not an octonion-valued metric space.

Example 2.7. Let $\Omega^b_{\mathbb{O}} : \mathbb{O} \times \mathbb{O} \to \mathbb{O}$ be an octonion valued function defined as $\Omega^b_{\mathbb{O}}(\rtimes, \rtimes') = |o_0 - o'_0|^2 + |o_1 - o'_1|^2 e_1 + |o_2 - o'_2|^2 e_2 + |o_3 - o'_3|^2 e_3 + |o_4 - o'_4|^2 e_4 + |o_5 - o'_5|^2 e_5 + |o_6 - o'_6|^2 e_6 + |o_7 - o'_7|^2 e_7$, where $\rtimes, \rtimes' \in \mathbb{O}$ with

$$\rtimes = o_0 + o_1 e_1 + o_2 e_2 + o_3 e_3 + o_4 e_4 + o_5 e_5 + o_6 e_6 + o_7 e_7,$$

$$\rtimes' = o'_0 + o'_1 e_1 + o'_2 e_2 + o'_3 e_3 + o'_4 e_4 + o'_5 e_5 + o'_6 e_6 + o'_7 e_7;$$

$$o_i, o'_i \in \mathbb{R};\ i = 0, 1, 2, 3, 4, 5, 6, 7.$$

Then $(\mathbb{O}, \Omega_{\mathbb{O}})$ defines an octonion valued $b-$metric space. Indeed, note that if we take

$$\rtimes = 3 + 3e_1 + 3e_2 + 3e_3 + 3e_4 + 3e_5 + 3e_6 + 3e_7$$
$$\rtimes' = 2 + 2e_1 + 2e_2 + 2e_3 + 2e_4 + 2e_5 + 2e_6 + 2e_7$$
$$\rtimes'' = 1 + e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7,$$

although they are comparable under the partial ordering relation defined on octonions in [6],

$$\Omega^b_{\mathbb{O}}(\rtimes, \rtimes'') = 4 + 4e_1 + 4e_2 + 4e_3 + 4e_4 + 4e_5 + 4e_6 + 4e_7$$
$$\Omega^b_{\mathbb{O}}(\rtimes, \rtimes') = 1 + e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7$$
$$\Omega^b_{\mathbb{O}}(\rtimes', \rtimes'') = 1 + e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7$$
$$\Omega^b_{\mathbb{O}}(\rtimes, \rtimes') + \Omega^b_{\mathbb{O}}(\rtimes', \rtimes'') = 2 + 2e_1 + 2e_2 + 2e_3 + 2e_4 + 2e_5 + 2e_6 + 2e_7,$$

which would violate the third property of the axioms for being an octonion-valued metric space as stated in Definition 2.1, making it not an octonion-valued metric space. However, if we take $b = 2$, in this case, the partial ordering $\preceq$ satisfies the axioms in Definition 2.4.

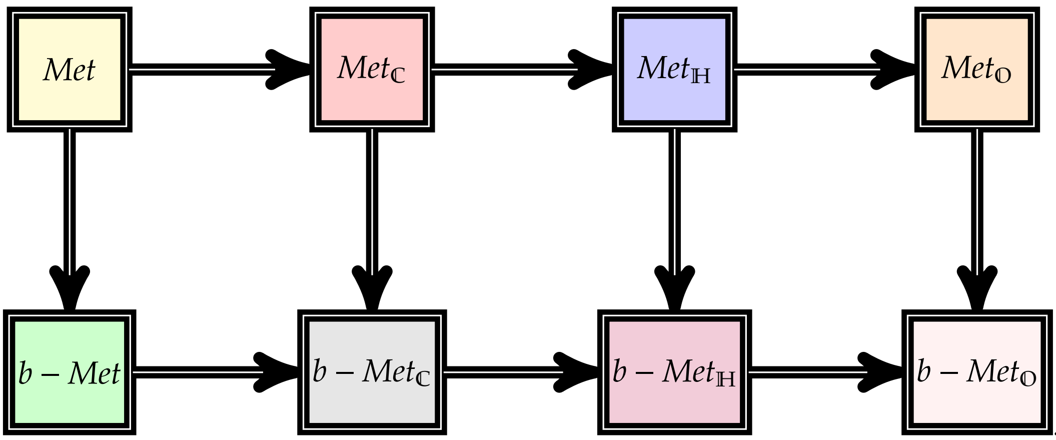
As can be seen from the definitions and example above, the definition we provided is a natural generalization of the classical $b-$metric definition, as well as complex and quaternion-valued $b-$ metrics. To express the connections between them, let us present the following propositions.

Proposition 2.8. *Every quaternion-valued $b-$metric space can be embedded into an octonion-valued $b-$metric space.*

Proposition 2.9. *Every complex-valued $b-$metric space can be embedded into a quaternion-valued $b-$metric space and an octonion-valued $b-$metric space.*

Proposition 2.10. *Every $b-$metric space can be embedded into a complex-valued $b-$metric space, a quaternion-valued $b-$metric space and an octonion-valued $b-$metric space.*

Categorically speaking, the diagrammatic representation of the above propositions and the transitions between these different metric space categories are as follows:



From usual metric spaces, the transition to complex-valued metric spaces is achieved through the generalization of scalar fields. Further generalization to quaternion-valued metric spaces extends the integral domain, and non-associative, higher-dimensional extensions lead to octonion-valued metric spaces. Relaxing the triangle inequality for $b \geq 1$ introduces the categories of classical, complex-valued, quaternion-valued, and octonion-valued $b-$metric spaces. These transitions are facilitated by inclusion functors, while reverse transitions occur through forgetful functors. Here, we focus on the calculus aspects of octonion-valued $b-$metric spaces rather than their algebraic and categorical properties.

Thus, we can now proceed to define some fundamental concepts related to the definition above (see [5, 28]).

Definition 2.11. Any point $s \in S$ is called an interior point of set $A \subset S$ whenever there exists $0_{\mathbb{O}} \prec r \in \mathbb{O}$ such that

$$B(s, r) = \{t \in S : \Omega_{\mathbb{O}}(s, t) \prec r\} \subset A.$$

Definition 2.12. Any point $s \in S$ is called a limit point of $A \subset S$ whenever for every $0_{\mathbb{O}} \prec r \in \mathbb{O}$

$$B(s, r) \cap (A - \{s\}) \neq \emptyset.$$

Definition 2.13. Set O is said to be an open set whenever each element of O is an interior point of O. Subset $C \subset S$ is called a closed set whenever each limit point of C belongs to C. The family

$$F = \{B(s, r) : s \in S, 0_{\mathbb{O}} \prec r\}$$

is a subbase for Hausdorff topology τ on S.

Definition 2.14. Let $s \in S$ and s_k be a sequence in the set S. If for each $\rtimes \in \mathbb{O}$ with $0_{\mathbb{O}} \prec \rtimes$ there is $k_0 \in \mathbb{N}$ such that for all $k > k_0, \Omega_{\mathbb{O}}(s_k, s) \prec \rtimes$, then (s_k) is called convergence sequence. Then, in this case (s_k) sequence converges to the limit point s; as notation, $s_k \to s$ as $k \to \infty$ or $\lim_k s_k = s$.

Definition 2.15. If there exists $k_0 \in \mathbb{N}$ such that for all $k > k_0, \Omega_{\mathbb{O}}(s_{k+m}, s_k) \prec \rtimes$, then (s_k) is said to be a Cauchy sequence in the octonion-valued $b-$metric space $(S, \Omega_{\mathbb{O}})$. If every Cauchy sequence is convergent in $(S, \Omega_{\mathbb{O}})$, then $(S, \Omega_{\mathbb{O}})$ is said to be a complete octonion valued $b-$metric space.

Let's review the meanings of statistical convergence, statistical Cauchy, and natural density (for more information, read the cited sources above). Here

$$\delta(K) = \lim_{N\to\infty} \frac{1}{N} |\{k \le N : k \in K\}|$$

is the definition of the asymptotic (or natural) density for a set K of positive integers. If

$$\lim_N \frac{1}{N} |\{k \le N : |s_k - s| \ge \varepsilon\}| = 0$$

for any $\varepsilon > 0$, then a sequence (s_k) is statistically convergent to s, Also, if there is a positive integer $M = M(\varepsilon)$ such that

$$\lim_N \frac{1}{N} |\{k \le N : |s_k - s_M| \ge \varepsilon\}| = 0$$

for all $\varepsilon > 0$, then the sequence (s_k) is statistically Cauchy.

3. Main result

In this section, we present some definitions for octonion valued b-metric spaces as provided in [28], in the context of concepts like convergent sequences, Cauchy sequences, and bounded sequences. These definitions are statistically generalized versions of their classical counterparts. These concepts apply when not all, but only a significant majority of the terms of a sequence exhibit behavior such as convergence, Cauchy properties, or boundedness.

Definition 3.1. A sequence (s_k) in an octonion valued b-metric space $(S, \Omega_{\mathbb{O}})$ is said to converge statistically to a point $s \in S$ (denoted as $s_k \xrightarrow{stg} s$), if as for all $0_{\mathbb{O}} \prec \rtimes$, we have

$$\lim_{N\to\infty} \frac{1}{N} \left| \left\{ k \le N : \Omega_{\mathbb{O}}(s_k, s) \nprec \rtimes \right\} \right| = 0.$$

In this definition,

$$\left| \left\{ k \le N : \Omega_{\mathbb{O}}(s_k, s) \nprec \rtimes \right\} \right|$$

represents the number of terms in the sequence (s_k) for which the octonion value indicating the distance to s does not precede $\rtimes$ according to the partial ordering relation given in Definition 2.4 above. The ratio of these terms to the number of terms N must approach zero as $N \to \infty$. That is,

$$\frac{\left|\left\{k \leq N : \Omega_{\mathbb{O}}(s_k, s) \nprec \rtimes\right\}\right|}{N} \to 0 \quad \text{as} \quad N \to \infty.$$

This is a necessary condition for the sequence to be statistically convergent to s.

In classical convergence, for all $0_{\mathbb{O}} \prec \rtimes \in \mathbb{O}$, there exists $N \in \mathbb{N}$ such that for $k \geq N$, $\Omega_{\mathbb{O}}(s_k, s) \prec \rtimes$ holds. In statistical convergence, $\Omega_{\mathbb{O}}(s_k, s) \prec \rtimes$ must hold only for the majority of the terms in the sequence; some terms are allowed to be far from s.

Statistical convergence is a generalized version of the classical convergence concept, indicating that the majority of the terms in a sequence converge to a point. This concept can be considered a weaker form of classical convergence.

Example 3.2. Let S be a set, (s_k) a sequence in S, and let $\Omega_{\mathbb{O}}$ an arbitrary octonion valued b-metric on this set. Define the elements of the sequence as follows:

$$f(k) = s_k = \begin{cases} s_8, & \text{if } k = n^3, \exists n \in \mathbb{N}, \\ s_2, & \text{in the other cases.} \end{cases}$$

Here, the function $f : \mathbb{N} \longrightarrow S$ specifies the sequence. Since the asymptotic density of the set $A = \{k : k = n^3 \ \exists n \in \mathbb{N}\} \subset \mathbb{N}$, is determined by $N^{\frac{-2}{3}}$, we find that $\lim_{N\to\infty} \frac{1}{N^{\frac{2}{3}}} = 0$. Consequently, the sequence (s_k) is statistically convergent.The sequence is statistically convergent to s_2.

Theorem 3.3. *In a given octonion valued b-metric space $(S, \Omega_{\mathbb{O}})$, every convergent sequence is also statistically convergent.*

Proof. From the definition of convergence provided in Definition 2.14, for every $\rtimes \in \mathbb{O}$ with $0_{\mathbb{O}} \prec \rtimes$, there exists $k_0 \in \mathbb{N}$ such that for all $k > k_0, \Omega_{\mathbb{O}}(s_k, s) \prec \rtimes$. Consequently, the number of terms that fail to satisfy this condition must be finite. Since the asymptotic density of any finite subset of the natural numbers is zero, it follows from Definition 3.1 that the sequence is statistically convergent. □

Statistical convergence, unlike standard convergence, requires that the majority of the terms, rather than all of them, are close to s. This concept is particularly significant in the analysis of large data sets and complex structures.

Theorem 3.4. *Given an octonion valued b-metric space $(S, \Omega_{\mathbb{O}})$ and a sequence (s_k) in this space, a necessary and sufficient condition for the sequence (s_k) to converge statistically to s is*

$$\|\Omega_{\mathbb{O}}(s_k, s)\| \xrightarrow{stg} 0 \quad as \quad k \to \infty.$$

Proof. Let the sequence (s_k) statistical converge to point s. Given a real number $\varepsilon > 0$, suppose that

$$\rtimes = \frac{\varepsilon}{2\sqrt{2}} + e_1\frac{\varepsilon}{2\sqrt{2}} + e_2\frac{\varepsilon}{2\sqrt{2}} + e_3\frac{\varepsilon}{2\sqrt{2}} + e_4\frac{\varepsilon}{2\sqrt{2}} + e_5\frac{\varepsilon}{2\sqrt{2}} + e_6\frac{\varepsilon}{2\sqrt{2}} + e_7\frac{\varepsilon}{2\sqrt{2}}.$$

From the definition of statistical convergence, for $\forall\, 0_{\mathbb{O}} \prec \rtimes' \in \mathbb{O}$, we have

$$\lim_{N\to\infty} \frac{1}{N}\left|\left\{k \leq N : \Omega_{\mathbb{O}}(s_k, s) \nprec \rtimes'\right\}\right| = 0.$$

In this case, specially for $0_{\mathbb{O}} \prec \rtimes \in \mathbb{O}$ and there exists

$$\lim_{N\to\infty} \frac{1}{N}\left|\left\{k \le N : \Omega_{\mathbb{O}}(s_n, s) \nprec \rtimes\right\}\right| = \lim_{N\to\infty} \frac{1}{N}\left|\left\{k \le N : \|\Omega_{\mathbb{O}}(s_k, s)\| \ge \|\rtimes\| = \varepsilon\right\}\right| = 0.$$

Thus, we obtain

$$\lim_{N\to\infty} \frac{1}{N}\left|\left\{k \le N : \|\Omega_{\mathbb{O}}(s_k, s)\| < \|\rtimes\| = \varepsilon\right\}\right| = 1.$$

Hence, $\|\Omega_{\mathbb{O}}(s_k, s)\| \xrightarrow{stg} 0$ as $k \to \infty$.

On the other hand, suppose that $\|\Omega_{\mathbb{O}}(s_k, s)\| \xrightarrow{stg} 0$ as $k \to \infty$. In this case, for a given $\rtimes \in \mathbb{O}$ with $0_{\mathbb{O}} \prec \rtimes$, there exists a real number $\delta > 0$, such that for any $\rtimes' \in \mathbb{O}$, the following holds:

$$\|\rtimes'\| < \delta \Longrightarrow \rtimes' \prec \rtimes.$$

For this δ, we find

$$\lim_{N\to\infty} \frac{1}{N}\left|\left\{k \le N : \|\Omega_{\mathbb{O}}(s_k, s)\| \ge \|\rtimes\| = \varepsilon \ge \delta \ge \|\rtimes'\|\right\}\right| = \lim_{N\to\infty} \frac{1}{N}\left|\left\{k \le N : \Omega_{\mathbb{O}}(s_k, s) \nprec \rtimes'\right\}\right| = 0.$$

This leads to

$$\lim_{N\to\infty} \frac{1}{N}\left|\left\{k \le N : \|\Omega_{\mathbb{O}}(s_k, s)\| < \|\rtimes\| = \varepsilon\right\}\right| = 1.$$

which implies

$$\lim_{N\to\infty} \frac{1}{N}\left|\left\{k \le N : \Omega_{\mathbb{O}}(s_k, s) \nprec \rtimes\right\}\right| = 0.$$

Hence, the sequence (s_k) converges statistically to point s. □

Theorem 3.5. *Let (s_k) be a sequence in the octonion valued b-metric space $(S, \Omega_{\mathbb{O}})$. Let both $s_k \xrightarrow{stg} s_0$ and $s_k \xrightarrow{stg} t_0$ hold in this metric space. In that case, $s_0 = t_0$.*

Proof. Assume that $s_k \xrightarrow{stg} s_0$ and $s_k \xrightarrow{stg} t_0$. Conjunction from here, by the definition of statistical convergence given in Definition 3.1, for any $\varepsilon > 0$ and $0_{\mathbb{O}} \prec \rtimes \in \mathbb{O}$, let us take

$$\rtimes = \frac{\varepsilon}{4\sqrt{2}} + e_1\frac{\varepsilon}{4\sqrt{2}} + e_2\frac{\varepsilon}{4\sqrt{2}} + e_3\frac{\varepsilon}{4\sqrt{2}} + e_4\frac{\varepsilon}{4\sqrt{2}} + e_5\frac{\varepsilon}{4\sqrt{2}} + e_6\frac{\varepsilon}{4\sqrt{2}} + e_7\frac{\varepsilon}{4\sqrt{2}}.$$

The equalities

$$\lim_{N\to\infty} \frac{1}{N}\left|\left\{k \le N : \Omega_{\mathbb{O}}(s_k, s_0) \nprec \rtimes\right\}\right| = 0,$$

and

$$\lim_{N\to\infty} \frac{1}{N}\left|\left\{k \le N : \Omega_{\mathbb{O}}(s_k, t_0) \nprec \rtimes\right\}\right| = 0$$

hold. From the third axiom of the octonion valued b-metric space, we have

$$0_{\mathbb{O}} \preceq \Omega_{\mathbb{O}}(s_0, t_0) \preceq b \cdot \left(\Omega_{\mathbb{O}}(s_0, s_k) + \Omega_{\mathbb{O}}(s_k, t_0)\right),$$

and as a result,

$$\lim_{N\to\infty}\frac{1}{N}\left|\left\{k\le N:\Omega_{\mathbb{O}}(s_k,s_0)+\Omega_{\mathbb{O}}(s_k,t_0)\nprec\rtimes\right\}\right|=0.$$

By the partial ordering property, it follows that

$$0\le\|\Omega_{\mathbb{O}}(s_0,t_0)\|\le\|b\cdot\left(\Omega_{\mathbb{O}}(s_0,s_k)+\Omega_{\mathbb{O}}(s_k,t_0)\right)\|\le b\cdot\left(\|\Omega_{\mathbb{O}}(s_0,s_k)\|+\|\Omega_{\mathbb{O}}(s_k,t_0)\|\right)=b\cdot\left(0+0\right)=0.$$

From this, we conclude that $\|\Omega_{\mathbb{O}}(s_0,t_0)\|=0$,which implies $\Omega_{\mathbb{O}}(s_0,t_0)=0_{\mathbb{O}}$. Finally, by the first axiom of the octonion valued b-metric space, we deduce $s_0=t_0$. This completes the proof. □

Proposition 3.6. *In both cases the quaternion valued b-metric space $(S,\Omega_{\mathbb{H}})$ and the complex valued b-metric space $(S,\Omega_{\mathbb{C}})$, the statistical limit is unique.*

Proof. This can be directly seen from Proposition 2.8, Proposition 2.9 and Theorem 3.5, respectively. □

Definition 3.7. Let (s_k) be a sequence in the octonion valued b-metric space $(S,\Omega_{\mathbb{O}})$. For a sequence (s_k), a subsequence (s_{k_n}) is called a statistical cluster subsequence if:

$$\forall\, 0_{\mathbb{O}}\prec\rtimes,\quad \exists s\in S\quad\text{such that}\quad \limsup_{N\to\infty}\frac{1}{N}\left|\{n\le N:\Omega_{\mathbb{O}}(s_{k_n},s)\nprec\rtimes\}\right|=0.$$

Theorem 3.8. *Let (s_k) and (t_k) be two sequences in the octonion valued b-metric space $(S,\Omega_{\mathbb{O}})$. If $t_k\xrightarrow{stg}s$, and $\Omega_{\mathbb{O}}(s_k,s)\le\Omega_{\mathbb{O}}(t_k,s)$ for each $k\in\mathbb{N}$, then $s_k\xrightarrow{stg}s$.*

Proof. Since $t_k\xrightarrow{stg}s$, it follows from Theorem 3.4 that

$$\|\Omega_{\mathbb{O}}(t_k,s)\|\xrightarrow{stg}0\quad\text{as}\quad k\to\infty.$$

For each $0_{\mathbb{O}}\prec\rtimes\in\mathbb{O}$ and $k\in\mathbb{N}$, we observe that

$$\{k\le N:\Omega_{\mathbb{O}}(t_k,s)\prec\rtimes\}\subseteq\{k\le N:\Omega_{\mathbb{O}}(s_k,s)\prec\rtimes\}.$$

Thus,

$$1=\lim_{N\to\infty}\frac{1}{N}\left|\left\{k\le N:\Omega_{\mathbb{O}}(t_k,s)\prec\rtimes\right\}\right|\le\lim_{N\to\infty}\frac{1}{N}\left|\left\{k\le N:\Omega_{\mathbb{O}}(s_k,s)\prec\rtimes\right\}\right|.$$

Since the asymptotic density can be at most 1, by Definition 3.1, for all $0_{\mathbb{O}}\prec\rtimes$, we have

$$\lim_{N\to\infty}\frac{1}{N}\left|\left\{k\le N:\Omega_{\mathbb{O}}(s_k,s)\nprec\rtimes\right\}\right|=0.$$

Consequently, $s_k\xrightarrow{stg}s$. □

Definition 3.9. ([11, 14]) A subsequence (s_{k_n}) of a sequence (s_k) is statistically dense in (s_k) if the index set $\{k_n:n\in\mathbb{N}\}$ is a statistically dense subset of $\mathbb{N}$, in other words,

$$\lim_{N\to\infty}\frac{1}{N}\left|\left\{k_n\le N:n\in\mathbb{N}\right\}\right|=1.$$

Theorem 3.10. *In an octonion valued b-metric space $(S,\Omega_{\mathbb{O}})$, let (s_k) be an arbitrary sequence. In this case, the following conditions are equivalent:*

1. *The sequence (s_k) is statistically convergent in the octonion-valued space $(S,\Omega_{\mathbb{O}})$.*
2. *There exists a sequence (t_k) in S that converges such that $s_k = t_k$ for almost all $k \in \mathbb{N}$.*
3. *The sequence (s_k) contains a statistically dense subsequence (s_{k_n}), which is a convergent sequence.*
4. *The sequence (s_k) contains a statistically dense subsequence (s_{k_n}), which is statistically convergent.*

Proof. **(1) ⇒ (2)** Assume that $s_k \xrightarrow{stg} s$. By Definition 3.1, for all $0_{\mathbb{O}} \prec \rtimes$, we have

$$\lim_{N\to\infty} \frac{1}{N}\left|\left\{k \le N : \Omega_{\mathbb{O}}(s_k, s) \nprec \rtimes\right\}\right| = 0.$$

Specially, let $\|\rtimes'\| = 1$ for a chosen element $\rtimes' \in \mathbb{O}$. Then

$$\lim_{N\to\infty} \frac{1}{N}\left|\left\{k \le N : \Omega_{\mathbb{O}}(s_k, s) \prec \frac{\rtimes'}{3}\right\}\right| = 1.$$

This implies that there exists $N_1 \in \mathbb{N}$ such that

$$\lim_{N\to\infty} \frac{1}{N}\left|\left\{k \le N : \Omega_{\mathbb{O}}(s_k, s) \prec \frac{\rtimes'}{3}\right\}\right| > 1 - \frac{1}{3}$$

for all $N > N_1$. We can construct a sequence (N_n) of natural numbers satisfying

$$\lim_{N\to\infty} \frac{1}{N}\left|\left\{l \le N : \Omega_{\mathbb{O}}(s_l, s) \prec \frac{\rtimes'}{3^n}\right\}\right| > 1 - \frac{1}{3^n}$$

for all $N > N_n$. Suppose that $N_n < N_{n+1}$ for each $n \in \mathbb{N}$. Let t_l be defined as

$$t_l = \begin{cases} s_l, & 1 \le l \le N_1, \\ s_l, & N_n < l \le N_{n+1},\ \ \Omega_{\mathbb{O}}(s_l, s) \prec \frac{\rtimes'}{3^n}, \\ s, & \text{otherwise.} \end{cases}$$

Given $0_{\mathbb{O}} \prec \rtimes \in \mathbb{O}$, choose $n \in \mathbb{N}$ such that $\frac{\rtimes'}{3^n} \prec \rtimes$. Then, $\Omega_{\mathbb{O}}(t_l, s) \prec \rtimes$ for all $l > N_n$, indicating that the sequence (t_l) converges to s.

For all $0_{\mathbb{O}} \prec \rtimes \in \mathbb{O}$, there exists $n \in \mathbb{N}$ with $\frac{\rtimes'}{3^n} \prec \rtimes$. Let $N \in \mathbb{N}$. If $N_n < N \le N_{n+1}$, then

$$\{l \le N : t_l \ne s_l\} \subset \{1, 2, \ldots, N\} - \{l \le N : \Omega_{\mathbb{O}}(s_l, s) \prec \frac{\rtimes'}{3^n}\},$$

so

$$\frac{1}{N}|\{l \le N : t_l \ne s_l\}| \le 1 - \frac{1}{N}|\{l \le N : \Omega_{\mathbb{O}}(s_l, s) \prec \frac{\rtimes'}{3^n}\}| < \frac{1}{3^n} < \|\rtimes\|.$$

Therefore $\lim_{N\to\infty} \frac{1}{N}\left|\left\{l \le N : t_l \ne s_l\right\}\right| = 0$. Hence $s_l = t_l$ for almost every $l \in \mathbb{N}$.

(2) ⇒ (3) Assume that (t_k) is a convergent sequence in S with $s_k = t_k$ for almost every $k \in \mathbb{N}$. In this situation, $\lim_{N\to\infty} \frac{1}{N}\left|\left\{l \le N : t_l \ne s_l\right\}\right| = 0$. If we take $(t_k) = (s_{k_n})$, in this situation, from Definition 2.14 and Definition 3.9, (t_k) is both a convergent sequence and a statistically dense subsequence of (s_k).

(3) ⇒ (4) If we take $(t_k) = (s_{k_n})$ as a subsequence, it can be directly seen from the definition of a statistically dense subsequence (Definition 3.9) and the definition of statistical convergence (Definition 3.1).

(4) ⇒ (1) Assume that there exists a statistically dense subsequence (s_{k_n}) of the sequence (s_k) with the sequence (s_{k_n}) is statistically convergent. By the definition of statistical convergence, we have for all $0_{\mathbb{O}} \prec \rtimes$, we have

$$\lim_{N\to\infty} \frac{1}{N}\left|\left\{k_n \le N : \Omega_{\mathbb{O}}(s_{k_n}, s) \nprec \rtimes\right\}\right| = 0.$$

and let

$$s_{k_n} \xrightarrow{stg} s \quad \text{as} \quad n \to \infty.$$

It follows that

$$\lim_{N\to\infty} \frac{1}{N}\left|\left\{k_n \le N : \Omega_{\mathbb{O}}(s_{k_n}, s) \prec \rtimes\right\}\right| = 1.$$

Because it happens that for each $0_{\mathbb{O}} \prec \rtimes \in \mathbb{O}$,

$$\{k_n \in \mathbb{N} : \Omega_{\mathbb{O}}(s_{k_n}, s) \prec \rtimes\} \subset \{k \in \mathbb{N} : \Omega_{\mathbb{O}}(s_k, s) \prec \rtimes\},$$

and

$$\lim_{N\to\infty} \frac{1}{N}\left|\left\{k \le N : \Omega_{\mathbb{O}}(s_k, s) \prec \rtimes\right\}\right| \ge \lim_{N\to\infty} \frac{1}{N}\left|\left\{k_n \le N : \Omega_{\mathbb{O}}(s_{k_n}, s) \prec \rtimes\right\}\right| = 1.$$

we have

$$s_k \xrightarrow{stg} s \quad \text{as} \quad k \to \infty.$$

□

Corollary 3.11. *In the octonion valued b-metric space $(S, \Omega_{\mathbb{O}})$, every statistically convergent sequence has a convergent subsequence within this space.*

Proof. The desired result can be directly seen through Definition 2.14, Definition 3.1, and Theorem 3.10. □

Definition 3.12. A sequence (s_k) in an octonion valued b-metric space $(S, \Omega_{\mathbb{O}})$ is said to be statistical Cauchy sequence, if as for all $0_{\mathbb{O}} \prec \rtimes$, we have $l \in \mathbb{N}^+$ depending on the norm of $\rtimes \in \mathbb{O}$

$$\lim_{N} \frac{1}{N}\left|\left\{k \le N : \Omega_{\mathbb{O}}(s_k, s_l) \nprec \rtimes\right\}\right| = 0.$$

If we carefully examine this definition,

$$\left|\left\{k \le N : \Omega_{\mathbb{O}}(s_k, s_l) \nprec \rtimes\right\}\right|$$

represents the number of terms in the sequence (s_k) in S whose octonion value, indicating the distance between the elements of the sequence, does not precede $\rtimes$ according to the partial ordering relation given in Definition 2.1. The ratio of these terms to the total number of terms N must approach zero as $N \to \infty$. In other words,

$$\frac{\left|\left\{k, l \le N : \Omega_{\mathbb{O}}(s_k, s_l) \nprec \rtimes\right\}\right|}{N} \to 0 \quad \text{as} \quad N \to \infty.$$

This is a necessary condition for the sequence to be statistically Cauchy.

In accustomed definition Cauchy sequence, for every $0_{\mathbb{O}} \prec \rtimes \in \mathbb{O}$, We have $N \in \mathbb{N}$ with as $k, l \ge N$, $\Omega_{\mathbb{O}}(s_k, s_l) \prec \rtimes$ satisfies. In statistical Cauchy sequence, $\Omega_{\mathbb{O}}(s_k, s_l) \prec \rtimes$ must satisfy only for the majority of the terms in the sequence; It is acceptable for the distances between some terms to follow after $\rtimes$.

The concept of a statistical Cauchy sequence is a generalized version of the classical Cauchy sequence and can be understood as a sequence where the distances between the majority of its terms precede $\rtimes$ in the ordering.

Theorem 3.13. *Given an octonion valued b-metric space $(S, \Omega_{\mathbb{O}})$ and let (s_k) be a sequence in S. Then (s_k) is a statistically Cauchy sequence if and only if*

$$||\Omega_{\mathbb{O}}(s_k, s_{k+m})|| \xrightarrow{stg} 0 \quad as \quad k \to \infty.$$

Proof. We assume that (s_k) is a statistically Cauchy sequence in S. From Definition 3.12, as for all $0_{\mathbb{O}} \prec \rtimes$, we have $l \in \mathbb{N}^+$ depending on the norm of $\rtimes \in \mathbb{O}$

$$\lim_N \frac{1}{N}\left|\left\{k, l \leq N : \Omega_{\mathbb{O}}(s_k, s_l) \nprec \rtimes\right\}\right| = 0.$$

As a given real number $\varepsilon > 0$, suppose that

$$\rtimes = \frac{\varepsilon}{2\sqrt{2}} + e_1\frac{\varepsilon}{2\sqrt{2}} + e_2\frac{\varepsilon}{2\sqrt{2}} + e_3\frac{\varepsilon}{2\sqrt{2}} + e_4\frac{\varepsilon}{2\sqrt{2}} + e_5\frac{\varepsilon}{2\sqrt{2}} + e_6\frac{\varepsilon}{2\sqrt{2}} + e_7\frac{\varepsilon}{2\sqrt{2}}.$$

$$\lim_N \frac{1}{N}\left|\left\{k, l \leq N : \Omega_{\mathbb{O}}(s_k, s_l) \nprec \rtimes\right\}\right| = \lim_N \frac{1}{N}\left|\left\{k, l \leq N : ||\Omega_{\mathbb{O}}(s_k, s_l)|| \geq ||\rtimes|| = \varepsilon\right\}\right| = 0.$$

In this case, we have

$$||\Omega_{\mathbb{O}}(s_k, s_l)|| \xrightarrow{stg} 0 \quad \text{as} \quad k \to \infty.$$

by Theorem 3.4 and Definition 3.1.

On the other hand, we assume that $||\Omega_{\mathbb{O}}(s_k, s_{k+m})|| < ||\rtimes|| \xrightarrow{stg} 0$ as $k \to \infty$. So, given $\rtimes \in \mathbb{O}$ with $0_{\mathbb{O}} \prec \rtimes$, there is a real number $\delta > 0$ such that as $\rtimes' \in \mathbb{O}$,

$$\lim_{N\to\infty} \frac{1}{N}\left|\left\{k \leq N : ||\Omega_{\mathbb{O}}(s_k, s_{k+m})|| \geq ||\rtimes|| = \varepsilon \geq \delta \geq ||\rtimes'||\right\}\right| = \lim_{N\to\infty} \frac{1}{N}\left|\left\{k \leq N : \Omega_{\mathbb{O}}(s_k, s_{k+m}) \nprec \rtimes'\right\}\right| = 0.$$

Corresponding to this δ, there exists $l \in \mathbb{N}^+$ depending on the norm of $\rtimes \in \mathbb{O}$, so, we get

$$\lim_N \frac{1}{N}\left|\left\{k \leq N : ||\Omega_{\mathbb{O}}(s_k, s_l)|| < ||\rtimes|| = \varepsilon\right\}\right| = 1.$$

This implies that

$$\lim_N \frac{1}{N}\left|\left\{k \leq N : \Omega_{\mathbb{O}}(s_k, s) \nprec \rtimes\right\}\right| = 0.$$

Hence the sequence (s_k) is statistical Cauchy sequence. Thus, the proof is complete. □

Theorem 3.14. *Every statistically convergent sequence in an octonion valued b-metric space is a statistical Cauchy sequence.*

Proof. Let (s_k) be a sequence in the octonion valued b-metric space $(S, \Omega_{\mathbb{O}})$. Suppose that $s_k \xrightarrow{stg} s$. In this case, for all $0_{\mathbb{O}} \prec \rtimes$, we have

$$\lim_{N\to\infty} \frac{1}{N}\left|\left\{k \leq N : \Omega_{\mathbb{O}}(s_k, s) \nprec \rtimes\right\}\right| = 0.$$

Additionally, by the independence of the representation of statistical convergence and by its definition, for every $0_{\mathbb{O}} \prec \rtimes'$, there exists a $K \in \mathbb{N}$ such that when $k, l > K$, and given the partial ordering definition above and the fact that $0_{\mathbb{O}} \prec \rtimes' \in \mathbb{O}$, it follows for the octonion $\frac{\rtimes'}{2}$ that $0_{\mathbb{O}} \prec \frac{\rtimes'}{2b} \in \mathbb{O}$. Furthermore,

$$\lim_{N\to\infty} \frac{1}{N}\left|\left\{k \leq N : \Omega_{\mathbb{O}}(s_k, s) \nprec \frac{\rtimes'}{2b}\right\}\right| = 0.$$

and

$$\lim_{N\to\infty}\frac{1}{N}\left|\left\{k\le N:\Omega_{\mathbb{O}}(s_l,s)\not\prec\frac{\rtimes'}{2b}\right\}\right|=0.$$

hold. Thus, for $k,l>K$, by the third axiom of the octonion valued b-metric space, we have:

$$\Omega_{\mathbb{O}}(s_k,s_l)\preceq b\cdot\left(\Omega_{\mathbb{O}}(s_k,s)+\Omega_{\mathbb{O}}(s,s_l)\right)\prec b\cdot\left(\frac{\rtimes'}{2\cdot b}+\frac{\rtimes'}{2\cdot b}\right)=\rtimes',$$

so for each $N\in\mathbb{N}$,

$$\left\{k\le N:\Omega_{\mathbb{O}}(s_k,s)\prec\rtimes'\right\}\subset\left\{k,l\le N:\Omega_{\mathbb{O}}(s_k,s_l)\prec\rtimes'\right\},$$

and

$$\lim_{N\to\infty}\frac{1}{N}\left|\left\{k\le N:\Omega_{\mathbb{O}}(s_l,s_k)\not\prec\rtimes'\right\}\right|=0.$$

Therefore, since $\Omega_{\mathbb{O}}(s_k,s_l)\prec\rtimes'$ holds for every $0_{\mathbb{O}}\prec\rtimes'\in\mathbb{O}$, the sequence (s_k) is a statistical Cauchy sequence. The proof is complete. □

Proposition 3.15. *Every statistically convergent sequence is also a statistical Cauchy sequence in both quaternion valued b-metric spaces and complex valued b-metric spaces.*

Proof. This can be directly seen from Theorem 3.14, Proposition 2.8 and Proposition 2.9, respectively. □

Definition 3.16. If every statistically Cauchy sequence is statistically convergent in $(S,\Omega_{\mathbb{O}})$, then $(S,\Omega_{\mathbb{O}})$ is said to be a statistically complete octonion valued b-metric space.

Corollary 3.17. *Every statistically complete octonion valued b-metric space is a complete.*

Proof. This can be directly observed from Definition 2.15 and Definition 3.16. □

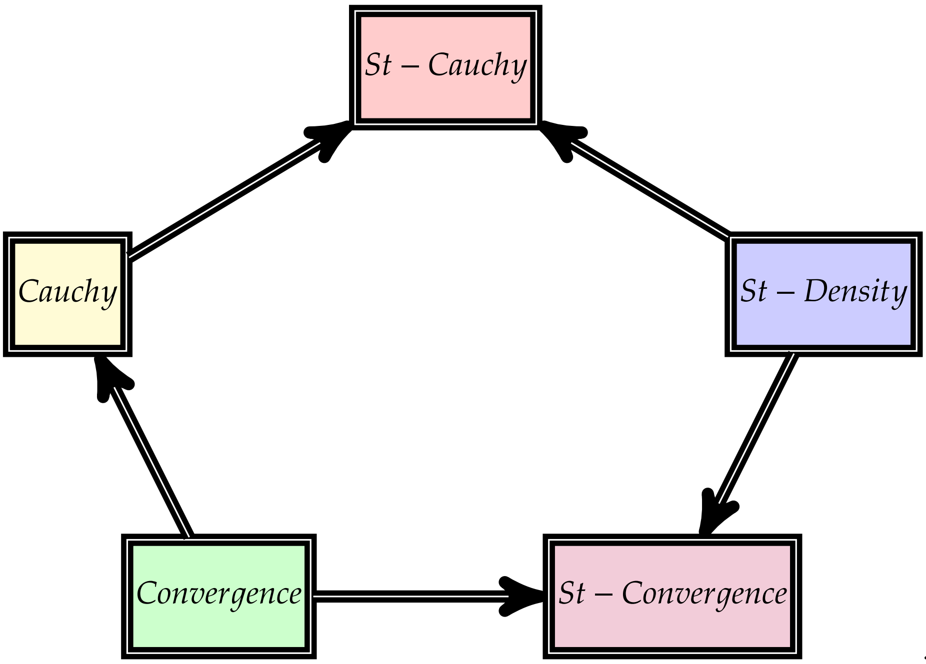
Note that not every octonion valued b-metric space must be statistically complete. The following example of an octonion valued b-metric space supports this.

Example 3.18. Let $d_{\mathbb{O}}:\mathbb{N}^+\times\mathbb{N}^+\to\mathbb{O}$ be an octonion valued function defined as

$$d_{\mathbb{O}}(n,m)=\begin{cases}1_{\mathbb{O}}, & m \text{ is prime},\\ \Omega_{\mathbb{O}}(n,m), & \text{otherwise},\end{cases}$$

where $n,m\in\mathbb{N}^+$. Then $(\mathbb{N}^+,d_{\mathbb{O}})$ defines an octonion valued b-metric space. However, since it is $0\notin\mathbb{N}^+$, this octonion valued b-metric space is not statistically complete.

A statistically dense subsequence is not necessarily statistically Cauchy. Statistical density and statistical Cauchy-ness are distinct concepts, and their relationship depends on the structure of the sequence. Statistical density implies that the sequence clusters around certain points or values, while statistical Cauchyness indicates that the distances between terms of the sequence decrease in a controlled manner. However, if a sequence has a statistically dense subsequence, then this subsequence is statistically convergent within the sequence, and thus it is also a statistically Cauchy subsequence.This and the results we have obtained can roughly be represented diagrammatically in the form of



Remark 3.19. Every ring forms a module over itself, and every field forms a vector space over itself, as is commonly known. Let's be clear, though, that octonions cannot form a module over themselves since they lack multiplicative associativity, which makes them ineligible even as rings. Because of this, our established metric spaces and the associated conclusions are of special importance.

4. Conclusion

In this paper, we have introduced and studied statistical convergence and related concepts in octonion-valued b-metric spaces. We established fundamental properties such as the uniqueness of statistical limits, the relation between statistical Cauchy and convergence sequences, and the existence of statistically dense subsequences. The presented results extend the framework of convergence beyond associative settings and demonstrate how octonionic algebra affects these notions. Future work may focus on fixed point results, completeness criteria, and functional analytic extensions in non-associative metric structures. These results may find potential applications in areas involving non-associative data structures, hypercomplex analysis, and theoretical physics, where octonionic frameworks naturally arise.

References

[1] A. A. Albert, *On a certain algebra of quantum mechanics*, Ann. Math. **35** (1934), 65–73.
[2] A. El-Sayed Ahmed, S. Omran, A. J. Asad, *Fixed point theorems in quaternion valued metric spaces*, Abstract Appl. Analysis Article ID 258958, (2014), 1–9.
[3] A. Azam, B. Fisher, M. Khan, *Common fixed point theorems in complex valued metric spaces*, Numer. Funct. Anal. Optim. **32** (2011), 243–253.
[4] J. C. Baez, *The octonions*, Bull. Amer. Math. Soc. **39** (2002), 145–205.
[5] S. Çetin, S., Ö. Kişi, M. Gürdal, *Deferred statistical convergence of sequences in octonion-valued metric spaces*, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat. **74** (2025), 375–394.
[6] S. Çetin, M. Mursaleen, Ö. Kişi, M. Gürdal, *Octonion-valued metric spaces and the concept of convergence in these spaces*, Preprint. 2025.
[7] J. H. Conway , D. A. Smith, *On quaternions and octonions: their geometry, arithmetic, and symmetry*, Bull. Amer. Math. Soc. **42** (2005), 229–243.
[8] B. G. Dhage, *Generalized metric spaces and mapping with fixed point*, Bull. Calcutta Math. Soc. **84** (1992), 329–336.

[9] T. Dray, C. Manogue, *The Geometry of the Octonions*, World Scientific, 2015.
[10] Ö. Ege, *Complex valued rectangular b-metric spaces and an application to linear equations*, J. Nonlinear Sci. Appl. **6** (2015), 1014–1021
[11] H. Fast, *Sur la convergence statistique*, Colloq. Math. **10** (1951), 142–149.
[12] D. Fiorenza, H. Sati, U. Schreiber, *Super-exceptional embedding construction of the heterotic M5: Emergence of SU(2)-flavor sector*, J. Geom. Phys. **170** (2021), 104349.
[13] M. M. Frechet, *Sur quelques points du calcul fonctionnel*, Rend. Circolo Mat. Palermo (1884-1940) **22** (1906), 1–72.
[14] J. A. Fridy, *On statistical convergence*, Analysis **5** (1985), 301-313.
[15] S. Gahler, *2-metrische Räume und ihre topologische struktur*, Math. Nachr. **26** (1963), 115–148.
[16] M. Gürdal, *Description of extended eigenvalues and extended eigenvectors of integration operators on the Wiener algebra*, Expo. Math. **27** (2009), 153–160.
[17] M. Gürdal, M. T. Garayev, S. Saltan, *Some concrete operators and their properties*, Turkish J. Math. **39** (2015), 970–989.
[18] M. Gürdal, N. Sarı, E. Savaş, *A-statistically localized sequences in n-normed spaces*, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat. **69**(2) (2020), 1484–1497.
[19] M. Gürdal, U. Yamancı, *Statistical convergence and some questions of operator theory*, Dynam. Systems Appl. **24** (2015), 305–311.
[20] O. Hadzić, L. Gajić, *Coincidence points for set-valued mappings in convex metric spaces*, Univerzitet u Novom Sadu. Zbornik Radova Prirodno-Matematićkog Fakulteta **16** (1986), 13–25.
[21] M. T. Karaev, M. Gürdal, M. B. Huban, *Reproducing kernels, Englis algebras and some applications*, Studia Math. **232**(2) (2016), 113–141.
[22] M. T. Karaev, M. Gürdal, S. Saltan, *Some applications of Banach algebra techniques*, Math. Nachr. **84** (2011), 1678–1689.
[23] M. Kansu, E.M. Tanışlı, S. Demir, *Octonion form of duality-invariant field equations for dyons*, Turk. J. Phys. **44** (2020), 10–23.
[24] M. A. Khamsi, *Generalized metric spaces: A survey*, J. Fixed Point Theory Appl. **17** (2015), 455–475.
[25] A. A. Nabiev, E. Savaş, M. Gürdal, *Statistically localized sequences in metric spaces*, J. Appl. Anal. Comput. **9** (2019), 739–746.
[26] S. Okubo, *Introduction to Octonion and Other Non-Associative Algebras in Physics*, Cambridge Univ. Press, 1995.
[27] R. P. Pant, R. K. Bisht, *Occasionally weakly compatible mappings and fixed points*, Bull. Belg. Math. Soc. **19** (2012), 655–661.
[28] X. -L. Qiu, S. Çetin, S., Ö. Kişi, M. Gürdal, Q. B. Cai, *Octonion-valued b-metric spaces and results on its application*, AIMS Mathematics **10** (2025), 10504–10527.
[29] K. Rao, P. Swamy, J. Prasad, *A common fixed point theorem in complex valued b-metric spaces*, Bull. Math. Stat. Research **1** (2013), 1–8.
[30] T. Salat, *On Statistically convergent sequences of real numbers*, Math. Slovaca **30** (1980), 139–150.
[31] E. Savaş, M. Gürdal, *Generalized statistically convergent sequences of functions in fuzzy 2-normed spaces*, J. Intell. Fuzzy Systems **27** (2014), 2067–2075.
[32] E. Savaş, Ö. Kişi, M. Gürdal, *On statistical convergence in credibility space*, Numer. Funct. Anal. Optim. **43** (2022), 987–1008.
[33] I. J. Schoenberg, *The integrability of certain functions and related summability methods*, Amer. Math. Monthly **66** (1959) 361-375.
[34] H. Steinhaus, *Sur la convergence ordiniaire et la convergence asymptotique*, Colloq. Math. **2** (1951), 73–84.
[35] K. Takahashi, M. Fujita, M. Hashimoto, *Remarks on octonion-valued neural networks with application to robot manipulator control*, IEEE International Conference on Mechatronics (ICM), Kashiwa, Japan, (2021), pp. 1–6.
[36] J. Wu, L. Xu, F. Wu, Y. Kong, L. Senhadji, H. Shu, *Deep octonion networks*, Neurocomputing **397** (2020), 179–191.
[37] U. Yamancı, M. Gürdal, *Statistical convergence and operators on Fock space*, New York J. Math. **22** (2016), 199–207.
[38] A. Zygmund, *Trigonometric Series*, Cambridge Univ. Press, UK, 1979.