Filomat 40:3 (2026), 817–848
https://doi.org/10.2298/FIL2603817Y

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Multiplicative fractional integral inequalities for multiplicative s-convex functions: A multi-parameter approach

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Abstract. Within the multiplicative Riemann–Liouville (RL) fractional framework, we develop multi-parameter inequalities for multiplicatively differentiable s-convex positive functions. Using a multi-parameter multiplicative fractional integral identity, we establish several inequalities under the conditions: (i) Ψ^* exhibits multiplicative s-convexity, and (ii) $(\ln \Psi^*)^q$ maintains s-convexity ($q > 1$). Numerical examples and graphical visualizations verify the proposed inequalities.

1. Motivation and background

Breakthroughs in mathematical analysis often give rise to revolutionary theoretical frameworks. The multiplicative calculus, also recognized as non-Newtonian calculus, proposed by Grossman and Katz in Ref. [18], serves as such a paradigm. By substituting linear operations in classical calculus with multiplicative ones, this theoretical framework can effectively models nonlinear systems with exponential and multiplicative characteristics. The core of this framework lies in the definition for the multiplicative derivative:

$$\Psi^*(\gamma) = \lim_{\rho \to 0} \left(\frac{\Psi(\gamma + \rho)}{\Psi(\gamma)} \right)^{\frac{1}{\rho}}, \tag{1}$$

where $\Psi(\gamma) \neq 0$ and $\frac{\Psi(\gamma+\rho)}{\Psi(\gamma)} > 0$ for sufficiently small values of ρ. The derivative exists provided that this limit converges, and its non-zero value signifies the instantaneous logarithmic growth rate, thereby encapsulating the intensity of exponential variation. This differs from the classical derivative:

$$\Psi'(\gamma) = \lim_{\rho \to 0} \frac{\Psi(\gamma + \rho) - \Psi(\gamma)}{\rho}. \tag{2}$$

2020 *Mathematics Subject Classification*. Primary 26A33; Secondary 26A51, 26D10, 26D15
Keywords. Multi-parameter inequalities; s-Convex functions; Multiplicative fractional integrals.
Received: 03 August 2025; Revised: 28 September 2025; Accepted: 22 November 2025
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Upon comparing the equations (1) and (2), it becomes evident that the multiplicative derivative replaces the operations of subtraction and multiplication with division and exponentiation, respectively. The relationship between Ψ^* and Ψ' is represented as:

$$\Psi^*(\gamma) = \exp\{(\ln\Psi)'(\gamma)\}. \tag{3}$$

Bashirov et al. [5], drawing from the multiplicative derivative, formulated the definition of the multiplicative integral:

$$\int_\kappa^\sigma \left(\Psi(\gamma)\right)^{\mathrm{d}\gamma} = \exp\left\{\int_\kappa^\sigma \ln\Psi(\gamma)\mathrm{d}\gamma\right\}, \tag{4}$$

where the function $\Psi(\gamma) > 0$ is required. The multiplicative integral represents accumulation through continuous exponentiation, with $\mathrm{d}\gamma$ acting as a multiplicative infinitesimal, whereas the Riemann integral $\int_\kappa^\sigma \left(\Psi(\gamma)\right)\mathrm{d}\gamma$ computes additive summation.

As a formidable mathematical tool, the multiplicative integral derives the cumulative geometric mean through the exponential reconstruction of logarithmic integrals, thus rendering it apt for exponential modeling in contexts such as financial compounding and biological population dynamics. Let us illustrate its application using a bacterial growth model. Suppose $\mathfrak{I}(\varepsilon) > 0$ represents the size of the population at time ε. The population's evolution can be described by the following dynamical model:

$$\mathfrak{I}'(\varepsilon) = R(\varepsilon)\mathfrak{I}(\varepsilon), \tag{5}$$

where $R(\varepsilon) > 0$ denotes the relative growth rate of the population, and $\mathfrak{I}'(\varepsilon)$ is the rate of change of the population over time. This idealized model, though neglecting carrying capacity limitations, serves as a foundational case for generalized growth models (e.g., Logistic growth).

Through the equation (3), we derive the multiplicative reformulation of the equation (5):

$$\mathfrak{I}^*(\varepsilon) = \exp\{R(\varepsilon)\}, \tag{6}$$

with a multiplicative integral solution given by

$$\mathfrak{I}(\varepsilon) = \mathfrak{I}(\varepsilon_0)\int_{\varepsilon_0}^{\varepsilon} \left(\exp\{R(\varepsilon)\}\right)^{\mathrm{d}\varepsilon}, \tag{7}$$

where $\mathfrak{I}(\varepsilon_0)$ is the initial population size.

This example illustrates the application value of multiplicative calculus in differential equations, with the equation (6) providing a more direct representation of exponential growth dynamics than the equation (5).

After analyzing multiplicative operations and functional properties, we find that convexity is essential for the theory of inequalities. An important result in this field is the following Hermite–Hadamard (HH) inequality

$$\Psi\left(\frac{\kappa+\sigma}{2}\right) \le \frac{1}{\sigma-\kappa}\int_\kappa^\sigma \Psi(\gamma)\,\mathrm{d}\gamma \le \frac{\Psi(\kappa)+\Psi(\sigma)}{2}, \tag{8}$$

holds for any convex function Ψ defined on $[\kappa, \sigma]$. The inequality (8) reflects the geometric properties of convexity and contributes to the advancement of integral inequality theory. In particular, Xi and Qi [52] generalized the HH-type inequality through the following parameterized identity.

Lemma 1.1. *[52] Assume that the function $\Psi : [\kappa, \sigma] \to \mathbb{R}$ exhibits differentiability on (κ, σ). If $\Psi' \in L_1[\kappa, \sigma]$ and $\lambda, \mu \in \mathbb{R}$, then the following identity holds:*

$$\begin{aligned}&\frac{\lambda\Psi(\kappa)+\mu\Psi(\sigma)}{2} + \frac{2-\lambda-\mu}{2}\Psi\left(\frac{\kappa+\sigma}{2}\right) - \frac{1}{\sigma-\kappa}\int_\kappa^\sigma \Psi(u)\,\mathrm{d}u \\ &= \frac{\sigma-\kappa}{4}\int_0^1 \left[(1-\lambda-\varepsilon)\Psi'\left(\varepsilon\kappa+(1-\varepsilon)\frac{\kappa+\sigma}{2}\right) + (\mu-\varepsilon)\Psi'\left(\varepsilon\frac{\kappa+\sigma}{2}+(1-\varepsilon)\sigma\right)\right]\mathrm{d}\varepsilon.\end{aligned}$$

Later, Toseef et al. [47] proposed the following four-parameter identity extending the HH-type inequality:

Lemma 1.2. *[47] Consider a differentiable function $\Psi : [\kappa, \sigma] \to \mathbb{R}$ where $\kappa < \sigma$, with $\Psi' \in L_1[\kappa, \sigma]$. For any non-negative parameters γ, φ, z and w, the following identity holds:*

$$\begin{aligned}&(1+\gamma-w)\left[\Psi(\kappa)+\Psi(\sigma)\right]+(\varphi-\gamma+w-z)\left[\Psi\left(\frac{3\kappa+\sigma}{4}\right)+\Psi\left(\frac{\kappa+3\sigma}{4}\right)\right]\\&\quad+2(z-\varphi)\Psi\left(\frac{\kappa+\sigma}{2}\right)-\frac{2}{\sigma-\kappa}\int_\kappa^\sigma\Psi(u)\mathrm{d}u\\&=(\sigma-\kappa)\int_0^1 P(\gamma,\varphi,z,w,\varepsilon)\left[\Psi'(\varepsilon\sigma+(1-\varepsilon)\kappa)-\Psi'(\varepsilon\kappa+(1-\varepsilon)\sigma)\right]\mathrm{d}\varepsilon,\end{aligned}$$

where

$$P(\gamma,\varphi,z,w,\varepsilon)=\begin{cases}\varepsilon-\gamma, & \varepsilon\in\left[0,\frac{1}{4}\right),\\ \varepsilon-\varphi, & \varepsilon\in\left[\frac{1}{4},\frac{1}{2}\right),\\ \varepsilon-z, & \varepsilon\in\left[\frac{1}{2},\frac{3}{4}\right),\\ \varepsilon-w, & \varepsilon\in\left[\frac{3}{4},1\right].\end{cases}$$

For example, by choosing $\gamma = \varphi = z = w = \frac{1}{2}$, the identity in Lemma 1.2 degenerates into the following trapezoid form:

$$\frac{\Psi(\kappa)+\Psi(\sigma)}{2}-\frac{1}{\sigma-\kappa}\int_\kappa^\sigma\Psi(u)\mathrm{d}u=\frac{\sigma-\kappa}{2}\int_0^1\left(\varepsilon-\frac{1}{2}\right)\left[\Psi'(\varepsilon\sigma+(1-\varepsilon)\kappa)-\Psi'(\varepsilon\kappa+(1-\varepsilon)\sigma)\right]\mathrm{d}\varepsilon. \quad (9)$$

And by choosing $\gamma = \varphi = 0$ and $z = w = 1$, the identity in Lemma 1.2 degenerates into the following midpoint form:

$$\Psi\left(\frac{\kappa+\sigma}{2}\right)-\frac{1}{\sigma-\kappa}\int_\kappa^\sigma\Psi(u)\mathrm{d}u=\frac{\sigma-\kappa}{2}\int_0^1\phi(\varepsilon)\left[\Psi'(\varepsilon\sigma+(1-\varepsilon)\kappa)-\Psi'(\varepsilon\kappa+(1-\varepsilon)\sigma)\right]\mathrm{d}\varepsilon, \quad (10)$$

where

$$\phi(\varepsilon)=\begin{cases}\varepsilon, & 0\le\varepsilon<\frac{1}{2},\\ \varepsilon-1, & \frac{1}{2}\le\varepsilon\le 1.\end{cases}$$

Currently, the exploration of inequalities has expanded into fractional calculus. To proceed, we recall the definition of RL-fractional integral operators.

Definition 1.3. *[24] Assume that $\Psi \in L_1([\kappa, \sigma])$. For $\alpha > 0$, the RL-fractional integrals, expressed as $\mathcal{I}^\alpha_{\kappa^+}\Psi(\gamma)$ and $\mathcal{I}^\alpha_{\sigma^-}\Psi(\gamma)$, are defined as follows:*

$$\mathcal{I}^\alpha_{\kappa^+}\Psi(\gamma)=\frac{1}{\Gamma(\alpha)}\int_\kappa^\gamma(\gamma-\varepsilon)^{\alpha-1}\Psi(\varepsilon)\mathrm{d}\varepsilon,\quad \gamma>\kappa,$$

and

$$\mathcal{I}^\alpha_{\sigma^-}\Psi(\gamma)=\frac{1}{\Gamma(\alpha)}\int_\gamma^\sigma(\varepsilon-\gamma)^{\alpha-1}\Psi(\varepsilon)\mathrm{d}\varepsilon,\quad \gamma<\sigma.$$

Here, $\Gamma(\cdot)$ represents the Euler gamma function, and it is defined through the integral representation:

$$\Gamma(\alpha)=\int_0^\infty\varepsilon^{\alpha-1}e^{-\varepsilon}\mathrm{d}\varepsilon,\quad \mathrm{Re}(\alpha)>0,$$

with $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$.

Recent advances in fractional calculus have stimulated growing interest in various integral inequalities involving fractional integrals, yielding important progress. In the context of RL-fractional integrals, Merad et al. [31] formulated parameterized symmetric inequalities of Simpson-like type for *s*-*tgs*-convex functions, while Nasri et al. [37] derived Newton-type inequalities for *s*-convex functions, and Sitthiwirattham et al. [46] yielded parameterized HH–Mercer type inequalities for convex functions. Besides the RL-fractional integral operators, Zhao et al. [55] constructed a two-parameter identity via generalized fractional integrals, which can be reduced to Simpson-, midpoint- and trapezoid-type inequalities. Utilizing the local fractional integrals, Butt and Khan [11] developed parameterized integral inequalities, while Meftah et al. [34] derived Newton-type inequalities. Additionally, the parameterized local fractional inequalities were proposed by Zhang and Sun [57] for generalized *h*-preinvex functions. Further contributions include Qi and Li [42] on Katugampola fractional inequalities for *s*-convex functions, Yuan et al. [54] on parameterized fractal-fractional integral inequalities for fractal (P, m)-convex functions, Benaissa and Azzouz [10] on HH–Fejér inequalities for ψ-Hilfer fractional integrals, and Butt et al. [12] on HH–Mercer inequalities involving Atangana–Baleanu Katugampola fractional integrals, along with related works on parameteized fractional schemes in Refs. [19, 27, 28, 45, 50]. However, all the aforementioned studies have been conducted within the framework of fractional analyses, rather than multiplicative integrals.

Building upon the RL-fractional integrals, the multiplicative form was proposed in Ref. [1] as a extension.

Definition 1.4. *[1] Let $\alpha > 0$. For a positive function Ψ on $[\kappa, \sigma]$, the left- and right-sided multiplicative RL-fractional integrals, denoted by ${}_{\kappa}\mathcal{I}_{*}^{\alpha}\Psi(\gamma)$ and ${}_{*}\mathcal{I}_{\sigma}^{\alpha}\Psi(\gamma)$, respectively, are defined as follows:*

$$
\begin{aligned}
{}_{\kappa}\mathcal{I}_{*}^{\alpha}\Psi(\gamma) &= \exp\left\{\mathcal{I}_{\kappa^+}^{\alpha}\ln\Psi(\gamma)\right\} \\
&= \exp\left\{\frac{1}{\Gamma(\alpha)}\int_{\kappa}^{\gamma}(\gamma-\varepsilon)^{\alpha-1}\ln\Psi(\varepsilon)\mathrm{d}\varepsilon\right\}, \quad \gamma > \kappa,
\end{aligned}
$$

and

$$
\begin{aligned}
{}_{*}\mathcal{I}_{\sigma}^{\alpha}\Psi(\gamma) &= \exp\left\{\mathcal{I}_{\sigma^-}^{\alpha}\ln\Psi(\gamma)\right\} \\
&= \exp\left\{\frac{1}{\Gamma(\alpha)}\int_{\gamma}^{\sigma}(\varepsilon-\gamma)^{\alpha-1}\ln\Psi(\varepsilon)\mathrm{d}\varepsilon\right\}, \quad \gamma < \sigma,
\end{aligned}
$$

where the function $\Psi(\gamma)$ is positive for every $\gamma \in [\kappa, \sigma]$, and $\Gamma(\cdot)$ denotes the Euler gamma function.

Multiplicative calculus has been extended to inequality analysis involving both integer-order and fractional integrals. In the integer-order field, the midpoint- and trapezoid-type inequalities were explored by Khan and Budak [22] and Xie et al. [53]. Additionally, Meftah et al. derived Maclaurin-type [32] and dual Simpson-type [33] inequalities for multiplicatively convex functions, and Boole-type inequalities were established by Mateen and Zhang [29]. Moreover, Berkane et al. [9] developed Right-Radau-type inequalities for multiplicatively *s*-convex functions. In 2024, Frioui et al. [17] proposed a dual-parameter multiplicative integral identity, from which one-point, two-point, and Newton-Cotes type inequalities can be derived.

In the realm of fractional calculus, through the application of the multiplicative RL-fractional integrals, Budak and Özçelik [7] studied HH-type inequalities in 2020. This subsequently motivated research by Boulares et al. [6] on Bullen-type inequalities, by Peng and Du [40] on Maclaurin-type inequalities, by Moumen et al. [35] on Simpson-type inequalities, and by Lakhdari et al. [25] on Newton-type inequalities. Later, Mateen et al. [30] established fractional midpoint-Mercer-, trapezoid-Mercer- and HH–Mercer-type inequalities. Especially, Almatrafi et al. [4] and Zhu et al. [60] proposed parameterized fractional integral inequalities for multiplicatively *s*-convex functions, whereas Du and Long [14] developed a multi-parameter integral identity that enabled the derivation of three-point Newton-Cotes type inequalities, and Zhou and Du [59] established multi-parameterized inequalities, generalizing traditional inequalities such as Bullen-type, Boole-type, and other inequalities. Additionally, significant progress has been achieved in research on other multiplicative fractional integral inequalities, such as multiplicative conformable fractional integrals [8], multiplicative tempered fractional integrals [16], multiplicative (k,s)-fractional integrals [56], multiplicative fractional integrals with exponential kernels [41], multiplicative Atangana–Baleanu fractional integrals [15], and multiplicative k-Atangana–Baleanu fractional integrals [26]. For readers seeking deeper insights into other multiplicative fractional operators, we suggest consulting Refs. [21, 23, 44, 49, 58] and the supplementary references cited therein.

Drawing on prior research findings, this paper aims to explore the parameterized fractional integral inequalities for multiplicatively once-differentiable s-convex positive functions. The work is structured as follows: After Sec. 2, a multiplicative RL-fractional identity with two parameters is established for *differentiable functions in Sec. 3. Leveraging this identity, the corresponding parameterized multiplicative fractional inequalities are derived, and it is demonstrated that the proposed inequalities outperform the midpoint- and trapezoid-type inequalities under certain conditions. Sec. 4 provides numerical validations of the results with examples. Finally. Sec. 5 summarizes the findings and concludes the paper.

2. Preliminaries

The organization of this section includes two subsections. Subsection 2.1 states fundamental definitions on convexity and beta functions, whereas Subsection 2.2 concentrates on key properties and theorems relevant to multiplicative calculus. Henceforth, let $I \subseteq \mathbb{R}$ denote a real interval, and let $\mathbb{R}^+ = (0, +\infty)$ throughout this work.

2.1. Convexities and beta functions

The definitions of s-convexity, multiplicative convexity and multiplicative s-convexity are essential to our main results. To this end, we envoke them as follows.

Definition 2.1. *[20] A function $\Psi : I \to \mathbb{R}^+$ exhibits s-convexity for some $s \in (0, 1]$, if it fulfills the following inequality*

$$\Psi(\varepsilon\gamma + (1-\varepsilon)\varphi) \le \varepsilon^s \Psi(\gamma) + (1-\varepsilon)^s \Psi(\varphi)$$

for any $\gamma, \varphi \in I$ and $\varepsilon \in [0, 1]$.

Definition 2.2. *[36] Consider $\Psi : I \to \mathbb{R}^+$ satisfying multiplicative convexity (logarithmical convexity) if, for all $\gamma, \varphi \in I$ and $\varepsilon \in [0, 1]$, the following inequality*

$$\Psi\Big(\varepsilon\gamma + (1-\varepsilon)\varphi\Big) \le [\Psi(\gamma)]^{\varepsilon} [\Psi(\varphi)]^{1-\varepsilon}$$

is satisfied.

Definition 2.3. *[51] A function $\Psi : I \to \mathbb{R}^+$ possesses multiplicative s-convexity (logarithmical s-convexity) for some $s \in (0, 1]$, provided that it fulfills the subsequent inequality*

$$\Psi(\varepsilon\gamma + (1-\varepsilon)\varphi) \le [\Psi(\gamma)]^{\varepsilon^s} [\Psi(\varphi)]^{(1-\varepsilon)^s}$$

for all $\gamma, \varphi \in I$ along with $\varepsilon \in [0, 1]$.

For $s = 1$, the multiplicative s-convexity coincides with the multiplicative convexity in Definition 2.2. We now revisit the definitions of the beta and incomplete beta functions used in this work.

Definition 2.4. *[43] For complex numbers ϖ_1, ϖ_2 with $\mathrm{Re}(\varpi_1) > 0$ and $\mathrm{Re}(\varpi_2) > 0$, the beta function $B(\cdot, \cdot)$ admits the following representation:*

$$B(\varpi_1, \varpi_2) = \int_0^1 \varepsilon^{\varpi_1 - 1}(1-\varepsilon)^{\varpi_2 - 1}\,\mathrm{d}\varepsilon = \frac{\Gamma(\varpi_1)\Gamma(\varpi_2)}{\Gamma(\varpi_1 + \varpi_2)},$$

in which $\Gamma(\cdot)$ denotes the Euler gamma function.

Definition 2.5. *[24] Let $\varpi_1, \varpi_2 \in \mathbb{C}$ satisfy $\mathrm{Re}(\varpi_1) > 0$ and $\mathrm{Re}(\varpi_2) > 0$. The incomplete beta function takes the form:*

$$B_{\gamma}(\varpi_1, \varpi_2) = \int_0^{\gamma} \varepsilon^{\varpi_1 - 1} (1-\varepsilon)^{\varpi_2 - 1} \, \mathrm{d}\varepsilon, \quad 0 \le \gamma < 1.$$

2.2. Multiplicative calculus and related results

Building upon the investigation conducted in Ref. [5], the authors investigated analytical properties of *integrable operators.

Proposition 2.6. *[5] Consider the positive and *integrable functions Ψ and $\Im$ defined on $[\kappa, \sigma]$. Then the following properties hold*

$$(i) \int_{\kappa}^{\sigma} ((\Psi(\varepsilon))^{\nu})^{\mathrm{d}\varepsilon} = \left(\int_{\kappa}^{\sigma} (\Psi(\varepsilon))^{\mathrm{d}\varepsilon} \right)^{\nu}, \ \nu \in \mathbb{R},$$

$$(ii) \int_{\kappa}^{\sigma} (\Psi(\varepsilon)\Im(\varepsilon))^{\mathrm{d}\varepsilon} = \int_{\kappa}^{\sigma} (\Psi(\varepsilon))^{\mathrm{d}\varepsilon} \cdot \int_{\kappa}^{\sigma} (\Im(\varepsilon))^{\mathrm{d}\varepsilon},$$

$$(iii) \int_{\kappa}^{\sigma} \left(\frac{\Psi(\varepsilon)}{\Im(\varepsilon)} \right)^{\mathrm{d}\varepsilon} = \frac{\int_{\kappa}^{\sigma} (\Psi(\varepsilon))^{\mathrm{d}\varepsilon}}{\int_{\kappa}^{\sigma} (\Im(\varepsilon))^{\mathrm{d}\varepsilon}},$$

$$(iv) \int_{\kappa}^{\sigma} (\Psi(\varepsilon))^{\mathrm{d}\varepsilon} = \int_{\kappa}^{\xi} (\Psi(\varepsilon))^{\mathrm{d}\varepsilon} \cdot \int_{\xi}^{\sigma} (\Psi(\varepsilon))^{\mathrm{d}\varepsilon}, \quad \kappa \le \xi \le \sigma,$$

$$(v) \int_{\kappa}^{\kappa} (\Psi(\varepsilon))^{\mathrm{d}\varepsilon} = 1 \quad \text{and} \quad \int_{\kappa}^{\sigma} (\Psi(\varepsilon))^{\mathrm{d}\varepsilon} = \left(\int_{\sigma}^{\kappa} (\Psi(\varepsilon))^{\mathrm{d}\varepsilon} \right)^{-1}.$$

The n-th *derivative and n-th derivative ($n \in \mathbb{N}$) have the subsequent relationships.

Proposition 2.7. *[5] Consider the function $\Psi : I \to \mathbb{R}^+$ with n-th derivative $\Psi^{(n)}$. Then the following relationships hold*

$$(i)\ \Psi^*(\varepsilon) = \exp\left\{ (\ln \circ \Psi)'(\varepsilon) \right\} = \exp\left\{ \frac{\Psi'(\varepsilon)}{\Psi(\varepsilon)} \right\},$$

$$(ii)\ \Psi^{**}(\varepsilon) = \exp\{(\ln \circ \Psi^*)'(\varepsilon)\} = \exp\{(\ln \circ \Psi)''(\varepsilon)\},$$

$$(iii)\ \Psi^{*(n)}(\varepsilon) = \exp\left\{ (\ln \circ \Psi)^{(n)}(\varepsilon) \right\}, \quad n = 1, 2, 3, \cdots.$$

Proposition 2.8. *[58] Suppose that the positive function Ψ is multiplicatively differentiable on the interval I. If Ψ is increasing on I, then it holds that $\Psi^* \ge 1$.*

The partial-integral formula of *integrable operators was developed by Ali *et al.* in Ref. [2].

Theorem 2.9. *[2] Consider the multiplicatively differentiable function $\Psi : [\kappa, \sigma] \to \mathbb{R}^+$, together with differentiable functions $\Phi : I \to [\kappa, \sigma]$ and $\Im : [\kappa, \sigma] \to \mathbb{R}$. Under these conditions, we obtain the identity*

$$\int_{\kappa}^{\sigma} \left(\Psi^*(\Phi(\gamma))^{\Im(\gamma)\Phi'(\gamma)} \right)^{\mathrm{d}\gamma} = \frac{[\Psi(\Phi(\sigma))]^{\Im(\sigma)}}{[\Psi(\Phi(\kappa))]^{\Im(\kappa)}} \cdot \frac{1}{\int_{\kappa}^{\sigma} \left(\Psi(\Phi(\gamma))^{\Im'(\gamma)} \right)^{\mathrm{d}\gamma}}.$$

Definition 2.10. *[5] Let the function $\Psi \in \mathbb{R}^+$. The multiplicative modulus (or the multiplicative absolute value) of Ψ, denoted by $|\Psi|^*$, is defined as follows:*

$$|\Psi|^* = \begin{cases} \Psi, & \Psi \geq 1, \\ \dfrac{1}{\Psi}, & 0 < \Psi < 1, \end{cases}$$

which can alternatively be expressed as $|\Psi|^* = \exp\{|\ln \circ \Psi|\}$, with $|\cdot|$ representing the standard absolute value.

Proposition 2.11. *The following multiplicative triangle inequality is readily verified:*

$$|\Psi\mathfrak{J}|^* \leq |\Psi|^*|\mathfrak{J}|^*,$$

which holds for all $\Psi, \mathfrak{J} \in \mathbb{R}^+$.

Proof. From Def. 2.10, combined with the triangle inequality, the intended conclusion follows. □

Proposition 2.12. *[5] Assume that $\Psi : [\kappa, \sigma] \to \mathbb{R}^+$ is multiplicatively differentiable.*
(1) If $\Psi^*(\gamma) \geq 1$ for all $\gamma \in [\kappa, \sigma]$, then Ψ is increasing on $[\kappa, \sigma]$.
(2) If $\Psi^*(\gamma) \leq 1$ for all $\gamma \in [\kappa, \sigma]$, then Ψ is decreasing on $[\kappa, \sigma]$.

We now introduce the *increasing concept and demonstrate one of its significant properties.

Definition 2.13. *A function $\Psi : [\kappa, \sigma] \to \mathbb{R}^+$ is called *increasing, if the inequality $\frac{\Psi(\gamma_1)}{\Psi(\gamma_2)} \leq 1$ holds for any two points $\gamma_1, \gamma_2 \in [\kappa, \sigma]$ with $\gamma_1 \leq \gamma_2$. The function is termed *decreasing provided that the inequality reverses.*

Proposition 2.14. *For a multiplicatively differentiable function $\Psi : [\kappa, \sigma] \to \mathbb{R}^+$, we have the following equivalent assertions:*
*(i) The function Ψ is *increasing on $[\kappa, \sigma]$,*
*(ii) The *differentiable satisfies $\Psi^*(\gamma) \geq 1$ for all $\gamma \in [\kappa, \sigma]$.*

Proof. Combining Prop. 2.8 with part (1) of Prop. 2.12 yields the desired result: (i) ⇒ (ii) ⇒ (i). □

To conclude, we state error estimates for the different types of multiplicative quadrature rules: midpoint, Simpson, Bullen, trapezoid and Milne types.

Theorem 2.15. *[38] Assuming that the multiplicatively differentiable function $\Psi : [\kappa, \sigma] \to \mathbb{R}^+$ is increasing on $[\kappa, \sigma]$ with $\kappa < \sigma$, and Ψ^* is multiplicatively convex over $[\kappa, \sigma]$. Then, we have the multiplicative midpoint-type integral inequality:*

$$\left|\Psi\left(\frac{\kappa+\sigma}{2}\right)\left(\int_\kappa^\sigma (\Psi(u))^{\mathrm{d}u}\right)^{\frac{1}{\kappa-\sigma}}\right| \leq \left[\Psi^*(\kappa)\left(\Psi^*\left(\frac{\kappa+\sigma}{2}\right)\right)^4 \Psi^*(\sigma)\right]^{\frac{\sigma-\kappa}{24}}$$
$$\leq \left[\Psi^*(\kappa)\Psi^*(\sigma)\right]^{\frac{\sigma-\kappa}{8}}.$$

Theorem 2.16. *[13] If the requirements of Theorem 2.15 are fulfilled, then the subsequent multiplicative Simpson-type inequality holds:*

$$\left|\left[\Psi(\kappa)\left(\Psi\left(\frac{\kappa+\sigma}{2}\right)\right)^4 \Psi(\sigma)\right]^{\frac{1}{6}}\left(\int_\kappa^\sigma (\Psi(u))^{\mathrm{d}u}\right)^{\frac{1}{\kappa-\sigma}}\right| \leq \left[\Psi^*(\kappa)\Psi^*(\sigma)\right]^{\frac{5(\sigma-\kappa)}{72}}.$$

Theorem 2.17. *[6] From the hypotheses of Theorem 2.15, it follows that the multiplicative inequalities of Bullen type hold:*

$$\left|\left[\Psi(\kappa)\left(\Psi\left(\frac{\kappa+\sigma}{2}\right)\right)^2 \Psi(\sigma)\right]^{\frac{1}{4}}\left(\int_\kappa^\sigma (\Psi(u))^{\mathrm{d}u}\right)^{\frac{1}{\kappa-\sigma}}\right| \leq \left[\Psi^*(\kappa)\left(\Psi^*\left(\frac{\kappa+\sigma}{2}\right)\right)^2 \Psi^*(\sigma)\right]^{\frac{\sigma-\kappa}{32}}$$
$$\leq \left[\Psi^*(\kappa)\Psi^*(\sigma)\right]^{\frac{\sigma-\kappa}{16}}.$$

Theorem 2.18. *[22] Provided that the hypotheses of Theorem 2.15 are fulfilled, it follows that the subsequent multiplicative trapezoid-type inequality is satisfied:*

$$\left|\sqrt{\Psi(\kappa)\Psi(\sigma)}\left(\int_\kappa^\sigma \left(\Psi(u)\right)^{\mathrm{d}u}\right)^{\frac{1}{\kappa-\sigma}}\right| \leq \left[\Psi^*(\kappa)\Psi^*(\sigma)\right]^{\frac{\sigma-\kappa}{8}}.$$

Theorem 2.19. *[17] Under the conditions specified in Theorem 2.15, the subsequent multiplicative inequality of Milne type holds:*

$$\begin{aligned}\left|\left[(\Psi(\kappa))^2\left(\Psi\left(\frac{\kappa+\sigma}{2}\right)\right)^{-1}(\Psi(\sigma))^2\right]^{\frac{1}{3}}\left(\int_\kappa^\sigma (\Psi(u))^{\mathrm{d}u}\right)^{\frac{1}{\kappa-\sigma}}\right| &\leq [\Psi^*(\kappa)\Psi^*(\sigma)]^{\frac{\sigma-\kappa}{8}}\left(\Psi^*\left(\frac{\kappa+\sigma}{2}\right)\right)^{\frac{\sigma-\kappa}{6}}\\ &\leq \left[\Psi^*(\kappa)\Psi^*(\sigma)\right]^{\frac{5(\sigma-\kappa)}{24}}.\end{aligned}$$

3. Main results

This section focuses on establishing multi-parameter inequalities concerning multiplicative RL-fractional integrals. As a foundation, we first demonstrate the subsequent fractional identity.

Lemma 3.1. *Consider a function $\Psi : [\kappa,\sigma] \to \mathbb{R}^+$ with multiplicative differentiability on (κ,σ). Suppose Ψ^* is the multiplicatively integrable function on $[\kappa,\sigma]$. Then, for $\lambda,\mu \in [0,+\infty)$, we establish the following identity concerning the multiplicative RL-fractional integrals:*

$$\begin{aligned}&[\Psi(\kappa)]^{\frac{\lambda}{2}}\left[\Psi\left(\frac{\kappa+\sigma}{2}\right)\right]^{1-\frac{\lambda+\mu}{2}}[\Psi(\sigma)]^{\frac{\mu}{2}}\left[{}_*\mathcal{I}^\alpha_{\frac{\kappa+\sigma}{2}}\Psi(\kappa)\cdot{}_{\frac{\kappa+\sigma}{2}}\mathcal{I}^\alpha_*\Psi(\sigma)\right]^{-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\sigma-\kappa)^\alpha}}\\ &=\left(\int_0^{\frac{1}{2}}\left(\left(\Psi^*(\varepsilon\kappa+(1-\varepsilon)\sigma)\right)^{\mu-2^\alpha\varepsilon^\alpha}\right)^{\mathrm{d}\varepsilon}\right)^{\frac{\sigma-\kappa}{2}}\left(\int_{\frac{1}{2}}^1\left(\left(\Psi^*(\varepsilon\kappa+(1-\varepsilon)\sigma)\right)^{2^\alpha(1-\varepsilon)^\alpha-\lambda}\right)^{\mathrm{d}\varepsilon}\right)^{\frac{\sigma-\kappa}{2}}.\end{aligned}$$

Proof. To streamline our presentation, we employ the notations

$$I_1=\left(\int_0^{\frac{1}{2}}\left(\left(\Psi^*(\varepsilon\kappa+(1-\varepsilon)\sigma)\right)^{\mu-2^\alpha\varepsilon^\alpha}\right)^{\mathrm{d}\varepsilon}\right)^{\frac{\sigma-\kappa}{2}},$$

and

$$I_2=\left(\int_{\frac{1}{2}}^1\left(\left(\Psi^*(\varepsilon\kappa+(1-\varepsilon)\sigma)\right)^{2^\alpha(1-\varepsilon)^\alpha-\lambda}\right)^{\mathrm{d}\varepsilon}\right)^{\frac{\sigma-\kappa}{2}}.$$

Considering Proposition 2.6, one obtains that

$$\begin{aligned}I_1&=\left(\int_0^{\frac{1}{2}}\left(\left(\Psi^*(\varepsilon\kappa+(1-\varepsilon)\sigma)\right)^{\mu-2^\alpha\varepsilon^\alpha}\right)^{\mathrm{d}\varepsilon}\right)^{\frac{\sigma-\kappa}{2}}\\ &=\int_0^{\frac{1}{2}}\left(\left(\Psi^*(\varepsilon\kappa+(1-\varepsilon)\sigma)\right)^{\left(\mu-2^\alpha\varepsilon^\alpha\right)\frac{\sigma-\kappa}{2}}\right)^{\mathrm{d}\varepsilon}.\end{aligned}\tag{11}$$

Leveraging the integration by parts formula in multiplicative form (see Theorem 2.9), we derive from I_1

that

$$
\begin{aligned}
I_1 &= \frac{\left(\Psi\left(\frac{\kappa+\sigma}{2}\right)\right)^{-\frac{1}{2}(\mu-1)}}{\left(\Psi(\sigma)\right)^{-\frac{\mu}{2}}} \times \frac{1}{\int_0^{\frac{1}{2}} \left(\left(\Psi(\varepsilon\kappa+(1-\varepsilon)\sigma)\right)^{\alpha 2^{\alpha-1}\varepsilon^{\alpha-1}}\right)^{\mathrm{d}\varepsilon}} \\
&= \left[\Psi\left(\frac{\kappa+\sigma}{2}\right)\right]^{\frac{1-\mu}{2}} \left(\Psi(\sigma)\right)^{\frac{\mu}{2}} \times \frac{1}{\exp\left\{\alpha 2^{\alpha-1}\int_0^{\frac{1}{2}} \varepsilon^{\alpha-1}\ln\Psi(\varepsilon\kappa+(1-\varepsilon)\sigma)\,\mathrm{d}\varepsilon\right\}} \\
&= \left[\Psi\left(\frac{\kappa+\sigma}{2}\right)\right]^{\frac{1-\mu}{2}} \left(\Psi(\sigma)\right)^{\frac{\mu}{2}} \times \frac{1}{\exp\left\{\frac{\alpha 2^{\alpha-1}}{(\sigma-\kappa)^{\alpha}}\int_{\frac{\kappa+\sigma}{2}}^{\sigma} (\sigma-\varepsilon)^{\alpha-1}\ln\Psi(\varepsilon)\,\mathrm{d}\varepsilon\right\}} \\
&= \left[\Psi\left(\frac{\kappa+\sigma}{2}\right)\right]^{\frac{1-\mu}{2}} \left(\Psi(\sigma)\right)^{\frac{\mu}{2}} \left[{}_{\frac{\kappa+\sigma}{2}}\mathcal{I}_{*}^{\alpha}\Psi(\sigma)\right]^{-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\sigma-\kappa)^{\alpha}}}. \qquad (12)
\end{aligned}
$$

Applying the analogous procedure to I_2 yields the following conclusion

$$
\begin{aligned}
I_2 &= \int_{\frac{1}{2}}^{1} \left(\left(\Psi^{*}(\varepsilon\kappa+(1-\varepsilon)\sigma)\right)^{\left[2^{\alpha}(1-\varepsilon)^{\alpha}-\lambda\right]\frac{\sigma-\kappa}{2}}\right)^{\mathrm{d}\varepsilon} \\
&= \frac{\left(\Psi(\kappa)\right)^{\frac{\lambda}{2}}}{\left[\Psi\left(\frac{\kappa+\sigma}{2}\right)\right]^{-\frac{1-\lambda}{2}}} \times \frac{1}{\int_{\frac{1}{2}}^{1} \left(\left(\Psi(\varepsilon\kappa+(1-\varepsilon)\sigma)\right)^{\alpha 2^{\alpha-1}(1-\varepsilon)^{\alpha-1}}\right)^{\mathrm{d}\varepsilon}} \\
&= \left(\Psi(\kappa)\right)^{\frac{\lambda}{2}} \left[\Psi\left(\frac{\kappa+\sigma}{2}\right)\right]^{\frac{1-\lambda}{2}} \times \frac{1}{\exp\left\{\alpha 2^{\alpha-1}\int_{\frac{1}{2}}^{1} (1-\varepsilon)^{\alpha-1}\ln\Psi(\varepsilon\kappa+(1-\varepsilon)\sigma)\,\mathrm{d}\varepsilon\right\}} \\
&= \left(\Psi(\kappa)\right)^{\frac{\lambda}{2}} \left[\Psi\left(\frac{\kappa+\sigma}{2}\right)\right]^{\frac{1-\lambda}{2}} \times \frac{1}{\exp\left\{\frac{\alpha 2^{\alpha-1}}{(\sigma-\kappa)^{\alpha}}\int_{\kappa}^{\frac{\kappa+\sigma}{2}} (\varepsilon-\kappa)^{\alpha-1}\ln\Psi(\varepsilon)\,\mathrm{d}\varepsilon\right\}} \\
&= \left(\Psi(\kappa)\right)^{\frac{\lambda}{2}} \left[\Psi\left(\frac{\kappa+\sigma}{2}\right)\right]^{\frac{1-\lambda}{2}} \left[{}_{*}\mathcal{I}_{\frac{\kappa+\sigma}{2}}^{\alpha}\Psi(\kappa)\right]^{-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\sigma-\kappa)^{\alpha}}}. \qquad (13)
\end{aligned}
$$

From the equations (12) and (13), it can be captured that

$$
I_1 \times I_2 = \left[\Psi(\kappa)\right]^{\frac{\lambda}{2}} \left[\Psi\left(\frac{\kappa+\sigma}{2}\right)\right]^{1-\frac{\lambda+\mu}{2}} \left[\Psi(\sigma)\right]^{\frac{\mu}{2}} \left[{}_{*}\mathcal{I}_{\frac{\kappa+\sigma}{2}}^{\alpha}\Psi(\kappa)\cdot {}_{\frac{\kappa+\sigma}{2}}\mathcal{I}_{*}^{\alpha}\Psi(\sigma)\right]^{-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\sigma-\kappa)^{\alpha}}}. \qquad (14)
$$

Consequently, we establish the desired identity. The proof is thus concluded.

Corollary 3.2. *In Lemma 3.1, by choosing $\lambda = \mu$, we have the subsequent identity:*

$$
\begin{aligned}
&\left[\Psi(\kappa)\Psi(\sigma)\right]^{\frac{\mu}{2}} \left[\Psi\left(\frac{\kappa+\sigma}{2}\right)\right]^{1-\mu} \left[{}_{*}\mathcal{I}_{\frac{\kappa+\sigma}{2}}^{\alpha}\Psi(\kappa)\cdot {}_{\frac{\kappa+\sigma}{2}}\mathcal{I}_{*}^{\alpha}\Psi(\sigma)\right]^{-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\sigma-\kappa)^{\alpha}}} \\
&= \left(\int_0^{\frac{1}{2}} \left(\left(\Psi^{*}(\varepsilon\kappa+(1-\varepsilon)\sigma)\right)^{\mu-2^{\alpha}\varepsilon^{\alpha}}\right)^{\mathrm{d}\varepsilon}\right)^{\frac{\sigma-\kappa}{2}} \left(\int_{\frac{1}{2}}^{1} \left(\left(\Psi^{*}(\varepsilon\kappa+(1-\varepsilon)\sigma)\right)^{2^{\alpha}(1-\varepsilon)^{\alpha}-\mu}\right)^{\mathrm{d}\varepsilon}\right)^{\frac{\sigma-\kappa}{2}}.
\end{aligned}
$$

Remark 3.3. *From Corollary 3.2, the following specific cases can be derived.*
(i) Considering $\mu = 0$, we obtain the multiplicative midpoint-type fractional identity

$$\Psi\left(\frac{\kappa+\sigma}{2}\right)\left[{}_{*}\mathcal{I}^{\alpha}_{\frac{\kappa+\sigma}{2}}\Psi(\kappa)\cdot {}_{\frac{\kappa+\sigma}{2}}\mathcal{I}^{\alpha}_{*}\Psi(\sigma)\right]^{-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\sigma-\kappa)^{\alpha}}} = \left(\int_0^{\frac{1}{2}}\left(\left(\Psi^{*}\left(\varepsilon\kappa+(1-\varepsilon)\sigma\right)\right)^{-2^{\alpha}\varepsilon^{\alpha}}\right)^{\mathrm{d}\varepsilon}\right)^{\frac{\sigma-\kappa}{2}}\left(\int_{\frac{1}{2}}^{1}\left(\left(\Psi^{*}\left(\varepsilon\kappa+(1-\varepsilon)\sigma\right)\right)^{2^{\alpha}(1-\varepsilon)^{\alpha}}\right)^{\mathrm{d}\varepsilon}\right)^{\frac{\sigma-\kappa}{2}}.$$

For $\alpha = 1$, this reduces to the multiplicative midpoint-type identity of integer order

$$\Psi\left(\frac{\kappa+\sigma}{2}\right)\left(\int_{\kappa}^{\sigma}(\Psi(\gamma))^{\mathrm{d}\gamma}\right)^{\frac{1}{\kappa-\sigma}} = \left(\int_0^{\frac{1}{2}}\left(\left(\Psi^{*}\left(\varepsilon\kappa+(1-\varepsilon)\sigma\right)\right)^{-\varepsilon}\right)^{\mathrm{d}\varepsilon}\right)^{\sigma-\kappa}\left(\int_{\frac{1}{2}}^{1}\left(\left(\Psi^{*}\left(\varepsilon\kappa+(1-\varepsilon)\sigma\right)\right)^{1-\varepsilon}\right)^{\mathrm{d}\varepsilon}\right)^{\sigma-\kappa}.$$

(ii) Considering $\mu = \frac{1}{3}$, we obtain the multiplicative fractional equality of Simpson type

$$\left[\Psi(\kappa)\left(\Psi\left(\frac{\kappa+\sigma}{2}\right)\right)^{4}\Psi(\sigma)\right]^{\frac{1}{6}}\left[{}_{*}\mathcal{I}^{\alpha}_{\frac{\kappa+\sigma}{2}}\Psi(\kappa)\cdot {}_{\frac{\kappa+\sigma}{2}}\mathcal{I}^{\alpha}_{*}\Psi(\sigma)\right]^{-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\sigma-\kappa)^{\alpha}}} = \left(\int_0^{\frac{1}{2}}\left(\left(\Psi^{*}\left(\varepsilon\kappa+(1-\varepsilon)\sigma\right)\right)^{\frac{1}{3}-2^{\alpha}\varepsilon^{\alpha}}\right)^{\mathrm{d}\varepsilon}\right)^{\frac{\sigma-\kappa}{2}}\left(\int_{\frac{1}{2}}^{1}\left(\left(\Psi^{*}\left(\varepsilon\kappa+(1-\varepsilon)\sigma\right)\right)^{2^{\alpha}(1-\varepsilon)^{\alpha}-\frac{1}{3}}\right)^{\mathrm{d}\varepsilon}\right)^{\frac{\sigma-\kappa}{2}}.$$

For $\alpha = 1$, this reduces to the multiplicative Simpson-type identity of integer order

$$\left[\Psi(\kappa)\left(\Psi\left(\frac{\kappa+\sigma}{2}\right)\right)^{4}\Psi(\sigma)\right]^{\frac{1}{6}}\left(\int_{\kappa}^{\sigma}(\Psi(\gamma))^{\mathrm{d}\gamma}\right)^{\frac{1}{\kappa-\sigma}} = \left(\int_0^{\frac{1}{2}}\left(\left(\Psi^{*}\left(\varepsilon\kappa+(1-\varepsilon)\sigma\right)\right)^{\frac{1}{6}-\varepsilon}\right)^{\mathrm{d}\varepsilon}\right)^{\sigma-\kappa}\left(\int_{\frac{1}{2}}^{1}\left(\left(\Psi^{*}\left(\varepsilon\kappa+(1-\varepsilon)\sigma\right)\right)^{\frac{5}{6}-\varepsilon}\right)^{\mathrm{d}\varepsilon}\right)^{\sigma-\kappa}.$$

(iii) Considering $\mu = \frac{1}{2}$, we obtain the multiplicative fractional equality of Bullen type

$$\left[\Psi(\kappa)\left(\Psi\left(\frac{\kappa+\sigma}{2}\right)\right)^{2}\Psi(\sigma)\right]^{\frac{1}{4}}\left[{}_{*}\mathcal{I}^{\alpha}_{\frac{\kappa+\sigma}{2}}\Psi(\kappa)\cdot {}_{\frac{\kappa+\sigma}{2}}\mathcal{I}^{\alpha}_{*}\Psi(\sigma)\right]^{-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\sigma-\kappa)^{\alpha}}} = \left(\int_0^{\frac{1}{2}}\left(\left(\Psi^{*}\left(\varepsilon\kappa+(1-\varepsilon)\sigma\right)\right)^{\frac{1}{2}-2^{\alpha}\varepsilon^{\alpha}}\right)^{\mathrm{d}\varepsilon}\right)^{\frac{\sigma-\kappa}{2}}\left(\int_{\frac{1}{2}}^{1}\left(\left(\Psi^{*}\left(\varepsilon\kappa+(1-\varepsilon)\sigma\right)\right)^{2^{\alpha}(1-\varepsilon)^{\alpha}-\frac{1}{2}}\right)^{\mathrm{d}\varepsilon}\right)^{\frac{\sigma-\kappa}{2}}.$$

For $\alpha = 1$, this reduces to the multiplicative Bullen-type identity of integer order

$$\left[\Psi(\kappa)\left(\Psi\left(\frac{\kappa+\sigma}{2}\right)\right)^{2}\Psi(\sigma)\right]^{\frac{1}{4}}\left(\int_{\kappa}^{\sigma}(\Psi(\gamma))^{\mathrm{d}\gamma}\right)^{\frac{1}{\kappa-\sigma}} = \left(\int_0^{\frac{1}{2}}\left(\left(\Psi^{*}\left(\varepsilon\kappa+(1-\varepsilon)\sigma\right)\right)^{\frac{1}{4}-\varepsilon}\right)^{\mathrm{d}\varepsilon}\right)^{\sigma-\kappa}\left(\int_{\frac{1}{2}}^{1}\left(\left(\Psi^{*}\left(\varepsilon\kappa+(1-\varepsilon)\sigma\right)\right)^{\frac{3}{4}-\varepsilon}\right)^{\mathrm{d}\varepsilon}\right)^{\sigma-\kappa}.$$

(iv) Considering $\mu = \frac{2}{3}$, we obtain the multiplicative fractional equality of Simpson-like type

$$\left[\Psi(\kappa)\Psi\left(\frac{\kappa+\sigma}{2}\right)\Psi(\sigma)\right]^{\frac{1}{3}}\left[{}_{*}\mathcal{I}^{\alpha}_{\frac{\kappa+\sigma}{2}}\Psi(\kappa)\cdot {}_{\frac{\kappa+\sigma}{2}}\mathcal{I}^{\alpha}_{*}\Psi(\sigma)\right]^{-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\sigma-\kappa)^{\alpha}}} = \left(\int_0^{\frac{1}{2}}\left(\left(\Psi^{*}\left(\varepsilon\kappa+(1-\varepsilon)\sigma\right)\right)^{\frac{2}{3}-2^{\alpha}\varepsilon^{\alpha}}\right)^{\mathrm{d}\varepsilon}\right)^{\frac{\sigma-\kappa}{2}}\left(\int_{\frac{1}{2}}^{1}\left(\left(\Psi^{*}\left(\varepsilon\kappa+(1-\varepsilon)\sigma\right)\right)^{2^{\alpha}(1-\varepsilon)^{\alpha}-\frac{2}{3}}\right)^{\mathrm{d}\varepsilon}\right)^{\frac{\sigma-\kappa}{2}}.$$

For $\alpha = 1$, this reduces to the multiplicative Simpson-like type identity of integer order

$$\begin{aligned}&\left[\Psi(\kappa)\Psi\left(\frac{\kappa+\sigma}{2}\right)\Psi(\sigma)\right]^{\frac{1}{3}}\left(\int_{\kappa}^{\sigma}(\Psi(\gamma))^{\mathrm{d}\gamma}\right)^{\frac{1}{\kappa-\sigma}}\\&=\left(\int_{0}^{\frac{1}{2}}\left(\left(\Psi^{*}\left(\varepsilon\kappa+(1-\varepsilon)\sigma\right)\right)^{\frac{1}{3}-\varepsilon}\right)^{\mathrm{d}\varepsilon}\right)^{\sigma-\kappa}\left(\int_{\frac{1}{2}}^{1}\left(\left(\Psi^{*}\left(\varepsilon\kappa+(1-\varepsilon)\sigma\right)\right)^{\frac{2}{3}-\varepsilon}\right)^{\mathrm{d}\varepsilon}\right)^{\sigma-\kappa}.\end{aligned}$$

(v) Considering $\mu = 1$, we obtain the multiplicative fractional equality of trapezoid type

$$\begin{aligned}&\left[\Psi(\kappa)\Psi(\sigma)\right]^{\frac{1}{2}}\left[{}_{*}\mathcal{I}^{\alpha}_{\frac{\kappa+\sigma}{2}}\Psi(\kappa)\cdot{}_{\frac{\kappa+\sigma}{2}}\mathcal{I}^{\alpha}_{*}\Psi(\sigma)\right]^{-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\sigma-\kappa)^{\alpha}}}\\&=\left(\int_{0}^{\frac{1}{2}}\left(\left(\Psi^{*}\left(\varepsilon\kappa+(1-\varepsilon)\sigma\right)\right)^{1-2^{\alpha}\varepsilon^{\alpha}}\right)^{\mathrm{d}\varepsilon}\right)^{\frac{\sigma-\kappa}{2}}\left(\int_{\frac{1}{2}}^{1}\left(\left(\Psi^{*}\left(\varepsilon\kappa+(1-\varepsilon)\sigma\right)\right)^{2^{\alpha}(1-\varepsilon)^{\alpha}-1}\right)^{\mathrm{d}\varepsilon}\right)^{\frac{\sigma-\kappa}{2}}.\end{aligned}$$

For $\alpha = 1$, this reduces to the multiplicative trapezoid-type identity of integer order

$$\begin{aligned}&\left[\Psi(\kappa)\Psi(\sigma)\right]^{\frac{1}{2}}\left(\int_{\kappa}^{\sigma}(\Psi(\gamma))^{\mathrm{d}\gamma}\right)^{\frac{1}{\kappa-\sigma}}\\&=\left(\int_{0}^{\frac{1}{2}}\left(\left(\Psi^{*}\left(\varepsilon\kappa+(1-\varepsilon)\sigma\right)\right)^{\frac{1}{2}-\varepsilon}\right)^{\mathrm{d}\varepsilon}\right)^{\sigma-\kappa}\left(\int_{\frac{1}{2}}^{1}\left(\left(\Psi^{*}\left(\varepsilon\kappa+(1-\varepsilon)\sigma\right)\right)^{\frac{1}{2}-\varepsilon}\right)^{\mathrm{d}\varepsilon}\right)^{\sigma-\kappa}\\&=\left(\int_{0}^{1}\left(\left(\Psi^{*}\left(\varepsilon\kappa+(1-\varepsilon)\sigma\right)\right)^{\frac{1}{2}-\varepsilon}\right)^{\mathrm{d}\varepsilon}\right)^{\sigma-\kappa}.\end{aligned}$$

(vi) Considering $\mu = \frac{4}{3}$, we obtain the multiplicative fractional equality of Milne type

$$\begin{aligned}&\left[\left(\Psi(\kappa)\right)^{2}\left(\Psi\left(\frac{\kappa+\sigma}{2}\right)\right)^{-1}\left(\Psi(\sigma)\right)^{2}\right]^{\frac{1}{3}}\left[{}_{*}\mathcal{I}^{\alpha}_{\frac{\kappa+\sigma}{2}}\Psi(\kappa)\cdot{}_{\frac{\kappa+\sigma}{2}}\mathcal{I}^{\alpha}_{*}\Psi(\sigma)\right]^{-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\sigma-\kappa)^{\alpha}}}\\&=\left(\int_{0}^{\frac{1}{2}}\left(\left(\Psi^{*}\left(\varepsilon\kappa+(1-\varepsilon)\sigma\right)\right)^{\frac{4}{3}-2^{\alpha}\varepsilon^{\alpha}}\right)^{\mathrm{d}\varepsilon}\right)^{\frac{\sigma-\kappa}{2}}\left(\int_{\frac{1}{2}}^{1}\left(\left(\Psi^{*}\left(\varepsilon\kappa+(1-\varepsilon)\sigma\right)\right)^{2^{\alpha}(1-\varepsilon)^{\alpha}-\frac{4}{3}}\right)^{\mathrm{d}\varepsilon}\right)^{\frac{\sigma-\kappa}{2}}.\end{aligned}$$

For $\alpha = 1$, this reduces to the multiplicative Milne-type identity of integer order

$$\begin{aligned}&\left[\left(\Psi(\kappa)\right)^{2}\left(\Psi\left(\frac{\kappa+\sigma}{2}\right)\right)^{-1}\left(\Psi(\sigma)\right)^{2}\right]^{\frac{1}{3}}\left(\int_{\kappa}^{\sigma}(\Psi(\gamma))^{\mathrm{d}\gamma}\right)^{\frac{1}{\kappa-\sigma}}\\&=\left(\int_{0}^{\frac{1}{2}}\left(\left(\Psi^{*}\left(\varepsilon\kappa+(1-\varepsilon)\sigma\right)\right)^{\frac{2}{3}-\varepsilon}\right)^{\mathrm{d}\varepsilon}\right)^{\sigma-\kappa}\left(\int_{\frac{1}{2}}^{1}\left(\left(\Psi^{*}\left(\varepsilon\kappa+(1-\varepsilon)\sigma\right)\right)^{\frac{1}{3}-\varepsilon}\right)^{\mathrm{d}\varepsilon}\right)^{\sigma-\kappa}.\end{aligned}$$

Since the computation of the following two definite integrals is essential for proving subsequent theorems, we now state them as two lemmas.

Lemma 3.4. *Let $\alpha, s \in (0,1]$ and $\gamma \in \{\lambda, \mu\}$ with $0 \le \gamma \le +\infty$. Following this, the result presented below holds*

$$\int_{0}^{\frac{1}{2}}\varepsilon^{s}\left|\gamma-2^{\alpha}\varepsilon^{\alpha}\right|\mathrm{d}\varepsilon=\begin{cases}K_{1}(\alpha,s,\gamma), & \gamma>1,\\ K_{2}(\alpha,s,\gamma), & 0\le\gamma\le1,\end{cases}\tag{15}$$

where

$$K_1(\alpha,s,\gamma)=\frac{1}{2^{s+1}}\left(\frac{\gamma}{s+1}-\frac{1}{\alpha+s+1}\right),$$

and

$$K_2(\alpha,s,\gamma)=\frac{\gamma^{\frac{\alpha+s+1}{\alpha}}}{2^s}\left(\frac{1}{s+1}-\frac{1}{\alpha+s+1}\right)-\frac{1}{2^{s+1}}\left(\frac{\gamma}{s+1}-\frac{1}{\alpha+s+1}\right).$$

Proof. (i) We observe that for $\gamma>1$, the inequality $\gamma-2^\alpha\varepsilon^\alpha\geq 0$ holds for all $\varepsilon\in[0,\frac{1}{2}]$, from which it follows that

$$\int_0^{\frac{1}{2}}\varepsilon^s\left|\gamma-2^\alpha\varepsilon^\alpha\right|\mathrm{d}\varepsilon=\frac{1}{2^{s+1}}\left(\frac{\gamma}{s+1}-\frac{1}{\alpha+s+1}\right).\tag{16}$$

(ii) For $0\leq\gamma\leq 1$, we define $\zeta_{(\alpha,\gamma)}=\frac{1}{2}\gamma^{\frac{1}{\alpha}}$. Then, it follows that $\gamma-2^\alpha\varepsilon^\alpha\geq 0$ on $[0,\zeta_{(\alpha,\gamma)})$ and $\gamma-2^\alpha\varepsilon^\alpha<0$ on $[\zeta_{(\alpha,\gamma)},\frac{1}{2})$. Consequently, we have that

$$\begin{aligned}\int_0^{\frac{1}{2}}\varepsilon^s\left|\gamma-2^\alpha\varepsilon^\alpha\right|\mathrm{d}\varepsilon&=\int_0^{\zeta_{(\alpha,\gamma)}}\varepsilon^s\left(\gamma-2^\alpha\varepsilon^\alpha\right)\mathrm{d}\varepsilon+\int_{\zeta_{(\alpha,\gamma)}}^{\frac{1}{2}}\varepsilon^s\left(2^\alpha\varepsilon^\alpha-\gamma\right)\mathrm{d}\varepsilon\\&=\frac{\gamma^{\frac{\alpha+s+1}{\alpha}}}{2^s}\left(\frac{1}{s+1}-\frac{1}{\alpha+s+1}\right)-\frac{1}{2^{s+1}}\left(\frac{\gamma}{s+1}-\frac{1}{\alpha+s+1}\right).\end{aligned}\tag{17}$$

This concludes the proof of Lemma 3.4.

Lemma 3.5. *Provided that the hypotheses of Lemma 3.4 are fulfilled, the result presented below holds*

$$\int_0^{\frac{1}{2}}(1-\varepsilon)^s\left|\gamma-2^\alpha\varepsilon^\alpha\right|\mathrm{d}\varepsilon=\begin{cases}K_3(\alpha,s,\gamma), & \gamma>1,\\ K_4(\alpha,s,\gamma), & 0\leq\gamma\leq 1,\end{cases}\tag{18}$$

where

$$K_3(\alpha,s,\gamma)=\frac{\gamma}{s+1}\left(1-\frac{1}{2^{s+1}}\right)-2^\alpha B_{\frac{1}{2}}(\alpha+1,s+1),$$

and

$$K_4(\alpha,s,\gamma)=\frac{\gamma}{s+1}\left[1+\frac{1}{2^{s+1}}-2\left(1-\frac{1}{2}\gamma^{\frac{1}{\alpha}}\right)^{s+1}\right]+2^\alpha\left[B_{\frac{1}{2}}(\alpha+1,s+1)-2B_{\frac{1}{2}\gamma^{\frac{1}{\alpha}}}(\alpha+1,s+1)\right].$$

Proof. Following the proof method of Lemma 3.5, we deduce the required result.

To facilitate subsequent results, we define the following expression

$$\begin{aligned}&\Lambda_\Psi(\alpha,\lambda,\mu;\kappa,\sigma)\\&:=\left(\Psi(\kappa)\right)^{\frac{\lambda}{2}}\left(\Psi\left(\frac{\kappa+\sigma}{2}\right)\right)^{1-\frac{\lambda+\mu}{2}}\left(\Psi(\sigma)\right)^{\frac{\mu}{2}}\left[{}_*\mathcal{I}^\alpha_{\frac{\kappa+\sigma}{2}}\Psi(\kappa)\cdot{}_{\frac{\kappa+\sigma}{2}}\mathcal{I}^\alpha_*\Psi(\sigma)\right]^{-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\sigma-\kappa)^\alpha}}.\end{aligned}$$

For the case $\lambda=\mu$, we obtain that

$$\Lambda_\Psi(\alpha,\mu;\kappa,\sigma):=\left[\Psi(\kappa)\Psi(\sigma)\right]^{\frac{\mu}{2}}\left[\Psi\left(\frac{\kappa+\sigma}{2}\right)\right]^{1-\mu}\left[{}_*\mathcal{I}^\alpha_{\frac{\kappa+\sigma}{2}}\Psi(\kappa)\cdot{}_{\frac{\kappa+\sigma}{2}}\mathcal{I}^\alpha_*\Psi(\sigma)\right]^{-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\sigma-\kappa)^\alpha}}.$$

Based on Lemma 3.1 together with the multiplicative s-convexity of Ψ^*, we obtain the subsequent theorem.

Theorem 3.6. *Suppose $\Psi : [\kappa,\sigma] \to \mathbb{R}^+$ is both multiplicatively differentiable on (κ,σ) and *increasing on $[\kappa,\sigma]$. And let the function Ψ^* be multiplicatively s-convex on $[\kappa,\sigma]$ with some $s \in (0,1]$. Then, for $\alpha \in (0,1]$ and $\lambda,\mu \in [0,+\infty)$, the multiplicative fractional integral inequalities presented below are satisfied.*
(i) If $\mu > 1, \lambda > 1$, then we have that

$$\left|\Lambda_\Psi(\alpha,\lambda,\mu;\kappa,\sigma)\right|^* \le [\Psi^*(\kappa)]^{\frac{\sigma-\kappa}{2}\left[K_1(\alpha,s,\mu)+K_3(\alpha,s,\lambda)\right]}[\Psi^*(\sigma)]^{\frac{\sigma-\kappa}{2}\left[K_3(\alpha,s,\mu)+K_1(\alpha,s,\lambda)\right]}.$$

(ii) If $\mu > 1, 0 \le \lambda \le 1$, then we have that

$$\left|\Lambda_\Psi(\alpha,\lambda,\mu;\kappa,\sigma)\right|^* \le [\Psi^*(\kappa)]^{\frac{\sigma-\kappa}{2}\left[K_1(\alpha,s,\mu)+K_4(\alpha,s,\lambda)\right]}[\Psi^*(\sigma)]^{\frac{\sigma-\kappa}{2}\left[K_3(\alpha,s,\mu)+K_2(\alpha,s,\lambda)\right]}.$$

(iii) If $0 \le \mu \le 1, \lambda > 1$, then we have that

$$\left|\Lambda_\Psi(\alpha,\lambda,\mu;\kappa,\sigma)\right|^* \le [\Psi^*(\kappa)]^{\frac{\sigma-\kappa}{2}\left[K_2(\alpha,s,\mu)+K_3(\alpha,s,\lambda)\right]}[\Psi^*(\sigma)]^{\frac{\sigma-\kappa}{2}\left[K_4(\alpha,s,\mu)+K_1(\alpha,s,\lambda)\right]}.$$

(iv) If $0 \le \mu \le 1, 0 \le \lambda \le 1$, then we have that

$$\left|\Lambda_\Psi(\alpha,\lambda,\mu;\kappa,\sigma)\right|^* \le [\Psi^*(\kappa)]^{\frac{\sigma-\kappa}{2}\left[K_2(\alpha,s,\mu)+K_4(\alpha,s,\lambda)\right]}[\Psi^*(\sigma)]^{\frac{\sigma-\kappa}{2}\left[K_4(\alpha,s,\mu)+K_2(\alpha,s,\lambda)\right]}.$$

Here, $K_1(\alpha,s,\cdot)$ and $K_2(\alpha,s,\cdot)$ are given in Lemma 3.4, while $K_3(\alpha,s,\cdot)$ and $K_4(\alpha,s,\cdot)$ are given in Lemma 3.5.

Proof. By applying the multiplicative integral definition (4) to the identity in Lemma 3.1, we derive that

$$\begin{aligned}\left|\Lambda_\Psi(\alpha,\lambda,\mu;\kappa,\sigma)\right|^* &= \left|\begin{array}{c}\left(\int_0^{\frac{1}{2}}\left(\left[\Psi^*\left(\varepsilon\kappa+(1-\varepsilon)\sigma\right)\right]^{\mu-2^\alpha\varepsilon^\alpha}\right)^{\mathrm{d}\varepsilon}\right)^{\frac{\sigma-\kappa}{2}}\\ \times\left(\int_{\frac{1}{2}}^1\left(\left[\Psi^*\left(\varepsilon\kappa+(1-\varepsilon)\sigma\right)\right]^{2^\alpha(1-\varepsilon)^\alpha-\lambda}\right)^{\mathrm{d}\varepsilon}\right)^{\frac{\sigma-\kappa}{2}}\end{array}\right|^*\\ &= \left|\begin{array}{c}\exp\left\{\frac{\sigma-\kappa}{2}\int_0^{\frac{1}{2}}(\mu-2^\alpha\varepsilon^\alpha)\ln\Psi^*\left(\varepsilon\kappa+(1-\varepsilon)\sigma\right)\mathrm{d}\varepsilon\right\}\\ \times\exp\left\{\frac{\sigma-\kappa}{2}\int_{\frac{1}{2}}^1\left(2^\alpha(1-\varepsilon)^\alpha-\lambda\right)\ln\Psi^*\left(\varepsilon\kappa+(1-\varepsilon)\sigma\right)\mathrm{d}\varepsilon\right\}\end{array}\right|^*. \end{aligned} \tag{19}$$

Utilizing Prop. 2.11 and Def. 2.10, we conclude that

$$\begin{aligned}&\left|\Lambda_\Psi(\alpha,\lambda,\mu;\kappa,\sigma)\right|^*\\ &\le\left|\exp\left\{\frac{\sigma-\kappa}{2}\int_0^{\frac{1}{2}}(\mu-2^\alpha\varepsilon^\alpha)\ln\Psi^*\left(\varepsilon\kappa+(1-\varepsilon)\sigma\right)\mathrm{d}\varepsilon\right\}\right|^*\\ &\quad\times\left|\exp\left\{\frac{\sigma-\kappa}{2}\int_{\frac{1}{2}}^1\left(2^\alpha(1-\varepsilon)^\alpha-\lambda\right)\ln\Psi^*\left(\varepsilon\kappa+(1-\varepsilon)\sigma\right)\mathrm{d}\varepsilon\right\}\right|^*\\ &=\exp\left\{\left|\frac{\sigma-\kappa}{2}\int_0^{\frac{1}{2}}(\mu-2^\alpha\varepsilon^\alpha)\ln\Psi^*\left(\varepsilon\kappa+(1-\varepsilon)\sigma\right)\mathrm{d}\varepsilon\right|\right\}\\ &\qquad\times\exp\left\{\left|\frac{\sigma-\kappa}{2}\int_{\frac{1}{2}}^1\left(2^\alpha(1-\varepsilon)^\alpha-\lambda\right)\ln\Psi^*\left(\varepsilon\kappa+(1-\varepsilon)\sigma\right)\mathrm{d}\varepsilon\right|\right\}\\ &\le\exp\left\{\begin{array}{l}\frac{\sigma-\kappa}{2}\int_0^{\frac{1}{2}}\left|\mu-2^\alpha\varepsilon^\alpha\right|\left|\ln\Psi^*\left(\varepsilon\kappa+(1-\varepsilon)\sigma\right)\right|\mathrm{d}\varepsilon\\ +\frac{\sigma-\kappa}{2}\int_0^{\frac{1}{2}}\left|\lambda-2^\alpha\varepsilon^\alpha\right|\left|\ln\Psi^*\left((1-\varepsilon)\kappa+\varepsilon\sigma\right)\right|\mathrm{d}\varepsilon\end{array}\right\}.\end{aligned} \tag{20}$$

According to Prop. 2.14, i.e., the fact that Ψ is *increasing ensures $\Psi^* \geq 1$, and equivalently, $\ln \Psi^* \geq 0$. Therefore, we obtain the following result

$$\left|\Lambda_\Psi(\alpha,\lambda,\mu;\kappa,\sigma)\right|^* \leq \exp\left\{ \begin{array}{l} \frac{\sigma-\kappa}{2}\int_0^{\frac{1}{2}} \left|\mu - 2^\alpha \varepsilon^\alpha\right| \ln \Psi^*\Big(\varepsilon\kappa + (1-\varepsilon)\sigma\Big) d\varepsilon \\ +\frac{\sigma-\kappa}{2}\int_0^{\frac{1}{2}} \left|\lambda - 2^\alpha \varepsilon^\alpha\right| \ln \Psi^*\Big((1-\varepsilon)\kappa + \varepsilon\sigma\Big) d\varepsilon \end{array}\right\}. \tag{21}$$

The multiplicative s-convexity of Ψ^* on $[\kappa,\sigma]$ implies that

$$\ln \Psi^*\Big(\varepsilon\kappa + (1-\varepsilon)\sigma\Big) \leq \varepsilon^s \ln \Psi^*(\kappa) + (1-\varepsilon)^s \ln \Psi^*(\sigma), \tag{22}$$

and

$$\ln \Psi^*\Big((1-\varepsilon)\kappa + \varepsilon\sigma\Big) \leq (1-\varepsilon)^s \ln \Psi^*(\kappa) + \varepsilon^s \ln \Psi^*(\sigma). \tag{23}$$

The combination of the inequalities (22) and (23) with the inequality (21) produces the subsequent result

$$\begin{aligned} &\left|\Lambda_\Psi(\alpha,\lambda,\mu;\kappa,\sigma)\right|^* \\ &\leq \exp\left\{ \begin{array}{l} \frac{\sigma-\kappa}{2}\int_0^{\frac{1}{2}} \left|\mu - 2^\alpha \varepsilon^\alpha\right|\Big[\varepsilon^s \ln \Psi^*(\kappa) + (1-\varepsilon)^s \ln \Psi^*(\sigma)\Big] d\varepsilon \\ +\frac{\sigma-\kappa}{2}\int_0^{\frac{1}{2}} \left|\lambda - 2^\alpha \varepsilon^\alpha\right|\Big[(1-\varepsilon)^s \ln \Psi^*(\kappa) + \varepsilon^s \ln \Psi^*(\sigma)\Big] d\varepsilon \end{array}\right\} \\ &= \exp\left\{ \begin{array}{l} \frac{\sigma-\kappa}{2}\left(\int_0^{\frac{1}{2}} \varepsilon^s\left|\mu - 2^\alpha \varepsilon^\alpha\right| d\varepsilon + \int_0^{\frac{1}{2}} (1-\varepsilon)^s \left|\lambda - 2^\alpha \varepsilon^\alpha\right| d\varepsilon\right) \ln \Psi^*(\kappa) \\ +\frac{\sigma-\kappa}{2}\left(\int_0^{\frac{1}{2}} (1-\varepsilon)^s \left|\mu - 2^\alpha \varepsilon^\alpha\right| d\varepsilon + \int_0^{\frac{1}{2}} \varepsilon^s\left|\lambda - 2^\alpha \varepsilon^\alpha\right| d\varepsilon\right) \ln \Psi^*(\sigma) \end{array}\right\} \\ &= \Big[\Psi^*(\kappa)\Big]^{\frac{\sigma-\kappa}{2}\left(\int_0^{\frac{1}{2}} \varepsilon^s\left|\mu - 2^\alpha \varepsilon^\alpha\right| d\varepsilon + \int_0^{\frac{1}{2}} (1-\varepsilon)^s \left|\lambda - 2^\alpha \varepsilon^\alpha\right| d\varepsilon\right)} \\ &\quad \times \Big[\Psi^*(\sigma)\Big]^{\frac{\sigma-\kappa}{2}\left(\int_0^{\frac{1}{2}} (1-\varepsilon)^s \left|\mu - 2^\alpha \varepsilon^\alpha\right| d\varepsilon + \int_0^{\frac{1}{2}} \varepsilon^s\left|\lambda - 2^\alpha \varepsilon^\alpha\right| d\varepsilon\right)}. \end{aligned} \tag{24}$$

By employing Lemma 3.4 and Lemma 3.5 in the inequality (24), we establish the required result, thereby concluding the proof of Theorem 3.6.

Corollary 3.7. *In Theorem 3.6, by taking $\lambda = \mu$, it yields the subsequent result*

$$\left|\Lambda_\Psi(\alpha,\mu;\kappa,\sigma)\right|^* \leq \begin{cases} \Big[\Psi^*(\kappa)\Psi^*(\sigma)\Big]^{\frac{\sigma-\kappa}{2}\left[K_1(\alpha,s,\mu)+K_3(\alpha,s,\mu)\right]}, & \mu > 1, \\ \Big[\Psi^*(\kappa)\Psi^*(\sigma)\Big]^{\frac{\sigma-\kappa}{2}\left[K_2(\alpha,s,\mu)+K_4(\alpha,s,\mu)\right]}, & 0 \leq \mu \leq 1. \end{cases}$$

Remark 3.8. *From Corollary 3.7, the following specific cases can be derived.*
(i) Considering $\mu = 0$, we obtain the multiplicative fractional inequality of midpoint type

$$\left|\Psi\left(\frac{\kappa+\sigma}{2}\right)\left[{}_*\mathcal{I}^\alpha_{\frac{\kappa+\sigma}{2}}\Psi(\kappa)\cdot {}_{\frac{\kappa+\sigma}{2}}\mathcal{I}^\alpha_*\Psi(\sigma)\right]^{-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\sigma-\kappa)^\alpha}}\right|^* \leq \Big[\Psi^*(\kappa)\Psi^*(\sigma)\Big]^{\frac{\sigma-\kappa}{2}[K_2(\alpha,s,0)+K_4(\alpha,s,0)]}, \tag{25}$$

where

$$K_2(\alpha,s,0)+K_4(\alpha,s,0)=\frac{1}{2^{s+1}(\alpha+s+1)}+2^{\alpha}B_{\frac{1}{2}}(\alpha+1,s+1).$$

Furthermore, if we take $s=1$, then we get that

$$\left|\Psi\left(\frac{\kappa+\sigma}{2}\right)\left[{}_*\mathcal{I}^{\alpha}_{\frac{\kappa+\sigma}{2}}\Psi(\kappa)\cdot{}_{\frac{\kappa+\sigma}{2}}\mathcal{I}^{\alpha}_{*}\Psi(\sigma)\right]^{-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\sigma-\kappa)^{\alpha}}}\right|^{*}\leq\left[\Psi^{*}(\kappa)\Psi^{*}(\sigma)\right]^{\frac{\sigma-\kappa}{4(\alpha+1)}}.$$

In particular, for $\alpha=1$, we arrive at the following inequality

$$\left|\Psi\left(\frac{\kappa+\sigma}{2}\right)\left(\int_{\kappa}^{\sigma}(\Psi(u))^{\mathrm{d}u}\right)^{\frac{1}{\kappa-\sigma}}\right|^{*}\leq\left[\Psi^{*}(\kappa)\Psi^{*}(\sigma)\right]^{\frac{\sigma-\kappa}{8}}. \tag{26}$$

It is noteworthy that the second inequality in Theorem 2.15 is of midpoint type under the classical absolute value, whereas the inequality (26) is of midpoint type under the multiplicative modulus. However, both inequalities admit the same upper bound.
(ii) Considering $\mu=\frac{1}{3}$, we obtain the multiplicative fractional inequality of Simpson type

$$\left|\left[\Psi(\kappa)\left(\Psi\left(\frac{\kappa+\sigma}{2}\right)\right)^{4}\Psi(\sigma)\right]^{\frac{1}{6}}\left[{}_*\mathcal{I}^{\alpha}_{\frac{\kappa+\sigma}{2}}\Psi(\kappa)\cdot{}_{\frac{\kappa+\sigma}{2}}\mathcal{I}^{\alpha}_{*}\Psi(\sigma)\right]^{-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\sigma-\kappa)^{\alpha}}}\right|^{*}\leq\left[\Psi^{*}(\kappa)\Psi^{*}(\sigma)\right]^{\frac{\sigma-\kappa}{2}\left[K_2\left(\alpha,s,\frac{1}{3}\right)+K_4\left(\alpha,s,\frac{1}{3}\right)\right]}, \tag{27}$$

where

$$\begin{aligned}&K_2\left(\alpha,s,\frac{1}{3}\right)+K_4\left(\alpha,s,\frac{1}{3}\right)\\&=\frac{1}{2^{s+1}(\alpha+s+1)}\left(1-2\cdot3^{-\frac{\alpha+s+1}{\alpha}}\right)+\frac{1}{3(s+1)}\left[1-2\left(1-\frac{1}{2}3^{-\frac{1}{\alpha}}\right)^{s+1}+\frac{1}{2^{s}}3^{-\frac{s+1}{\alpha}}\right]\\&\quad+2^{\alpha}\left[B_{\frac{1}{2}}(\alpha+1,s+1)-2B_{\frac{1}{2}\left(\frac{1}{3}\right)^{\frac{1}{\alpha}}}(\alpha+1,s+1)\right].\end{aligned}$$

Furthermore, if we take $s=1$, then we get that

$$\left|\left[\Psi(\kappa)\left(\Psi\left(\frac{\kappa+\sigma}{2}\right)\right)^{4}\Psi(\sigma)\right]^{\frac{1}{6}}\left[{}_*\mathcal{I}^{\alpha}_{\frac{\kappa+\sigma}{2}}\Psi(\kappa)\cdot{}_{\frac{\kappa+\sigma}{2}}\mathcal{I}^{\alpha}_{*}\Psi(\sigma)\right]^{-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\sigma-\kappa)^{\alpha}}}\right|^{*}\leq\left[\Psi^{*}(\kappa)\Psi^{*}(\sigma)\right]^{\frac{\sigma-\kappa}{2}\left(\frac{\alpha}{\alpha+1}\cdot3^{-\frac{\alpha+1}{\alpha}}+\frac{1}{2(\alpha+1)}-\frac{1}{6}\right)}.$$

In particular, for $\alpha=1$, the following inequality can be obtained

$$\left|\left[\Psi(\kappa)\left(\Psi\left(\frac{\kappa+\sigma}{2}\right)\right)^{4}\Psi(\sigma)\right]^{\frac{1}{6}}\left(\int_{\kappa}^{\sigma}(\Psi(u))^{\mathrm{d}u}\right)^{\frac{1}{\kappa-\sigma}}\right|^{*}\leq\left[\Psi^{*}(\kappa)\Psi^{*}(\sigma)\right]^{\frac{5(\sigma-\kappa)}{72}}. \tag{28}$$

It should be noted that Theorem 2.16 presents a Simpson-type inequality in the setting of the classical absolute value, while the inequality (28) is formulated in the setting of the multiplicative modulus. However, they share the same upper bound.
(iii) Considering $\mu=\frac{1}{2}$, we obtain the multiplicative fractional inequality of Bullen type

$$\left|\left[\Psi(\kappa)\left(\Psi\left(\frac{\kappa+\sigma}{2}\right)\right)^{2}\Psi(\sigma)\right]^{\frac{1}{4}}\left[{}_*\mathcal{I}^{\alpha}_{\frac{\kappa+\sigma}{2}}\Psi(\kappa)\cdot{}_{\frac{\kappa+\sigma}{2}}\mathcal{I}^{\alpha}_{*}\Psi(\sigma)\right]^{-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\sigma-\kappa)^{\alpha}}}\right|^{*}\leq\left[\Psi^{*}(\kappa)\Psi^{*}(\sigma)\right]^{\frac{\sigma-\kappa}{2}\left[K_2\left(\alpha,s,\frac{1}{2}\right)+K_4\left(\alpha,s,\frac{1}{2}\right)\right]}, \tag{29}$$

where

$$K_2\left(\alpha,s,\frac{1}{2}\right)+K_4\left(\alpha,s,\frac{1}{2}\right)$$
$$=\frac{1}{2^{s+1}}\left[\frac{1}{\alpha+s+1}+\left(\frac{1}{s+1}-\frac{1}{\alpha+s+1}\right)2^{-\frac{s+1}{\alpha}}\right]+\frac{1}{2(s+1)}\left[1-2\left(1-2^{-\frac{\alpha+1}{\alpha}}\right)^{s+1}\right]$$
$$+2^{\alpha}\left[B_{\frac{1}{2}}(\alpha+1,s+1)-2B_{\left(\frac{1}{2}\right)^{\frac{1}{\alpha}+1}}(\alpha+1,s+1)\right].$$

Furthermore, if we take $s=1$*, then we get that*

$$\left|\left[\Psi(\kappa)\left(\Psi\left(\frac{\kappa+\sigma}{2}\right)\right)^2\Psi(\sigma)\right]^{\frac{1}{4}}\left[{}_*\mathcal{I}^{\alpha}_{\frac{\kappa+\sigma}{2}}\Psi(\kappa)\cdot{}_{\frac{\kappa+\sigma}{2}}\mathcal{I}^{\alpha}_*\Psi(\sigma)\right]^{-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\sigma-\kappa)^{\alpha}}}\right|^*\leq\left[\Psi^*(\kappa)\Psi^*(\sigma)\right]^{\frac{\sigma-\kappa}{2}\left(\frac{\alpha}{\alpha+1}\cdot2^{-\frac{\alpha+1}{\alpha}}+\frac{1}{2(\alpha+1)}-\frac{1}{4}\right)}.$$

In particular, for $\alpha=1$*, we obtain that*

$$\left|\left[\Psi(\kappa)\left(\Psi\left(\frac{\kappa+\sigma}{2}\right)\right)^2\Psi(\sigma)\right]^{\frac{1}{4}}\left(\int_{\kappa}^{\sigma}(\Psi(u))^{\mathrm{d}u}\right)^{\frac{1}{\kappa-\sigma}}\right|^*\leq\left[\Psi^*(\kappa)\Psi^*(\sigma)\right]^{\frac{\sigma-\kappa}{16}}. \tag{30}$$

It is worth mentioning that the second inequality in Theorem 2.17 is of Bullen type under the classical absolute value, whereas the inequality (30) *is of Bullen type under the multiplicative modulus. However, both inequalities share the same upper bound.*

(iv) Considering $\mu=\frac{2}{3}$*, we obtain the multiplicative fractional inequality of Simpson-like type*

$$\left|\left[\Psi(\kappa)\Psi\left(\frac{\kappa+\sigma}{2}\right)\Psi(\sigma)\right]^{\frac{1}{3}}\left[{}_*\mathcal{I}^{\alpha}_{\frac{\kappa+\sigma}{2}}\Psi(\kappa)\cdot{}_{\frac{\kappa+\sigma}{2}}\mathcal{I}^{\alpha}_*\Psi(\sigma)\right]^{-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\sigma-\kappa)^{\alpha}}}\right|^*\leq\left[\Psi^*(\kappa)\Psi^*(\sigma)\right]^{\frac{\sigma-\kappa}{2}\left[K_2\left(\alpha,s,\frac{2}{3}\right)+K_4\left(\alpha,s,\frac{2}{3}\right)\right]}, \tag{31}$$

where

$$K_2\left(\alpha,s,\frac{2}{3}\right)+K_4\left(\alpha,s,\frac{2}{3}\right)$$
$$=\frac{1}{2^s}\left(\frac{2}{3}\right)^{\frac{\alpha+s+1}{\alpha}}\left(\frac{1}{s+1}-\frac{1}{\alpha+s+1}\right)+\frac{1}{2^{s+1}(\alpha+s+1)}$$
$$+\frac{2}{3(s+1)}\left[1-2\left(1-\frac{1}{2}\left(\frac{2}{3}\right)^{\frac{1}{\alpha}}\right)^{s+1}\right]+2^{\alpha}\left[B_{\frac{1}{2}}(\alpha+1,s+1)-2B_{\frac{1}{2}\left(\frac{2}{3}\right)^{\frac{1}{\alpha}}}(\alpha+1,s+1)\right].$$

Furthermore, if we take $s=1$*, then we get that*

$$\left|\left[\Psi(\kappa)\Psi\left(\frac{\kappa+\sigma}{2}\right)\Psi(\sigma)\right]^{\frac{1}{3}}\left[{}_*\mathcal{I}^{\alpha}_{\frac{\kappa+\sigma}{2}}\Psi(\kappa)\cdot{}_{\frac{\kappa+\sigma}{2}}\mathcal{I}^{\alpha}_*\Psi(\sigma)\right]^{-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\sigma-\kappa)^{\alpha}}}\right|^*\leq\left[\Psi^*(\kappa)\Psi^*(\sigma)\right]^{\frac{\sigma-\kappa}{2}\left(\frac{\alpha}{\alpha+1}\cdot\left(\frac{2}{3}\right)^{\frac{\alpha+1}{\alpha}}+\frac{1}{2(\alpha+1)}-\frac{1}{3}\right)}.$$

In particular, if we take $\alpha=1$*, then we have the multiplicative Simpson-like type inequality of integer order*

$$\left|\left[\Psi(\kappa)\Psi\left(\frac{\kappa+\sigma}{2}\right)\Psi(\sigma)\right]^{\frac{1}{3}}\left(\int_{\kappa}^{\sigma}(\Psi(\gamma))^{\mathrm{d}\gamma}\right)^{\frac{1}{\kappa-\sigma}}\right|^*\leq\left[\Psi^*(\kappa)\Psi^*(\sigma)\right]^{\frac{5(\sigma-\kappa)}{72}}. \tag{32}$$

(v) Considering $\mu=1$*, we obtain the multiplicative fractional inequality of trapezoid type*

$$\left|\left[\Psi(\kappa)\Psi(\sigma)\right]^{\frac{1}{2}}\left[{}_*\mathcal{I}^{\alpha}_{\frac{\kappa+\sigma}{2}}\Psi(\kappa)\cdot{}_{\frac{\kappa+\sigma}{2}}\mathcal{I}^{\alpha}_*\Psi(\sigma)\right]^{-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\sigma-\kappa)^{\alpha}}}\right|^*\leq\left[\Psi^*(\kappa)\Psi^*(\sigma)\right]^{\frac{\sigma-\kappa}{2}\left[K_2(\alpha,s,1)+K_4(\alpha,s,1)\right]}, \tag{33}$$

where

$$K_2(\alpha,s,1)+K_4(\alpha,s,1)=\frac{1}{s+1}-\frac{1}{2^{s+1}(\alpha+s+1)}-2^{\alpha}B_{\frac{1}{2}}(\alpha+1,s+1).$$

Furthermore, if we take $s=1$, then we get that

$$\left|[\Psi(\kappa)\Psi(\sigma)]^{\frac{1}{2}}\left[{}_*\mathcal{I}^{\alpha}_{\frac{\kappa+\sigma}{2}}\Psi(\kappa)\cdot{}_{\frac{\kappa+\sigma}{2}}\mathcal{I}^{\alpha}_{*}\Psi(\sigma)\right]^{-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\sigma-\kappa)^{\alpha}}}\right|^{*}\le\left[\Psi^{*}(\kappa)\Psi^{*}(\sigma)\right]^{\frac{\alpha(\sigma-\kappa)}{4(\alpha+1)}}.$$

In particular, for $\alpha=1$, the following inequality can be derived

$$\left|\sqrt{\Psi(\kappa)\Psi(\sigma)}\left(\int_{\kappa}^{\sigma}\left(\Psi(u)\right)^{\mathrm{d}u}\right)^{\frac{1}{\kappa-\sigma}}\right|^{*}\le\left[\Psi^{*}(\kappa)\Psi^{*}(\sigma)\right]^{\frac{\sigma-\kappa}{8}}. \tag{34}$$

It is noteworthy that the inequality in Theorem 2.18 is of trapezoid type under the classical absolute value, whereas the inequality (34) *is of trapezoid type under the multiplicative modulus. However, both inequalities admit the same upper bound.*
(vi) Considering $\mu=\frac{4}{3}$, we obtain the multiplicative fractional inequality of Milne type

$$\left|\left[\left(\Psi(\kappa)\right)^{2}\left(\Psi\Big(\frac{\kappa+\sigma}{2}\Big)\right)^{-1}\left(\Psi(\sigma)\right)^{2}\right]^{\frac{1}{3}}\left[{}_*\mathcal{I}^{\alpha}_{\frac{\kappa+\sigma}{2}}\Psi(\kappa)\cdot{}_{\frac{\kappa+\sigma}{2}}\mathcal{I}^{\alpha}_{*}\Psi(\sigma)\right]^{-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\sigma-\kappa)^{\alpha}}}\right|^{*}\le\left[\Psi^{*}(\kappa)\Psi^{*}(\sigma)\right]^{\frac{\sigma-\kappa}{2}\left[K_1\left(\alpha,s,\frac{4}{3}\right)+K_3\left(\alpha,s,\frac{4}{3}\right)\right]}, \tag{35}$$

where

$$K_1\left(\alpha,s,\frac{4}{3}\right)+K_3\left(\alpha,s,\frac{4}{3}\right)=\frac{4}{3(s+1)}-\frac{1}{2^{s+1}(\alpha+s+1)}-2^{\alpha}B_{\frac{1}{2}}(\alpha+1,s+1).$$

Furthermore , if we take $s=1$, then we get that

$$\left|\left[\left(\Psi(\kappa)\right)^{2}\left(\Psi\Big(\frac{\kappa+\sigma}{2}\Big)\right)^{-1}\left(\Psi(\sigma)\right)^{2}\right]^{\frac{1}{3}}\left[{}_*\mathcal{I}^{\alpha}_{\frac{\kappa+\sigma}{2}}\Psi(\kappa)\cdot{}_{\frac{\kappa+\sigma}{2}}\mathcal{I}^{\alpha}_{*}\Psi(\sigma)\right]^{-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\sigma-\kappa)^{\alpha}}}\right|^{*}\le\left[\Psi^{*}(\kappa)\Psi^{*}(\sigma)\right]^{\frac{\sigma-\kappa}{2}\left(\frac{2}{3}-\frac{1}{2(\alpha+1)}\right)}.$$

In particular, for $\alpha=1$, the subsequent inequality can be deduced

$$\left|\left[(\Psi(\kappa))^{2}\left(\Psi\left(\frac{\kappa+\sigma}{2}\right)\right)^{-1}(\Psi(\sigma))^{2}\right]^{\frac{1}{3}}\left(\int_{\kappa}^{\sigma}(\Psi(u))^{\mathrm{d}u}\right)^{\frac{1}{\kappa-\sigma}}\right|^{*}\le\left[\Psi^{*}(\kappa)\Psi^{*}(\sigma)\right]^{\frac{5(\sigma-\kappa)}{24}}. \tag{36}$$

It is noteworthy that the second inequality in Theorem 2.19 is of Milne type under the classical absolute value, whereas the inequality (36) *is of Milne type under the multiplicative modulus. However, both inequalities admit the same upper bound.*

Theorem 3.9. *Consider $\Psi:[\kappa,\sigma]\to\mathbb{R}^{+}$ satisfying *increasing on $[\kappa,\sigma]$ and multiplicatively differentiable on (κ,σ), and assume that Ψ^* possesses multiplicative convexity on $[\kappa,\sigma]$. Then, for $\alpha\in(0,1]$ and the parameters $\lambda,\mu\in[0,1]$, the subsequent multiplicative midpoint-type inequality holds:*

$$\left|\Psi\left(\frac{\kappa+\sigma}{2}\right)\left[{}_*\mathcal{I}^{\alpha}_{\frac{\kappa+\sigma}{2}}\Psi(\kappa)\cdot{}_{\frac{\kappa+\sigma}{2}}\mathcal{I}^{\alpha}_{*}\Psi(\sigma)\right]^{-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\sigma-\kappa)^{\alpha}}}\right|^{*}\le\Theta_1(\kappa,\sigma;\lambda,\mu)\left[\Psi^{*}(\kappa)\right]^{\frac{\sigma-\kappa}{2}\Delta_1(\alpha;\lambda,\mu)}\left[\Psi^{*}(\sigma)\right]^{\frac{\sigma-\kappa}{2}\Delta_2(\alpha;\lambda,\mu)},$$

where

$$\Theta_1(\kappa,\sigma;\lambda,\mu)=\left(\Psi(\kappa)\right)^{-\frac{\lambda}{2}}\left(\Psi\left(\frac{\kappa+\sigma}{2}\right)\right)^{\frac{\lambda+\mu}{2}}\left(\Psi(\sigma)\right)^{-\frac{\mu}{2}},$$

$$\Delta_1(\alpha;\lambda,\mu)=\frac{\alpha}{4(\alpha+2)}\left(\mu^{\frac{\alpha+2}{\alpha}}-\lambda^{\frac{\alpha+2}{\alpha}}\right)-\frac{\mu+3\lambda}{8}+\frac{\alpha}{\alpha+1}\lambda^{\frac{\alpha+1}{\alpha}}+\frac{1}{2(\alpha+1)},$$

and

$$\Delta_2(\alpha;\lambda,\mu)=\frac{\alpha}{4(\alpha+2)}\left(\lambda^{\frac{\alpha+2}{\alpha}}-\mu^{\frac{\alpha+2}{\alpha}}\right)-\frac{\lambda+3\mu}{8}+\frac{\alpha}{\alpha+1}\mu^{\frac{\alpha+1}{\alpha}}+\frac{1}{2(\alpha+1)}.$$

Proof. Upon multiplying the inequality (24) through by

$$\left(\Psi(\kappa)\right)^{-\frac{\lambda}{2}}\left(\Psi\left(\frac{\kappa+\sigma}{2}\right)\right)^{\frac{\lambda+\mu}{2}}\left(\Psi(\sigma)\right)^{-\frac{\mu}{2}}$$

with $s=1$ and $\lambda,\mu\in[0,1]$, we can derive that

$$\begin{aligned}&\left|\Psi\left(\frac{\kappa+\sigma}{2}\right)\left[{}_*\mathcal{I}^{\alpha}_{\frac{\kappa+\sigma}{2}}\Psi(\kappa)\cdot{}_{\frac{\kappa+\sigma}{2}}\mathcal{I}^{\alpha}_*\Psi(\sigma)\right]^{-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\sigma-\kappa)^{\alpha}}}\right|^*\\&\quad\le\left(\Psi(\kappa)\right)^{-\frac{\lambda}{2}}\left(\Psi\left(\frac{\kappa+\sigma}{2}\right)\right)^{\frac{\lambda+\mu}{2}}\left(\Psi(\sigma)\right)^{-\frac{\mu}{2}}\left[\Psi^*(\kappa)\right]^{\frac{\sigma-\kappa}{2}\left[K_2(\alpha,1,\mu)+K_4(\alpha,1,\lambda)\right]}\left[\Psi^*(\sigma)\right]^{\frac{\sigma-\kappa}{2}\left[K_2(\alpha,1,\lambda)+K_4(\alpha,1,\mu)\right]}.\end{aligned}\tag{37}$$

Based on Lemma 3.4 and Lemma 3.5 , we can readily obtain that

$$K_2(\alpha,1,\gamma)=\frac{\gamma^{\frac{\alpha+2}{\alpha}}}{2}\left(\frac{1}{2}-\frac{1}{\alpha+2}\right)-\frac{1}{4}\left(\frac{\gamma}{2}-\frac{1}{\alpha+2}\right),\quad\gamma\in\{\lambda,\mu\},\tag{38}$$

and

$$K_4(\alpha,1,\gamma)=-\frac{3}{8}\gamma+\frac{\alpha}{\alpha+1}\gamma^{\frac{\alpha+1}{\alpha}}+\frac{1}{4}\gamma^{\frac{\alpha+2}{\alpha}}+\frac{1}{2(\alpha+1)}-\frac{1}{4(\alpha+2)},\quad\gamma\in\{\lambda,\mu\}.\tag{39}$$

By substituting the equalities (38) and (39) into the inequality (37), we can obtain the expected result. Thus the proof is concluded.

Corollary 3.10. *In Theorem 3.9, by setting $\alpha=1$, it leads to the subsequent midpoint-type inequality of integer order:*

$$\left|\Psi\left(\frac{\kappa+\sigma}{2}\right)\left(\int_\kappa^\sigma(\Psi(\gamma))^{\mathrm{d}\gamma}\right)^{\frac{1}{\kappa-\sigma}}\right|^*\le\Theta_1(\kappa,\sigma;\lambda,\mu)\left[\Psi^*(\kappa)\right]^{\frac{\sigma-\kappa}{2}\Delta_1(1;\lambda,\mu)}\left[\Psi^*(\sigma)\right]^{\frac{\sigma-\kappa}{2}\Delta_2(1;\lambda,\mu)},\tag{40}$$

where

$$\Delta_1(1;\lambda,\mu)=\frac{1}{12}\left(\mu^3-\lambda^3\right)-\frac{\mu+3\lambda}{8}+\frac{1}{2}\lambda^2+\frac{1}{4},$$

and

$$\Delta_2(1;\lambda,\mu)=\frac{1}{12}\left(\lambda^3-\mu^3\right)-\frac{\lambda+3\mu}{8}+\frac{1}{2}\mu^2+\frac{1}{4}.$$

Remark 3.11. *We note that, if $\Theta_1(\kappa,\sigma;1,1)<1$, the inequality* (40) *improves upon the inequality* (26) *with a smaller right-hand side value. Now, we present an example to illustrate this fact.*

Example 3.12. From the inequality (40), if $\lambda=1$ and $\mu=1$, it follows that

$$\left|\Psi\left(\frac{\kappa+\sigma}{2}\right)\left(\int_\kappa^\sigma\left(\Psi(\gamma)\right)^{\mathrm{d}\gamma}\right)^{\frac{1}{\kappa-\sigma}}\right|^*\le\Theta_1(\kappa,\sigma;1,1)\left[\Psi^*(\kappa)\Psi^*(\sigma)\right]^{\frac{\sigma-\kappa}{8}},\tag{41}$$

where

$$\Theta_1(\kappa,\sigma;1,1)=\Big[\Psi(\kappa)\Psi(\sigma)\Big]^{-\frac{1}{2}}\Psi\Big(\frac{\kappa+\sigma}{2}\Big).$$

Consider the function $\Psi(\gamma)=\exp\left\{\gamma^3\right\}$ for $\gamma\geq 0$, and let $[\kappa,\sigma]=[\varepsilon,\varepsilon+1]$ with $\varepsilon\geq 0$. Then, its multiplicatively differentiable function $\Psi^*(\gamma)=\exp\left\{3\gamma^2\right\}$ is multiplicatively convex on $[\varepsilon,\varepsilon+1]$. The requirements of Corollary 3.10 are fully met, and we derive that

$$\Theta_1(\varepsilon,\varepsilon+1;1,1)=\Big[\Psi(\varepsilon)\Psi(\varepsilon+1)\Big]^{-\frac{1}{2}}\Psi\Big(\varepsilon+\frac{1}{2}\Big)=\exp\left\{-\frac{3}{4}\varepsilon-\frac{3}{8}\right\}.$$

Obviously, for all $\varepsilon\geq 0$, the inequality $\Theta_1(\varepsilon,\varepsilon+1;1,1)<1$ holds, which verifies the conclusion in Remark 3.11.

Theorem 3.13. *Provided that the hypotheses of Theorem 3.9 hold, the subsequent trapezoid-type inequality holds:*

$$\left|\sqrt{\Psi(\kappa)\Psi(\sigma)}\left[{}_*\mathcal{I}^{\alpha}_{\frac{\kappa+\sigma}{2}}\Psi(\kappa)\cdot{}_{\frac{\kappa+\sigma}{2}}\mathcal{I}^{\alpha}_{*}\Psi(\sigma)\right]^{-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\sigma-\kappa)^{\alpha}}}\right|^{*}\leq\Theta_2(\kappa,\sigma;\lambda,\mu)\Big[\Psi^*(\kappa)\Big]^{\frac{\sigma-\kappa}{2}\Delta_1(\alpha;\lambda,\mu)}\Big[\Psi^*(\sigma)\Big]^{\frac{\sigma-\kappa}{2}\Delta_2(\alpha;\lambda,\mu)},$$

where

$$\Theta_2(\kappa,\sigma;\lambda,\mu)=\Big(\Psi(\kappa)\Big)^{\frac{1-\lambda}{2}}\left(\Psi\Big(\frac{\kappa+\sigma}{2}\Big)\right)^{\frac{\lambda+\mu}{2}-1}\Big(\Psi(\sigma)\Big)^{\frac{1-\mu}{2}},$$

and the expressions $\Delta_1(\alpha;\lambda,\mu)$ and $\Delta_2(\alpha;\lambda,\mu)$ are defined in Theorem 3.9, respectively.

Proof. Upon multiplying the inequality (24) through by

$$\Big(\Psi(\kappa)\Big)^{\frac{1-\lambda}{2}}\left(\Psi\Big(\frac{\kappa+\sigma}{2}\Big)\right)^{\frac{\lambda+\mu}{2}-1}\Big(\Psi(\sigma)\Big)^{\frac{1-\mu}{2}}$$

with $s=1$ and $\lambda,\mu\in[0,1]$, and applying the equalities presented in (38) and (39) to derive that

$$\begin{aligned}&\left|\sqrt{\Psi(\kappa)\Psi(\sigma)}\left[{}_*\mathcal{I}^{\alpha}_{\frac{\kappa+\sigma}{2}}\Psi(\kappa)\cdot{}_{\frac{\kappa+\sigma}{2}}\mathcal{I}^{\alpha}_{*}\Psi(\sigma)\right]^{-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\sigma-\kappa)^{\alpha}}}\right|^{*}\\&\quad\leq\Big(\Psi(\kappa)\Big)^{\frac{1-\lambda}{2}}\left(\Psi\Big(\frac{\kappa+\sigma}{2}\Big)\right)^{\frac{\lambda+\mu}{2}-1}\Big(\Psi(\sigma)\Big)^{\frac{1-\mu}{2}}\Big[\Psi^*(\kappa)\Big]^{\frac{\sigma-\kappa}{2}\left[K_2(\alpha,1,\mu)+K_4(\alpha,1,\lambda)\right]}\\&\quad\quad\times\Big[\Psi^*(\sigma)\Big]^{\frac{\sigma-\kappa}{2}\left[K_2(\alpha,1,\lambda)+K_4(\alpha,1,\mu)\right]},\end{aligned}\tag{42}$$

which conclude the proof of Theorem 3.13.

Corollary 3.14. *In Theorem 3.13, by setting $\alpha=1$, it leads to the subsequent integer-order inequality of trapezoid type:*

$$\left|\sqrt{\Psi(\kappa)\Psi(\sigma)}\left(\int_{\kappa}^{\sigma}(\Psi(\gamma))^{\mathrm{d}\gamma}\right)^{\frac{1}{\kappa-\sigma}}\right|^{*}\leq\Theta_2(\kappa,\sigma;\lambda,\mu)\Big[\Psi^*(\kappa)\Big]^{\frac{\sigma-\kappa}{2}\Delta_1(1;\lambda,\mu)}\Big[\Psi^*(\sigma)\Big]^{\frac{\sigma-\kappa}{2}\Delta_2(1;\lambda,\mu)},\tag{43}$$

where $\Delta_1(1;\lambda,\mu)$ and $\Delta_2(1;\lambda,\mu)$ are defined in Corollary 3.7.

Remark 3.15. *Under the conditions of Corollary 3.14, we find that the right term of the inequality* (43) *is better than the inequality* (34) *under certain conditions. Now, we present an example to illustrate this fact.*

Example 3.16. Consider the function $\Psi(\gamma) = 3^{\gamma^4}$ defined on $[\kappa, \sigma]$, and $[\kappa, \sigma] = [\varepsilon, \varepsilon + 1]$ with $\varepsilon \geq 0$. Its multiplicatively differentiable function $\Psi^*(\gamma) = 3^{4\gamma^3}$ is multiplicatively convex on $[\varepsilon, \varepsilon + 1]$. This satisfies all requirements of Corollary 3.14, from which we obtain that

$$\Theta_2(\varepsilon, \varepsilon+1; \lambda, \mu)\left[\Psi^*(\varepsilon)\right]^{\frac{1}{2}\Delta_1(1;\lambda,\mu)}\left[\Psi^*(\varepsilon+1)\right]^{\frac{1}{2}\Delta_2(1;\lambda,\mu)} = 3^{\left[\lambda^2+(\mu-1)^2\right]\varepsilon^3+\frac{\lambda^3-\mu^3+6\mu^2-9\mu+6}{2}\varepsilon^2+\frac{\lambda^3-\mu^3-\lambda+6\mu^2-8\mu+6}{2}\varepsilon+\frac{\lambda^3-\mu^3}{6}+\mu^2-\frac{7\lambda+39\mu}{32}+\frac{15}{16}}.$$

Let

$$M(\lambda, \mu; \varepsilon) = \left[\lambda^2 + (\mu - 1)^2\right]\varepsilon^3 + \frac{\lambda^3 - \mu^3 + 6\mu^2 - 9\mu + 6}{2}\varepsilon^2 + \frac{\lambda^3 - \mu^3 - \lambda + 6\mu^2 - 8\mu + 6}{2}\varepsilon + \frac{\lambda^3 - \mu^3}{6} + \mu^2 - \frac{7\lambda + 39\mu}{32} + \frac{15}{16},$$

where $\lambda, \mu \in [0, 1]$ and some fixed $\varepsilon \geq 0$. For some fixed $\varepsilon \geq 0$, we find that $M(\lambda, \mu; \varepsilon)$ is minimized when

$$\lambda = \lambda_0(\varepsilon) = \frac{-2\varepsilon^3 + \sqrt{4\varepsilon^6 + \left(\varepsilon + \frac{7}{16}\right)(3\varepsilon^2 + 3\varepsilon + 1)}}{3\varepsilon^2 + 3\varepsilon + 1},$$

and

$$\mu = \mu_0(\varepsilon) = \frac{2(\varepsilon + 1)^3 - \sqrt{4\varepsilon^6 + 12\varepsilon^5 + 12\varepsilon^4 + 16\varepsilon^3 + \frac{267}{16}\varepsilon^2 + \frac{139}{16}\varepsilon + \frac{25}{16}}}{3\varepsilon^2 + 3\varepsilon + 1}.$$

After calculations, the right term of the inequality (34) is $3^{\frac{1}{2}\left(2\varepsilon^3+3\varepsilon^2+3\varepsilon+1\right)}$.
Let

$$G(\varepsilon) = \frac{1}{2}\left(2\varepsilon^3 + 3\varepsilon^2 + 3\varepsilon + 1\right),$$

and

$$H(\varepsilon) = M(\lambda_0(\varepsilon), \mu_0(\varepsilon); \varepsilon) - G(\varepsilon), \ \varepsilon \geq 0.$$

The derivative function of $H(\varepsilon)$ is obtained as

$$H'(\varepsilon) = 3\left(\lambda_0^2 + \mu_0^2 - 2\mu_0\right)\varepsilon^2 + \left(\lambda_0^3 - \mu_0^3 + 6\mu_0^2 - 9\mu_0 + 3\right)\varepsilon + \frac{\lambda_0^3 - \mu_0^3 - \lambda_0 + 6\mu_0^2 - 8\mu_0 + 3}{2}.$$

For $\varepsilon \geq 0$, then $H'(\varepsilon) < 0$, which ensures that $H(\varepsilon)$ is monotonically decreasing on $[0, +\infty)$. Also, we have that

$$H(0) = \frac{3 - 7\sqrt{7}}{192} < 0.$$

It can be readily concluded that $H(\varepsilon) < 0$ for all $\varepsilon \geq 0$. Consequently, the inequality (43) is more precise than the inequality (34) under the conditions $\lambda = \lambda_0(\varepsilon)$ and $\mu = \mu_0(\varepsilon)$.

Given the s-convexity of $(\ln \Psi^*)^q$ for $q > 1$, where p and q adheres to the conjugate condition $p + q = pq$ with $p > 1$, the subsequent theorem is established.

Theorem 3.17. *Suppose the function $\Psi : [\kappa, \sigma] \to \mathbb{R}^+$ is multiplicatively differentiable on (κ, σ) and *increasing on $[\kappa, \sigma]$, and assume that $(\ln \Psi^*)^q$ exhibits s-convexity for some $s \in (0, 1]$ on the closed interval $[\kappa, \sigma]$, where $q > 1$ and*

$p>1$ satisfies $p+q=pq$. Then, for $\alpha\in(0,1]$ and $\lambda,\mu\in[0,+\infty)$, the subsequent multiplicative fractional integral inequality holds:

$$\left|\Lambda_\Psi(\alpha,\lambda,\mu;\kappa,\sigma)\right|^* \le \left[\Psi^*(\kappa)\right]^{\frac{\sigma-\kappa}{2}C_1(\lambda,\mu,\alpha,s;p,q)}\left[\Psi^*(\sigma)\right]^{\frac{\sigma-\kappa}{2}C_2(\lambda,\mu,\alpha,s;p,q)},$$

where

$$C_1(\lambda,\mu,\alpha,s;p,q)=\left(\frac{1}{(s+1)\,2^{s+1}}\right)^{\frac{1}{q}}\left[\left(\int_0^{\frac{1}{2}}|\mu-2^\alpha\varepsilon^\alpha|^p\mathrm{d}\varepsilon\right)^{\frac{1}{p}}+\left(2^{s+1}-1\right)^{\frac{1}{q}}\left(\int_0^{\frac{1}{2}}|\lambda-2^\alpha\varepsilon^\alpha|^p\mathrm{d}\varepsilon\right)^{\frac{1}{p}}\right],$$

and

$$C_2(\lambda,\mu,\alpha,s;p,q)=\left(\frac{1}{(s+1)\,2^{s+1}}\right)^{\frac{1}{q}}\left[\left(2^{s+1}-1\right)^{\frac{1}{q}}\left(\int_0^{\frac{1}{2}}|\mu-2^\alpha\varepsilon^\alpha|^p\mathrm{d}\varepsilon\right)^{\frac{1}{p}}+\left(\int_0^{\frac{1}{2}}|\lambda-2^\alpha\varepsilon^\alpha|^p\mathrm{d}\varepsilon\right)^{\frac{1}{p}}\right].$$

Proof. The inequality (21) in the proof of Theorem 3.6, combined with the Hölder's integral inequality, leads to the subsequent result

$$\begin{aligned}&\left|\Lambda_\Psi\left(\alpha,\lambda,\mu;\kappa,\sigma\right)\right|^*\\ &\le\exp\left\{\begin{array}{l}\dfrac{\sigma-\kappa}{2}\left(\displaystyle\int_0^{\frac{1}{2}}\left|\mu-2^\alpha\varepsilon^\alpha\right|^p\mathrm{d}\varepsilon\right)^{\frac{1}{p}}\left(\displaystyle\int_0^{\frac{1}{2}}\left[\ln\Psi^*\big(\varepsilon\kappa+(1-\varepsilon)\,\sigma\big)\right]^q\mathrm{d}\varepsilon\right)^{\frac{1}{q}}\\ +\dfrac{\sigma-\kappa}{2}\left(\displaystyle\int_0^{\frac{1}{2}}|\lambda-2^\alpha\varepsilon^\alpha|^p\,\mathrm{d}\varepsilon\right)^{\frac{1}{p}}\left(\displaystyle\int_0^{\frac{1}{2}}\left[\ln\Psi^*\big((1-\varepsilon)\,\kappa+\varepsilon\sigma\big)\right]^q\mathrm{d}\varepsilon\right)^{\frac{1}{q}}\end{array}\right\}.\end{aligned}\tag{44}$$

By leveraging the s-convexity of $(\ln\Psi^*)^q$ on the interval $[\kappa,\sigma]$, we infer that

$$\begin{aligned}&\int_0^{\frac{1}{2}}\left[\ln\Psi^*\big(\varepsilon\kappa+(1-\varepsilon)\,\sigma\big)\right]^q\mathrm{d}\varepsilon\\ &\le\big(\ln\Psi^*(\kappa)\big)^q\int_0^{\frac{1}{2}}\varepsilon^s\mathrm{d}\varepsilon+\big(\ln\Psi^*(\sigma)\big)^q\int_0^{\frac{1}{2}}(1-\varepsilon)^s\mathrm{d}\varepsilon\\ &=\frac{1}{(s+1)\,2^{s+1}}\big(\ln\Psi^*(\kappa)\big)^q+\frac{1}{s+1}\left(1-\frac{1}{2^{s+1}}\right)\big(\ln\Psi^*(\sigma)\big)^q\\ &=\left[\left(\frac{1}{(s+1)\,2^{s+1}}\right)^{\frac{1}{q}}\ln\Psi^*(\kappa)\right]^q+\left[\left(\frac{2^{s+1}-1}{(s+1)\,2^{s+1}}\right)^{\frac{1}{q}}\ln\Psi^*(\sigma)\right]^q.\end{aligned}\tag{45}$$

Similarly, we have the following result

$$\int_0^{\frac{1}{2}}\left[\ln\Psi^*\big((1-\varepsilon)\,\kappa+\varepsilon\sigma\big)\right]^q\mathrm{d}\varepsilon\le\left[\left(\frac{2^{s+1}-1}{(s+1)\,2^{s+1}}\right)^{\frac{1}{q}}\ln\Psi^*(\kappa)\right]^q+\left[\left(\frac{1}{(s+1)\,2^{s+1}}\right)^{\frac{1}{q}}\ln\Psi^*(\sigma)\right]^q.\tag{46}$$

Applying the inequalities (45) and (46) to the inequality (44), and noticing the fact that $\psi^\tau+\delta^\tau\le(\psi+\delta)^\tau$

for $\psi \geq 0, \delta \geq 0$ with $\tau \geq 1$, it yields that

$$\begin{aligned}
&\left|\Lambda_{\Psi}(\alpha,\lambda,\mu;\kappa,\sigma)\right|^{*} \\
&\leq \exp\left\{\begin{array}{l}
\frac{\sigma-\kappa}{2}\left(\int_0^{\frac{1}{2}}\left|\mu-2^{\alpha}\varepsilon^{\alpha}\right|^{p}\mathrm{d}\varepsilon\right)^{\frac{1}{p}}\left[\left(\frac{1}{(s+1)2^{s+1}}\right)^{\frac{1}{q}}\ln\Psi^{*}(\kappa)+\left(\frac{2^{s+1}-1}{(s+1)2^{s+1}}\right)^{\frac{1}{q}}\ln\Psi^{*}(\sigma)\right] \\
+\frac{\sigma-\kappa}{2}\left(\int_0^{\frac{1}{2}}\left|\lambda-2^{\alpha}\varepsilon^{\alpha}\right|^{p}\mathrm{d}\varepsilon\right)^{\frac{1}{p}}\left[\left(\frac{2^{s+1}-1}{(s+1)2^{s+1}}\right)^{\frac{1}{q}}\ln\Psi^{*}(\kappa)+\left(\frac{1}{(s+1)2^{s+1}}\right)^{\frac{1}{q}}\ln\Psi^{*}(\sigma)\right]
\end{array}\right\} \\
&= \exp\left\{\begin{array}{l}
\frac{\sigma-\kappa}{2}\left[\begin{array}{l}\left(\int_0^{\frac{1}{2}}\left|\mu-2^{\alpha}\varepsilon^{\alpha}\right|^{p}\mathrm{d}\varepsilon\right)^{\frac{1}{p}}\left(\frac{1}{(s+1)2^{s+1}}\right)^{\frac{1}{q}} \\ +\left(\int_0^{\frac{1}{2}}\left|\lambda-2^{\alpha}\varepsilon^{\alpha}\right|^{p}\mathrm{d}\varepsilon\right)^{\frac{1}{p}}\left(\frac{2^{s+1}-1}{(s+1)2^{s+1}}\right)^{\frac{1}{q}}\end{array}\right]\ln\Psi^{*}(\kappa) \\
+\frac{\sigma-\kappa}{2}\left[\begin{array}{l}\left(\int_0^{\frac{1}{2}}\left|\lambda-2^{\alpha}\varepsilon^{\alpha}\right|^{p}\mathrm{d}\varepsilon\right)^{\frac{1}{p}}\left(\frac{1}{(s+1)2^{s+1}}\right)^{\frac{1}{q}} \\ +\left(\int_0^{\frac{1}{2}}\left|\mu-2^{\alpha}\varepsilon^{\alpha}\right|^{p}\mathrm{d}\varepsilon\right)^{\frac{1}{p}}\left(\frac{2^{s+1}-1}{(s+1)2^{s+1}}\right)^{\frac{1}{q}}\end{array}\right]\ln\Psi^{*}(\sigma)
\end{array}\right\}. \qquad (47)
\end{aligned}$$

This completes the proof of Theorem 3.17.

Corollary 3.18. *In Theorem 3.17, by choosing $\lambda = \mu$, we obtain the subsequent result:*

$$\left|\Lambda_{\Psi}(\alpha,\mu;\kappa,\sigma)\right|^{*} \leq \left[\Psi^{*}(\kappa)\Psi^{*}(\sigma)\right]^{\frac{\sigma-\kappa}{2}C_1^1(\mu,\alpha,s;p,q)},$$

where

$$C_1^1(\mu,\alpha,s;p,q) = \left(\frac{1}{(s+1)2^{s+1}}\right)^{\frac{1}{q}}\left[\left(2^{s+1}-1\right)^{\frac{1}{q}}+1\right]\left(\int_0^{\frac{1}{2}}\left|\mu-2^{\alpha}\varepsilon^{\alpha}\right|^{p}\mathrm{d}\varepsilon\right)^{\frac{1}{p}}.$$

Especially, for $\mu = 0$, we have the subsequent midpoint-type fractional inequality:

$$\left|\Psi\left(\frac{\kappa+\sigma}{2}\right)\left[{}_{*}\mathcal{I}^{\alpha}_{\frac{\kappa+\sigma}{2}}\Psi(\kappa)\cdot{}_{\frac{\kappa+\sigma}{2}}\mathcal{I}^{\alpha}_{*}\Psi(\sigma)\right]^{-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\sigma-\kappa)^{\alpha}}}\right|^{*} \leq \left[\Psi^{*}(\kappa)\Psi^{*}(\sigma)\right]^{\frac{\sigma-\kappa}{4}\left(\frac{1}{(s+1)2^{s}}\right)^{\frac{1}{q}}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left[\left(2^{s+1}-1\right)^{\frac{1}{q}}+1\right]}.$$

Exploiting the s-convexity of $(\ln\Psi^{*})^{q}$ with $q > 1$, the subsequent theorem can be established.

Theorem 3.19. *Consider the multiplicatively differentiable function $\Psi : [\kappa,\sigma] \to \mathbb{R}^{+}$ that is *increasing on $[\kappa,\sigma]$, and suppose that $(\ln\Psi^{*})^{q}$ possesses s-convexity on $[\kappa,\sigma]$ with some $s \in (0,1]$, where $q > 1$. Then, for $\alpha \in (0,1]$ and $\lambda,\mu \in [0,+\infty)$, the following inequalities hold for multiplicative RL-fractional integrals.*
(i) If $\mu > 1, \lambda > 1$, then we have that

$$\begin{aligned}
\left|\Lambda_{\Psi}(\alpha,\lambda,\mu;\kappa,\sigma)\right|^{*} \leq{}& \left[\Psi^{*}(\kappa)\right]^{\frac{\sigma-\kappa}{2}\left[\left(\eta_1(\alpha,\mu)\right)^{1-\frac{1}{q}}\left(K_1(\alpha,s,\mu)\right)^{\frac{1}{q}}+\left(\eta_1(\alpha,\lambda)\right)^{1-\frac{1}{q}}\left(K_3(\alpha,s,\lambda)\right)^{\frac{1}{q}}\right]} \\
&\times\left[\Psi^{*}(\sigma)\right]^{\frac{\sigma-\kappa}{2}\left[\left(\eta_1(\alpha,\mu)\right)^{1-\frac{1}{q}}\left(K_3(\alpha,s,\mu)\right)^{\frac{1}{q}}+\left(\eta_1(\alpha,\lambda)\right)^{1-\frac{1}{q}}\left(K_1(\alpha,s,\lambda)\right)^{\frac{1}{q}}\right]}.
\end{aligned}$$

(ii) If $\mu > 1, 0 \leq \lambda \leq 1$, then we have that

$$\left|\Lambda_{\Psi}(\alpha,\lambda,\mu;\kappa,\sigma)\right|^{*} \leq \left[\Psi^{*}(\kappa)\right]^{\frac{\sigma-\kappa}{2}\left[\left(\eta_1(\alpha,\mu)\right)^{1-\frac{1}{q}}\left(K_1(\alpha,s,\mu)\right)^{\frac{1}{q}}+\left(\eta_2(\alpha,\lambda)\right)^{1-\frac{1}{q}}\left(K_4(\alpha,s,\lambda)\right)^{\frac{1}{q}}\right]} \times\left[\Psi^{*}(\sigma)\right]^{\frac{\sigma-\kappa}{2}\left[\left(\eta_1(\alpha,\mu)\right)^{1-\frac{1}{q}}\left(K_3(\alpha,s,\mu)\right)^{\frac{1}{q}}+\left(\eta_2(\alpha,\lambda)\right)^{1-\frac{1}{q}}\left(K_2(\alpha,s,\lambda)\right)^{\frac{1}{q}}\right]}.$$

(iii) If $0 \leq \mu \leq 1, \lambda > 1$, then we have that

$$\left|\Lambda_{\Psi}(\alpha,\lambda,\mu;\kappa,\sigma)\right|^{*} \leq \left[\Psi^{*}(\kappa)\right]^{\frac{\sigma-\kappa}{2}\left[\left(\eta_2(\alpha,\mu)\right)^{1-\frac{1}{q}}\left(K_2(\alpha,s,\mu)\right)^{\frac{1}{q}}+\left(\eta_1(\alpha,\lambda)\right)^{1-\frac{1}{q}}\left(K_3(\alpha,s,\lambda)\right)^{\frac{1}{q}}\right]} \times\left[\Psi^{*}(\sigma)\right]^{\frac{\sigma-\kappa}{2}\left[\left(\eta_2(\alpha,\mu)\right)^{1-\frac{1}{q}}\left(K_4(\alpha,s,\mu)\right)^{\frac{1}{q}}+\left(\eta_1(\alpha,\lambda)\right)^{1-\frac{1}{q}}\left(K_1(\alpha,s,\lambda)\right)^{\frac{1}{q}}\right]}.$$

(iv) If $0 \leq \mu \leq 1, 0 \leq \lambda \leq 1$, then we have that

$$\left|\Lambda_{\Psi}(\alpha,\lambda,\mu;\kappa,\sigma)\right|^{*} \leq \left[\Psi^{*}(\kappa)\right]^{\frac{\sigma-\kappa}{2}\left[\left(\eta_2(\alpha,\mu)\right)^{1-\frac{1}{q}}\left(K_2(\alpha,s,\mu)\right)^{\frac{1}{q}}+\left(\eta_2(\alpha,\lambda)\right)^{1-\frac{1}{q}}\left(K_4(\alpha,s,\lambda)\right)^{\frac{1}{q}}\right]} \times\left[\Psi^{*}(\sigma)\right]^{\frac{\sigma-\kappa}{2}\left[\left(\eta_2(\alpha,\mu)\right)^{1-\frac{1}{q}}\left(K_4(\alpha,s,\mu)\right)^{\frac{1}{q}}+\left(\eta_2(\alpha,\lambda)\right)^{1-\frac{1}{q}}\left(K_2(\alpha,s,\lambda)\right)^{\frac{1}{q}}\right]}.$$

Here, for $\gamma \in \{\lambda, \mu\}$,

$$\eta_1(\alpha,\gamma) = \frac{1}{2}\left(\gamma - \frac{1}{\alpha+1}\right),$$
$$\eta_2(\alpha,\gamma) = \frac{\alpha}{\alpha+1}\gamma^{\frac{\alpha+1}{\alpha}} - \frac{1}{2}\left(\gamma - \frac{1}{\alpha+1}\right),$$

and $K_1(\alpha,s,\gamma)$ and $K_2(\alpha,s,\gamma)$ are given in Lemma 3.4, while $K_3(\alpha,s,\gamma)$ and $K_4(\alpha,s,\gamma)$ are given in Lemma 3.5.

Proof. The inequality (21) in the proof of Theorem 3.6, combined with the power-mean integral inequality, leads to the subsequent result

$$\begin{aligned}&\left|\Lambda_{\Psi}(\alpha,\lambda,\mu;\kappa,\sigma)\right|^{*}\\&\leq \exp\left\{\begin{array}{l}\frac{\sigma-\kappa}{2}\left(\int_0^{\frac{1}{2}}\left|\mu-2^{\alpha}\varepsilon^{\alpha}\right|\mathrm{d}\varepsilon\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}\left|\mu-2^{\alpha}\varepsilon^{\alpha}\right|\left[\ln\Psi^{*}\left(\varepsilon\kappa+(1-\varepsilon)\sigma\right)\right]^{q}\mathrm{d}\varepsilon\right)^{\frac{1}{q}}\\ +\frac{\sigma-\kappa}{2}\left(\int_0^{\frac{1}{2}}\left|\lambda-2^{\alpha}\varepsilon^{\alpha}\right|\mathrm{d}\varepsilon\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}\left|\lambda-2^{\alpha}\varepsilon^{\alpha}\right|\left[\ln\Psi^{*}\left((1-\varepsilon)\kappa+\varepsilon\sigma\right)\right]^{q}\mathrm{d}\varepsilon\right)^{\frac{1}{q}}\end{array}\right\}.\end{aligned} \tag{48}$$

By leveraging the s-convexity of $(\ln\Psi^{*})^{q}$ on the interval $[\kappa,\sigma]$, we deduce that

$$\begin{aligned}&\int_0^{\frac{1}{2}}\left|\mu-2^{\alpha}\varepsilon^{\alpha}\right|\left[\ln\Psi^{*}\left(\varepsilon\kappa+(1-\varepsilon)\sigma\right)\right]^{q}\mathrm{d}\varepsilon\\&\leq\left(\ln\Psi^{*}(\kappa)\right)^{q}\int_0^{\frac{1}{2}}\varepsilon^{s}\left|\mu-2^{\alpha}\varepsilon^{\alpha}\right|\mathrm{d}\varepsilon+\left(\ln\Psi^{*}(\sigma)\right)^{q}\int_0^{\frac{1}{2}}(1-\varepsilon)^{s}\left|\mu-2^{\alpha}\varepsilon^{\alpha}\right|\mathrm{d}\varepsilon\\&=\left[\left(\int_0^{\frac{1}{2}}\varepsilon^{s}\left|\mu-2^{\alpha}\varepsilon^{\alpha}\right|\mathrm{d}\varepsilon\right)^{\frac{1}{q}}\ln\Psi^{*}(\kappa)\right]^{q}+\left[\left(\int_0^{\frac{1}{2}}(1-\varepsilon)^{s}\left|\mu-2^{\alpha}\varepsilon^{\alpha}\right|\mathrm{d}\varepsilon\right)^{\frac{1}{q}}\ln\Psi^{*}(\sigma)\right]^{q}.\end{aligned} \tag{49}$$

Similarly, we have that

$$\begin{aligned}&\int_0^{\frac{1}{2}}\left|\lambda-2^\alpha\varepsilon^\alpha\right|\left[\ln\Psi^*\Big((1-\varepsilon)\kappa+\varepsilon\sigma\Big)\right]^q\mathrm{d}\varepsilon\\&\le\left[\left(\int_0^{\frac{1}{2}}(1-\varepsilon)^s\left|\lambda-2^\alpha\varepsilon^\alpha\right|\mathrm{d}\varepsilon\right)^{\frac{1}{q}}\ln\Psi^*(\kappa)\right]^q+\left[\left(\int_0^{\frac{1}{2}}\varepsilon^s\left|\lambda-2^\alpha\varepsilon^\alpha\right|\mathrm{d}\varepsilon\right)^{\frac{1}{q}}\ln\Psi^*(\sigma)\right]^q.\end{aligned}\tag{50}$$

Applying the inequalities (49) and (50) to the inequality (48), and noticing the fact that $\psi^\tau+\delta^\tau\le(\psi+\delta)^\tau$ for $\psi\ge 0,\delta\ge 0$ with $\tau\ge 1$, we derive that

$$\begin{aligned}&\left|\Lambda_\Psi(\alpha,\lambda,\mu;\kappa,\sigma)\right|^*\\&\le\exp\left\{\begin{array}{l}\frac{\sigma-\kappa}{2}\left(\int_0^{\frac{1}{2}}\left|\mu-2^\alpha\varepsilon^\alpha\right|\mathrm{d}\varepsilon\right)^{1-\frac{1}{q}}\\ \quad\times\left[\Big(\ln\Psi^*(\kappa)\Big)\left(\int_0^{\frac{1}{2}}\varepsilon^s\left|\mu-2^\alpha\varepsilon^\alpha\right|\mathrm{d}\varepsilon\right)^{\frac{1}{q}}+\Big(\ln\Psi^*(\sigma)\Big)\left(\int_0^{\frac{1}{2}}(1-\varepsilon)^s\left|\mu-2^\alpha\varepsilon^\alpha\right|\mathrm{d}\varepsilon\right)^{\frac{1}{q}}\right]\\ +\frac{\sigma-\kappa}{2}\left(\int_0^{\frac{1}{2}}|\lambda-2^\alpha\varepsilon^\alpha|\mathrm{d}\varepsilon\right)^{1-\frac{1}{q}}\\ \quad\times\left[\Big(\ln\Psi^*(\kappa)\Big)\left(\int_0^{\frac{1}{2}}(1-\varepsilon)^s|\lambda-2^\alpha\varepsilon^\alpha|\mathrm{d}\varepsilon\right)^{\frac{1}{q}}+\Big(\ln\Psi^*(\sigma)\Big)\left(\int_0^{\frac{1}{2}}\varepsilon^s|\lambda-2^\alpha\varepsilon^\alpha|\mathrm{d}\varepsilon\right)^{\frac{1}{q}}\right]\end{array}\right\}\\&=\exp\left\{\begin{array}{l}\frac{\sigma-\kappa}{2}\Big(\ln\Psi^*(\kappa)\Big)\left[\begin{array}{l}\left(\int_0^{\frac{1}{2}}\left|\mu-2^\alpha\varepsilon^\alpha\right|\mathrm{d}\varepsilon\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}\varepsilon^s\left|\mu-2^\alpha\varepsilon^\alpha\right|\mathrm{d}\varepsilon\right)^{\frac{1}{q}}\\ +\left(\int_0^{\frac{1}{2}}|\lambda-2^\alpha\varepsilon^\alpha|\mathrm{d}\varepsilon\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}(1-\varepsilon)^s|\lambda-2^\alpha\varepsilon^\alpha|\mathrm{d}\varepsilon\right)^{\frac{1}{q}}\end{array}\right]\\ +\frac{\sigma-\kappa}{2}\Big(\ln\Psi^*(\sigma)\Big)\left[\begin{array}{l}\left(\int_0^{\frac{1}{2}}\left|\mu-2^\alpha\varepsilon^\alpha\right|\mathrm{d}\varepsilon\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}(1-\varepsilon)^s\left|\mu-2^\alpha\varepsilon^\alpha\right|\mathrm{d}\varepsilon\right)^{\frac{1}{q}}\\ +\left(\int_0^{\frac{1}{2}}|\lambda-2^\alpha\varepsilon^\alpha|\mathrm{d}\varepsilon\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}\varepsilon^s|\lambda-2^\alpha\varepsilon^\alpha|\mathrm{d}\varepsilon\right)^{\frac{1}{q}}\end{array}\right]\end{array}\right\}.\end{aligned}\tag{51}$$

Also, we have that

$$\int_0^{\frac{1}{2}}\left|\gamma-2^\alpha\varepsilon^\alpha\right|\mathrm{d}\varepsilon=\begin{cases}\frac{1}{2}\left(\gamma-\frac{1}{\alpha+1}\right), & \gamma>1,\\[2ex] \frac{\alpha}{\alpha+1}\gamma^{\frac{\alpha+1}{\alpha}}-\frac{1}{2}\left(\gamma-\frac{1}{\alpha+1}\right), & 0\le\gamma\le 1,\quad \gamma\in\{\lambda,\mu\}.\end{cases}\tag{52}$$

Applying Lemma 3.4, Lemma 3.5 and the equality (52) to the inequality (51) produces the required conclusion, thereby concluding the proof of Theorem 3.19.

Corollary 3.20. *In Theorem 3.19, by choosing $\lambda = \mu$, we have the subsequent result*

$$\left|\Lambda_\Psi(\alpha,\mu;\kappa,\sigma)\right|^* \le \begin{cases} \left[\Psi^*(\kappa)\Psi^*(\sigma)\right]^{\frac{\sigma-\kappa}{2}\left(\eta_1(\alpha,\mu)\right)^{1-\frac{1}{q}}\left[\left(K_1(\alpha,s,\mu)\right)^{\frac{1}{q}}+\left(K_3(\alpha,s,\mu)\right)^{\frac{1}{q}}\right]}, & \mu > 1, \\ \left[\Psi^*(\kappa)\Psi^*(\sigma)\right]^{\frac{\sigma-\kappa}{2}\left(\eta_2(\alpha,\mu)\right)^{1-\frac{1}{q}}\left[\left(K_2(\alpha,s,\mu)\right)^{\frac{1}{q}}+\left(K_4(\alpha,s,\mu)\right)^{\frac{1}{q}}\right]}, & 0 \le \mu \le 1. \end{cases}$$

Furthermore, we have the following results:
(i) If $\mu = 0$, then we have the subsequent midpoint-type fractional inequality

$$\left|\Psi\left(\frac{\kappa+\sigma}{2}\right)\left[{}_*\mathcal{I}^\alpha_{\frac{\kappa+\sigma}{2}}\Psi(\kappa)\cdot {}_{\frac{\kappa+\sigma}{2}}\mathcal{I}^\alpha_*\Psi(\sigma)\right]^{-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\sigma-\kappa)^\alpha}}\right|^* \le \left[\Psi^*(\kappa)\Psi^*(\sigma)\right]^{\frac{\sigma-\kappa}{4}\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left[\left(\frac{1}{(\alpha+s+1)2^s}\right)^{\frac{1}{q}}+\left(2^{\alpha+1}B_{\frac{1}{2}}(\alpha+1,s+1)\right)^{\frac{1}{q}}\right]}.$$

(ii) If $\mu = 1$, then we have the subsequent trapezoid-type fractional inequality

$$\begin{aligned}&\left|\sqrt{\Psi(\kappa)\Psi(\sigma)}\left[{}_*\mathcal{I}^\alpha_{\frac{\kappa+\sigma}{2}}\Psi(\kappa)\cdot {}_{\frac{\kappa+\sigma}{2}}\mathcal{I}^\alpha_*\Psi(\sigma)\right]^{-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\sigma-\kappa)^\alpha}}\right|^* \\ &\quad\le \left[\Psi^*(\kappa)\Psi^*(\sigma)\right]^{\frac{\sigma-\kappa}{4}\left(\frac{\alpha}{\alpha+1}\right)^{1-\frac{1}{q}}\left[\left(\frac{1}{2^s}\left(\frac{1}{s+1}-\frac{1}{\alpha+s+1}\right)\right)^{\frac{1}{q}}+\left(\frac{2^{s+1}-1}{(s+1)2^s}-2^{\alpha+1}B_{\frac{1}{2}}(\alpha+1,s+1)\right)^{\frac{1}{q}}\right]}.\end{aligned}$$

4. Numerical examples

Theoretical conclusions are computationally verified in this section through 2-D and 3-D plots of the inequalities. The resulting plots demonstrate numerical tendencies, reinforcing both correctness and significance.

Example 4.1. Given the function $\Psi(\gamma) = \exp\left\{\gamma^{s+1}\right\}$ defined for $\gamma \in [0,\infty)$ with fixed $s \in (0,1]$, we can infer that the function $\Psi^*(\gamma) = \exp\{(s+1)\gamma^s\}$ is multiplicatively s-convex on $[0,\infty)$ with $s \in (0,1]$. By selecting $\kappa = 0, \sigma = 1$ and $\alpha \in (0,1]$, the assumptions of Theorem 3.6 are fulfilled.
(i) For $\lambda = 4$ and $\mu = 2$, it follows that

$$\begin{aligned}&\exp\left\{\left|1-\frac{1}{2^s}-\frac{\alpha}{\alpha+s+1}\cdot\frac{1}{2^{s+2}}-\alpha\cdot 2^{\alpha-1}B_{\frac{1}{2}}(\alpha,s+2)\right|\right\} \\ &\quad\le \exp\left\{\frac{1}{2^{s+1}}-\frac{s+1}{\alpha+s+1}\cdot\frac{1}{2^{s+2}}+1-(s+1)\cdot 2^{\alpha-1}B_{\frac{1}{2}}(\alpha+1,s+1)\right\}.\end{aligned} \tag{53}$$

(ii) For $\lambda = 1$ and $\mu = 3$, it follows that

$$\begin{aligned}&\exp\left\{\left|\frac{3}{2}-\frac{1}{2^{s+1}}-\frac{\alpha}{\alpha+s+1}\cdot\frac{1}{2^{s+2}}-\alpha\cdot 2^{\alpha-1}B_{\frac{1}{2}}(\alpha,s+2)\right|\right\} \\ &\quad\le \exp\left\{\frac{3}{2}-\frac{1}{2^{s+1}}-\frac{s+1}{\alpha+s+1}\cdot\frac{1}{2^{s+2}}-(s+1)\cdot 2^{\alpha-1}B_{\frac{1}{2}}(\alpha+1,s+1)\right\}.\end{aligned} \tag{54}$$

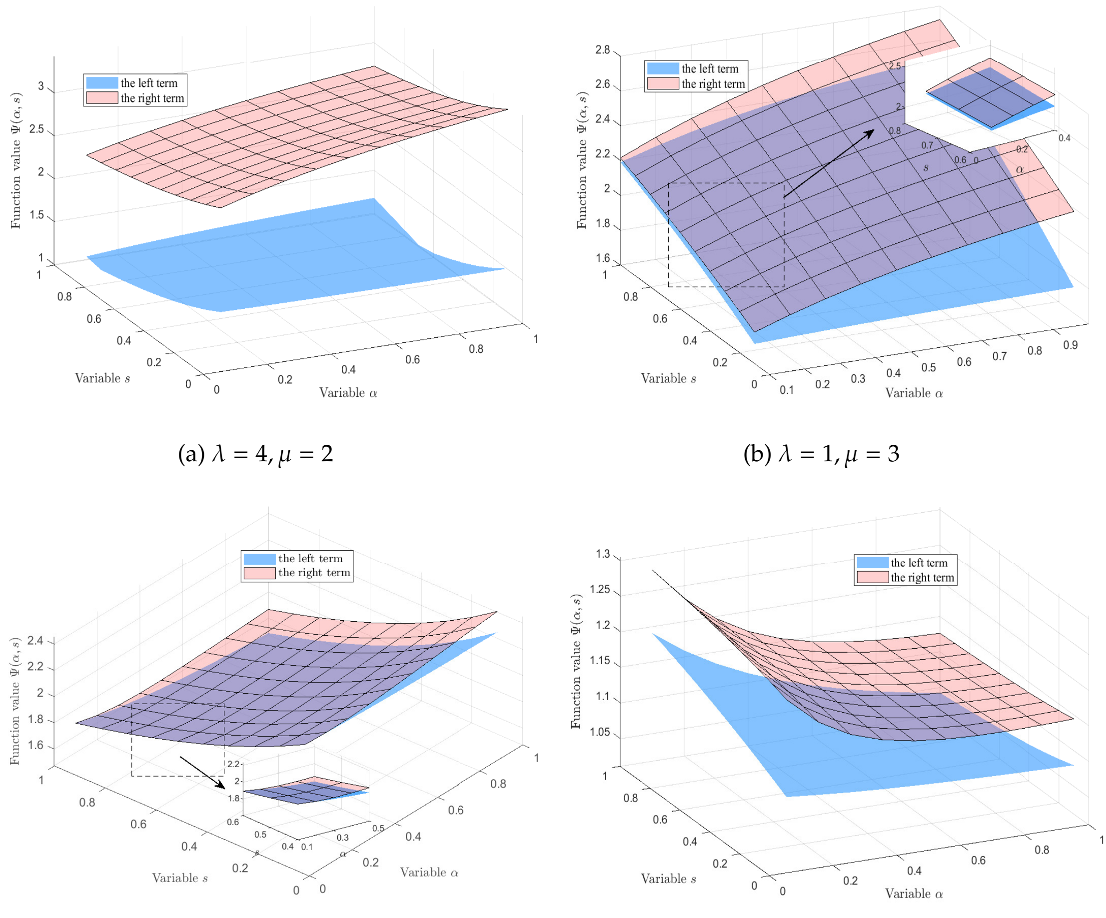
(iii) For $\lambda = 4$ and $\mu = \frac{1}{2}$, it follows that

$$\begin{aligned}&\exp\left\{\left|\frac{1}{4}-\frac{5}{4}\cdot\frac{1}{2^{s+1}}-\frac{\alpha}{\alpha+s+1}\cdot\frac{1}{2^{s+2}}-\alpha\cdot 2^{\alpha-1}B_{\frac{1}{2}}(\alpha,s+2)\right|\right\} \\ &\quad\le \exp\left\{\begin{array}{l}\frac{1}{2^{s+2}}\left(\frac{9}{2}-\frac{s+1}{\alpha+s+1}\right)+\frac{1}{4}\left[1-\frac{1}{2^s}\left(2-2^{-\frac{1}{\alpha}}\right)^{s+1}\right] \\ \quad+(s+1)\cdot 2^{\alpha-1}\left[B_{\frac{1}{2}}(\alpha+1,s+1)-2B_{2^{-\frac{\alpha+1}{\alpha}}}(\alpha+1,s+1)\right]\end{array}\right\}.\end{aligned} \tag{55}$$

(iv) For $\lambda = \frac{2}{3}$ and $\mu = \frac{1}{3}$, it follows that

$$\exp\left\{\left|\frac{1}{6} + \frac{1}{2^{s+2}} - \frac{\alpha}{\alpha+s+1}\cdot\frac{1}{2^{s+2}} - \alpha\cdot 2^{\alpha-1}B_{\frac{1}{2}}(\alpha, s+2)\right|\right\}$$
$$\leq \exp\left\{\begin{array}{c}\frac{1}{\alpha+s+1}\cdot\frac{1}{2^{s+2}}\left[2\alpha\cdot\left(\frac{2}{3}\right)^{\frac{\alpha+s+1}{\alpha}} + \frac{2}{3}(s+1) - \frac{1}{3}\alpha\right] + \frac{1}{6}\left[1 - \frac{1}{2^s}\left(2 - 3^{-\frac{1}{\alpha}}\right)^{s+1}\right] \\ +(s+1)\cdot 2^{\alpha-1}\left[B_{\frac{1}{2}}(\alpha+1, s+1) - 2B_{\frac{1}{2}\cdot 3^{-\frac{1}{\alpha}}}(\alpha+1, s+1)\right]\end{array}\right\}. \tag{56}$$

The comparative analysis in Fig. 1 shows that the left-sided values are clearly smaller than the right-sided values, thereby providing numerical support for Theorem 3.6.



(a) $\lambda = 4, \mu = 2$

(b) $\lambda = 1, \mu = 3$

(c) $\lambda = 4, \mu = \frac{1}{2}$

(d) $\lambda = \frac{2}{3}, \mu = \frac{1}{3}$

Figure 1: Numerical comparison of the inequalities (53)–(56) in Theorem 3.6 versus Example 4.1, illustrated via 3-D plot for $\alpha, s \in (0, 1]$.

Example 4.2. For $\gamma \in \mathbb{R}$, the hyperbolic functions are defined as $\sinh\gamma = \frac{1}{2}(e^{\gamma} - e^{-\gamma})$ and $\cosh\gamma = \frac{1}{2}(e^{\gamma} + e^{-\gamma})$. For $\Psi(\gamma) = \exp\{\sinh\gamma\}$ defined on $[0, \infty)$, it can be deduced that its multiplicatively differentiable function $\Psi^*(\gamma) = \exp\{\cosh\gamma\}$. Notably, the function $(\ln\Psi^*(\gamma))^q = (\cosh\gamma)^q$ possesses s-convexity on $[0, \infty)$ with

$q > 1$. By taking $\kappa = 0, \sigma = 1, p = \frac{3}{2}, q = 3$, $s = 1$ and $\alpha \in (0, 1]$, all hypotheses specified in Theorem 3.17 are fulfilled. Consequently, this simplifies the inequality stated in Theorem 3.17:

$$\begin{aligned}
&\left|\Lambda_{\Psi}(\alpha, \lambda, \mu; 0, 1)\right|^{*} \\
&= \exp\left\{\left|\left(1 - \frac{\lambda+\mu}{2}\right)\sinh\frac{1}{2} + \frac{\mu}{2}\sinh 1 - \alpha \cdot 2^{\alpha-1}\int_0^{\frac{1}{2}} \varepsilon^{\alpha-1}\Big[\sinh\varepsilon + \sinh(1-\varepsilon)\Big]\mathrm{d}\varepsilon\right|\right\} \\
&\le \left[\Psi^{*}(0)\right]^{\frac{1}{2}C_1(\lambda,\mu,\alpha,1;\frac{3}{2},3)}\left[\Psi^{*}(1)\right]^{\frac{1}{2}C_2(\lambda,\mu,\alpha,1;\frac{3}{2},3)} \\
&= \exp\left\{\begin{aligned}&\frac{1}{4}\left[\left(\int_0^{\frac{1}{2}}\left|\mu - 2^{\alpha}\varepsilon^{\alpha}\right|^{\frac{3}{2}}\mathrm{d}\varepsilon\right)^{\frac{2}{3}} + \sqrt[3]{3}\left(\int_0^{\frac{1}{2}}\left|\lambda - 2^{\alpha}\varepsilon^{\alpha}\right|^{\frac{3}{2}}\mathrm{d}\varepsilon\right)^{\frac{2}{3}}\right] \\
&+\frac{\cosh 1}{4}\left[\left(\int_0^{\frac{1}{2}}\left|\lambda - 2^{\alpha}\varepsilon^{\alpha}\right|^{\frac{3}{2}}\mathrm{d}\varepsilon\right)^{\frac{2}{3}} + \sqrt[3]{3}\left(\int_0^{\frac{1}{2}}\left|\mu - 2^{\alpha}\varepsilon^{\alpha}\right|^{\frac{3}{2}}\mathrm{d}\varepsilon\right)^{\frac{2}{3}}\right]\end{aligned}\right\}.
\end{aligned} \tag{57}$$

From the numerical comparison in Tabs. 1–4 and Figs. 2–5, it is clear that left-side values are smaller than right-side values, confirming Theorem 3.17 numerically.

Table 1: Numerical comparison of the inequality (57) in Theorem 3.17 with $\lambda = 3$ and $\mu = 2$

α	the left term	the right term
0.1	1.2028	4.6571
0.2	1.1946	5.0254
0.3	1.1875	5.3609
0.4	1.1817	5.6660
0.5	1.1768	5.9438
0.6	1.1727	6.1972
0.7	1.1692	6.4289
0.8	1.1662	6.6415
0.9	1.1636	6.8369
1.0	1.1613	7.0172

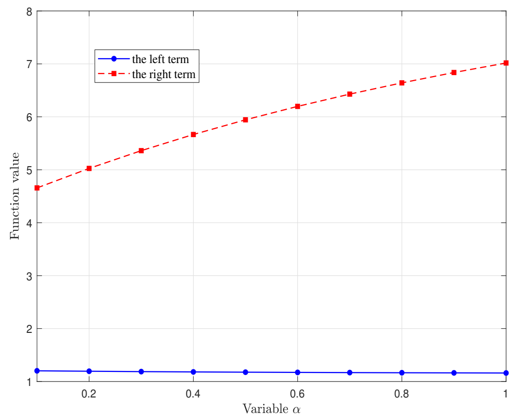


Figure 2: Numerical comparison of the inequality (57) in Theorem 3.17 versus Example 4.2, illustrated via 2-D plot across $\alpha \in (0, 1]$ with $\lambda = 3$ and $\mu = 2$

Table 2: Numerical comparison of the inequality (57) in Theorem 3.17 with $\lambda = \frac{3}{5}$ and $\mu = 4$

α	the left term	the right term
0.1	2.9884	5.5774
0.2	3.0090	5.6558
0.3	3.0269	5.7743
0.4	3.0418	5.9168
0.5	3.0544	6.0696
0.6	3.0651	6.2239
0.7	3.0743	6.3748
0.8	3.0823	6.5194
0.9	3.0893	6.6567
1.0	3.0953	6.7862

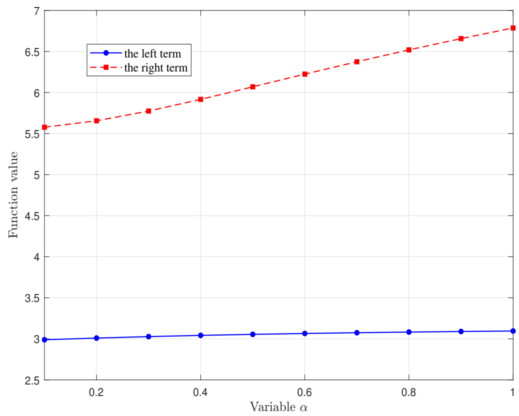


Figure 3: Numerical comparison of the inequality (57) in Theorem 3.17 versus Example 4.2, illustrated via 2-D plot across $\alpha \in (0, 1]$ with $\lambda = \frac{3}{5}$ and $\mu = 4$

Table 3: Numerical comparison of the inequality (57) in Theorem 3.17 with $\lambda = 2$ and $\mu = \frac{2}{3}$

α	the left term	the right term
0.1	1.4336	1.8987
0.2	1.4238	1.9277
0.3	1.4154	1.9777
0.4	1.4084	2.0378
0.5	1.4026	2.1010
0.6	1.3977	2.1634
0.7	1.3935	2.2233
0.8	1.3899	2.2799
0.9	1.3868	2.3330
1.0	1.3841	2.3825

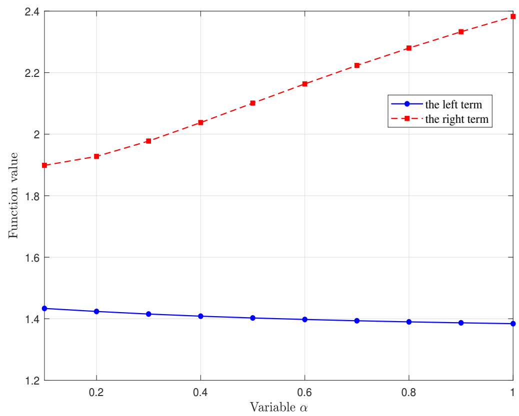


Figure 4: Numerical comparison of the inequality (57) in Theorem 3.17 versus Example 4.2, illustrated via 2-D plot across $\alpha \in (0, 1]$ with $\lambda = 2$ and $\mu = \frac{2}{3}$

Table 4: Numerical comparison of the inequality (57) in Theorem 3.17 with $\lambda = \frac{1}{2}$ and $\mu = \frac{1}{3}$

α	the left term	the right term
0.1	1.0815	1.6299
0.2	1.0741	1.5294
0.3	1.0678	1.4605
0.4	1.0626	1.4139
0.5	1.0582	1.3823
0.6	1.0545	1.3610
0.7	1.0513	1.3468
0.8	1.0486	1.3376
0.9	1.0462	1.3318
1.0	1.0442	1.3284

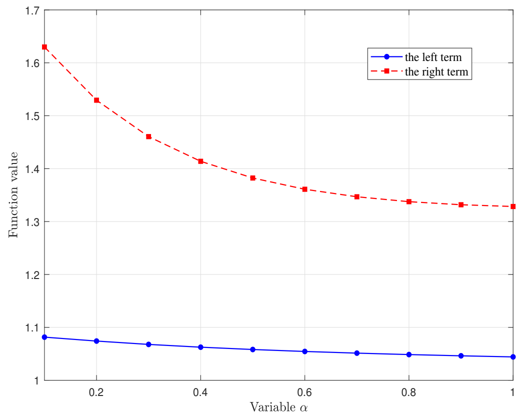


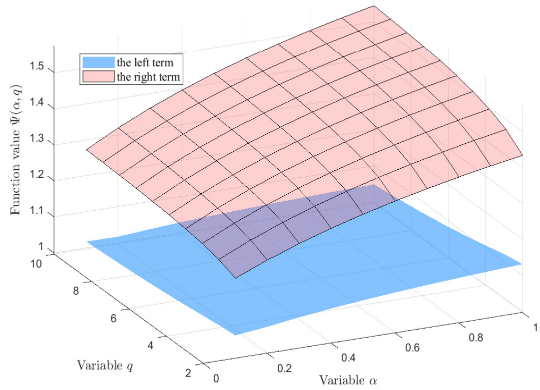
Figure 5: Numerical comparison of the inequality (57) in Theorem 3.17 versus Example 4.2, illustrated via 2-D plot across $\alpha \in (0,1]$ with $\lambda = \frac{1}{2}$ and $\mu = \frac{1}{3}$

Example 4.3. Consider the function $\Psi(\gamma) = \exp\left\{\frac{q}{q+3}\gamma^{\frac{q+3}{q}}\right\}$ defined for $\gamma \in [0,\infty)$ with $q > 1$. We deduce its multiplicatively differentiable function $\Psi^*(\gamma) = \exp\left\{\gamma^{\frac{3}{q}}\right\}$. Additionally, $(\ln \Psi^*(\gamma))^q = \gamma^3$ is shown to be s-convex for $\gamma \geq 0$ and $q > 1$. As a result, all requirements of Theorem 3.19 are met. Fixing $\kappa = 0, \sigma = 1$ and $s = \frac{1}{2}$ across $\alpha \in (0,1]$, we examine four parameter configurations: (i) $\lambda = \frac{3}{2}, \mu = \frac{4}{3}$; (ii) $\lambda = 1, \mu = 2$; (iii) $\lambda = \frac{5}{3}, \mu = \frac{2}{3}$; (iv) $\lambda = \mu = \frac{1}{3}$. As demonstrated in Fig. 6, the left-hand values are always smaller than the right-hand values, thereby offering numerical support for Theorem 3.19.

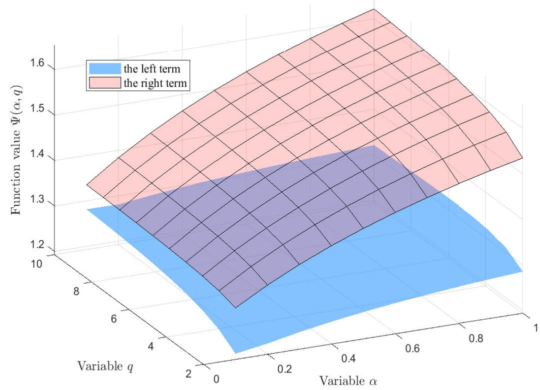
5. Conclusions

This study establishes multi-parameter inequalities for multiplicatively s-convex functions using multiplicative RL-fractional integrals. By deriving a two-parameter identity for multiplicatively differentiable functions, we obtain multiplicative integral inequalities. Under certain conditions, we can further optimize the upper bounds for the midpoint and trapezoidal inequalities developed in this paper. Our work provides a unified framework for multiple inequality types—including midpoint-, Simpson-, Bullen-, trapezoid- and Milne-type inequalities—advancing the theory of multiplicative calculus, particularly multiplicative fractional integrals.

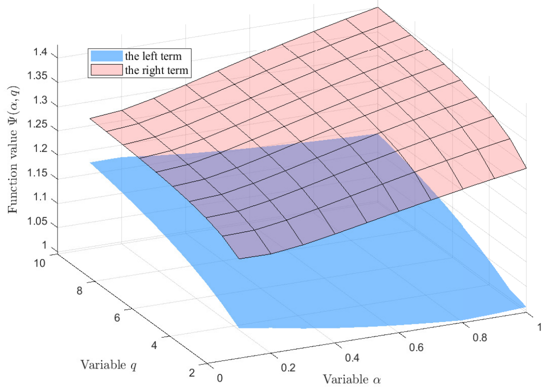
Beyond the multiplicative RL-fractional integrals, future research may focus on multi-parameter inequalities for other types of multiplicative fractional integrals, as exemplified by multiplicative Katugampola fractional integrals [3], multiplicative ψ-Hilfer fractional integrals [48], and multiplicative tempered fractional integrals [39]. These directions could advance multiplicative fractional calculus theory.



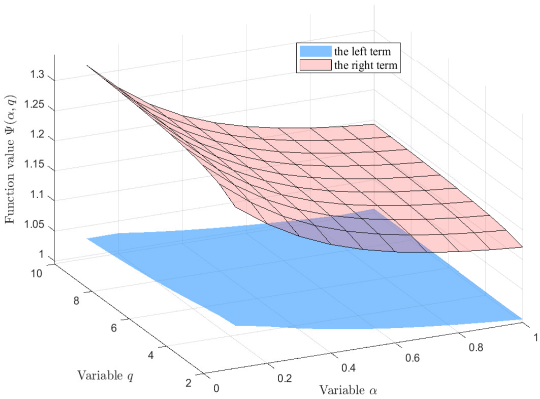
(a) $\lambda = \frac{3}{2}, \mu = \frac{4}{3}$



(b) $\lambda = 1, \mu = 2$



(c) $\lambda = \frac{5}{3}, \mu = \frac{2}{3}$



(d) $\lambda = \frac{1}{3}, \mu = \frac{1}{3}$

Figure 6: Numerical comparison of the Example 4.3, illustrated via 3-D plot for $\alpha \in (0, 1]$ and $q \in [2, 10]$

References

[1] T. Abdeljawad, M. Grossman, On geometric fractional calculus, *J. Semigroup Theory Appl.*, **2016** (2016) 2.

[2] M. A. Ali, H. Budak, M. Z. Sarikaya, Z. Y. Zhang, *Ostrowski and Simpson type inequalities for multiplicative integrals*, Proyecciones, **40** (3) (2021) 743–763.

[3] M. A. Ali, Fečkan, C. Promsakon, T. Sitthiwirattham, *A new approach of generalized fractional integrals in multiplicative calculus and related HermiteHadamard-type inequalities with applications*, Math. Slovaca, **74** (6) (2024) 1445–1456.

[4] M. B. Almatrafi, W. Saleh, A. Lakhdari, F. Jarad, B. Meftah, *On the multiparameterized fractional multiplicative integral inequalities*, J. Inequal. Appl., **2024** (2024) 52.

[5] A. E. Bashirov, E. M. Kurpınar, A. Özyapıcı, *Multiplicative calculus and its applications*, J. Math. Anal. Appl., **337** (2008) 36–48.

[6] H. Boulares, B. Meftah, A. Moumen, R. Shafqat, H. Saber, T. Alraqad, E. E. Ali, *Fractional multiplicative Bullen-type inequalities for multiplicative differentiable functions*, Symmetry, **15** (2) (2023) 451.

[7] H. Budak, K. Özçelik, *On Hermite–Hadamard type inequalities for multiplicative fractional integrals*, Miskolc Math. Notes, **21** (1) (2020) 91–99.

[8] H. Budak, B. B. Ergün, *On multiplicative conformable fractional integrals: theory and applications*, Bound. Value Probl., **2025** (1) (2025) 30.

[9] A. Berkane, B. Meftah, A. Lakhdari, *Right-Radau-type inequalities for multiplicative differentiable s-convex functions*, J. Appl. Math. Informatics, **42** (4) (2024) 785–800.

[10] B. Benaissa, N. Azzouz, *Hermite–Hadamard–Fejér type inequalities for h-convex functions involving ψ-Hilfer operators*, J. Inequal. Math. Anal., **1** (2) (2025) 113–123.

[11] S. I. Butt, A. Khan, *New fratal-fractional parametric inequalities with applications*, Chaos Solitons Fractals, **172** (2023) 113529.

[12] S. I. Butt, P. Agarwal, J. J. Nieto, *New Hadamard–Mercer inequalities pertaining Atangana–Baleanu operator in Katugampola sense with applications*, Mediterr. J. Math., **21** (2024) 9.

[13] S. Chasreechai, M. A. Ali, S. Naowarat, T. Sitthiwirattham, K. Nonlaopon, *On some Simpson's and Newton's type of inequalities in multiplicative calculus with applications*, AIMS Math., **8** (2) (2023) 3885–3896.

[14] T. S. Du, Y. Long, *The multi-parameterized integral inequalities for multiplicative Riemann–Liouville fractional integrals*, J. Math. Anal. Appl., **541** (2025) 128692.

[15] T. S. Du, Y. Long, J. G, Liao, *Multiplicative fractional HH-type inequalities via multiplicative AB-fractional integral operators*, J. Comput. Appl. Math., **474** (2026) 116970.

[16] H. Fu, Y. Peng, T. S. Du, *Some inequalities for multiplicative tempered fractional integrals involving the λ-incomplete gamma functions*, AIMS Math., **6** (7) (2021) 7456–7478.

[17] A. Frioui, B. Meftah, A. Shokri, A. Lakhdari, H. Mukalazi, *Parametrized multiplicative integral inequalities*, Adv. Continuous Disc. Models, **2024** (2024) 12.

[18] M. Grossman, R. Katz, Non-Newtonian calculus, Lee press, Pigeon Cove, MA, 1972.

[19] F. Hezenci, H. Budak, *An extensive study on parameterized inequalities for conformable fractional integrals*, Anal. Math. Phys., **13** (5) (2023) 82.

[20] H. Hudzik, L. Maligranda, *Some remarks on s-convex functions*, Aequationes Math, **48** (1994) 100–111.

[21] A. Kashuri, S. K. Sahoo, M. Aljuaid, M. Tariq, M. D. L. Sen, *Some new Hermite–Hadamard type inequalities pertaining to generalized multiplicative fractional integrals*, Symmetry, **15** (2023) 868.

[22] S. Khan, H. Budak, *On midpoint and trapezoid type inequalities for multiplicative integrals*, Mathematica, **64** (87) (2022) 95–108.

[23] D. Khan, S. I. Butt, Y. Seol, *Analysis on multiplicatively (P, m)-superquadratic functions and related fractional inequalities with applications*, Fractals, **33** (3) (2025) 2450129.

[24] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, Theory and Applications of Fractional Differentiable Equations, North-Holland Mathematics Studies., **204**, Elsevier, Amsterdam (2006).

[25] A. Lakhdari, D. C. Benchettah, B. Meftah, *Fractional multiplicative Newton-type inequalities for multiplicative s-convex positive functions with application*, J. Comput. Appl. Math., **456** (2025) 116600.

[26] Y. Long, T. S. Du, *Analysis on multiplicative k-Atangana–Baleanu fractional integrals with application to various Mercer-type inequalities*, Fractals, **33** (1) (2025) 2550003.

[27] C. E. T. Ledesma, H. A. C. Gutierrez, J. P. A. Rodriguez, W. Z. Vera, *Some boundedness results for Riemann–Liouville tempered fractional integrals*, Fract. Calc. Appl. Anal., **27** (2024) 818–847.

[28] C. Y. Luo, B. Yu, Y. Zhang, T. S. Du, *Certain bounds related to multi-parameterized k-fractional integral inequalities and their applications*, IEEE Access, **7** (2019) 124662–124673.

[29] A. Mateen, Z. Y. Zhang, M. Toseef, M. A. Ali, *A new version of Boole's formula type inequalities in multiplicative calculus with application to quadrature formula*, Bull. Belg. Math. Soc. Simon Stevin, **31** (4) (2024) 541–562.

[30] A. Mateen, Z. Y. Zhang, S. Özcan, M. A. Ali, *Generalization of Hermite–Hadamard, trapezoid, and midpoint Mercer type inequalities for fractional integrals in multiplicative calculus*, Bound. Value Probl., **2025** (1) (2025) 22.

[31] M. Merad, B. Meftah, H. Boulares, A. Moumen, M. Bouye, *Fractional Simpson-like inequalities with parameter for differential s-tgs-convex functions*, Fractal Fract., **7** (2023) 772.

[32] B. Meftah, *Maclaurin type inequalities for multiplicatively convex functions*, Proc. Amer. Math. Soc., **151** (5) (2023) 2115–2125.

[33] B. Meftah, A. Lakhdari, *Dual Simpson type inequalities for multiplicatively convex functions*, Filomat, **37** (22) (2023) 7673–7683.

[34] B. Meftah, D. C. Benchettah, A. Lakhdari, *Some new local fractional Newton-type inequalities*, J. Inequal. Math. Anal., **1** (2) (2025) 79–96.

[35] A. Moumen, H. Boulares, B. Meftah, R. Shafqat, T. Alraqad, E. E. Ali, Z. Khaled, *Multiplicatively Simpson type inequalities via fractional integral*, Symmetry, **15** (2023) 460.

[36] C. Niculescu, L.E. Persson, Convex Functions and Their Applications, New York: Springer, 2006.

[37] N. Nasri, B. Meftah, A. Moumen, H. Saber, *Fractional 3/8-Simpson type inequalities for differentiable convex functions*, AIMS Math., **9** (3) (2024) 5349–5375.

[38] S. Özcan, *Simpson, midpoint, and trapezoid-type inequalities for multiplicatively s-convex functions*, Demonstr. Math., **58** (1) (2025) 20240060.

[39] Y. Peng, T. S. Du, *Hermite–Hadamard-type inequalities for *differentiable multiplicative m-preinvexity and (s, m)-preinvexity via the multiplicative tempered fractional integrals*, J. Math. Inequal., **17** (3) (2023) 1179–1201.

[40] Y. Peng, T. S. Du, *Fractional Maclaurin-type inequalities for multiplicatively convex functions and multiplicatively P-functions*, Filomat, **37** (28) (2023) 9497–9509.

[41] Y. Peng, H. Fu , T. S. Du, *Estimations of bounds on the multiplicative fractional integral inequalities having exponential kernels*, Commun. Math. Stat., **12** (2) (2024) 187–211.

[42] Y. F. Qi, G. P. Li, *Katugampola fractional parameterized inequalities for s-convex function in the fourth sense*, Fractals, **32** (6) (2024) 2450108.

[43] E. D. Rainville, Special Functions, Reprint of 1960 first edition. Chelsea Publishing Co., Bronx, N.Y. 1971.

[44] W. Saleh, B. Meftah, M. U. Awan, A. Lakhdari, *On Katugampola fractional multiplicative Hermite–Hadamard-type inequalities*, Mathematics, **13** (2025) 1575.

[45] H. M. Srivastava, S. K. Sahoo, P. O. Mohammed, A. Kashuri, N. Chorfi, *Results on Minkowski-type inequalities for weighted fractional integral operators*, Symmetry, **2023** (15) (2023) 1522.

[46] T. Sitthiwirattham, M. Vivas–Cortez, M. A. Ali, H. Budak, I. Avci, *A study of fractional Hermite–Hadamard–Mercer inequalities for differentiable functions*, Fractals, **32** (2) (2024) 2440016.

[47] M. Toseef, A. Mateen, M. A. Ali, Z. Y. Zhang, *A family of quadrature formulas with their error bounds for convex functions and applications*, Math. Meth. Appl. Sci., **3** (48) (2025) 2766–2783.

[48] Z. R. Tan, T. S. Du, *Analytical properties on multiplicatively $M_\psi A$-p-functions and related multiplicative ψ-Hilfer fractional trapezoid-type inequalities*. Fractals, **33** (2025) 2550059.
[49] M. Umar, S. I. Butt, Y. Seol, *Hybrid fractional integral inequalities in multiplicative calculus with applications*, Fractals, **33** (1) (2025) 2550019.
[50] C. Unal, F. Hezenci, H. Budak, *Some remarks on parameterized inequalities involving conformable fractional operators*, Turk. J. Math., **47** (2023) 590–607.
[51] B. Y. Xi, F. Qi, *Some integral inequalities of Hermite–Hadamard type for s-logarithmically convex functions*, Acta Math. Sci. Ser. B, **35A** (3) (2015) 515-526.
[52] B. Y. Xi, F. Qi, *Some integral inequalities of Hermite–Hadamard type for convex functions with applications to means*, J. Funct. Space Appl., **2012** (2012) 980438.
[53] J. Q. Xie, M. A. Ali, T. Sitthiwirattham, *Some new midpoint and trapezoidal type inequalities in multiplicative calculus with applications*, Filomat, **37** (20) (2023) 6665–6675.
[54] X. M. Yuan, H. Budak, T. S. Du, *The multi-parameter fractal-fractional inequalities for fractal (P, m)-convex functions*, Fractals, **32** (1) (2024) 2450025.
[55] D. F. Zhao, M. A. Ali, C. Promsakon, T. Sitthiwirattham, *Some generalized fractional integral inequalities for convex functions with applications*, Fractal Fract., **6** (2022) 64.
[56] X. H. Zhang, Y. Peng, T. S. Du, *(k, s)-fractional integral operators in multiplicative calculus*, Chaos Solitons Fractals, **195** (2025) 116303.
[57] Y. Zhang, W. B. Sun, *On general local fractional integral inequalities for generalized h-preinvex functions on Yang's fractal sets*, Fractals, **32** (4) (2024) 2440025.
[58] Z. Y. Zhou, T. S. Du, *Analytical properties and related inequalities derived from multiplicative Hadamard k-fractional integrals*, Chaos Solitons Fractals, **189** (2024) 115715.
[59] Z. Y. Zhou and T. S. Du, *Error estimation for a broad class of fractional multiplicative quadrature formulas*, Exp. Math., (2025), 1-23, https://doi.org/10.1080/10586458.2025.2519603.
[60] W. S. Zhu, B. Meftah, H. Y. Xu, F. Jarad, A. Lakhdari, *On parameterized inequalities for fractional multiplicative integrals*, Demonstr. Math., **57** (1) (2024) 20230155.