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Ideal deferred statistical convergence of multisequencesc

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Abstract. This paper introduces the concept of deferred statistical convergence of multisequences, extending classical statistical convergence methods to a framework that accommodates repeated elements in multisets. Multisequences, which allow repeated elements within a set, are widely applicable in various fields such as computer science, chemistry, and telecommunications.

The study revisits foundational concepts, including ideal convergence, deferred mean, and statistical convergence, providing a thorough theoretical framework. Main definitions and properties of deferred statistical convergence for multisequences are presented, followed by the introduction of ideal deferred statistical limit superior and inferior, which extend the classical notions of limit points.

Furthermore, the paper establishes several inclusion theorems, demonstrating the relationships between ideal deferred statistical convergence and strong summability within the context of multisequences. Special cases are analyzed under specific conditions, offering insights into the behavior of multisequences. Moreover, the paper introduces original definitions and results regarding ideal deferred statistical cluster and limit points, as well as ideal deferred statistical limit supremum and infimum.

By bridging the gap between classical convergence theories and multisequence analysis, this work provides a new perspective on the study of multisequences and their convergence properties, encouraging further research in this evolving field.

1. Introduction and background

In classical theory, each element in a set is listed only once, which does not reflect the real world. In real life, an element may appear more than once in a set. For example, there are 4 Mondays, 4 Tuesdays, 5 Wednesdays, 5 Thursdays, 5 Fridays, 4 Saturdays and 4 Sundays in January 2025. Therefore, the set {Monday, Tuesday, Wednesday, Thursday, Friday, Saturday, Sunday} cannot represent January 2025. This type of repetition is important in our daily lives and particularly prevalent in various contexts, such as computer programming, where specific code elements may need to appear multiple times to ensure functionality. Similarly, in chemical formulas, certain elements are repeated to represent the accurate composition of compounds. Additionally, in phone numbers, digits are often repeated to complete the sequence necessary for communication. If we use two bits to represent 'a b c d' the code can be '00 01 10 11'. Another illustrative example is given by Pachilangode and John in [21]: " The prime factorizes n completely, and let $\mathscr{F}_n$ be the sets of these factors, including 1. Then, $\mathscr{F}_1 = \{1\}$, $\mathscr{F}_2 = \{1, 2\}$, $\mathscr{F}_4 = \{1, 2, 2\}$, $\mathscr{F}_5 = \{1, 5\}$, $\mathscr{F}_{20} = \{1, 2, 2, 5\}$ and $\mathscr{F}_{100} = \{1, 2, 2, 5, 5\}$."

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A multiset of real numbers is a collection where elements can appear multiple times. For example,

$$\{1, 1, \sqrt{2}, \frac{1}{2}, \frac{1}{2}, \sqrt{2}, \sqrt{2}, 1, 1\}$$

is a multiset of real numbers. In multisets, the order in which elements are written is not important. However, the number of times each element is repeated in the set is highly significant. To simplify notation, elements in a multiset are denoted as "element|number of repetitions". So, the multiset $\{1, 1, \sqrt{2}, \frac{1}{2}, \frac{1}{2}, \sqrt{2}, \sqrt{2}, 1, 1\}$ denoted by $\{1|4, \sqrt{2}|3, \frac{1}{2}|2\}$. Multiset of real numbers denoted by $m\mathbb{R} := \{x_i|m_i : x_i \in \mathbb{R} \text{ and } m_i \in \mathbb{Z}^+\}$.

Pachilangode and John introduced the concept of multiset sequences and their convergence in the Wijsman sense. They also provided impressive examples of multiset sequences in 2021 [21]. In the same year, Debnath and Debnath defined the multisequences and provided the definition of statistical convergence of multisequences [8]. They also generalized some results from Fridy [13] and Fridy and Orhan [14]. Demir and Gümüş introduced the ideal convergence of multisequences in [9], considering the definition of multisequences from [8]. Also they investigated some basic algebraic and topological properties of multisequences. Gümüş et al. defined the lacunary statistical convergence of multisequences and they investigated some related results in [15].

Statistical convergence has a considerable range of studies in the field of mathematical analysis [12], [13], [22], [25]. Deferred statistical convergence provides a refined approach that extends Cesàro means and statistical convergence [1], [2], [11], [19]. Şengül et al. [23] extended the concept of ideal statistical convergence by incorporating the notion of deferred mean, thereby introducing the idea of ideal deferred statistical convergence. Furthermore, several subsequent studies have investigated various aspects of this concept [4], [5], [7], [24].

Nowadays, the study of statistical convergence has expanded beyond traditional sequences to encompass more complex structures such as multisequences and multisets [8], [9], [15], [21]. These developments have sparked new avenues of exploration, particularly in extending classical convergence concepts to accommodate the rich nature of multisets, where elements may repeat and possess additional structures.

This paper focuses on the concept of ideal deferred statistical convergence in the context of multisequences, providing a thorough examination of its definitions and exploring various inclusion theorems that bridge the gap between strong summability and statistical convergence in this more generalized framework. Building on the work of previous researchers in this field, we aim to contribute to the understanding of multiset sequences and their convergence properties, establishing connections between different types of ideal deferred statistical convergence and their implications.

2. Definitions and preliminaries

In this section, we first provide some fundamental definitions and results about ideal, ideal convergence, deferred mean, deferred convergence etc. which are already in literature and useful for our study. Also we mention some mathematical results about this concept. Then we recall the definitions and results about multisequences.

2.1. *Ideal convergence*

Ideal convergence is one of the most popular generalized forms of convergence. Kostyroko et al. [16] introduced this concept to the literature. Ideal statistical convergence was defined by Das et al. in [6].

The statistical limit superior and inferior concepts were studied by Fridy and Orhan [14]. Demirci generalized statistical limit superior and inferior using ideals [10]. Mursaleen et al. defined ideal statistical limit superior and limit inferior [20]. Also, Altinok and Kucukaslan gave an effectice result between ideal limit supremum-infimum and ideal convergence in [3].

Now, let us recall some relevant definitions related to ideal convergence.

Definition 2.1. A collection $I \subset 2^{\mathbb{N}}$ is called an ideal if the following conditions hold:
(i) $\mathscr{A}, \mathscr{B} \in I$ implies that $\mathscr{A} \cup \mathscr{B} \in I$,
(ii) $\mathscr{A} \in I, \mathscr{B} \subset \mathscr{A}$ implies that $\mathscr{B} \in I$.

I is non-trivial if $\mathbb{N} \notin I$ and I is admissible if $\{n\} \in I$ for each $\mathbb{N}$.

Definition 2.2. A collection $F \subset 2^{\mathbb{N}}$ is called a filter if the following conditions hold:
(i) $\mathscr{A}, \mathscr{B} \in F$ implies that $\mathscr{A} \cap \mathscr{B} \in F$,
(ii) $\mathscr{A} \in F, \mathscr{A} \subset \mathscr{B} \subset \mathbb{N}$ implies that $\mathscr{B} \in F$.

Definition 2.3. ([16]) Let $x = (x_i)$ be a real valued sequence. $x = (x_i)$ is ideal convergent to $x_0 \in \mathbb{R}$ (denoted by $I - \lim x_i = x_0$) if and only if for every $\varepsilon > 0$ the set $\{n \in \mathbb{N} : |x_i - x_0| \geq \varepsilon\}$ belongs to ideal (I).

Definition 2.4. ([6]) Let $x = (x_i)$ be a real valued sequence. $x = (x_i)$ is ideal statistical convergent to $x_0 \in \mathbb{R}$ (denoted by $I - \mathfrak{S} \lim x_i = x_0$) if for each $\varepsilon > 0$ and $\delta > 0$ the set $\{n \in \mathbb{N} : \frac{1}{n}|\{i \leq n : |x_i - x_0| \geq \varepsilon\}| \geq \delta\}$ belongs to ideal (I).

The reason for the popularity of ideal convergence is that many convergences can be obtained with the specially chosen ideals. Now we will give some examples for some special ideals (more examples are in [17]).

Let I_f be the set of all finite subsets of natural numbers. Then, I_f-convergence is the usual convergence.

Let $I_d = \{A \subset \mathbb{N} : d(A) = 0\}$, where d is the natural density. Then, I_d-convergence is statistical convergence.

Definition 2.5. ([10]) Let $x = (x_i)$ be a real valued sequence and I be an ideal. Ideal limit superior and ideal limit inferior of $x = (x_i)$ defined as follows:

$$I - \limsup x_i = \begin{cases} \sup U_x, & U_x \neq \emptyset, \\ -\infty, & U_x = \emptyset. \end{cases}$$

and

$$I - \liminf x_i = \begin{cases} \inf L_x, & L_x \neq \emptyset, \\ +\infty, & L_x = \emptyset. \end{cases}$$

where $U_x = \{b \in R : \{i \in \mathbb{N} : x_i > b\} \notin I\}$, $L_x = \{a \in R : \{i \in \mathbb{N} : x_i < a\} \notin I\}$.

2.2. *Deferred convergence*

The concept of deferred statistical convergence was introduced by Küçükaslan and Yılmaztürk [19] in 2016. They used deferred Cesàro mean which was given by Agnew [1] in 1932. There are several papers about this concept such as [2, 11], etc.

Definition 2.6. ([1]) Deferred Cesàro mean of $x = (x_k)$ (real valued sequence) is

$$\mathfrak{D}(\mathfrak{p}, \mathfrak{q})(x)_n := (\mathfrak{q} - \mathfrak{p})^{-1} \sum_{k=\mathfrak{p}+1}^{\mathfrak{q}} x_k, \ \ n = 1, 2, ...,$$

where $\mathfrak{p} = \{\mathfrak{p}(n)\}_{n \in \mathbb{N}}$ and $\mathfrak{q} = \{\mathfrak{q}(n)\}_{n \in \mathbb{N}}$ (for brevity $\mathfrak{p}$ and $\mathfrak{q}$, respectively) are the sequences of positive integers satisfying

$$0 \leq \mathfrak{p} < \mathfrak{q} \text{ and } \mathfrak{q} \to \infty \text{ when } n \to \infty. \quad (1)$$

A sequence $x = (x_k)$ is deferred strongly summable to $x_0 \in \mathbb{R}$ (denoted by $\mathfrak{D} \lim x_n = x_0$) if the following ratio

$$(\mathfrak{q} - \mathfrak{p})^{-1} \sum_{k=\mathfrak{p}+1}^{\mathfrak{q}} |x_k - x_0|$$

tends to zero when $n \to \infty$.

Definition 2.7. ([19]) A sequence $x = (x_k)$ is deferred statistical convergent to x_0 (denoted by $\mathfrak{D}\mathfrak{S} \lim x_n = x_0$) if

$$\lim_{n\to\infty} (\mathfrak{q} - \mathfrak{p})^{-1} |\{k : \mathfrak{p} < k \leq \mathfrak{q}, |x_k - x_0| \geq \varepsilon\}| = 0$$

holds.

2.3. Multisequences

In our daily lives, we frequently encounter situations where an element of a set must be repeated multiple times within that set. This is particularly common in various contexts, such as computer programming, where specific code elements may need to appear multiple times to ensure functionality. Similarly, in chemical formulas, certain elements are repeated to accurately represent the composition of compounds. Additionally, in phone numbers, digits are often repeated to complete the sequence necessary for communication. These examples illustrate the necessity and practicality of repeating elements in different settings. For this reason, it is clear that this concept will be popular in mathematics.

Definition 2.8. ([8]) A sequence whose range is a set of $m\mathbb{R}$ is called multisequence. A multisequence $mx = (x_i|m_i)$ is defined as $mx := \{x_i|m_i : x_i \in \mathbb{R}$ and $m_i \in \mathbb{Z}^+\}$ such that $x = (x_i)$ is a real valued sequence.

Due to the repetition of elements in a multiset, it is necessary to define a new metric space. Let M be a multiset and the metric $d_M : M \times M \to [0, \infty)$ defined as

$$d_M(mx, my) = d_M(x_i|m_i^1, y_i|m_i^2) = \left((x_i - y_i)^2 + (m_i^1 - m_i^2)^2\right)^{1/2}$$

for each $i \in \mathbb{N}$.

Definition 2.9. ([9]) A multisequence $mx = (x_i|m_i)$ is convergent to $x_0|m$ (denoted by $\lim x_i|m_i = x_0|m$) if

$$\lim_{i\to\infty} \left((x_i - x_0)^2 + (m_i - m)^2\right)^{1/2} = 0$$

holds.

In this case, for any $\varepsilon > 0$, it is clear from definition that $|x_i - x_0| < \varepsilon$ and $|m_i - m| < \varepsilon$ hold.

Definition 2.10. ([9]) A multisequence $mx = (x_i|m_i)$ is ideal convergent to $x_0|m$ (denoted by $I - \lim x_i|m_i = x_0|m$) if for each $\varepsilon > 0$

$$\{i \in \mathbb{N} : \left((x_i - x_0)^2 + (m_i - m)^2\right)^{1/2} \geq \varepsilon\} \in I.$$

If we consider $I = I_d$, then $mx = (x_i|m_i)$ is called statistical convergent to $x_0|m$ (denoted by $\mathfrak{S} - \lim x_i|m_i = x_0|m$) [8].

3. Main results

In this section, the core contributions of the paper are presented, focusing on the definitions and properties of ideal deferred strongly summability for multisequences and ideal deferred statistical convergence for multisequences. Main inclusion theorems are established, demonstrating the relationships between ideal deferred statistical convergence and their implications. Additionally, the section explores special cases obtained by specific choices of the deferred parameters $\mathfrak{p}$ and $\mathfrak{q}$ highlighting their influence on convergence behavior. The properties of ideal deferred statistical limit and cluster points are thoroughly examined, with theoretical insights supported by concrete examples. A significant focus is placed on the ideal deferred statistical limit superior and limit inferior, where new definitions and propositions are introduced to characterize the extremal behavior of multisequences under deferred statistical convergence. The relationships between these new concepts and their role in determining the boundedness and convergence of multisequences are analyzed in detail.

Definition 3.1. A multisequence $mx = (x_i|m_i)$ is ideal statistical convergent to $x_0|m$ (denoted by $I-\mathfrak{S}\lim x_i|m_i = x_0|m$) if for every $\varepsilon > 0$ and $\delta > 0$

$$\left\{n : \frac{1}{n}|\{i \le n : ((x_i - x_0)^2 + (m_i - m)^2)^{1/2} \ge \varepsilon\}| \ge \delta\right\} \in I.$$

Definition 3.2. A multisequence $mx = (x_i|m_i)$ of $m\mathbb{R}$ is ideal Deferred strongly summable to $x_0|m$ of $m\mathbb{R}$ (denoted by $I - \mathfrak{D}\lim x_i|m_i = x_0|m$) if for any $\varepsilon > 0$

$$\left\{n : (\mathfrak{q} - \mathfrak{p})^{-1} \sum_{i=\mathfrak{p}+1}^{\mathfrak{q}} ((x_i - x_0)^2 + (m_i - m)^2)^{1/2} \ge \varepsilon\right\} \in I.$$

Definition 3.3. A multisequence $mx = (x_i|m_i)$ is ideal Deferred statistical convergent to $x_0|m$ (denoted by $I - \mathfrak{D}\mathfrak{S}\lim x_i|m_i = x_0|m$) if for every $\varepsilon > 0$ and $\delta > 0$

$$\left\{n : (\mathfrak{q} - \mathfrak{p})^{-1}|\{\mathfrak{p} < i \le \mathfrak{q} : ((x_i - x_0)^2 + (m_i - m)^2)^{1/2} \ge \varepsilon\}| \ge \delta\right\} \in I.$$

3.1. Inclusion theorems

Theorem 3.4. *Let $mx = (x_i|m_i)$ be a multisequence. If $I - \mathfrak{D}\lim x_i|m_i = x_0|m$, then $I - \mathfrak{D}\mathfrak{S}\lim x_i|m_i = x_0|m$.*

Proof. Let $I - \mathfrak{D}\lim x_i|m_i = x_0|m$. So, for any arbitrary $\varepsilon > 0$, we have

$$\begin{aligned} & (\mathfrak{q} - \mathfrak{p})^{-1} \sum_{i=\mathfrak{p}+1}^{\mathfrak{q}} ((x_i - x_0)^2 + (m_i - m)^2)^{1/2} \\ \ge & \sum_{\substack{i=\mathfrak{p}+1 \\ ((x_i-x_0)^2+(m_i-m)^2)^{1/2} \ge \varepsilon}}^{\mathfrak{q}} ((x_i - x_0)^2 + (m_i - m)^2)^{1/2} \\ \ge & \ \varepsilon|\{\mathfrak{p} < i \le \mathfrak{q} : ((x_i - x_0)^2 + (m_i - m)^2)^{1/2} \ge \varepsilon\}| \end{aligned}$$

and it implies that following inequality

$$\begin{aligned} & \frac{(\mathfrak{q} - \mathfrak{p})^{-1}}{\varepsilon} \sum_{i=\mathfrak{p}+1}^{\mathfrak{q}} ((x_i - x_0)^2 + (m_i - m)^2)^{1/2} \\ \ge & \ (\mathfrak{q} - \mathfrak{p})^{-1}|\{\mathfrak{p} < i \le \mathfrak{q} : ((x_i - x_0)^2 + (m_i - m)^2)^{1/2} \ge \varepsilon\}| \end{aligned}$$

holds. Then, for any $\delta > 0$

$$\left\{n : (\mathfrak{q}-\mathfrak{p})^{-1}|\{\mathfrak{p} < i \leq \mathfrak{q} : ((x_i - x_0)^2 + (m_i - m)^2)^{1/2} \geq \varepsilon\}| > \delta\right\}$$
$$\subset \left\{n : (\mathfrak{q}-\mathfrak{p})^{-1}\sum_{i=\mathfrak{p}+1}^{\mathfrak{q}}((x_i - x_0)^2 + (m_i - m)^2)^{1/2} \geq \varepsilon\delta\right\} \in I.$$

So, $I - \mathfrak{D}\mathfrak{S}\lim x_i|m_i = x_0|m$. □

Corollary 3.5. *If* $\lim x_i|m_i = x_0|m$ *then* $I - \mathfrak{D}\mathfrak{S}\lim x_i|m_i = x_0|m$ *holds.*

Let us note that the converse of Theorem 3.4 (or Corollary 3.5) is not true, in general. For this, let us take account the sequences $\mathfrak{q}(n) = n$, $\mathfrak{p}(n) = 0$ and the multisequence $mx = (x_i|m_i)$, as follows:

$$x_i = \begin{cases} n, & i = n^2 \\ 0, & i \neq n^2 \end{cases}$$

and

$$m_i = \begin{cases} n, & i = n^3 \\ 0, & i \neq n^3. \end{cases}$$

Then for every $\varepsilon > 0$, we get

$$(\mathfrak{q}-\mathfrak{p})^{-1}|\{\mathfrak{p} < i \leq \mathfrak{q} : ((x_i - 0)^2 + (m_i - 0)^2)^{1/2} \geq \varepsilon\}| \leq \frac{n^{1/2} + n^{1/3} - n^{1/6}}{n},$$

and for any $\delta > 0$

$$\left\{n : (\mathfrak{q}-\mathfrak{p})^{-1}|\{\mathfrak{p} < i \leq \mathfrak{q} : ((x_i)^2 + (m_i)^2)^{1/2} \geq \varepsilon\}| \geq \delta\right\}$$
$$\subseteq \left\{n : \frac{n^{1/2} + n^{1/3} - n^{1/6}}{n} \geq \delta\right\} \in I$$

holds because the last set has finitely many element. Hence $I - \mathfrak{D}\mathfrak{S}\lim x_i|m_i = 0|0$.

On the other hand

$$(\mathfrak{q}-\mathfrak{p})^{-1}\sum_{\mathfrak{p}+1}^{\mathfrak{q}}\sqrt{(x_i - 0)^2 + (m_i - 0)^2} = \frac{1}{n}(\sqrt{2} + 2.2^2 + 2.3^2 + ... + 2.n^2)$$
$$\geq \frac{n(n+1)(2n+1)}{6n} \to \infty,$$

Then,

$$\left\{n : (\mathfrak{q}-\mathfrak{p})^{-1}\sum_{i=\mathfrak{p}+1}^{\mathfrak{q}}((x_i - x_0)^2 + (m_i - m)^2)^{1/2} \geq r\right\} \supseteq \{n : \frac{n(n+1)(2n+1)}{6n} \geq A\} \in F$$

for some $A \in \mathbb{R}^+$. So, left-hand side set belongs to F, since I is admissible. Hence, $I - \mathfrak{D}\lim x_i|m_i \neq 0|0$.

Definition 3.6. ([8]) A multisequence $mx = (x_i|m_i)$ is bounded if there exists $B > 0$ such that $(x_i^2 + (m_i - 1)^2)^{1/2} \leq B$.

Theorem 3.7. *Let* $mx = (x_i|m_i)$ *be a bounded multisequence. If* $I - \mathfrak{D}\mathfrak{S}\lim x_i|m_i = x_0|m$, *then* $I - \mathfrak{D}\lim x_i|m_i = x_0|m$.

Proof. Let us assume that $I-\mathfrak{D}\mathfrak{S}\lim x_i|m_i = x_0|m$ and mx is bounded. There exists $B>0$ such that $(x_i^2+(m_i-1)^2)^{1/2}\leq B$ for all $i\in\mathbb{N}$. Also, for $m,m_i\in\mathbb{N}_0$ and $x_i\to x_0$, we have

$$((x_i-x_0)^2+(m_i-m)^2)^{1/2}\leq (x_i^2+(m_i-1)^2)^{1/2}\leq B.$$

So,

$$\begin{aligned}
&(\mathfrak{q}-\mathfrak{p})^{-1}\sum_{i=\mathfrak{p}+1}^{\mathfrak{q}}((x_i-x_0)^2+(m_i-m)^2)^{1/2} = \\
=\;&(\mathfrak{q}-\mathfrak{p})^{-1}\sum_{\substack{i=\mathfrak{p}+1\\ ((x_i-x_0)^2+(m_i-m)^2)^{1/2}\geq\varepsilon/2}}^{\mathfrak{q}}((x_i-x_0)^2+(m_i-m)^2)^{1/2}\\
+\;&(\mathfrak{q}-\mathfrak{p})^{-1}\sum_{\substack{i=\mathfrak{p}+1\\ ((x_i-x_0)^2+(m_i-m)^2)^{1/2}\leq\varepsilon/2}}^{\mathfrak{q}}((x_i-x_0)^2+(m_i-m)^2)^{1/2}\\
\leq\;& B(\mathfrak{q}-\mathfrak{p})^{-1}|\{\mathfrak{p}<i\leq\mathfrak{q}:((x_i-x_0)^2+(m_i-m)^2)^{1/2}\geq\frac{\varepsilon}{2}\}|+\frac{\varepsilon}{2}.
\end{aligned}$$

Hence, for any $\delta>0$,

$$\begin{aligned}
&\left\{n:(\mathfrak{q}-\mathfrak{p})^{-1}\sum_{i=\mathfrak{p}+1}^{\mathfrak{q}}((x_i-x_0)^2+(m_i-m)^2)^{1/2}\geq\varepsilon\right\}\\
\subseteq\;&\left\{n:(\mathfrak{q}-\mathfrak{p})^{-1}|\{\mathfrak{p}<i\leq\mathfrak{q}:((x_i-x_0)^2+(m_i-m)^2)^{1/2}\geq\varepsilon\}|>\frac{2\delta-\varepsilon}{2B}\right\}\in I.
\end{aligned}$$

This gives that $I-\mathfrak{D}\lim x_i|m_i=x_0|m$. □

Theorem 3.8. *If the sequence $\{\frac{\mathfrak{p}}{\mathfrak{q}-\mathfrak{p}}\}$ is bounded, then $I-\mathfrak{S}\lim x_i|m_i=x_0|m$ implies $I-\mathfrak{D}\mathfrak{S}\lim x_i|m_i=x_0|m$.*

Proof. Assume that $I-\mathfrak{S}\lim x_i|m_i=x_0|m$. Then, for every $\varepsilon>0$ and $\delta'>0$, we have

$$A(\varepsilon,\delta')=\left\{n:\frac{1}{n}|\{i\leq n:((x_i-x_0)^2+(m_i-m)^2)^{1/2}\geq\varepsilon\}|\geq\delta'\right\}\in I. \tag{2}$$

Let us show that for every $\varepsilon>0$ and $\delta>0$,

$$B(\varepsilon,\delta)=\left\{n:(\mathfrak{q}-\mathfrak{p})^{-1}|\{\mathfrak{p}<i\leq\mathfrak{q}:((x_i-x_0)^2+(m_i-m)^2)^{1/2}\geq\varepsilon\}|\geq\delta\right\}\in I.$$

Since $\mathfrak{q}\to\infty$ (for $n\to\infty$), then for every $\varepsilon>0$ and $\delta'>0$

$$\left\{\mathfrak{q}:\mathfrak{q}^{-1}|\{i\leq\mathfrak{q}:((x_i-x_0)^2+(m_i-m)^2)^{1/2}\geq\varepsilon\}|\geq\delta'\right\}\subseteq A(\varepsilon,\delta') \tag{3}$$

holds. From (2) and the hereditary property of ideal the left-hand side set in (3) is also belongs to ideal. For each n, we clearly have

$$|\{\mathfrak{p}<i\leq\mathfrak{q}:((x_i-x_0)^2+(m_i-m)^2)^{1/2}\geq\varepsilon\}|\leq|\{i\leq\mathfrak{q}:((x_i-x_0)^2+(m_i-m)^2)^{1/2}\geq\varepsilon\}|$$

Multiplying both sides by $(\mathfrak{q}-\mathfrak{p})^{-1}$ yields

$$(\mathfrak{q}-\mathfrak{p})^{-1}|\{\mathfrak{p}<i\leq\mathfrak{q}:((x_i-x_0)^2+(m_i-m)^2)^{1/2}\geq\varepsilon\}|\leq(\mathfrak{q}-\mathfrak{p})^{-1}|\{i\leq\mathfrak{q}:((x_i-x_0)^2+(m_i-m)^2)^{1/2}\geq\varepsilon\}|$$

The right-hand side can be written as

$$(\mathfrak{q}-\mathfrak{p})^{-1}|\{i\leq\mathfrak{q}:((x_i-x_0)^2+(m_i-m)^2)^{1/2}\geq\varepsilon\}|=\frac{\mathfrak{q}}{\mathfrak{q}-\mathfrak{p}}\mathfrak{q}^{-1}|\{i\leq\mathfrak{q}:((x_i-x_0)^2+(m_i-m)^2)^{1/2}\geq\varepsilon\}|$$

Since $\{\frac{\mathfrak{p}}{\mathfrak{q}-\mathfrak{p}}\}$ is bounded, there exists $M > 0$ such that $\frac{\mathfrak{q}}{\mathfrak{q}-\mathfrak{p}} = 1 + \frac{\mathfrak{p}}{\mathfrak{q}-\mathfrak{p}} \leq 1 + M$ for all $n \in \mathbb{N}$. Thus

$$(\mathfrak{q}-\mathfrak{p})^{-1}|\{i \leq \mathfrak{q} : ((x_i - x_0)^2 + (m_i - m)^2)^{1/2} \geq \varepsilon\}| \leq (1+M)\mathfrak{q}^{-1}|\{i \leq \mathfrak{q} : ((x_i - x_0)^2 + (m_i - m)^2)^{1/2} \geq \varepsilon\}|.$$

Let $\delta' = \delta/(1+M)$. Hence,

$$B(\varepsilon,\delta) \subseteq \left\{n : \mathfrak{q}^{-1}|\{i \leq \mathfrak{q} : ((x_i - x_0)^2 + (m_i - m)^2)^{1/2} \geq \varepsilon\}| \geq \frac{\delta}{1+M}\right\}$$

holds. From (3), the set on the right belongs to I, and thus $B(\varepsilon,\delta) \in I$. Therefore, for all $\varepsilon > 0$ and $\delta > 0$, $B(\varepsilon,\delta) \in I$, which means that $I - \mathfrak{D}\mathfrak{S}\lim x_i|m_i = x_0|m$. □

Corollary 3.9. *Let us assume that* $\mathfrak{q} < n$ *holds for all* $n \in \mathbb{N}$ *and* $\{\frac{n}{\mathfrak{q}-\mathfrak{p}}\}$ *is a bounded sequence. Then,* $I - \mathfrak{S}\lim x_i|m_i = x_0|m$ *implies* $I - \mathfrak{D}\mathfrak{S}\lim x_i|m_i = x_0|m$.

Theorem 3.10. *Let* $\mathfrak{q} = \{n\}_{n\in\mathbb{N}}$. *Then,* $I - \mathfrak{D}\mathfrak{S}\lim x_i|m_i = x_0|m$ *if and only if* $I - \mathfrak{S}\lim x_i|m_i = x_0|m$.

Proof. ($\Rightarrow$) Assume that $I - \mathfrak{D}\mathfrak{S}\lim x_i|m_i = x_0|m$. For any $n \in \mathbb{N}$

$$... < n^{(3)} = \mathfrak{p}(n^{(2)}) < n^{(2)} = \mathfrak{p}(n^{(1)}) < n^{(1)} = \mathfrak{p}(n)$$

and we can write $\{i \leq n : ((x_i - x_0)^2 + (m_i - m)^2)^{1/2} \geq \varepsilon\}$ as

$$\begin{aligned} & \{i \leq n : ((x_i - x_0)^2 + (m_i - m)^2)^{1/2} \geq \varepsilon\} = \\ = \; & \{i \leq n^{(1)} : ((x_i - x_0)^2 + (m_i - m)^2)^{1/2} \geq \varepsilon\} \\ \cup \; & \{n^{(1)} < i \leq n : ((x_i - x_0)^2 + (m_i - m)^2)^{1/2} \geq \varepsilon\}, \end{aligned}$$

and the set $\{i \leq n^{(1)} : ((x_i - x_0)^2 + (m_i - m)^2)^{1/2} \geq \varepsilon\}$ as

$$\begin{aligned} & \{i \leq n^{(1)} : ((x_i - x_0)^2 + (m_i - m)^2)^{1/2} \geq \varepsilon\} = \\ = \; & \{i \leq n^{(2)} : ((x_i - x_0)^2 + (m_i - m)^2)^{1/2} \geq \varepsilon\} \\ \cup \; & \{n^{(2)} < i \leq n^{(1)} : ((x_i - x_0)^2 + (m_i - m)^2)^{1/2} \geq \varepsilon\}, \end{aligned}$$

and the set $\{i \leq n^{(2)} : ((x_i - x_0)^2 + (m_i - m)^2)^{1/2} \geq \varepsilon\}$ as

$$\begin{aligned} & \{i \leq n^{(2)} : ((x_i - x_0)^2 + (m_i - m)^2)^{1/2} \geq \varepsilon\} = \\ = \; & \{i \leq n^{(3)} : ((x_i - x_0)^2 + (m_i - m)^2)^{1/2} \geq \varepsilon\} \\ \cup \; & \{n^{(3)} < i \leq n^{(2)} : ((x_i - x_0)^2 + (m_i - m)^2)^{1/2} \geq \varepsilon\} \end{aligned}$$

if this process is carried on

$$\begin{aligned} & \{i \leq n^{(\gamma-1)} : ((x_i - x_0)^2 + (m_i - m)^2)^{1/2} \geq \varepsilon\} = \\ = \; & \{i \leq n^{(\gamma)} : ((x_i - x_0)^2 + (m_i - m)^2)^{1/2} \geq \varepsilon\} \\ \cup \; & \{n^{(\gamma)} < i \leq n^{(\gamma-1)} : ((x_i - x_0)^2 + (m_i - m)^2)^{1/2} \geq \varepsilon\}, \end{aligned}$$

is obtained for a certain positive integer $\gamma > 0$ depending on n such that $n^{(\gamma)} \geq 1$ and $n^{(\gamma+1)} = 0$. From the above process, for every $n \in \mathbb{N}$ following relation

$$\begin{aligned} & \left\{n : n^{-1}|\{i \leq n : ((x_i - x_0)^2 + (m_i - m)^2)^{1/2} \geq \varepsilon\}| \geq \delta\right\} \\ = \; & \left\{n : \sum_{t=0}^{\gamma} \frac{n^{(t)} - n^{(t+1)}}{n(n^{(t)} - n^{(t+1)})}|\{n^{(t+1)} < i \leq n^{(t)} : ((x_i - x_0)^2 + (m_i - m)^2)^{1/2} \geq \varepsilon\}| \geq \delta\right\} \end{aligned}$$

holds. This gives that ideal statistical convergence of the multisequence $x_i|m_i$ to $x_0|m$ can obtained from the following sequence

$$\left\{\frac{1}{n^{(t)} - n^{(t+1)}}|\{n^{(t+1)} < i \leq n^{(t)} : ((x_i - x_0)^2 + (m_i - m)^2)^{1/2} \geq \varepsilon\}|\right\}_{t\in\mathbb{N}}.$$

Let us consider the matrix

$$b_{n,t} := \begin{cases} \frac{n^{(t)}-n^{(t+1)}}{n}, & t = 0,1,2,...,\gamma \\ 0, & t \neq 0,1,2,...,\gamma \end{cases}, n^{(0)} := n.$$

Silverman Toeplitz theorem [18] is provided for the matrix $(b_{n,t})$. So we have,

$$\left\{n : n^{-1}|\{i \leq n : ((x_i - x_0)^2 + (m_i - m)^2)^{1/2} \geq \varepsilon\}| \geq \delta\right\} \in I$$

because of the assumption of

$$\left\{n : \frac{1}{n^{(t)} - n^{(t+1)}}|\{n^{(t+1)} < i \leq n^{(t)} : ((x_i - x_0)^2 + (m_i - m)^2)^{1/2} \geq \varepsilon\}| \geq \delta\right\} \in I.$$

(⇒) Since $\mathfrak{q}(n) = n$ holds (1), then the inverse is simply from the Theorem 3.8. □

Let we have following assumption for $\mathfrak{p}, \mathfrak{q}, \mathfrak{p}', \mathfrak{q}'$

$$\mathfrak{p} \leq \mathfrak{p}' < \mathfrak{q}' \leq \mathfrak{q} \tag{4}$$

for all $n \in \mathbb{N}$. We take into account the assumption (4) in the following two results for to compare $I - \mathfrak{D}\mathfrak{S}(\mathfrak{p}, \mathfrak{q})$ and $I - \mathfrak{D}\mathfrak{S}(\mathfrak{p}'\mathfrak{q}')$.

Theorem 3.11. *Let $\mathfrak{p}, \mathfrak{q}, \mathfrak{p}', \mathfrak{q}'$ be sequences of positive integers such that the sets*

$$\{k : \mathfrak{p} < k \leq \mathfrak{p}'\} \text{ and } \{k : \mathfrak{q}' < k \leq \mathfrak{q}\}$$

are finite subsets of $\mathbb{N}$ for all $n \in \mathbb{N}$ also (4) holds. Then, $I - \mathfrak{D}\mathfrak{S}(\mathfrak{p}', \mathfrak{q}')x_i|m_i \to x_0|m$ implies $I - \mathfrak{D}\mathfrak{S}(\mathfrak{p}, \mathfrak{q})x_i|m_i \to x_0|m$.

Proof. Let us consider $I - \mathfrak{D}\mathfrak{S}(\mathfrak{p}', \mathfrak{q}')x_i|m_i = x_0|m$ holds. For any $\varepsilon > 0$ and $\delta > 0$

$$\left\{n : (\mathfrak{q}' - \mathfrak{p}')^{-1}|\{\mathfrak{p}' < k \leq \mathfrak{q}' : ((x_i - x_0)^2 + (m_i - m)^2)^{1/2} \geq \varepsilon\}| \geq \delta\right\} \in I$$

holds. Also from (4)

$$\begin{aligned} & \{\mathfrak{p} < k \leq \mathfrak{q} : ((x_i - x_0)^2 + (m_i - m)^2)^{1/2} \geq \varepsilon\} \\ = & \{\mathfrak{p} < k \leq \mathfrak{p}' : ((x_i - x_0)^2 + (m_i - m)^2)^{1/2} \geq \varepsilon\} \\ \cup & \{\mathfrak{p}' < k \leq \mathfrak{q}' : ((x_i - x_0)^2 + (m_i - m)^2)^{1/2} \geq \varepsilon\} \\ \cup & \{\mathfrak{q}' < k \leq \mathfrak{q} : ((x_i - x_0)^2 + (m_i - m)^2)^{1/2} \geq \varepsilon\} \end{aligned}$$

and

$$\begin{aligned} & (\mathfrak{q} - \mathfrak{p})^{-1}|\{\mathfrak{p} < k \leq \mathfrak{q} : ((x_i - x_0)^2 + (m_i - m)^2)^{1/2} \geq \varepsilon\}| \\ \leq & (\mathfrak{q}' - \mathfrak{p}')^{-1}|\{\mathfrak{p} < k \leq \mathfrak{p}' : ((x_i - x_0)^2 + (m_i - m)^2)^{1/2} \geq \varepsilon\}| \\ + & (\mathfrak{q}' - \mathfrak{p}')^{-1}|\{\mathfrak{p}' < k \leq \mathfrak{q}' : ((x_i - x_0)^2 + (m_i - m)^2)^{1/2} \geq \varepsilon\}| \\ + & (\mathfrak{q}' - \mathfrak{p}')^{-1}|\{\mathfrak{q}' < k \leq \mathfrak{q} : ((x_i - x_0)^2 + (m_i - m)^2)^{1/2} \geq \varepsilon\}| \end{aligned}$$

are hold. So from the hypothesis we obtain

$$\left\{n : (\mathfrak{q} - \mathfrak{p})^{-1}|\{\mathfrak{p} < k \leq \mathfrak{q} : ((x_i - x_0)^2 + (m_i - m)^2)^{1/2} \geq \varepsilon\}| \geq \delta\right\} \in I$$

which gives that $I - \mathfrak{D}\mathfrak{S}(\mathfrak{p}, \mathfrak{q})x_i|m_i \to x_0|m$. □

Theorem 3.12. *Let $\mathfrak{p}, \mathfrak{q}, \mathfrak{p}', \mathfrak{q}'$ be sequences of positive natural numbers satisfying* (4) *such that following limit is a positive real number*

$$\lim_{n\to\infty} \frac{\mathfrak{q}-\mathfrak{p}}{\mathfrak{q}'-\mathfrak{p}'}.$$

Then, $I-\mathfrak{D}\mathfrak{S}(\mathfrak{p},\mathfrak{q})x_i|m_i \to x_0|m$ implies $I-\mathfrak{D}\mathfrak{S}(\mathfrak{p}',\mathfrak{q}')x_i|m_i \to x_0|m$.

Proof. From the hypothesis the inclusion

$$\begin{aligned} & \{\mathfrak{p}'+1 \le k \le \mathfrak{q}' : ((x_i-x_0)^2+(m_i-m)^2)^{1/2} \ge \varepsilon\} \\ \subset\ & \{\mathfrak{p}+1 \le k \le \mathfrak{q} : ((x_i-x_0)^2+(m_i-m)^2)^{1/2} \ge \varepsilon\} \end{aligned}$$

and the inequality

$$\begin{aligned} & |\{\mathfrak{p}'+1 \le k \le \mathfrak{q}' : ((x_i-x_0)^2+(m_i-m)^2)^{1/2} \ge \varepsilon\}| \\ \le\ & |\{\mathfrak{p}+1 \le k \le \mathfrak{q} : ((x_i-x_0)^2+(m_i-m)^2)^{1/2} \ge \varepsilon\}| \end{aligned}$$

are true. Thus, we have

$$\begin{aligned} & \left\{n : (\mathfrak{q}'-\mathfrak{p}')^{-1}|\{\mathfrak{p}'+1 \le k \le \mathfrak{q}' : ((x_i-x_0)^2+(m_i-m)^2)^{1/2} \ge \varepsilon\}| \ge \delta\right\} \\ \subseteq\ & \left\{n : \frac{\mathfrak{q}-\mathfrak{p}}{\mathfrak{q}-\mathfrak{p}}(\mathfrak{q}'-\mathfrak{p}')^{-1}|\{\mathfrak{p}+1 \le k \le \mathfrak{q} : ((x_i-x_0)^2+(m_i-m)^2)^{1/2} \ge \varepsilon\}| \ge \delta\right\} \in I. \end{aligned}$$

This proves our assertion. □

3.2. Ideal deferred statistical limit and cluster points of multisequences

Definition 3.13. The number $l|m$ of $m\mathbb{R}$ is an ideal deferred statistical limit point of the multisequence $mx = (x_i|m_i)$ if there is a set $S = \{s_1 < s_2 < ... < s_i < ...\} \subset \mathbb{N}$ such that $S \notin I$ and $\mathfrak{D}\mathfrak{S}\lim x_{s_i}|m_{s_i} = l|m$.

$\mathfrak{D}\mathfrak{S}\Lambda^I_{mx}$ denotes the set of all ideal deferred statistical limit points of the multisequence.

Definition 3.14. The number $c|m$ of $m\mathbb{R}$ is an ideal deferred statistical cluster point of the multisequence $mx = (x_i|m_i)$ if for any $\varepsilon > 0$,

$$\left\{n \in \mathbb{N} : (\mathfrak{q}-\mathfrak{p})^{-1}|\{\mathfrak{p} < i \le \mathfrak{q} : \left((x_i-c)^2+(m_i-m)^2\right)^{1/2} \ge \varepsilon\}| < \delta\right\} \notin I.$$

$\mathfrak{D}\mathfrak{S}\Gamma^I_{mx}$ denotes the set of all ideal deferred statistical cluster points of the multisequence $mx = (x_i|m_i)$.

Theorem 3.15. *For any multisequence $mx = (x_i|m_i)$, $\mathfrak{D}\mathfrak{S}\Lambda^I_{mx} \subset \mathfrak{D}\mathfrak{S}\Gamma^I_{mx}$ holds.*

Proof. Let us assume that $l|m \in \mathfrak{D}\mathfrak{S}\Lambda^I_{mx}$. So, there exists a set $S = \{s_1 < s_2 < ... < s_i < ...\} \subset \mathbb{N}$ such that $S \notin I$ and $I-\mathfrak{D}\mathfrak{S}\lim x_{s_i}|m_{s_i} = l|m$. From the Definition 3.3 for every $\varepsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that for $n > n_0$ and for any $\delta > 0$

$$\left\{n : (\mathfrak{q}-\mathfrak{p})^{-1}|\{\mathfrak{p} < i \le \mathfrak{q} : \left((x_{s_i}-l)^2+(m_{s_i}-m)^2\right)^{1/2} \ge \varepsilon\}| \ge \delta\right\} \in I$$

holds. Also,

$$\left\{n : (\mathfrak{q}-\mathfrak{p})^{-1}|\{\mathfrak{p} < i \le \mathfrak{q} : \left((x_i-l)^2+(m_i-m)^2\right)^{1/2} \ge \varepsilon\}| < \delta\right\} \supset S\backslash\{s_1, s_2, ..., s_{i_0}\}.$$

and so $\left\{n : (\mathfrak{q}-\mathfrak{p})^{-1}|\{\mathfrak{p} < i \le \mathfrak{q} : \left((x_i-l)^2+(m_i-m)^2\right)^{1/2} \ge \varepsilon\}| < \delta\right\} \notin I$ since I is admissible. Hence, $l|m \in \mathfrak{D}\mathfrak{S}\Gamma^I_{mx}$. □

3.3. Ideal deferred statistical limit superior, limit inferior of multisequences

Definition 3.16. For any multisequence $mx = (x_i|m_i)$ let $\mathfrak{D}\mathfrak{S}\mathcal{U}^I_{mx}$ denotes the following multiset

$$\left\{b|m : \left\{n \in \mathbb{N} : (\mathfrak{q}-\mathfrak{p})^{-1}|\{\mathfrak{p} < i \le \mathfrak{q} : \left(x_i^2 + (m_i-1)^2\right)^{1/2} > \left(b^2 + (m-1)^2\right)^{1/2}\}| > \delta\right\} \notin I\right\}$$

and $\mathfrak{D}\mathfrak{S}\mathcal{L}^I_{mx}$ denotes the following multiset

$$\left\{a|m : \left\{n \in \mathbb{N} : (\mathfrak{q}-\mathfrak{p})^{-1}|\{\mathfrak{p} < i \le \mathfrak{q} : \left(x_i^2 + (m_i-1)^2\right)^{1/2} < \left(a^2 + (m-1)^2\right)^{1/2}\}| > \delta\right\} \notin I\right\}$$

Let us note that $x_0|m$ is called supremum of $\mathfrak{D}\mathfrak{S}\mathcal{U}^I_{mx}$ (or infimum of $\mathfrak{D}\mathfrak{S}\mathcal{L}^I_{mx}$), if m is the greatest (or lowest) multiplicity in $\mathfrak{D}\mathfrak{S}\mathcal{U}^I_{mx}$ (or $\mathfrak{D}\mathfrak{S}\mathcal{L}^I_{mx}$) under the condition $m \le \max\{m_i\}$ (or $m \ge \max\{m_i\}$) in mx and x_0 denotes the supremum (or infimum) of the unique sets of real numbers associated with the multiplicity m in $\mathfrak{D}\mathfrak{S}\mathcal{U}^I_{mx}$ (or $\mathfrak{D}\mathfrak{S}\mathcal{L}^I_{mx}$), whenever it exists.

Definition 3.17. The ideal deferred statistical limit superior of the multisequence $mx = (x_i|m_i)$ is given by

$$I - \mathfrak{D}\mathfrak{S}\limsup x_i|m_i = \begin{cases} \sup \mathfrak{D}\mathfrak{S}\mathcal{U}^I_{mx}, & \mathfrak{D}\mathfrak{S}\mathcal{U}^I_{mx} \neq \emptyset, \\ -\infty, & \mathfrak{D}\mathfrak{S}\mathcal{U}^I_{mx} = \emptyset. \end{cases}$$

The ideal deferred statistical limit inferior of the multisequence mx is given by

$$I - \mathfrak{D}\mathfrak{S}\liminf x_i|m_i = \begin{cases} \inf \mathfrak{D}\mathfrak{S}\mathcal{L}^I_{mx}, & \mathfrak{D}\mathfrak{S}\mathcal{L}^I_{mx} \neq \emptyset, \\ +\infty, & \mathfrak{D}\mathfrak{S}\mathcal{L}^I_{mx} = \emptyset. \end{cases}$$

Proposition 3.18. *Let $mx = (x_i|m_i)$ be a multisequence of $m\mathbb{R}$. Let $I - \mathfrak{D}\mathfrak{S}\limsup x_i|m_i = b|m$ (finite), then for any $\varepsilon > 0$*

$$\left\{n : (\mathfrak{q}-\mathfrak{p})^{-1}|\{\mathfrak{p} < i \le \mathfrak{q} : \left(x_i^2 + (m_i-1)^2\right)^{1/2} > \left((b-\varepsilon)^2 + (m-1)^2\right)^{1/2}\}| > \delta\right\} \notin I$$

and

$$\left\{n : (\mathfrak{q}-\mathfrak{p})^{-1}|\{\mathfrak{p} < i \le \mathfrak{q} : \left(x_i^2 + (m_i-1)^2\right)^{1/2} > \left((b+\varepsilon)^2 + (m-1)^2\right)^{1/2}\}| > \delta\right\} \in I$$

hold.

Proposition 3.19. *Let $mx = (x_i|m_i)$ be a multisequence of $m\mathbb{R}$. Let $I - \mathfrak{D}\mathfrak{S}\liminf x_i|m_i = a|m$ (finite), then for any $\varepsilon > 0$*

$$\left\{n : (\mathfrak{q}-\mathfrak{p})^{-1}|\{\mathfrak{p} < i \le \mathfrak{q} : \left(x_i^2 + (m_i-1)^2\right)^{1/2} < \left((a+\varepsilon)^2 + (m-1)^2\right)^{1/2}\}| > \delta\right\} \notin I$$

and

$$\left\{n : (\mathfrak{q}-\mathfrak{p})^{-1}|\{\mathfrak{p} < i \le \mathfrak{q} : \left(x_i^2 + (m_i-1)^2\right)^{1/2} < \left((a-\varepsilon)^2 + (m-1)^2\right)^{1/2}\}| > \delta\right\} \in I$$

hold.

Theorem 3.20. *Let $mx = (x_i|m_i)$ be a multisequence of $m\mathbb{R}$.*

$$I - \mathfrak{D}\mathfrak{S}\liminf x_i|m_i \le I - \mathfrak{D}\mathfrak{S}\limsup x_i|m_i$$

holds.

Proof. We will consider three cases for the proof. Firstly, let us consider $I-\mathfrak{D}\mathfrak{S}\limsup x_i|m_i = -\infty$. So, $\mathfrak{D}\mathfrak{S}\mathcal{U}^I_{mx} = \emptyset$. Then, for every $b|m \in m\mathbb{R}$,

$$\left\{n : (\mathfrak{q}-\mathfrak{p})^{-1}|\{\mathfrak{p} < i \leq \mathfrak{q} : \left(x_i^2 + (m_i-1)^2\right)^{1/2} > \left(b^2 + (m-1)^2\right)^{1/2}\}| > \delta\right\} \in I$$

This implies that

$$\left\{n : (\mathfrak{q}-\mathfrak{p})^{-1}|\{\mathfrak{p} < i \leq \mathfrak{q} : \left(x_i^2 + (m_i-1)^2\right)^{1/2} \leq \left(b^2 + (m-1)^2\right)^{1/2}\}| > \delta\right\} \in F,$$

for any $\alpha|\beta \in m\mathbb{R}$,

$$\left\{n : (\mathfrak{q}-\mathfrak{p})^{-1}|\{\mathfrak{p} < i \leq \mathfrak{q} : \left(x_i^2 + (m_i-1)^2\right)^{1/2} < \left(\alpha^2 + (\beta-1)^2\right)^{1/2}\}| > \delta\right\} \in F$$

Hence, $I-\mathfrak{D}\mathfrak{S}\liminf x_i|m_i = -\infty$.

In case $I-\mathfrak{D}\mathfrak{S}\limsup x_i|m_i = +\infty$ the proof is obvious. Let us take account third case.

Let us assume that $I-\mathfrak{D}\mathfrak{S}\limsup x_i|m_i = \beta|m$ (finite), and let $I-\mathfrak{D}\mathfrak{S}\liminf x_i|m_i = \alpha|m'$. For given $\varepsilon > 0$ and $\delta > 0$ we show that $(\beta+\varepsilon)|m \in \mathfrak{D}\mathfrak{S}\mathcal{L}^I_{mx}$, so that

$$\left\{n : (\mathfrak{q}-\mathfrak{p})^{-1}|\{\mathfrak{p} < i \leq \mathfrak{q} : \left(\alpha^2 + (m'-1)^2\right)^{1/2} < \left((\beta+\varepsilon)^2 + (m-1)^2\right)^{1/2}\}| > \delta\right\} \in I.$$

By Proposition 3.18,

$$\left\{n : (\mathfrak{q}-\mathfrak{p})^{-1}|\{\mathfrak{p} < i \leq \mathfrak{q} : \left(x_i^2 + (m_i-1)^2\right)^{1/2} > \left((\beta+\varepsilon)^2 + (m-1)^2\right)^{1/2}\}| > \delta\right\} \in I,$$

because $\beta|m = I-\mathfrak{D}\mathfrak{S}\limsup x_i|m_i$. This implies that

$$\left\{n : (\mathfrak{q}-\mathfrak{p})^{-1}|\{\mathfrak{p} < i \leq \mathfrak{q} : \left(x_i^2 + (m_i-1)^2\right)^{1/2} \leq \left((\beta+\varepsilon)^2 + (m-1)^2\right)^{1/2}\}| > \delta\right\} \in F.$$

So,

$$\left\{n : (\mathfrak{q}-\mathfrak{p})^{-1}|\{\mathfrak{p} < i \leq \mathfrak{q} : \left(x_i^2 + (m_i-1)^2\right)^{1/2} < \left((\beta+\varepsilon)^2 + (m-1)^2\right)^{1/2}\}| > \delta\right\} \notin I.$$

Hence, $(\beta+\varepsilon)|m \in \mathfrak{D}\mathfrak{S}\mathcal{L}^I_{mx}$. By definition $\alpha|m' = I-\mathfrak{D}\mathfrak{S}\liminf x_i|m_i$, so we conclude that

$$\left(\alpha^2 + (m'-1)^2\right)^{1/2} \leq \left((\beta+\varepsilon)^2 + (m-1)^2\right)^{1/2}.$$

Since ε is arbitrary this gives that $\left(\alpha^2 + (m'-1)^2\right)^{1/2} \leq \left(\beta^2 + (m-1)^2\right)^{1/2}$, i.e., $I-\mathfrak{D}\mathfrak{S}\liminf x_i|m_i \leq I-\mathfrak{D}\mathfrak{S}\limsup x_i|m_i$. □

Definition 3.21. A multisequence $mx = (x_i|m_i)$ is ideal deferred statistical bounded if there exists a non-negative real number K such that for every $\delta > 0$

$$\left\{n : (\mathfrak{q}-\mathfrak{p})^{-1}|\{\mathfrak{p} < i \leq \mathfrak{q} : \left(x_i^2 + (m_i-1)^2\right)^{1/2} > K\}| > \delta\right\} \in I.$$

Theorem 3.22. *The ideal deferred statistical bounded multisequence $mx = (x_i|m_i)$ of $m\mathbb{R}$ is ideal deferred statistical convergent if and only if*

$$I-\mathfrak{D}\mathfrak{S}\liminf x_i|m_i = I-\mathfrak{D}\mathfrak{S}\limsup x_i|m_i.$$

Proof. Let us assume that $a|m_1 = I - \mathfrak{DS}\liminf x_i|m_i$, $b|m_2 = I - \mathfrak{DS}\limsup x_i|m_i$ and $I - \mathfrak{DS}\lim x_i|m_i = x_0|m$. So for every $\varepsilon > 0$ and $\delta > 0$

$$\left\{n : (\mathfrak{q}-\mathfrak{p})^{-1}|\{\mathfrak{p} < i \leq \mathfrak{q} : \left((x_i - x_0)^2 + (m_i - m)^2\right)^{1/2} \geq \varepsilon\}| > \delta\right\} \in I$$

and

$$\left\{n : (\mathfrak{q}-\mathfrak{p})^{-1}|\{\mathfrak{p} < i \leq \mathfrak{q} : \left((x_i - x_0)^2 + (m_i - m)^2\right)^{1/2} \geq \varepsilon\}| < \delta\right\} \in F$$

holds which implies that

$$\left\{n : (\mathfrak{q}-\mathfrak{p})^{-1}|\{\mathfrak{p} < i \leq \mathfrak{q} : |x_i - x_0| > \frac{\varepsilon}{\sqrt{2}} = \varepsilon_1,\ |m_i - m| > \frac{\varepsilon}{\sqrt{2}} = \varepsilon_1\}| < \delta\right\} \in F.$$

So,$\left\{n : (\mathfrak{q}-\mathfrak{p})^{-1}|\{\mathfrak{p} < i \leq \mathfrak{q} : |x_i - x_0| \geq \varepsilon_1,\ |m_i - m| \geq \varepsilon_1\}| > \delta\right\} \in I$. i.e.,

$$\left\{n : (\mathfrak{q}-\mathfrak{p})^{-1}|\{\mathfrak{p} < i \leq \mathfrak{q} : |x_i - x_0| > \varepsilon_1,\ |m_i - m| > \varepsilon_1\}| > \delta\right\} \in I.$$

Therefore,

$$\left\{n : (\mathfrak{q}-\mathfrak{p})^{-1}|\{\mathfrak{p} < i \leq \mathfrak{q} : x_i > x_0 + \varepsilon_1,\ m_i - 1 > m + \varepsilon_1 - 1\}| > \delta\right\} \in I$$

and so,

$$\left\{n : (\mathfrak{q}-\mathfrak{p})^{-1}|\{\mathfrak{p} < i \leq \mathfrak{q} : \left(x_i^2 + (m_i - 1)^2\right)^{1/2} > \left((x_0 + \varepsilon_1)^2 + (m + \varepsilon_1 - 1)^2\right)^{1/2}\}| > \delta\right\} \in I$$

which implies that $\left(b^2 + (m_2 - 1)^2\right)^{1/2} < \left((x_0 + \varepsilon_1)^2 + (m + \varepsilon_1 - 1)^2\right)^{1/2}$, i.e.,

$$\left(b^2 + (m_2 - 1)^2\right)^{1/2} \leq \left(x_0^2 + (m - 1)^2\right)^{1/2}.$$

Hence,

$$I - \mathfrak{DS}\limsup x_i|m_i \leq I - \mathfrak{DS}\lim x_i|m_i. \tag{5}$$

Also we have

$$\left\{n : (\mathfrak{q}-\mathfrak{p})^{-1}|\{\mathfrak{p} < i \leq \mathfrak{q} : x_i < x_0 - \varepsilon_1, m_i - 1 < m - 1 - \varepsilon_1\}| > \delta\right\} \in I$$

and so,

$$\left\{n : (\mathfrak{q}-\mathfrak{p})^{-1}|\{\mathfrak{p} < i \leq \mathfrak{q} : \left(x_i^2 + (m_i - 1)^2\right)^{1/2} < \left((x_0 - \varepsilon_1)^2 + (m - \varepsilon_1 - 1)^2\right)^{1/2}\}| > \delta\right\} \in I$$

which implies that $\left((x_0 - \varepsilon_1)^2 + (m - \varepsilon_1 - 1)^2\right)^{1/2} < \left(a^2 + (m_1 - 1)^2\right)^{1/2}$, i.e.,

$$\left(x_0^2 + (m - 1)^2\right)^{1/2} \leq \left(a^2 + (m_1 - 1)^2\right)^{1/2}.$$

Hence,

$$I - \mathfrak{DS}\lim x_i|m_i \leq I - \mathfrak{DS}\liminf x_i|m_i. \tag{6}$$

So, from (5) and (6) and Theorem 3.20 we obtained that

$$I - \mathfrak{DS}\liminf x_i|m_i = I - \mathfrak{DS}\limsup x_i|m_i.$$

Now let us assume that $a|m_1 = b|m_2$ and define $x_0|m = a|m_1 = b|m_2$. Let $I - \mathfrak{D}\mathfrak{S} \limsup x_i|m_i = x_0|m$. So, from Proposition 3.18 for given any $\varepsilon_1 > 0$,

$$\left\{n : (\mathfrak{q}-\mathfrak{p})^{-1}|\{\mathfrak{p} < i \leq \mathfrak{q} : \left(x_i^2 + (m_i - 1)^2\right)^{1/2} > \left((x_0 + \varepsilon_1)^2 + (m-1)^2\right)^{1/2}\}| > \delta\right\} \in I.$$

We can find any positive number ε' that

$$\left((x_0 + \varepsilon_1)^2 + (m-1)^2\right)^{1/2} = \left(x_0^2 + (m-1)^2\right)^{1/2} + \varepsilon',$$

i.e.,

$$\left\{n : (\mathfrak{q}-\mathfrak{p})^{-1}|\{\mathfrak{p} < i \leq \mathfrak{q} : \left(x_i^2 + (m_i - 1)^2\right)^{1/2} > \left(x_0^2 + (m-1)^2\right)^{1/2} + \varepsilon'\}| > \delta\right\} \in I.$$

Again from the hypothesis and Proposition 3.18 for given any $\varepsilon_2 > 0$,

$$\left\{n : (\mathfrak{q}-\mathfrak{p})^{-1}|\{\mathfrak{p} < i \leq \mathfrak{q} : \left(x_i^2 + (m_i - 1)^2\right)^{1/2} < \left((x_0 - \varepsilon_2)^2 + (m-1)^2\right)^{1/2}\}| > \delta\right\} \in I.$$

We can find a positive real number ε'' that

$$\left((x_0 - \varepsilon_2)^2 + (m-1)^2\right)^{1/2} = \left(x_0^2 + (m-1)^2\right)^{1/2} - \varepsilon'',$$

i.e.,

$$\left\{n : (\mathfrak{q}-\mathfrak{p})^{-1}|\{\mathfrak{p} < i \leq \mathfrak{q} : \left(x_i^2 + (m_i - 1)^2\right)^{1/2} < \left(x_0^2 + (m-1)^2\right)^{1/2} - \varepsilon''\}| > \delta\right\} \in I.$$

Hence if we choose ε as $\max(\varepsilon', \varepsilon'')$, then we have

$$\left\{n : (\mathfrak{q}-\mathfrak{p})^{-1}|\{\mathfrak{p} < i \leq \mathfrak{q} : \left(x_i^2 + (m_i - 1)^2\right)^{1/2} > \left(x_0^2 + (m-1)^2\right)^{1/2} + \varepsilon\}| > \delta\right\} \in I.$$

and

$$\left\{n : (\mathfrak{q}-\mathfrak{p})^{-1}|\{\mathfrak{p} < i \leq \mathfrak{q} : \left(x_i^2 + (m_i - 1)^2\right)^{1/2} < \left(x_0^2 + (m-1)^2\right)^{1/2} - \varepsilon\}| > \delta\right\} \in I.$$

i.e., $I - \mathfrak{D}\mathfrak{S} \lim x_i|m_i = x_0|m$. □

4. Conclusion

In this paper, we have introduced and analyzed the concept of deferred statistical convergence of multisequences, extending classical statistical convergence methods to accommodate the multisequences. By incorporating ideal theory and deferred Cesàro means, we have provided a comprehensive framework that bridges the gap between traditional and modern approaches to sequence convergence.

Main contributions of this study include the formal definitions of ideal deferred statistical convergence, limit superior, and limit inferior for multisequences, along with several inclusion theorems that establish relationships between various convergence types. Our results offer a deeper understanding of the structural properties of multisequences and their behavior under ideal conditions.

The results presented in this paper not only extend existing literature but also provide a foundation for further exploration. Future research could focus on generalizing the concept to more complex mathematical structures, exploring applications in optimization problems, and investigating connections with other types of generalized summability.

In conclusion, this study contributes to the knowledge in statistical convergence and multisequence analysis, offering valuable insights and potential applications across various domains of mathematical research.

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