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Convex fuzzifying bornologies and its duality

Conghua Yan[a]

[a]*Institute of Mathematics, School of Mathematical Sciences, Nanjing Normal University, Nanjing Jiangsu 210023, People's Republic of China*

Abstract. The main objective of this paper is to investigate convex fuzzifying bornological linear spaces and their duality. First, we introduce the notions of convex fuzzifying bornological linear spaces and locally convex fuzzifying topological linear spaces, along with several examples. Next, we study the relationship between convex fuzzifying bornological linear spaces and locally convex fuzzifying topological linear spaces, utilizing the mapping of fuzzifying bornivorous. Subsequently, we introduce the concepts of *fuzzifying bornological locally convex spaces* and *fuzzifying topological convex bornological linear spaces*. We demonstrate that the category of *fuzzifying bornological locally convex spaces* can be embedded as a reflective subcategory in the category of convex fuzzifying bornological linear spaces, and the category of *fuzzifying topological convex bornological linear spaces* can also be embedded as a reflective subcategory in the category of locally convex fuzzifying topological linear spaces. Moreover, the category of *fuzzifying topological convex bornological linear spaces* is topological over the category of linear spaces with respect to the expected forgetful functor. Lastly, we provide several characterizations of the fuzzifying bornological topologies.

1. Introduction

It is well-known that boundedness is a very important concept in the theory of functional analysis. However, the concept of bounded sets lacks clarity in topological spaces. In 1949, Hu [9, 10] first introduced the concepts of bornology and bornological spaces, which were later developed in the context of bornological vector spaces [8, 18]. In recent years, the theory of general bornological spaces has played a critical role in researching convergence structures on hyperspaces [17], optimization theory [4], and topologies on function spaces [6, 15].

It is worthy noting that Abel and Šostak [1] originally extended the theory of bornological spaces to the context of fuzzy sets in 2011. They discussed bornologies over an infinitely distributive complete lattice L and gave the concept of an L-bornology as an extension of that of crisp bornologies. Combing an L-bornology and a vector space, Paseka et al. [19] introduced the notions of lattice-valued bornological vector spaces and L-convex L-bornological vector spaces. Some categorical properties of L-bornological vector spaces are also studied by them. Subsequently, Jin and Yan [11] proposed L-Mackey convergence and separation in L-bornological vector spaces, and discussed an equivalent characterization of separation in

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Email address: chyan@njnu.edu.cn (Conghua Yan)
ORCID iD: https://orcid.org/0000-0002-6500-7807 (Conghua Yan)

terms of L-Mackey convergence. Just recently, Shen and Yan [23] studied the duality between L-bornologies and L-topologies. It is shown that the category of L-bornological locally convex L-topological vector spaces can be embedded in the category of L-convex L-bornological vector spaces as a subcategory, and the category of L-topological convex L-bornological vector spaces can be embedded in the category of locally convex L-topological vector spaces as a subcategory. Additionally, Liang et al. [16] introduced the concepts of (L, M)-fuzzy bornological spaces and (L, M)-fuzzy bornological vector spaces.

On the other hand, Šostak and Uļjane [24] develop an alternative approach to the " fuzzification" of the concept of bornology. Namely there they defined an L-valued bornology (or called L-fuzzifying bornology) on a set X. This L-fuzzifying bornology is a mapping $\mathcal{B} : 2^X \to L$ satisfying some certain L-valued axioms. Also, this mapping in a certain sense determines *the degree of boundedness* $\mathcal{B}(A) \in L$ of a set $A \subseteq X$. In 2022, Jin and Yan [13] introduced the concept of fuzzifying bornological linear spaces, inspired by the literature [24], while considering the necessary and sufficient conditions for compatibility between fuzzifying bornologies and linear structures. Recently, Shen and Yan [22] discussed the fuzzifying bornologies induced by fuzzy pseudo-norms and proved that v. Neumann fuzzifying bornology is separated in fuzzy pseudo-normed linear spaces if and only if the fuzzifying topology determined by a fuzzy pseudo-norm is Hausdorff. This study is the first to explore the relationship between fuzzifying topological linear spaces and fuzzifying bornological linear spaces in the frame of the semantical method of continuous-valued logic.

Locally convex spaces and convex bornological spaces are known to exhibit a duality relationship (reference [8]). Previous literature has only discussed the relationships between the properties of fuzzifying bornologies and fuzzifying topologies in linear spaces [22]. However, no progress has been made in the study of the theory of convex fuzzifying bornological linear spaces. Furthermore, the duality between convex fuzzifying bornological spaces locally convex fuzzifying spaces has not been studied. The main objective of this paper is to initiate research in this area. This study will establish the relationships between fuzzifying topological structure and fuzzifying bornological structure in fuzzy functional analysis. Additionally, it will promote further research on fuzzifying bornological linear spaces, specifically contributing to the establishment of the famous closed graph theorem. The focus of this study is to investigate the elementary properties of the duality between convex fuzzifying bornologies and locally convex topologies. This can be considered as the inaugural attempt to study this internal duality using the semantical method.

The introduction concludes with an outline of the subsequent sections of the paper. In section 2, we recall some basic definitions and fundamental results. In section 3, we introduce the concept of L-bornivorous sets and propose an approach to convex fuzzifying bornology that can generate a locally convex fuzzifying topology. Furthermore, we demonstrate that the v. Neumann fuzzifying bornology is the coarsest convex fuzzifying bornology that aligns with locally convex fuzzifying topology. In section 4, we introduce the concepts of fuzzifying topological bornologies and fuzzifying bornological topologies, and explore their equivalent conditions. Additionally, we provide some characterizations of fuzzifying bornological bornologies.

2. Preliminaries

In this section, we review some necessary notions and fundamental results which are used in the sequel.

Throughout this paper, $\mathbb{K}$ represents a field of real or complex numbers, X always denotes a linear space over $\mathbb{K}$. 2^X and $2^{(X)}$ denote the classes of all crisp and finite crisp subsets of X, respectively. The notation $N(X)$ denotes the set of all sequences in X.

According to the terminology [8], a subset A of X is a *disk* if A is both convex and circled. The notation *disk*(X) denotes the set of all disks in X. For any subset A of X, the symbols $co(A), \Gamma(A)$ denote the convex hull of A and *disk* hull of A, respectively.

Definition 2.1. ([20, 24, 25]) An $([0, 1], \wedge)$-valued bornology on a set X is a mapping $\mathcal{B} : 2^X \to [0, 1]$ satisfying the following conditions:

(B1) for all $x \in X$, $\mathcal{B}(\{x\}) = 1$;

(B2) if $U \subseteq V \subseteq X$, then $\mathcal{B}(V) \leq \mathcal{B}(U)$;

(B3) for all $U, V \subset X$, $\mathcal{B}(U \cup V) \geq \mathcal{B}(U) \wedge \mathcal{B}(V)$ holds. The pair $(X, \mathcal{B})$ is called an $([0,1], \wedge)$-valued bornological space and the value $\mathcal{B}(A)$ is interpreted as the degree of boundedness of a set A in the space $(X, \mathcal{B})$.

From now on, an $([0,1], \wedge)$-valued bornology on a set X is also called a fuzzifying bornology on a set X.

Remark 2.2. The axiom (B3) is stated that for a continuous t-norm $*$ instead of $\wedge$ in the original paper [24]. As A. Šostak et al. [24] pointed out, in case of $* = \wedge$, the axiom (B2) is redundant since it follows from axiom (B3). Hence the axioms (B2) and (B3) may be replaced by the following axiom (B3)$'$:
(B3)$'$: $\forall U, V \subseteq X$, $\mathcal{B}(U \cup V) = \mathcal{B}(U) \wedge \mathcal{B}(V)$.

Let $\mathcal{B}(X, [0,1], \wedge)$ stand for the family of all fuzzifying bornologies on X. A partial order relation $\leq$ on $\mathcal{B}(X, [0,1], \wedge)$ by setting for $\mathcal{B}_1, \mathcal{B}_2 \in \mathcal{B}(X, [0,1], \wedge)$:

$\mathcal{B}_1 \leq \mathcal{B}_2$ if and only if $\mathcal{B}_1(A) \geq \mathcal{B}_2(A)$, $\forall A \in 2^X$,

and say in this case that $\mathcal{B}_1$ is coarser, or smaller than $\mathcal{B}_2$, and $\mathcal{B}_2$ is finer, or larger than $\mathcal{B}_1$.

Definition 2.3. ([24]) A mapping $f : (X, \mathcal{B}_X) \to (Y, \mathcal{B}_Y)$ of fuzzifying bornological spaces is called bounded if $\mathcal{B}_X(A) \leq \mathcal{B}_Y(f(A))$ for all $A \in 2^X$.

Definition 2.4. ([13]) Let X be a linear space over $\mathbb{K}$. A fuzzifying bornology $\mathcal{B}$ on X is said to be a linear fuzzifying bornology on X, if the following two mappings are bounded: $f : X \times X \to X$, defined by $(x, y) \to x + y$; $g : \mathbb{K} \times X \to X$, defined by $(k, x) \to kx$, where $X \times X$ and $\mathbb{K} \times X$ are equipped with the corresponding product fuzzifying bornologies $\mathcal{B} \times \mathcal{B}$ and $\mathcal{B}_{\mathbb{K}} \times \mathcal{B}$ (here $\mathcal{B}_{\mathbb{K}}$ is the fuzzifying bornology determined by the crisp bornology on $\mathbb{K}$, i.e., $\mathcal{B}_{\mathbb{K}}(A) = 1$ whenever A is a crisp bounded set in $\mathbb{K}$, and $\mathcal{B}_{\mathbb{K}}(A) = 0$ if A is not bounded in $\mathbb{K}$) which is defined as

$$(\mathcal{B} \times \mathcal{B})(A \times B) = \mathcal{B}(A) \wedge \mathcal{B}(B) \text{ for all } A, B \subseteq X.$$

We call any pair $(X, \mathcal{B})$ consisting of a linear space and a linear fuzzifying bornology a fuzzifying bornological linear space on X.

Theorem 2.5. ([13]) *Let $\mathcal{B}$ be a fuzzifying bornology on X. Then $\mathcal{B}$ is a linear fuzzifying bornology if and only if satisfies the following conditions: for all $U, V \subseteq X$*
(B4) $\mathcal{B}(U + V) \geq \mathcal{B}(U) \wedge \mathcal{B}(V)$;
(B5) $\mathcal{B}(\lambda U) \geq \mathcal{B}(U)$, *for all* $\lambda \in \mathbb{K}$;
(B6) $\mathcal{B}\left(\bigcup_{|\alpha| \leq 1} \alpha U\right) \geq \mathcal{B}(U)$.

According to the terminology adopted in [8], a crisp bornological linear space $(X, \mathcal{B})$ is separated if $\{\theta\}$ is the only bounded linear subspace of X. Naturally, one can generalize this property from the crisp case to the fuzzy setting as follows.

Definition 2.6. ([13]) Let $(X, \mathcal{B})$ be a fuzzifying bornological linear space. Then the degree to which $(X, \mathcal{B})$ is separated is defined by

$$[\mathrm{S}(X, \mathcal{B})] = \bigwedge_{\substack{M \neq \{\theta\} \\ M \in Svec(X)}} (1 - \mathcal{B}(M)),$$

where the notation $Svec(X)$ denotes the set of all linear subspaces of X.

Definition 2.7. ([13]) Let $(X, \mathcal{B})$ be a fuzzifying bornological linear space and $\{x_n\} \in N(X)$. The degree to which $\{x_n\}$ is convergent to x bornologically is

$$[x_n \xrightarrow{M} x] = \bigvee_{\substack{A \in Bal(X) \\ \lambda_n \to 0}} \{\mathcal{B}(A) : \forall n \in \mathbb{N}, x_n - x \in \lambda_n A\},$$

where $Bal(X)$ means the family of all balanced sets in X(A subset $B \subseteq X$ is called balanced if $\lambda B \subseteq B$ whenever $\lambda \in \mathbb{K}$ and $|\lambda| \leq 1$).

Definition 2.8. ([13]) Let $(X,\mathscr{B})$ be a fuzzifying bornological linear space and $A\subseteq X$. Then the degree to which A is bornologically closed is defined as follows:

$$[BC(A)]=\bigwedge_{\substack{\{x_n\}\subseteq A\\ x\notin A}}\ \bigwedge_{\substack{B\in Bal(X)\\ \lambda_n\to 0}}\{1-\mathscr{B}(B):\forall n\in\mathbb{N}, x_n-x\in\lambda_n B\}.$$

If $[BC(A)]=1$, we also called A is bornologically closed.

Definition 2.9. ([26]) Let (X,τ) be a fuzzifying topological space. For any $x\in X$, $\mathscr{N}_x:2^X\to[0,1]$ is called a fuzzifying neighborhood system of x which is defined as follows: for any $A\in 2^X$, $\mathscr{N}_x(A)=\bigvee_{x\in B\subseteq A}\tau(B)$.

Definition 2.10. ([26]) Let (X,τ) be a fuzzifying topological space. Then the degree to which (X,τ) is T_2 (Hausdorff) is defined as follows:

$$[T_2(X,\tau)]=\bigwedge_{x\neq y}\ \bigvee_{U\bigcap V=\emptyset}(\mathscr{N}_x(U)\wedge\mathscr{N}_y(V)).$$

Definition 2.11. ([2, 3, 5, 7]) Let X be a linear space over a field $\mathbb{K}$. A fuzzy set N in $X\times[0,+\infty)$ is said to be a fuzzy pseudo-norm on X if the following conditions are satisfied:

(N1) $N(x,0)=0,(\forall)x\in X$;
(N2) $N(\theta,t)=1,(\forall)t>0$;
(N3) $N(kx,t)=N(x,\frac{t}{|k|}),(\forall)x\in X,k\in\mathbb{K},k\neq 0$;
(N4) $N(x+y,t+s)\geq N(x,t)\wedge N(y,s),(\forall)x,y\in X$;
(N5) $\forall x\in X, N(x,\cdot)$ is left continuous and $\lim_{t\to\infty}N(x,t)=1$.

The pair (X,N) is called a fuzzy pseudo-normed linear space.
If a fuzzy pseudo-norm N also satisfies the following condition
(N2$'$) $N(x,t)=1,(\forall)t>0$ if and only if $x=\theta$,
then it will be called a fuzzy norm on X.

By the axioms of (N2) and (N4) of fuzzy pseudo-norm N, it follows that $N(x,\cdot):[0,\infty)\to[0,1]$ is non-decreasing for all $x\in X$.

Remark 2.12. For any fuzzy pseudo-norm N, we will use the notation $N(x,0+)$ denotes the right limit of the real function $N(x,t)$ at 0. Since $N(x,\cdot)$ is non-decreasing and left continuous, we have $N(x,0+)=\bigwedge_{t>0}N(x,t)$.

Theorem 2.13. ([21, 28]) *Let (X,τ) be a fuzzifying topological linear space on $\mathbb{K}$ and $\mathscr{N}_\theta(\cdot)$ be its corresponding fuzzifying neighborhood system of the neutral element. Then it has the following properties:*

(P1) $\mathscr{N}_\theta(X)=1$;
(P2) $\forall U\subseteq X,\mathscr{N}_\theta(U)>0\Rightarrow\theta\in U$;
(P3) $\forall U,V\subseteq X,\mathscr{N}_\theta(U\cap V)=\mathscr{N}_\theta(U)\wedge\mathscr{N}_\theta(V)$;
(P4) $\forall W\subseteq X,\mathscr{N}_\theta(W)\leq\bigvee_{U+V\subseteq W}\mathscr{N}_\theta(U)\wedge\mathscr{N}_\theta(V)$;
(P5) $\forall U\subseteq X,x\in X,\mathscr{N}_\theta(U)>0\Rightarrow\exists\varepsilon>0$ *such that* $kx\in U$ *for all* $|k|<\varepsilon$;
(P6) $\forall U\subseteq X,\mathscr{N}_\theta(U)>a$ *implies there exists a circled set* $V\subseteq U$ *such that* $\mathscr{N}_\theta(V)>a$.

Conversely, let X be a linear space over $\mathbb{K}$ and consider a set-valued function $\mathscr{N}_\theta(\cdot):2^X\to[0,1]$ which satisfies the conditions (P1)-(P6). Then there exists a fuzzifying topology $\tau_{\mathscr{N}}$ on X such that $(X,\tau_{\mathscr{N}})$ be a fuzzifying topological linear space and $\mathscr{N}_\theta(\cdot)$ is a fuzzifying neighborhood system of the neutral element.

Definition 2.14. ([21]) Let (X,τ) be a fuzzifying topological linear space. Then the unary fuzzy predicates $Bd\in\mathscr{F}\left(2^X\right)$, called fuzzy boundedness, is defined as follows:

$$Bd(A)\triangleq\left(\forall V\in 2^X\right)(V\in\mathscr{N}_\theta\to_L(\exists\lambda\in\mathbb{K})(A\subseteq\lambda V))$$

for any $A\in 2^X$. Where the notation $\to_L$ means the Łukasiewicz residuum.

Intuitively, the degree to which A is bounded is

$$Bd(A) = \bigwedge_{U\subseteq X}\{1 - \mathscr{N}_\theta(U) : A \notin Abs(U)\}.$$

Where $Abs(U) \underset{def}{=} \{A : \exists\delta > 0, \forall\lambda \in \mathbb{K}, |\lambda| \geq \delta, A \subseteq \lambda U\}$.

Theorem 2.15. ([13]) *Let* (X, τ) *be a fuzzifying topological linear space, then Bd given by Definition* 2.13 *is a linear fuzzifying bornology.*

In classical topological linear spaces, it is well recognized that the collection of all bounded sets constitutes a linear bornology known as the von Neumann bornology [8]. Following the convention for ordinary bornological linear spaces, we refer to $Bd(\cdot)$ as a fuzzifying von Neumann bornology, denoted as b_τ.

Definition 2.16. ([14]) Let $(X, \mathcal{B})$ be a fuzzifying bornological linear space. Then the mapping $Bv : 2^X \to [0, 1]$ is called fuzzifying bornivorous if it defined as follows:

$$P \in Bv \triangleq (\forall A \subseteq X)(A \in \mathcal{B}) \to_L (A \in Abs(P)).$$

Moreover, the degree to which P is a bornivorous set is

$$Bv(P) = \bigwedge_{A\subseteq X}\{1 - \mathcal{B}(A) : A \notin Abs(P)\}.$$

Theorem 2.17. ([14]) *Let* $(X, \mathscr{B})$ *be a fuzzifying bornological linear space. For all* $P, Q \subseteq X$, *the following statements holds:*

(1) $\forall P \in 2^X, Bv(P) > 0 \Longrightarrow \theta \in P$;
(2) $Bv(P \bigcap Q) \geq Bv(P) \wedge Bv(Q)$;
(3) if $P \subseteq Q$, *then* $Bv(P) \leq Bv(Q)$;
(4) for all $\alpha \in \mathbb{K} \setminus \{0\}$, $Bv(\alpha P) = Bv(P)$;
(5) $Bv\left(\bigcup_{|\alpha|\leq 1} \alpha P\right) \geq Bv(P)$.

3. The relationships between convex fuzzifying bornological linear spaces and locally convex fuzzifying topological linear spaces

In this section, we introduce the concepts of convex fuzzifying bornological linear spaces and locally convex fuzzifying topological linear spaces, along with providing some examples. Next, we prove that the v. Neumann fuzzifying bornology b_τ induced by the locally convex fuzzifying topology τ is indeed the coarsest convex fuzzifying bornology that is compatible with τ. Finally, we utilize the mapping of fuzzifying bornivorous to deduce the finest locally convex fuzzifying topology, denoted as $(X, \tau_{\mathcal{B}})$, which is compatible with the convex fuzzifying bornology $\mathcal{B}$.

Definition 3.1. A fuzzifying bornological linear space $(X, \mathcal{B})$ is called a convex fuzzifying bornological linear space if it satisfies the following condition:

$$\mathcal{B}(A) \leq \mathcal{B}(co(A)) \text{ for all } A \in 2^X.$$

Remark 3.2. By the condition (B6), for all $A \in 2^X$, $\mathcal{B}(A) \leq \mathcal{B}(\Gamma(A))$. Moreover, if we restrict $\mathcal{B} : 2^X \to \{0, 1\}$, it follows that any crisp convex bornological linear space must be a convex fuzzifying bornological linear space.

Example 3.3. Let $(X, N, \wedge)$ be a fuzzy pseudo-normed space, the mapping $\mathcal{B}(\cdot) : 2^X \to [0,1]$ be defined as follows:

$$\mathcal{B}(A) = \bigvee_{t>0} \bigwedge_{x\in A} N(x,t), \qquad \forall A \in 2^X.$$

Then the pair $(X, \mathcal{B})$ is a convex fuzzifying linear bornological space.

In fact, by [22, Theorem 3.1], the pair $(X, \mathcal{B})$ is a fuzzifying linear bornological space. It suffices to prove that $\mathcal{B}(A) \le \mathcal{B}(co(A))$ for all $A \in 2^X$. At first, we may prove that $\bigwedge_{p,q\in A} N(p-q,t) \le \bigwedge_{x,y\in co(A)} N(x-y,(t)$ for all $t > 0$. Otherwise, there is $t_0 > 0$ such that

$$\bigwedge_{p,q\in A} N(p-q,t_0) > \bigwedge_{x,y\in co(A)} N(x-y,t_0).$$

Then there exist $x_1, x_2 \in co(A)$ such that $N(x_1 - x_2, t_0) < \bigwedge_{p,q\in A} N(p-q,t_0)$. For $x \in X$, let $S_x(\bigwedge_{p,q\in A} N(p-q,t_0)) = \{y \in X : N_{x-y}(t_0) \ge \bigwedge_{p,q\in A} N(p-q,t_0)\}$.

Clearly, $S_x(\bigwedge_{p,q\in A} N(p-q,t_0)) \ne \emptyset$. In addition, $S_x(\bigwedge_{p,q\in A} N(p-q,t_0))$ is a convex set. Since for $y_1, y_2 \in S_x(\bigwedge_{p,q\in A} N(p-q,t_0))$ and $\alpha \in [0,1]$, we have

$$\begin{aligned} N(x-\alpha y_1-(1-\alpha)y_2,t_0) &\ge \min\{N(\alpha(x-y_1),\alpha t_0), N((1-\alpha)(x-y_2),(1-\alpha)t_0)\} \\ &= \min\{N_{x-y_1}(t_0), N_{x-y_2}(t_0)\} \\ &\ge \min\{\bigwedge_{p,q\in A} N(p-q,t_0), \bigwedge_{p,q\in A} N(p-q,t_0)\} \\ &= \bigwedge_{p,q\in A} N(p-q,t_0). \end{aligned}$$

It follows that $\alpha y_1 + (1-\alpha)y_2 \in S_x(\bigwedge_{p,q\in A} N(p-q,t_0))$, i.e., $S_x(\bigwedge_{p,q\in A} N(p-q,t_0))$ is a convex set. Assuming that $A \nsubseteq S_{x_1}(\bigwedge_{p,q\in A} N(p-q,t_0))$, which means that there exists $q \in A \setminus S_{x_1}(\bigwedge_{p,q\in A} N(p-q,t_0))$. Thus for every $p \in A$, $N(q-p,t_0) = N(p-q,t_0) \ge \bigwedge_{p,q\in A} N(p-q,t_0)$. we have $A \subseteq S_q(\bigwedge_{p,q\in A} N(p-q,t_0))$. From the convexity of $S_q(\bigwedge_{p,q\in A} N(p-q,t_0))$, it deduces that $co(A) \subseteq S_q(\bigwedge_{p,q\in A} N(p-q,t_0))$. Further, from $q \notin S_{x_1}(\bigwedge_{p,q\in A} N(p-q,t_0))$, we obtain that $N_{x_1-q}(t_0) < \bigwedge_{p,q\in A} N(p-q,t_0)$ and so $x_1 \notin S_q(\bigwedge_{p,q\in A} N(p-q,t_0))$, which is a contradiction since $x_1 \in co(A)$.

If we suppose that $A \subseteq S_{x_1}(\bigwedge_{p,q\in A} N(p-q,t_0))$, then $co(A) \subseteq S_{x_1}(\bigwedge_{p,q\in A} N(p-q,t_0))$. But from $N_{x_1-x_2}(t_0) < \bigwedge_{p,q\in A} N(p-q,t_0)$ it follows that $x_2 \in co(A) \setminus S_{x_1}(\bigwedge_{p,q\in A} N(p-q,t_0))$ and this is a contradiction.

Hence the conclusion $\bigwedge_{p,q\in A} N(p-q,t) \le \bigwedge_{x,y\in co(A)} N(x-y,t)$ for all $t > 0$ holds. So,

$$\mathcal{B}(A) = \bigvee_{t>0} \bigwedge_{p,q\in A} N(p-q,t) \le \bigvee_{t>0} \bigwedge_{x,y\in co(A)} N(x-y,t) = \mathcal{B}(co(A)).$$

Theorem 3.4. *Let X be a linear space, $\mathscr{P} = \{N_j\}_{j\in J}$ be a family of fuzzy pseudo-norms on X indexed by a non-empty set J and the mapping $\mathcal{B} : 2^X \to [0,1]$ be defined by $\mathcal{B}(A) = \bigwedge_{j\in J} Bd_j(A) = \bigwedge_{j\in J} \bigvee_{t>0} \bigwedge_{x,y\in A} N_j(x-y,t)$. Then $(X, \mathcal{B})$ is a convex fuzzifying bornological linear space. Moreover, $[S(X, \mathcal{B})] = \bigwedge_{x\ne\theta} \bigvee_{j\in J}(1 - N_j(x, 0+))$.*

Proof. First, it is easy to check that $(X, \mathcal{B})$ is a fuzzifying bornological linear space. By Example 3.3, it is clear $\mathcal{B}(A) = \bigwedge_{j\in J} Bd_j(A) \le \bigwedge_{j\in J} Bd_j(co(A)) = \mathcal{B}(co(A))$. Thus $(X, \mathcal{B})$ is a convex fuzzifying bornological linear space.

It is left to prove $[S(X, \mathcal{B})] = \bigwedge_{x\ne\theta} \bigvee_{j\in J}(1 - N_j(x, 0+))$. In fact, for any $a > [S(X, \mathcal{B})] = \bigwedge_{\substack{M\ne\{\theta\} \\ M\in Svec(X)}} \{1 - \mathcal{B}(M)\}$, there exists a linear subspace M of X with $M \ne \{\theta\}$ such that $1 - \mathcal{B}(M) < a$. Then $\mathcal{B}(M) = \bigwedge_{j\in J} \bigvee_{t>0} \bigwedge_{x\in M} N_j(x,t) > 1 - a$, it follows that there is $t_j > 0$ such that $N_j(x, t_j) > 1 - a$ for all $x \in M$ and $j \in J$. Let $x_0 \in M$, obviously,

$N_j(x_0, t_j) > 1 - a$. Since M is a linear subspace of X, it is clear $nx_0 \in M$ for all $n \in \mathbb{N}$, thus $N_j(nx_0, t_j) > 1 - a$. Equivalently, $N_j(x_0, \frac{t_j}{n}) > 1 - a$ for all $n \in \mathbb{N}$. Since $N_j(x_0, t)$ is increasing in the variable t, it deduces that $N_j(x_0, 0+) \geq 1 - a$, i.e., $a \geq 1 - N_j(x_0, 0+)$. Hence $[\mathrm{S}(X, \mathcal{B})] \geq \bigvee_{j\in J}(1 - N_j(x_0, 0+)) \geq \bigwedge_{x\neq\theta}\bigvee_{j\in J}(1 - N_j(x, 0+))$.

On the other hand, for every $a > \bigwedge_{x\neq\theta}\bigvee_{j\in J}(1 - N_j(x, 0+))$, there is $x_0 \neq \theta$ such that $a > \bigvee_{j\in J} 1 - N_j(x_0, 0+) = \bigvee_{j\in J}\bigvee_{t>0} 1 - N_j(x_0, t)$. This implies that $N_j(x_0, t) > 1 - a$ for all $j \in J$ and $t > 0$. Put $M_1 = \{kx_0 \mid k \in \mathbb{K}\} = \mathrm{Span}\{x_0\}$, clearly, $M_1 \neq \{\theta\}$ and M_1 is a linear subspace of X. For any fixed $t_0 > 0$, we have $\bigwedge_{z\in M_1} N_j(z, t_0) \geq 1 - a$. Furthermore, we have

$$\mathcal{B}(M_1) = \bigwedge_{j\in J}\bigvee_{t>0}\bigwedge_{z\in M_1} N_j(z, t) \geq \bigwedge_{j\in J}\bigwedge_{z\in M_1} N_j(z, t_0) \geq 1 - a.$$

Thus $[\mathrm{S}(X, \mathcal{B})] = \bigwedge_{\substack{M\neq\{\theta\}\\ M\in Svec(X)}} \{1 - \mathcal{B}(M)\} \leq 1 - \mathcal{B}(M_1) \leq a$. By the arbitrariness of a, we have $[\mathrm{S}(X, \mathcal{B})] \leq \bigwedge_{x\neq\theta}\bigvee_{j\in J}(1 - N_j(x, 0+))$. So, the proof of the equality $[\mathrm{S}(X, \mathcal{B})] = \bigwedge_{x\neq\theta}\bigvee_{j\in J}(1 - N_j(x, 0+))$ is completed. □

Definition 3.5. Let (X, τ) be a fuzzifying topological linear space. We say that (X, τ) is a locally convex fuzzifying topological linear space, if there is a mapping $\mathscr{U} : 2^X \to [0, 1]$ with $\mathscr{U} \leq \mathscr{N}$ such that $\mathscr{N}(U) \leq \bigvee_{\substack{W\subseteq U\\ W\in disk(X)}} \mathscr{U}(W)$ for all $U \in 2^X$.

Example 3.6. Let $(X, N, \wedge)$ be a fuzzy pseudo-normed space. Then there is a fuzzifying topology τ_N such that (X, τ_N) is a locally convex fuzzifying topological linear space.

In fact, let $\mathscr{N}(U) = \bigvee_{\varepsilon>0}\bigwedge_{z\notin U}(1 - N(z, \varepsilon))$, by Theorem 4.1 in [12], there exists a fuzzifying topology τ^N on X such that $\mathscr{N}(\cdot)$ is a fuzzifying neighborhood system of θ with respect to τ^N. Denote $\mathscr{A} = \{B(\theta, \frac{1}{n}, r) : n \in \mathbb{N},\ r \text{ is a rational number in } [0, 1]\}$, where $B(\theta, \frac{1}{n}, r) = \{x : N(x, \frac{1}{n}) \geq 1 - r\}$. Clearly, $B(\theta, \frac{1}{n}, r)$ is a absolute convex set for any $n \in \mathbb{N}$ and r. Put

$$\mathscr{U}(W) = \begin{cases} \mathscr{N}(W), & W \in \mathscr{A}, \\ 0, & \text{others} \end{cases}.$$

By Theorem 4.5 in [29], it follows that $\mathscr{N}(U) \leq \bigvee_{\substack{W\subseteq U\\ W\in disk(X)}} \mathscr{U}(W)$ for all $U \in 2^X$. Hence (X, τ_N) is a locally convex fuzzifying topological linear space.

Definition 3.7. Let (X, τ) be a fuzzifying topological linear space and $\mathcal{B}$ be a fuzzifying bornology on X. We say that $\mathcal{B}$ and τ are compatible if $\mathcal{B}(A) \leq b_\tau(A)$ for all $A \in 2^X$, where b_τ is v. Neumann fuzzifying bornology determined by (X, τ).

Theorem 3.8. *Let (X, τ) be a locally convex fuzzifying topological linear space and b_τ be its v. Neumann fuzzifying bornology. Then (X, b_τ) is a convex fuzzifying bornological linear space and b_τ is the coarsest convex fuzzifying bornology compatible with τ.*

Proof. First we will prove (X, b_τ) is a convex fuzzifying bornological linear space. It only needs to prove that $b_\tau(B) \leq b_\tau(co(B))$ for all $B \subseteq X$. Since $b_\tau(co(B)) = \bigwedge_{U\subseteq X}\{1 - \mathscr{N}(U) : co(B) \notin Abs(U)\}$, then for each $t > b_\tau(co(B)) = \bigwedge_{U\subseteq X}\{1 - \mathscr{N}(U) : co(B) \notin Abs(U)\}$, there is $U \subseteq X$ with $co(B) \notin Abs(U)$ such that $1 - \mathscr{N}(U) < t$, i.e., $\mathscr{N}(U) > 1 - t$. Because (X, τ) is a locally convex fuzzifying topological linear space, we have a $W \subseteq X$ with $W \in disk(X)$ satisfying $\mathscr{U}(W) > 1 - t$. It implies $\mathscr{N}(W) \geq \mathscr{U}(W) > 1 - t$. At this moment, we may claim that $B \notin Abs(W)$. Or else, there is $\delta > 0$ such that $B \subseteq sW$ for all $|s| \geq \delta$. Then we have $co(B) \subseteq sW \subseteq sU$. This contradicts with $co(B) \notin Abs(U)$. Thus $b_\tau(B) \leq 1 - \mathscr{N}(W) < t$. Hence $b_\tau(B) \leq b_\tau(co(B))$ holds.

On the other hand, clearly b_τ is a convex fuzzifying bornology compatible with τ. Assume $\mathcal{B}_1$ is a convex fuzzifying bornology compatible with τ. Then for each $B \subseteq X$, $\mathcal{B}_1(B) \leq b_\tau(B)$. Hence b_τ is the coarsest convex fuzzifying bornology compatible with τ. □

Theorem 3.9. *Let $(X, \mathcal{B})$ be a convex fuzzifying bornological linear space and the mapping $Bv : 2^X \to [0,1]$ be fuzzifying bornivorous. Then there exists the finest locally convex fuzzifying topology τ on X such that which is compatible with the fuzzifying bornology $\mathcal{B}$ and Bv is a base of fuzzifying neighborhood of θ.*

Proof. First we will prove that the mapping $\mathscr{N}(V) = \bigvee_{\substack{W \subseteq V \\ W \in disk(X)}} Bv(W)$ is fuzzifying neighborhood system of θ. It suffices to prove the set mapping $\mathscr{N}(\cdot)$ satisfies the all conditions in Theorem 2.13.

(1). If $\mathscr{N}(V) > 0$, there exists $W \subseteq V$ with $W \in disk(X)$ such that $Bv(W) > 0$. It follows that $\theta \in W \subseteq V$ by Theorem 2.17(1).

(2). $\mathscr{N}(X) = \bigvee_{\substack{W \subseteq V \\ W \in disk(X)}} Bv(W) \geq Bv(X) = 1$.

(3). For each $a < \mathscr{N}(U) \bigvee \mathscr{N}(V)$, there exist $W_1 \subseteq U, W_2 \subseteq V$ with $W_1, W_2 \in disk(X)$ such that $a < Bv(W_1)$ and $a < Bv(W_2)$. By Theorem 2.17(2), it follows that $a < Bv(W_1) \wedge Bv(W_2) \leq Bv(W_1 \cap W_2)$ and $W_1 \cap W_2 \in disk(X)$. Then $a \leq \mathscr{N}(U \cap V)$. Thus $\mathscr{N}(U) \bigvee \mathscr{N}(V) \leq \mathscr{N}(U \cap V)$. The converse is clear. So, $\mathscr{N}(U) \bigvee \mathscr{N}(V) = \mathscr{N}(U \cap V)$.

(4). For each $a < \mathscr{N}(V)$, there exists a subset $W \subseteq V$ that belongs to the disk of X, satisfying $a < Bv(W)$. Utilizing the fact that $\frac{1}{2}W + \frac{1}{2}W \subseteq W \subseteq V$ and referring to Theorem 2.17(4), we can conclude that $\mathscr{N}(\frac{1}{2}W) \geq Bv(\frac{1}{2}W) = Bv(W)$. Hence,

$$a \leq Bv(\tfrac{1}{2}W) \leq \bigvee_{V_1 + V_2 \subseteq V} (\mathscr{N}(V_1) \wedge \mathscr{N}(V_2)).$$

Then $\mathscr{N}(V) \leq \bigvee_{V_1 + V_2 \subseteq V} (\mathscr{N}(V_1) \wedge \mathscr{N}(V_2))$.

(5). For all $x \in X, U \subseteq X$, if $\mathscr{N}(V) > 0$, there exists $W \subseteq V$ with $W \in disk(X)$ such that $Bv(W) > 0$. Since $\mathcal{B}(\{x\}) = 1$, we have $\{x\} \in Abs(W)$. Then there exists $\delta > 0$ such that $x \in sW \subseteq sU$ for all $|s| \geq \delta$.

(6). For each $a < \mathscr{N}(V)$, there exists $W \subseteq V$ with $W \in disk(X)$ such that $a < Bv(W)$. Then $\mathscr{N}(W) \geq Bv(W) > a$.

Hence there exists a fuzzifying topology τ on X such that (X, τ) is a locally convex fuzzifying topological linear space and Bv is a base of fuzzifying neighborhood of θ.

Secondly we will prove that the vector fuzzifying topology τ is compatible with the fuzzifying bornology $\mathcal{B}$. The v. Neumann fuzzifying bornology determined by τ is denoted by b_τ. It needs to prove that $\mathcal{B}(A) \leq b_\tau(A)$ for all $A \subseteq X$. By the definition of V. Neumann fuzzifying bornology, $b_\tau(A) = \bigwedge_{U \subseteq X} \{1 - \mathscr{N}(U) : A \notin Abs(U)\}$. We only need to prove that $1 - \mathscr{N}(U) \geq \mathcal{B}(A)$ for all $U \in 2^X$ satisfying $A \notin Abs(U)$, i.e., $\mathscr{N}(U) \leq 1 - \mathcal{B}(A)$.

In fact, for each $a < \mathscr{N}(U)$, there exists $W \subseteq V$ with $W \in disk(X)$ such that $a < Bv(W)$. Then for any $B \notin Abs(W)$, we have $a < 1 - \mathcal{B}(B)$. At this point, we assert that $A \notin Abs(W)$. Otherwise, if $A \in Abs(W)$, it follows that $A \in Abs(U)$ from the fact $W \subseteq U$. It deduces a contradiction. Thus $A \notin Abs(W)$. So, $a < 1 - \mathcal{B}(A)$. Thus $\mathscr{N}(U) \leq 1 - \mathcal{B}(A)$. This means that τ is compatible with the fuzzifying bornology $\mathcal{B}$.

Finally, we will prove that τ is the finest locally convex fuzzifying topology τ on X such that which is compatible with the fuzzifying bornology $\mathcal{B}$. Let τ_1 is a locally convex fuzzifying topology which is compatible with $\mathcal{B}$, i.e. $\mathcal{B}(A) \leq b_{\tau_1}(A)$ for all $A \in 2^X$. Since $\tau(U) = \bigwedge_{x \in U} \mathscr{N}_x(U) = \bigwedge_{x \in U} \mathscr{N}(U - x)$, it is sufficient to prove $\mathscr{N}^1(U) \leq \mathscr{N}(U)$ for all $U \subseteq X$, where $\mathscr{N}^1(\cdot)$ is a fuzzifying neighborhood system of θ with respect to fuzzifying topology τ_1.

As a matter of fact, for each $a < \mathscr{N}^1(U)$, there is a $W \in disk(X)$ with $W \subseteq U$ such that $a < \mathscr{U}^1(W) \leq \mathscr{N}^1(W)$. For all $B \subseteq X$ with $B \notin Abs(W)$, since

$$\mathcal{B}(B) \leq b_{\tau_1}(B) = \bigwedge_{U \subseteq X} \{1 - \mathscr{N}^1(V) : B \notin Abs(V)\} \leq 1 - \mathscr{N}^1(W) < 1 - a.$$

Thus

$$Bv(W) = \bigwedge_{B \subseteq X} \{1 - \mathscr{B}(B) : B \notin Abs(W)\} \geq a.$$

It implies $\mathscr{N}(U) = \bigvee_{\substack{W \subseteq U \\ W \in disk(X)}} Bv(W) \geq a$. Furthermore, $\mathscr{N}^1(U) \leq \mathscr{N}(U)$. This completes the proof. □

Remark 3.10. The fuzzifying topology τ defined in Theorem 3.9 is referred to as the locally convex fuzzifying topology associated with the fuzzifying bornology $\mathcal{B}$ of X and is denoted by $\tau_{\mathcal{B}}$.

4. Relationships between categories of FT-CFBLS and LCFTLS, FB-LCFTLS and CFBLS

In this section, we will introduce the concepts of the *fuzzifying topological bornology* and the *fuzzifying bornological topology*. Through this discussion, we aim to explore the relationship between the category **FT-CFBLS** of *fuzzifying topological convex bornological linear spaces* and the category **LCFTLS** of locally convex fuzzifying topological linear spaces, as well as the relationship between the category **FB-LCFTLS** of *fuzzifying bornological locally convex spaces* and the category **CFBLS** of convex fuzzifying bornological linear spaces. We will show that the category **FT-CFBLS** can be embedded in the the category **LCFTLS** as a reflective subcategory, and the category **FB-LCFTLS** can also be embedded in the the category **CFBLS** as a reflective subcategory. Meanwhile, the category **FT-CFBLS** is topological over the category of linear spaces with respect to the expected forgetful functor.

At first, we will discuss the categorical relations between locally convex fuzzifying topological linear spaces and convex fuzzifying bornological linear spaces.

Theorem 4.1. *Let $(X,\tau_X),(Y,\tau_Y)$ be two fuzzifying topological linear spaces and f be a linear map from X to Y. Then $[C(f)]\le[Bd(f)]$, where $[Bd(f)]$ is corresponding to the v. Neumann fuzzifying bornologies b_{τ_X} and b_{τ_Y}.*

Proof. For each $0<t<[C(f)]$ and for every $A\subseteq X$. If $t_1<t-b_{\tau_Y}(f^{\rightarrow}(A))$, we have $1+t_1-t<1-b_{\tau_Y}(f^{\rightarrow}(A))=\bigvee_{W\subseteq Y}\{\mathcal{N}_Y(W):f^{\rightarrow}(A)\notin Abs(W)\}$. Then there exist $W\subseteq Y$ with $f^{\rightarrow}(A)\notin Abs(W)$ such that $1+t_1-t<\mathcal{N}_Y(W)$. On the other hand, form the hypothesis $t<[C(f)]$, we have $t<1-\mathcal{N}_Y(W)+\mathcal{N}_X(f^{\leftarrow}(W))$. That is to say $\mathcal{N}_Y(W)<1-t+\mathcal{N}_X(f^{\leftarrow}(W))$. So $t_1<\mathcal{N}_X(f^{\leftarrow}(W))$. In this case, we have $A\notin Abs(f^{\leftarrow}(W))$. Otherwise, there exists $\delta>0$ such that $A\subseteq sf^{\leftarrow}(W)$ for all $|s|\ge\delta$. It follows that $f^{\rightarrow}(A)=f^{\rightarrow}(sf^{\leftarrow}(W))=sf^{\rightarrow}(f^{\leftarrow}(W))\subseteq sW$ for all $|s|\ge\delta$. It means that $f^{\rightarrow}(A)\in Abs(W)$. This contradicts to the fact $f^{\rightarrow}(A)\notin Abs(W)$. This implies $t_1<1-b_{\tau_X}(A)$. By the arbitrariness of t_1, we have $t-b_{\tau_Y}(f^{\rightarrow}(A))\le 1-b_{\tau_X}(A)$. Hence $t\le 1-b_{\tau_X}(A)+b_{\tau_Y}(f^{\rightarrow}(A))$ for all $A\subseteq X$. Furthermore $t\le[Bd(f)]$. Therefore the conclusion holds. □

Corollary 4.2. *Let $(X,\tau_X),(Y,\tau_Y)$ be two fuzzifying topological linear spaces and f be a continuous linear map. Then $f:(X,b_{\tau_X})\to(Y,b_{\tau_Y})$ is bounded.*

Theorem 4.3. *Let $(X,\mathcal{B})$ and $(Y,\mathcal{B}_1)$ be two convex fuzzifying bornological linear spaces, and let $f:(X,\mathcal{B})\to(Y,\mathcal{B}_1)$ be a bounded linear mapping. Then $f:(X,\tau_{\mathcal{B}})\to(Y,\tau_{\mathcal{B}_1})$ is continuous, where $\tau_{\mathcal{B}}$ is defined as Remark 3.10.*

Proof. It suffices to prove that $f:(X,\tau_{\mathcal{B}})\to(Y,\tau_{\mathcal{B}_1})$ is continuous at θ^X. Let W be a subset of Y and $0<\mathcal{N}_{\theta^Y}(W)$, then for every $a<\mathcal{N}_{\theta^Y}(W)$, by Theorem 3.9, there exists $U\subseteq W$ with $U\in disk(Y)$ such that $a<Bv(U)$. It follows that $a<1-\mathcal{B}_1(A)$ for all $A\notin Abs(U)$. Clearly, $f^{\leftarrow}(U)\in disk(X)$ and $f^{\leftarrow}(U)\subseteq f^{\leftarrow}(W)$. For any $B\notin Abs(f^{\leftarrow}(U))$, we have $f^{\rightarrow}(B)\notin Abs(U)$. Otherwise, there exists $\delta>0$ such that $f^{\rightarrow}(B)\subseteq\lambda U$ for all $|\lambda|\ge\delta$. Then $B\subseteq f^{\leftarrow}(\lambda U)=\lambda f^{\leftarrow}(U)$, this contradicts to the fact $B\notin Abs(f^{\leftarrow}(U))$. Since f is bounded, we have $\mathcal{B}(B)\le\mathcal{B}_1(f^{\rightarrow}(B))<1-a$. So, $Bv(f^{\leftarrow}(U))=\bigwedge_{B\subseteq X}\{1-\mathscr{B}(B):B\notin Abs(f^{\leftarrow}(U))\}\ge a$. Hence $\mathcal{N}_{\theta^X}(f^{\leftarrow}(W))=\bigvee_{\substack{V\subseteq f^{\leftarrow}(W)\\ V\in disk(X)}}Bv(V)\ge Bv(f^{\leftarrow}(U))\ge a$. It deduces that $\mathcal{N}_{\theta^Y}(W)\le\mathcal{N}_{\theta^X}(f^{\leftarrow}(W))$ by the arbitrariness of a. This means that $f:(X,\tau_{\mathcal{B}})\to(Y,\tau_{\mathcal{B}_1})$ is continuous. □

Remark 4.4. The category of convex fuzzifying bornological linear spaces and its bounded linear mappings is denoted by **CFBLS**. Similarly, the category of locally convex fuzzifying topological linear spaces and its continuous linear mappings is denoted by **LCFTLS**. By Theorem 3.9 and Corollary 4.2, we can establish the existence of a functor $\mathbb{B}$ from **LCFTLS** to **CFBLS**. Here $\mathbb{B}$ is defined as $\mathbb{B}$: **LCFTLS** $\to$ **CFBLS**. For any $(X,\tau)\in$ |**LCFTLS**|, the value $\mathbb{B}((X,\tau))$ is given by (X,b_τ), and for any linear mapping $f:(X,\tau_X)\to(Y,\tau_Y)$, we have $\mathbb{B}(f)=f$. The notation |**LCFTLS**| represents the set of all objects in the category **LCFTLS**. Meanwhile, by Theorem 3.9 and Theorem 4.3, there exists a functor $\mathbb{T}$ from **CFBLS** to **LCFTLS**, denoted as $\mathbb{T}$: **CFBLS** $\to$ **LCFTLS**. For any $(X,\mathcal{B})\in$ |**CFBLS**|, we have $\mathbb{T}((X,\mathcal{B}))=(X,\tau_{\mathcal{B}})$ and for all linear mapping $f:(X,\mathcal{B})\to(Y,\mathcal{B}_1)$, $\mathbb{T}(f)=f$.

Lemma 4.5. *Let (X,τ) be a locally convex fuzzifying topological linear space. Then $b_\tau = b_{\tau_{b_\tau}}$.*

Proof. For each $B \subseteq X$ and any $t < b_\tau(B)$, since $b_{\tau_{b_\tau}}(B) = \bigwedge_{U\subseteq X}\{1 - \mathscr{N}^{\tau_{b_\tau}}(U) : B \notin Abs(U)\}$. If there is $U \subseteq X$ with $B \notin Abs(U)$ satisfying $1 - \mathscr{N}^{\tau_{b_\tau}}(U) < t$, we get $W \subseteq U, W \in disk(X)$ satisfying $(Bv)^{b_\tau}(W) > 1 - t$. This implies that $1 - b_\tau(V) > 1 - t$ whenever $V \notin Abs(W)$. We claim $B \notin Abs(W)$, if not, $B \in Abs(W)$, this deduces $B \in Abs(U)$. It is in conflict. Thus we have $b_\tau(B) < t$. This contradicts with the hypothesis $t < b_\tau(B)$ again. Hence $t < b_{\tau_{b_\tau}}(B)$. It follow that $b_\tau(B) \le b_{\tau_{b_\tau}}(B)$.

Conversely, we will prove the identity mapping $i : (X, \tau_{b_\tau}) \to (X,\tau)$ is fuzzifying continuous. For each $U \subseteq X$ and any $t < \mathscr{N}(U)$, there exists $W \subseteq U$ with $W \in disk(X)$ such that $t < \mathscr{U}(W)$. Since $(Bv)^{b_\tau}(W) = \bigwedge\{1 - b_\tau(V) : W \notin Abs(V)\}$ and $b_\tau(V) = \bigwedge\{1 - \mathscr{N}(C) : C \notin Abs(V)\}$, then for all $W \notin Abs(V)$, $b_\tau(V) \le 1 - \mathscr{N}(W) \le 1 - \mathscr{U}(W) < 1 - t$. Thus $(Bv)^{b_\tau}(W) \ge t$. Furthermore, $\mathscr{N}^{\tau_{b_\tau}}(U) = \bigvee_{\substack{G\subseteq U\\ G\in disk(X)}} (Bv)^{b_\tau}(G) \ge (Bv)^{b_\tau}(W) \ge t$. This means that the identity mapping $i : (X, \tau_{b_\tau}) \to (X,\tau)$ is fuzzifying continuous, then the identity mapping $i : (X, \tau_{b_\tau}) \to (X,\tau)$ is also fuzzifying bounded. It follow that $b_{\tau_{b_\tau}}(B) \le b_\tau(B)$ for all $B \in 2^X$. The proof is completed. □

Theorem 4.6. *Let $(X,\mathcal{B})$ be a convex fuzzifying bornological linear space. Then $\mathcal{B} = b_{\tau_\mathcal{B}}$ if and only if the fuzzifying bornology $\mathcal{B}$ is the v. Newnann fuzzifying bornology of a locally convex fuzzifying topology on X.*

Proof. The necessity is obvious, since then the fuzzifying bornology $\mathcal{B}$ is the v. Neumann fuzzifying bornology of $\tau_\mathcal{B}$. For the sufficiency , let $\mathcal{B}$ be the v. Newnann fuzzifying bornology of a locally convex fuzzifying topology τ on X, i.e., $\mathcal{B} = b_\tau$. By Lemma 4.5, we have $\mathcal{B} = b_\tau = b_{\tau_{b_\tau}} = b_{\tau_\mathcal{B}}$. □

Definition 4.7. Let $(X,\mathcal{B})$ be a convex fuzzifying bornological linear space. We say that the fuzzifying bornology $\mathcal{B}$ of X is a *fuzzifying topological bornology*, or that $(X,\mathcal{B})$ is a *fuzzifying topological convex bornological space*, if the following fuzzifying bornological identity holds: $\mathcal{B} = b_{\tau_\mathcal{B}}$.

Lemma 4.8. *Let $(X,\mathcal{B})$ be a convex fuzzifying bornological linear space. Then $\tau_\mathcal{B} = \tau_{b_{\tau_\mathcal{B}}}$.*

Proof. We prove the identity mapping $i : (X,\mathcal{B}) \to (X, b_{\tau_\mathcal{B}})$ is fuzzifying bounded at first. It needs to prove $\mathcal{B}(B) \le b_{\tau_\mathcal{B}}(B)$ for all $B \subseteq X$. For each $t < \mathcal{B}(B)$,if there is $U \subseteq X$ with $B \notin Abs(U)$ satisfying $1 - \mathscr{N}^{\tau_\mathcal{B}}(U) < t$, then we have $W \subseteq X$ with $W \in disk(X)$ satisfying $Bv(W) > 1 - t$. Since $Bv(W) = \bigwedge_{V\subseteq X}\{1 - \mathscr{B}(V) : V \notin Abs(W)\}$, it follows that $1 - \mathscr{B}(V) > 1 - t$ whenever $V \notin Abs(W)$, i.e., $\mathcal{B}(V) < t$. In this case, we have $B \notin Abs(W)$, otherwise, it may deduce $B \in Abs(U)$. This contradicts with $B \notin Abs(U)$. This implies $\mathcal{B}(B) < t$. This also contradicts to the hypothesis of $t < \mathcal{B}(B)$. Hence $t \le 1 - \mathscr{N}^{\tau_\mathcal{B}}(U)$ for all $B \notin Abs(U)$. From the fact $b_{\tau_\mathcal{B}}(B) = \bigwedge\{1 - \mathscr{N}^{\tau_\mathcal{B}}(U) : B \notin Abs(U)\}$, we get $t \le b_{\tau_\mathcal{B}}(B)$. So the identity mapping $i : (X,\mathcal{B}) \to (X, b_{\tau_\mathcal{B}})$ is fuzzifying bounded. It follows that $i : (X, \tau_\mathcal{B}) \to (X, \tau_{b_{\tau_\mathcal{B}}})$ is fuzzifying continuous. In fact, for each $s < \mathscr{N}^{\tau_{b_{\tau_\mathcal{B}}}}(U)$, there exists $V \subseteq X, V \in disk(X)$ such that $s < (Bv)^{b_{\tau_\mathcal{B}}}(V)$. Then for all $B \notin Abs(V)$, we have $s < 1 - b_{\tau_\mathcal{B}}(B) \le 1 - \mathcal{B}(B)$. This implies $s \le (Bv)(V) \le \mathscr{N}\tau_\mathcal{B}(U)$. Therefore $i : (X, \tau_\mathcal{B}) \to (X, \tau_{b_{\tau_\mathcal{B}}})$ is fuzzifying continuous, i.e., $\tau_\mathcal{B} \ge \tau_{b_{\tau_\mathcal{B}}}$.

On the other hand, for any $t < \mathscr{N}^{\tau_\mathcal{B}}(W)$, there ia $W_1 \in disk(X)$ with $W_1 \subseteq W$ such that $t < Bv(W_1)$. If $(Bv)^{b_{\tau_\mathcal{B}}}(W_1) < t$, we get $B \subseteq X, B \notin Abs(W_1)$ satisfying $1 - b_{\tau_\mathcal{B}}(B) < t$. Then $1 - t < b_{\tau_\mathcal{B}}(B) = \bigwedge\{1 - \mathscr{N}^{\tau_\mathcal{B}}(V) : B \notin Abs(V)\}$. This deduces $1 - \mathscr{N}^{\tau_\mathcal{B}}(W_1) > 1 - t$. Thus we obtain $t < Bv(W_1) \le \mathscr{N}^{\tau_\mathcal{B}}(W_1) < t$. It is in conflict. Hence $\mathscr{N}^{\tau_{b_{\tau_\mathcal{B}}}}(W) \ge (Bv)^{b_{\tau_\mathcal{B}}}(W_1) \ge t$. This means that $i : (X, \tau_{b_{\tau_\mathcal{B}}}) \to (X, \tau_\mathcal{B})$ is fuzzifying continuous, i.e., $\tau_\mathcal{B} \le \tau_{b_{\tau_\mathcal{B}}}$. The proof is completed. □

Theorem 4.9. *Let (X,τ) be a locally convex fuzzifying topological linear space. Then $\tau = \tau_{b_\tau}$ if and only if the fuzzifying topology τ is the locally convex fuzzfying topology associated with a convex fuzzifying bornology on X.*

Proof. The necessity is obvious, since then the fuzzifying topology τ is determined by convex fuzzifying bornology b_τ. As for the sufficiency, let $\mathcal{B}$ be a convex fuzzifying bornology which τ is determined by $\mathcal{B}$, i.e., $\tau = \tau_\mathcal{B}$. By Lemma 4.8, $\tau = \tau_\mathcal{B} = \tau_{b_{\tau_\mathcal{B}}} = \tau_{b_\tau}$. □

Definition 4.10. Let (X, τ) be a fuzzifying locally convex space. We say that the fuzzifying topology τ on X is a *fuzzifying bornological topology*, or that (X, τ) is a *fuzzifying bornological locally convex space*, if the following fuzzifying topological identity holds: $\tau = \tau_{b_\tau}$.

If we denote the category of fuzzifying topological convex bornological spaces as **FT-CFBLS**, we can conclude from Remark 4.4, Definition 4.7, and the proof process of Lemma 4.5 that $\mathbb{B} \circ \mathbb{T}((X, \mathscr{B})) = (X, \mathscr{B})$ and $\mathbb{T} \circ \mathbb{B}((X, \tau)) \geq (X, \tau)$. To sum up, we get the following Theorem.

Theorem 4.11. *The category* **FT-CFBLS** *can be embedded in the category* **LCFTLS** *as a reflective subcategory.*

Similarly, if we denote the category of *fuzzifying bornological locally convex spaces* as **FB-LCFTLS**, we can conclude from Remark 4.4, Definition 4.10, and the proof process of Lemma 4.8 that $\mathbb{T} \circ \mathbb{B}((X, \tau)) = (X, \tau)$ and $\mathbb{B} \circ \mathbb{T}((X, \mathscr{B})) \geq (X, \mathscr{B})$. To sum up, we get the following Theorem.

Theorem 4.12. *The category* **FB-LCFTLS** *can be embedded in the category* **CFBLS** *as a reflective subcategory.*

Let **LIS** denote the category of linear spaces with linear mappings. Then we have the following theorem.

Theorem 4.13. *The category* **FT-CFBLS** *is topological over* **LIS** *with respect to the expected forgetful functor.*

Proof. Let $\mathbb{U}$: **FT − CFBLS** → **LIS** be the forgetful functor, and let $\{f_i : X \to (Y_i, \mathscr{B}_i)\}_{i\in I}$ be a $\mathbb{U}$-source, i.e., X is a linear space over $\mathbb{K}$, $(Y_i, \mathscr{B}_i)$ is a family of a *fuzzifying topological convex bornological spaces*, and for all $i \in I$, $f_i : X \to Y_i$ is a linear mapping. Define the mapping

$$\mathscr{B}_X(A) = \bigwedge_{i\in I} \mathscr{B}_i(f_i^{\to}(A)),\ \forall A \subseteq X.$$

Refer to the proof of Theorem 6.15 in [16] or the proof of Theorem 3.7 in [13], $(X, \mathscr{B}_X)$ is a fuzzifying bornological linear space. For all $A \in 2^X$, we have

$$\mathscr{B}_X(A) = \bigwedge_{i\in I} \mathscr{B}_i(f_i^{\to}(A)) \leq \bigwedge_{i\in I} \mathscr{B}_i(co(f_i^{\to}(A))) = \bigwedge_{i\in I} \mathscr{B}_i(f_i^{\to}(co(A))) = \mathscr{B}_X(co(A)).$$

This means that $(X, \mathscr{B}_X)$ is a convex fuzzifying bornological linear space. It can be easily verified that $\mathscr{B}_X$ is the weakest linear fuzzifying bornology in which each f_i is bounded. The remaining task is to prove that $(X, \mathscr{B}_X)$ is a *fuzzifying topological convex bornological space*, i.e., $\mathscr{B}_X = b_{\tau_{\mathscr{B}_X}}$. By utilizing the proof Lemma 4.8, we can establish that $\mathscr{B}_X(A) \leq b_{\tau_{\mathscr{B}_X}}(A)$ for all $A \in 2^X$. On the other hand, for all $t > \mathscr{B}_X(A) = \bigwedge_{i\in I} \mathscr{B}_i(f_i^{\to}(A))$, there exists $i_0 \in I$ such that $t > \mathscr{B}_{i_0}(f_{i_0}^{\to}(A))$. Since $(Y_{i_0}, \mathscr{B}_{i_0})$ is a *fuzzifying topological convex bornological space*, we have $t > \mathscr{B}_{i_0}(f_{i_0}^{\to}(A)) = b_{\tau_{B_{i_0}}}(f_{i_0}^{\to}(A))$. Consequently, there exists $V_{i_0} \subseteq Y_{i_0}$ such that $V_{i_0} \notin Abs(f_{i_0}^{\to}(A))$ and $1 - \mathscr{N}^{i_0}(V_{i_0}) < t$. Moreover, there exists $W_{i_0} \subseteq V_{i_0}$ belonging to the disk of Y_{i_0}, such that $1 - Bv^{i_0}(W_{i_0}) < t$. It follows that $\mathscr{B}_{i_0}(C_{i_0}) < t$ for all $C_{i_0} \notin Abs(W_{i_0})$. For any $B \notin Abs(f_{i_0}^{\leftarrow}(W_{i_0}))$, we have $f_{i_0}^{\to}(B) \notin (W_{i_0})$, which implies $\mathscr{B}_{i_0}(f_{i_0}^{\to}(B)) < t$. Consequently, we deduce that $\mathscr{B}_X(B) = \bigwedge_{i\in I} \mathscr{B}_i(f_i^{\to}(B)) \leq \mathscr{B}_{i_0}(f_{i_0}^{\to}(B)) < t$. Referring to the fact hat $A \notin Abs(f_{i_0}^{\leftarrow}(V_{i_0}))$, $f_{i_0}^{\leftarrow}(W_{i_0}) \in disk(X)$ and $Bv(f_{i_0}^{\leftarrow}(W_{i_0})) \leq Bv(f_{i_0}^{\leftarrow}(V_{i_0}))$, we have

$$\begin{aligned} b_{\tau_{\mathscr{B}_X}}(A) &= \bigwedge\{1 - \mathscr{N}(V) : A \notin Abs(V)\} \\ &\leq 1 - \mathscr{N}(f_{i_0}^{\leftarrow}(V_{i_0})) = \bigwedge_{\substack{C \subseteq f_{i_0}^{\leftarrow}(V_{i_0}) \\ C \in disk(X)}} 1 - Bv(C) \\ &\leq 1 - Bv(f_{i_0}^{\leftarrow}(W_{i_0})) = \bigvee\{\mathscr{B}_X(B) :\ B \notin Abs(f_{i_0}^{\leftarrow}(W_{i_0}))\} \leq t. \end{aligned}$$

Hence, $b_{\tau_{\mathscr{B}_X}}(A) \leq \mathscr{B}_X(A)$ for all $A \subseteq X$. Therefore $b_{\tau_{\mathscr{B}_X}} = \mathscr{B}_X$. Consequently, $(X, \mathscr{B}_X)$ is a *fuzzifying topological convex bornological space*. Thus, the proof is complete. □

5. Characterization of fuzzifying bornological topologies

This section presents characterizations of fuzzifying bornological topologies based on the concepts of boundedness and continuity of linear mappings. It is asserted that the fuzzifying bornological topologies on X that ensure every bounded linear map of X into any locally convex fuzzifying space is continuous. Furthermore, additional characterizations of the fuzzifying bornological topologies are examined.

Definition 5.1. ([27]) Let (X,τ) be a fuzzifying topological space. The value $C_I(X,\tau)=\bigwedge_{x\in X}\bigvee_{\mathscr{U}_x\vdash\mathscr{N}_x} FC(\mathscr{U}_x)$ is called the degree to which (X,τ) is first countable, where $\mathscr{U}_x\vdash\mathscr{N}_x$ means that $\mathscr{U}_x$ is a mapping from $2^X\to[0,1]$ satisfying $\mathscr{N}_x(U)=\bigvee_{V\subseteq U}\mathscr{U}_x(V)$, and $FC(\mathscr{U}_x)=1-\bigwedge\{r\ :\ C((\mathscr{U}_x)_r)\}$, where $(\mathscr{U}_x)_r=\{A\subseteq X\ :\ \mathscr{U}_x(A)>r\}$ and the notation $C((\mathscr{U}_x)_r)$ means that the set $(\mathscr{U}_x)_r$ is at most countable.

Definition 5.2. ([26]) Let (X,τ) be a fuzzifying topological space. Then for any $x\in X$ and any $S\in N(X)$, we define $S\to x:=(\forall V\in 2^X)((V\in\mathscr{N}_x)\to_L S\sqsubseteq V)$.
Where the notation $S\sqsubseteq V$ means S almost in V, that is, there is $n_0\in\mathbb{N}$ such that $S(n)\in V$ for all $n\in\mathbb{N}$ with $n_0\leq n$.

Intuitively, the value of S converges to x, that is $[S\to x]$ is

$$[S\to x]=\bigwedge_{S\not\sqsubseteq V}(1-\mathscr{N}_x(V)).$$

Theorem 5.3. *Let (X,τ) be a locally convex fuzzifying topological linear space and $C_I(X,\tau)=1$. Then (X,τ) is a fuzzifying bornological locally convex space.*

Proof. According to the proof of Lemma 4.5, the identity mapping $id:(X,\tau_{b_\tau})\to(X,\tau)$ is fuzzifying continuous. It follows that $\tau(A)\leq\tau_{b_\tau}(A)$ for all $A\in 2^X$. It suffices to prove the identity mapping $id:(X,\tau)\to(X,\tau_{b_\tau})$ is fuzzifying continuous. For all $V\subseteq X$ and every $t>\mathscr{N}^\tau(V)=\bigvee_{\substack{W\subseteq V\\ W\in disk(X)}}\mathscr{U}(W)$, it follows that $\mathscr{U}(W)\geq t$ whenever $W\not\subseteq V$. Since $C_I(X,\tau)=1$, the set $\{W\ :\ \mathscr{U}(W)>t\}$ is countable. Denote $\{W\ :\ \mathscr{U}(W)>t\}=\{W_1,W_2,\cdots,W_n,\cdots\}$ and we may assume that $\{W_n\}_{n\in\mathbb{N}}$ is decreasing. For each $n\in\mathbb{N}$, $\mathscr{U}(\frac{1}{n}W_n)=\mathscr{U}(W_n)>t$, it follows $\frac{1}{n}W_n\not\subseteq V$. Thus there is a sequence $\{x_n\}$ of X with $x_n\in W_n$ such that $x_n\notin nV$ for all $n\in\mathbb{N}$. This means that $\{x_n\}\notin Abs(V)$. Moreover, since $\mathscr{N}^{\tau_{b_\tau}}(V)=\bigvee_{\substack{W\subseteq V\\ W\in disk(X)}}Bv(W)=\bigvee_{\substack{W\subseteq V\\ W\in disk(X)}}\bigwedge_{A\subseteq X}\{1-b_\tau(A)\ :\ A\notin Abs(W)\}$. Noting that $\{x_n\}\notin Abs(W)$ for all absolute convex set $W\subseteq V$, we have $\mathscr{N}^{\tau_{b_\tau}}(V)\leq 1-b_\tau(\{x_n\})\leq 1-[x_n\to\theta]$. In addition, it comes to the conclusion $1-[x_n\to\theta]=\bigvee_{\{x_n\}\not\sqsubseteq U}\mathscr{N}^\tau(U)\leq t$. Otherwise, if there is $U\subseteq X$ such that $\mathscr{N}^\tau(U)>t$, there must be $W_{n_0}\subseteq U$ such that $\mathscr{U}(W_{n_0})>t$. It is easy to find $\{x_n\}\sqsubseteq W_{n_0}\subseteq U$. That is to say $\mathscr{N}^\tau(U)\leq t$ for all $\{x_n\}\not\sqsubseteq U$. So, $\mathscr{N}^{\tau_{b_\tau}}(V)\leq 1-[x_n\to\theta]\leq t$. Hence $\mathscr{N}^{\tau_{b_\tau}}(V)\leq\mathscr{N}^\tau(V)$ for all $V\subseteq X$. It follows that the identity mapping $id:(X,\tau)\to(X,\tau_{b_\tau})$ is fuzzifying continuous. Therefore (X,τ) is a *fuzzifying bornological locally convex space.* □

Proposition 5.4. *Let (X,τ) be a locally convex fuzzifying topological linear space. The following assertions are equivalent:*

(i). (X,τ) is a fuzzifying bornological locally convex space;

(ii). For every linear map f from X into an arbitrary locally convex fuzzifying topological linear space (Y,τ_Y), the inequality $[Bd(f)]\leq[C(f)]$ holds.

Proof. (i) ⇒ (ii). For any $t>[C(f)]=\bigwedge_{U\subseteq Y}1-\mathscr{N}^Y+\mathscr{N}^X(f^\leftarrow(U))$, there is $U\subseteq Y$ such that $1-\mathscr{N}^Y(U)+\mathscr{N}^X(f^\leftarrow(U))<t$, i.e., $1-t+\mathscr{N}^X(f^\leftarrow(U))<\mathscr{N}^Y(U)$. Assume that $\mathscr{U}^Y$ is a fuzzifying neighborhood base of θ^Y, then there is a absolute convex set $W\subseteq U$ such that $1-t+\mathscr{N}^X(f^\leftarrow(U))<\mathscr{U}^Y(W)$. It implies the following:

$$1-t+\mathscr{N}^X(f^\leftarrow(W))\leq 1-t+\mathscr{N}^X(f^\leftarrow(U))<\mathscr{U}^Y(W).$$

Since $\tau=\tau_{b_\tau}$, it follows that $\mathscr{N}^X(f^\leftarrow(W))=\bigvee_{\substack{D\subseteq f^\leftarrow(W)\\ D\in disk(X)}}Bv(D)$. By f is linear map, we have $f^\leftarrow(W)\in disk(X)$. Thus $1-t+Bv(f^\leftarrow(W))<\mathscr{U}^Y(W)\leq\mathscr{N}^Y(W)$. So, there is $A\notin Abs(f^\leftarrow(W))$ such that $1-t+1-b_\tau(A)<\mathscr{N}^Y(W)$. At this case, it is easy to check that $f^\rightarrow(A)\notin Abs(W)$. Hence

$$1-b_\tau(A)+Bd^Y(f^\rightarrow(A))\leq 1-b_\tau(A)+1-\mathscr{N}^Y(W)<t.$$

So, $[Bd(f)] = \bigwedge_{V\subseteq X} 1 - b_\tau(V) + Bd(f^{\rightarrow}(V)) \leq 1 - b_\tau(A) + Bd^Y(f^{\rightarrow}(A)) < t$. By the arbitrariness of t, it follows that $[Bd(f)] \leq [C(f)]$.

(ii) ⇒ (i). According to the proof of Lemma 4.5, the identity mapping $id : (X, \tau_{b_\tau}) \to (X, \tau)$ is fuzzifying continuous. It follows that $\tau(A) \leq \tau_{b_\tau}(A)$ for all $A \in 2^X$. On the other hand, by the proof of Lemma 4.5, we have $b_\tau(B) \leq b_{\tau_{b_\tau}}(B)$ for all $B \in 2^X$. Then for the identity mapping $id : (X, \tau) \to (X, \tau_{b_\tau})$, $[Bd(id)] = 1$. By the condition of (ii), clearly, $[C(id)] = 1$. So, $\tau(A) \geq \tau_{b_\tau}(A)$ for all $A \in 2^X$. Thus $\tau_{b_\tau} = \tau$. This means that (X, τ) is a *fuzzifying bornological locally convex space*. □

Definition 5.5. Let $(X, \mathcal{B})$ be a fuzzifying bornological space. We say that a mapping $\mathcal{B}_0 : 2^X \to I$ is a base of a fuzzifying bornology $\mathcal{B}$ if $\mathcal{B}_0 \geq \mathcal{B}$ and $\mathcal{B}(U) = \bigvee_{U\subseteq V} \mathcal{B}_0(V)$ for all $U \in 2^X$.

Lemma 5.6. *A mapping $\mathcal{B}_0 : 2^X \to I$ is a base for a fuzzifying bornology of X if and only if $\bigvee_{V\in\dot{x}} \mathcal{B}_0(V) = 1$ for all $x \in X$ and $\bigvee_{U\cup V\subseteq W} \mathcal{B}_0(W) \geq \mathcal{B}_0(U) \wedge \mathcal{B}_0(V), \forall U, V \in 2^X$.*

Proof. Necessity. Suppose that $\mathcal{B}_0 : 2^X \to I$ is a base for a fuzzifying bornology $\mathcal{B}$ of X. For each $x \in X$, it is clear $\bigvee_{V\in\dot{x}} \mathcal{B}_0(V) = \mathcal{B}(\{x\}) = 1$.

Moreover, for all $U, V \in 2^X$ and any $t < \mathcal{B}_0(U) \wedge \mathcal{B}_0(V) \leq \mathcal{B}(U) \wedge \mathcal{B}(V) = \mathcal{B}(U \bigcup V)$. Then there is $W \supseteq U \bigcup V$ such that $\mathcal{B}_0(W) > t$. So, $\bigvee_{U\cup V\subseteq W} \mathcal{B}_0(W) \geq \mathcal{B}_0(U) \wedge \mathcal{B}_0(V)$.

Sufficiency. Let $\mathcal{B}(U) = \bigvee_{U\subseteq V} \mathcal{B}_0(V)$ for all $U \subseteq X$. For all $x \in X$ and each $n \in \mathbb{N}$, there is $V_n \in \dot{x}$ such that $\mathcal{B}_0(V_n) > 1 - \frac{1}{n}$. Then $\mathcal{B}(\{x\}) \geq \mathcal{B}(V_n) \geq \mathcal{B}_0(V_n) > 1 - \frac{1}{n}$. It follows that $\mathcal{B}(\{x\}) = 1$. In addition, for any $U_1, U_2 \in 2^X$ with $U_1 \subseteq U_2$, then for each $t < \mathcal{B}(U_2)$, there exists $W \supseteq U_2 \supseteq U_1$ such that $\mathcal{B}_0(W) > t$. It follows that $t < \mathcal{B}(U_1)$. So, we have $\mathcal{B}(U_2) \leq \mathcal{B}(U_1)$.

Furthermore, for all $U_1, U_2 \in 2^X$ and every $t < \mathcal{B}(U_1) \wedge \mathcal{B}(U_2)$. There are $V_1, V_2 \in 2^X$ with $U_1 \subseteq V_1, U_2 \subseteq V_2$ such that $t < \mathcal{B}_0(V_1)$ and $t < \mathcal{B}_0(V_2)$. By the hypothesis of Sufficiency, there is $W \supseteq V_1 \bigcup V_2 \supseteq U_1 \bigcup U_2$ such that $t < \mathcal{B}_0(W)$. It follows that $\mathcal{B}(U_1 \bigcup U_2) \geq \mathcal{B}_0(W) > t$. Thus $\mathcal{B}(U_1) \wedge \mathcal{B}(U_2) \leq \mathcal{B}(U_1 \bigcup U_2)$. Hence $\mathcal{B}_0$ is a base for a fuzzifying bornology of X. □

Definition 5.7. Let $(X, \mathcal{B})$ be a fuzzifying bornological space. The value $C(X, \mathcal{B}) = \bigvee_{\mathcal{B}_0 \vdash \mathcal{B}} FC(\mathcal{B}_0)$ is called the degree to which $\mathcal{B}$ has a countable base, where $\mathcal{B}_0 \vdash \mathcal{B}$ means $\mathcal{B}_0$ is a mapping from 2^X to $[0, 1]$ satisfying $\mathcal{B}(U) = \bigvee_{U\subseteq V} \mathcal{B}_0(V)$ for all $U \in 2^X$. If $C(X, \mathcal{B}) = 1$, we say that $(X, \mathcal{B})$ has a countable base.

It is easy to check the following Lemma holds.

Lemma 5.8. *Let $(X, \mathcal{B})$ be a fuzzifying bornological space. Then $C(X, \mathcal{B}) = 1$ if and only if for all $a \in [0, 1]$, the set $\{U : \mathcal{B}_0(U) > a\}$ is countable.*

Theorem 5.9. *Let $(X, \mathcal{B}_X)$ and $(Y, \mathcal{B}_Y)$ be fuzzifying bornological linear spaces and suppose that one of the following conditions is satisfied:*

(i). The fuzzifying bornology $\mathcal{B}_Y$ has a countable base, i.e., $C(Y, \mathcal{B}_Y) = 1$;

(ii). The fuzzifying bornology $\mathcal{B}_Y$ is the von Neumann bornology of a linear fuzzifying topology of Y.

If f satisfies the following relation: $[x_n \xrightarrow{M} \theta] \leq \mathcal{B}_Y(f(\{x_n\}))$ for all $\{x_n\} \subseteq X$. Then f is bounded.

Proof. (i). Let $C(Y, \mathcal{B}_Y) = 1$, and if f is unbounded, then there exists a subset $A \subseteq X$ such that $\mathcal{B}_Y(f^{\rightarrow}(A)) = \bigvee_{f^{\rightarrow}(A)\subseteq W} \mathcal{B}_0(W) < \mathcal{B}_X(A)$. If $\bigvee_{f^{\rightarrow}(A)\subseteq W} \mathcal{B}_0(W) < t < \mathcal{B}_X(A)$, then it can be concluded that $\mathcal{B}_0(W) \geq t$ whenever $f^{\rightarrow}(A) \nsubseteq W$. Since $C(Y, \mathcal{B}_Y) = 1$, by Lemma 5.8, the set $\{\mathcal{B}_0(W) > t\}$ is countable. By Lemma 5.6, we can assume that $\{\mathcal{B}_0(W) > 1 - t_1 = t\} = \{W_n : n \in \mathbb{N}\}$ and $\{W_n\}$ is a increasing sequence. For each $n \in \mathbb{N}$, $\mathcal{B}_0(W_n) = \mathcal{B}_0(nW_n) > t$, which implies that $f^{\rightarrow}(A) \nsubseteq nW_n$. Consequently, there exists a sequence $\{a_n\}$ in A

such that $f(\frac{1}{n}a_n) \notin W_n$. It is evident taht $[\frac{1}{n}a_n \stackrel{M}{\to} \theta] > t$. Additionally, it is possible that $\mathcal{B}_Y(f(\{\frac{1}{n}a_n\}) \leq t$, if not, assuming $\mathcal{B}_Y(f(\{\frac{1}{n}a_n\}) = \bigvee_{f(\{\frac{1}{n}a_n\}\subseteq W} \mathcal{B}_0(W) > t$, there exists $n_0 \in \mathbb{N}$ such that $f(\{\frac{1}{n}a_n\} \subseteq W_{n_0}$. However, this contradicts the fact that $f(\frac{1}{n_0}) \notin W_{n_0}$. Hence, we have a sequence $\{\frac{1}{n}a_n\}$ of X such that $\mathcal{B}_Y(f(\{\frac{1}{n}a_n\}) < [\frac{1}{n}a_n \stackrel{M}{\to} \theta]$. It conflicts with the condition. Hence f is bounded.

(ii). If the fuzzifying bornology $\mathcal{B}_Y$ is the von Neumann bornology of a linear fuzzifying topology of Y, i.e., there is a linear fuzzifying topology τ on Y such that $\mathcal{B}_Y = b_\tau$. Assuming taht f is unbounded, there must be a set $A \subseteq X$ such that $\mathcal{B}_Y(f^\to(A)) = b_\tau(f^\to(A)) < \mathcal{B}_X(A)$. Let $b_\tau(f^\to(A)) < t < \mathcal{B}_X(A)$, then there is $W \subseteq Y$ with $f^\to(A) \notin Abs(W)$ such that $\mathcal{N}^\tau(W) > 1 - t$. It follows that $f^\to(A) \nsubseteq n^2W$ for all $n \in \mathbb{N}$, where W may be balanced. Thus A contains a sequence $\{a_n\}$ such that $f(a_n) \notin n^2W$. So we have $[\frac{1}{n}a_n \stackrel{M}{\to} \theta] > t$. On the other hand, we claim that $f(\{\frac{1}{n}a_n\}) \notin Abs(W)$. Otherwise, if $f(\{\frac{1}{n}a_n\}) \in Abs(W)$, there is $t_0 > 0$ such that $f(\{\frac{1}{n}a_n\}) \subseteq t_0W$. Then we have $n_0 \in \mathbb{N}$ such that $f(\{\frac{1}{n}a_n\}) \subseteq t_0W \subseteq n_0W$. This contradicts the fact $f(a_{n_0}) \notin n_0^2W$. Hence $\mathcal{B}_Y(f(\{\frac{1}{n}a_n\}) \leq 1 - \mathcal{N}^\tau(W) < t$. It follows that $\mathcal{B}_Y(f(\{\frac{1}{n}a_n\}) < [\frac{1}{n}a_n \stackrel{M}{\to} \theta]$. This contradicts the condition $[x_n \stackrel{M}{\to} \theta] \leq \mathcal{B}_Y(f(\{x_n\}))$ for all $\{x_n\} \subseteq X$. Therefore f is bounded. □

Theorem 5.10. *Let (X, τ_X) be a locally convex fuzzifying topological linear space. The following assertions are equivalent:*

(i). (X, τ_X) is a fuzzifying bornological locally convex space;

(ii). For every linear map f from (X, τ_X) to an arbitrary locally convex fuzzifying topological linear space (Y, τ_Y), the implication of $[Bd(f)] = 1$ is that $[C(f)] = 1$;

(iii). Every linear map f of (X, τ_X) into a locally convex fuzzifying topological linear space (Y, τ_Y), if $[x_n \to \theta] \leq \mathcal{B}_Y(f(\{x_n\}))$ for all $\{x_n\} \subseteq X$. Then f is continuous.

(iv). Every linear map f of (X, τ_X) into a locally convex fuzzifying topological linear space (Y, τ_Y), if $[x_n \stackrel{b_{\tau_X}}{\to} \theta] \leq \mathcal{B}_Y(f(\{x_n\}))$ for all $\{x_n\} \subseteq X$. Then f is continuous.

Proof. By Proposition 5.4, the statements (i) and (ii) are equivalent. In fact, by the proof of Proposition 5.4, the relationship $[Bd(f)] \leq [C(f)]$ may be replaced by $[Bd(f)] = 1$ implies $[C(f)] = 1$.

(iii)⇒ (ii). For every linear map f from (X, τ_X) to an arbitrary locally convex fuzzifying topological linear space (Y, τ_Y) and $[Bd(f)] = 1$. For any $U \in 2^X$ with $\{x_n\} \notin Abs(U)$ and $\mathcal{N}^{\tau_X}(U) \neq 0$, it follows that $\{x_n\} \not\sqsubseteq U$. Otherwise, if $\{x_n\} \sqsubseteq U$, there exists $n_0 \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq n_0$. In addition, there is $k_0 > 0$ such that $\{x_1, x_2, \cdots, x_{n_0-1}\} \subseteq k_0U$ by Theorem 2.13(P5). This deduces that $\{x_n\} \in Abs(U)$. Contradicting to the fact $\{x_n\} \notin Abs(U)$. Thus

$$[x_n \to \theta] = \bigwedge_{\{x_n\}\not\sqsubseteq U} \{1 - \mathcal{N}^{\tau_X}(U)\} \leq \bigwedge_{\{x_n\}\notin Abs(U)} \{1 - \mathcal{N}^{\tau_X}(U)\} = b_{\tau_X}(\{x_n\}).$$

Since $[Bd(f)] = 1$, we have $[x_n \to \theta] \leq b_{\tau_X}(\{x_n\}) \leq \mathcal{B}_Y(f(\{x_n\}))$. Thus $[C(f)] = 1$.

(ii)⇒(iv). Suppose that $[x_n \stackrel{b_{\tau_X}}{\to} \theta] \leq \mathcal{B}_Y(f(\{x_n\}))$ for all $\{x_n\} \subseteq X$, by Theorem 5.9 (ii), $[Bd(f)] = 1$. Furthermore, $[C(f)] = 1$.

(iv)⇒ (iii). For all $\{x_n\} \subseteq X$ with $[x_n \to \theta] \leq \mathcal{B}_Y(f(\{x_n\}))$, by Theorem 4.7 in [13], $[x_n \stackrel{b_{\tau_X}}{\to} \theta] \leq [x_n \to \theta] \leq \mathcal{B}_Y(f(\{x_n\}))$. Then by the hypothesis of (iv), $[C(f)] = 1$. □

6. Conclusions and future work

In this article, we introduce two concepts: convex fuzzifying bornological linear spaces and locally convex fuzzifying topological linear spaces. Based on the mapping of fuzzifying bornivorous, we demonstrate that any convex fuzzifying bornological linear space can determine the finest locally convex fuzzifying topology that is compatible with the fuzzifying bornology. Similarly, each locally convex fuzzifying topological linear space can derive the coarsest convex fuzzifying bornology that is compatible with the fuzzifying topology. Furthermore, the relationships between some categories are studied. We show that the category **FT-CFBLS** can be embedded in the the category **LCFTLS** as a reflective subcategory, and the category

FB-LCFTLS can also be embedded in the the category **CFBLS** as a reflective subcategory. Meanwhile, the category **FT-CFBLS** is topological over the category of linear spaces with respect to the expected forgetful functor. Lastly, we discuss several characterizations of fuzzifying bornological topologies and present some equivalent conditions.

An interesting direction for future research is to study the theory of convex fuzzifying bornological linear spaces and locally convex fuzzifying topological linear spaces. It would also be worthwhile to generalize these concepts to the case of convex L-fuzzy bornological linear spaces.

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