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On the spectral singularities of Klein-Gordon equation under interface conditions

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Abstract. In this study, we begin by deriving the differential operator associated with the Klein-Gordon equation under interface conditions. Next, we introduce the transfer matrix and the resolvent operator for the Klein-Gordon operator with interface conditions, observing that the zeros of a component of the transfer matrix coincide with the poles of the resolvent operator. Building on this observation, we use the transfer matrix to characterize the eigenvalues and spectral singularities of mentioned operator through an alternative approach. Finally, we establish the finiteness of eigenvalues and spectral singularities, with finite multiplicities, under certain specific conditions.

1. Introduction

The study of the spectral properties of differential operators plays a crucial role in various branches of mathematical physics, especially in the analysis of quantum fields and wave propagation. Spectral theory provides a deeper understanding of the underlying structure of solutions of differential equations, which model a wide range of physical and engineering phenomena. By examining the spectrum of a differential operator—specifically, its eigenvalues and corresponding eigenfunctions—we can gain valuable insights into the stability, dynamics, and asymptotic behavior of physical systems described by these equations. This analysis is crucial for classifying solutions, understanding their long-term behavior, and predicting phenomena such as resonance, wave propagation, and quantum states. In quantum mechanics, for example, spectral theory helps describe the energy levels of particles while in wave theory, it aids in understanding the propagation of waves.

In spectral theory, the pioneering work in the field of non-selfadjoint singular differential operators on an infinite interval belongs to Naimark [17]. He studied the spectral theory of the non-selfadjoint Sturm-Liouville operator in the space

$$L^2[0,\infty) := \left\{ f : \int_0^\infty |f(x)|^2 dx < \infty \right\}$$

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using the differential expression

$$\ell(y) = -y'' + q(x)y, \quad x \in \mathbb{R}_+ := [0, \infty)$$

and the boundary condition

$$y'(0) - ay(0) = 0,$$

where q is a complex valued function, a is a complex number. In this work, it was shown that the spectrum of the operator consists of continuous and discrete spectrum, as well as spectral singularities. Moreover, as is well-known in spectral theory, it was proven that a spectral singularity corresponds to a pole of the resolvent operator's kernel and lies on the continuous spectrum, although it is not an eigenvalue. It was also shown that when the condition

$$\int_0^\infty e^{\epsilon x} \left|q(x)\right| dx < \infty, \quad \epsilon > 0$$

is satisfied, eigenvalues and spectral singularities of the operator are finite in number, each with finite multiplicities. Then, Pavlov demonstrated how the structure of spectral singularities in differential operators is influenced by the potential function's structure at infinity [21]. A key contribution to the spectral theory of these operators came from Marchenko, who described the Jost solutions, which are of significant importance in potential theory and especially in problems related to the Schrödinger equations [14]. This framework enables the characterization of both the eigenvalues set and the set of spectral singularities through the zeros of the Jost functions. It also provides insight into crucial aspects, such as the finiteness and multiplicities of eigenvalues, as well as the occurrence of spectral singularities. Understanding these singularities allows researchers to identify critical transitions in the system, such as the formation of bound states, resonances, or changes in stability. For these reasons, a great many problems related to the spectral analysis of Sturm-Liouville operators, as well as different types of operators exhibiting spectral singularities, have been investigated so far [2, 5, 10, 13, 19, 24].

Among the operators encountered in these contexts, the Klein-Gordon operator stands out due to its importance, especially in the study of relativistic quantum mechanics and field theory. The classical Klein-Gordon equation models scalar fields and serves as the relativistic counterpart to the wave equation, incorporating a mass term that introduces complex and non-trivial dynamics into the system. A thorough literature review shows that many spectral properties related to differential operators are also applicable to Klein-Gordon type operators, which will be examined under an impulsive effect in this manuscript. The most necessary and useful properties and results related to the Klein-Gordon operators on both the entire axis and the half-axis can be found in [4, 6, 11].

The notion of interface conditions appears in the literature under several equivalent formulations, commonly referred to as impulsive conditions, transmission conditions, jump conditions or point interactions, depending on the mathematical or physical context. For simplicity of exposition, we shall adopt the framework of impulsive operators in place of differential operators with interface conditions throughout this work. Prior to presenting the so-called impulsive operator, let's briefly review the existing literature on impulsive equations. In any equation, the processes involved in mathematical modeling are not always smooth and continuous. In physical or chemical processes, there may be moments when rapid and instantaneous changes occur. While the period of these sudden changes is usually negligible compared to the overall process, these changes can still significantly affect the system's overall behavior. These are called as *impulsive* effects, and the problems arising from them are known as *impulsive problems*. Such problems are studied in various scientific fields and engineering systems, and they present challenges that need to be addressed. In biological systems, sudden changes can occur when a certain threshold value is surpassed. For example, the response of cells to a signal or a biological system deviating from its equilibrium state is associated with impulsive effects. Such changes can quickly impact the health or functions of the organism. In medicine, rhythm disorders in biological systems, such as heart rhythms, are associated with impulsive effects that cause sudden changes. For example, a sudden increase or decrease in heart rate can create a "burst" effect. Such changes can lead to health issues like heart attacks or epilepsy. In economic systems, impulsive effects can be seen, especially in optimal control theory, as sudden changes in decisions or interventions. For example, policies suddenly implemented by the government during an economic crisis or sudden changes in market conditions can lead to impulsive effects. With the advancement of technology, attention in impulsive theory has increased, prompting both theoretical and experimental research in this area. The theory has been greatly developed in the context of differential equations [1, 12, 22, 23]. In terms of spectral theory, various papers can be highly instructive for readers, as they explore the spectral properties of different forms of equations with impulsive conditions. [3, 7, 8, 15, 16, 20, 25–28].

Building on the insights from these papers and theoretical investigations, we will focus on Klein-Gordon s-wave equation

$$y'' + [\eta - Q(x)]^2 y = 0, \quad x \in \mathbb{R}_+ \setminus \{h\} \tag{1}$$

subject to the condition

$$y(0) = 0 \tag{2}$$

and the interface condition

$$\begin{bmatrix} y_+(h) \\ y'_+(h) \end{bmatrix} = T \begin{bmatrix} y_-(h) \\ y'_-(h) \end{bmatrix}, \quad T = \begin{bmatrix} \theta_1 & \theta_2 \\ \theta_3 & \theta_4 \end{bmatrix}, \tag{3}$$

where

$$y_\pm(h) = \lim_{x \to h^\pm} y_\pm(x),$$

Q is a complex valued function, $\theta_1, \theta_2, \theta_3, \theta_4$ are complex numbers such that $\det T \neq 0$ and η is a spectral parameter.

Firstly, let us present the operator $\mathcal{L}$ generated in the Hilbert space $L^2[0, \infty)$ by the impulsive problem (1)-(3).

The primary aim of this article is to explore the spectral analysis of the impulsive operator $\mathcal{L}$. Section 2 provides a concise overview of key concepts related to the Klein Gordon s-wave equations without impulsive effects, which will be referenced throughout the study. In the subsequent sections, we will derive the Jost solutions by the help of the transfer matrices, establish asymptotic equalities, and examine the resolvent operator. Finally, we will discuss the finiteness and multiplicity of eigenvalues and spectral singularities, under appropriate conditions.

2. Preliminaries

Define $\varphi(x, \eta)$ be a solution of equation (1) that satisfies the initial conditions

$$\varphi(0, \eta) = 0, \qquad \varphi'(0, \eta) = 1$$

and is expressed as

$$\varphi(x, \eta) = \frac{\sin \eta x}{\eta} + \int_0^x \frac{\sin \eta(x - t)}{\eta} [-Q^2(t) + 2\eta Q(t)]\varphi(t, \eta) dt. \tag{4}$$

It is clear from [11] that $\varphi(x, \eta)$ exists, is unique, and is an entire function of η.

In a similar treatment, let us create a new solution of equation (1). Now, we present the following theorem.

Theorem 2.1. *The solution* $\psi(x,\eta)$ *of equation* (1) *fulfilling the conditions*

$$\psi(0,\eta) = 1, \qquad \psi'(0,\eta) = 0$$

exists, is unique, is an entire function of η *and can equivalently be characterized as the solution to the integral equation*

$$\psi(x,\eta) = cos\eta x + \int_0^x \frac{\sin \eta(x-t)}{\eta}[-Q^2(t) + 2\eta Q(t)]\psi(t,\eta)dt. \tag{5}$$

Proof. The solution $\psi(x,\eta)$ which satisfies the integral representation (5), is obtained by using the method of variation of parameters. Then, the existence and uniqueness of the solution $\psi(x,\eta)$ are established using the method of successive approximations. The final part of the theorem follows from Weierstrass Criteria for convergence of series. □

It is worthy of note that

$$W[\varphi,\psi](x,\eta) = -1, \quad \eta \in \mathbb{C},$$

where W refers to the Wronskian of the solutions φ and ψ that is constant with respect to the variable x.

Consider that the function Q meets the given condition

$$\int_0^\infty x\,\{|Q(x)| + |Q'(x)|\}\,dx < \infty. \tag{6}$$

Under the condition (6), equation (1) has the solutions [11]

$$e^+(x,\eta) = e^{i\alpha(x)}e^{i\eta x} + \int_x^\infty P^+(x,t)e^{i\eta t}dt \tag{7}$$

for $\eta \in \overline{\mathbb{C}}_+$ and

$$e^-(x,\eta) = e^{-i\alpha(x)}e^{-i\eta x} + \int_x^\infty P^-(x,t)e^{-i\eta t}dt \tag{8}$$

for $\eta \in \overline{\mathbb{C}}_-$, where

$$\alpha(x) = \int_x^\infty Q(t)dt,$$

$P^\pm$ are solutions of Volterra type integral equations. In addition, $P^\pm$ and $\frac{\partial}{\partial x}(P^\pm) := P^\pm_x$ satisfy the inequalities

$$\left|P^\pm(x,t)\right| \le cw\left(\frac{x+t}{2}\right)\exp\{\gamma(x)\}, \tag{9}$$

$$\left|P^\pm_x(x,t)\right| \le c\left[w^2\left(\frac{x+t}{2}\right) + \beta\left(\frac{x+t}{2}\right)\right] \tag{10}$$

with a positive constant number c, in which

$$w(x) = \int_x^\infty \left\{|Q(t)|^2 + |Q'(t)|\right\}dt,$$

$$\gamma(x) = \int_x^\infty \left\{t\,|Q(t)|^2 + 2\,|Q(t)|\right\}dt,$$

$$\beta(x)=\frac{1}{4}\left\{2\left|Q(x)\right|^{2}+\left|Q'(x)\right|\right\}.$$

Using inequality (9), it can be concluded that the solutions e^{+} and e^{-} are analytic with respect to variable η, in $\mathbb{C}_{+}$ and $\mathbb{C}_{-}$, respectively. They also exhibit continuity extending up to the real axis. Moreover, $e^{\pm}(x,\eta)$ and $e_{x}^{\pm}(x,\eta)$ satisfy the following asymptotics [6]:

$$e^{\pm}(x,\eta)=e^{\pm i\eta x}\left[1+o(1)\right],\ \eta\in\overline{\mathbb{C}}_{\pm},\ x\to\infty,$$

$$e_{x}^{\pm}(x,\eta)=e^{\pm i\eta x}\left[\pm i\eta+o(1)\right],\ \eta\in\overline{\mathbb{C}}_{\pm},\ x\to\infty,$$

$$e^{\pm}(x,\eta)=e^{\pm\left(i\alpha x+i\eta x\right)}+o(1),\ \eta\in\overline{\mathbb{C}}_{\pm},\ |\eta|\to\infty. \tag{11}$$

On the other side, equation (1) has an unbounded solution $\hat{e}^{+}(x,\eta)$ matching the asymptotic

$$\lim_{x\to\infty}\hat{e}^{+}(x,\eta)\,e^{i\eta x}=1$$

for $\eta\in\overline{\mathbb{C}}_{+}$. Similarly, (1) has an other unbounded solution $\hat{e}^{-}(x,\eta)$ that complies with the asymptotic condition

$$\lim_{x\to\infty}\hat{e}^{-}(x,\eta)\,e^{-i\eta x}=1,$$

for $\eta\in\overline{\mathbb{C}}_{-}$. It is important to note that followings

$$\begin{aligned}W[e^{+},\hat{e}^{+}](x,\eta)&=-2i\eta,\\ W[e^{-},\hat{e}^{-}](x,\eta)&=2i\eta\end{aligned} \tag{12}$$

hold for $\eta\in\overline{\mathbb{C}}_{+}$ and $\eta\in\overline{\mathbb{C}}_{-}$, repsectively.

3. Construction of the transfer matrices

Let us denote the solutions of equation (1) in the interval $[0,h)$ and $(h,\infty)$by y_{-} and y_{+}, respectively, namely

$$\begin{cases} y_{-}(x):=y(x), & 0\le x<h\\ y_{+}(x):=y(x), & x>h.\end{cases}$$

By the help of linearly independent solutions of (1) in the intervals $[0,h)$ and (h,∞), we can state the general solution of (1) for $\eta\in\overline{\mathbb{C}}_{+}$ as

$$\begin{cases} y_{-}(x,\eta)=A_{-}\varphi(x,\eta)+B_{-}\psi(x,\eta), & 0\le x<h\\ y_{+}(x,\eta)=A_{+}e^{+}(x,\eta)+B_{+}\hat{e}^{+}(x,\eta), & x>h,\end{cases}$$

where $A_{\pm}$ and $B_{\pm}$ are η dependent constant coefficients. Using (4), (5), (7) and (12), we obtain $y_{-}(h,\eta)$, $y_{+}(h,\eta)$, $\left.\frac{\partial y_{-}}{\partial x}\right|_{x=h}$ and $\left.\frac{\partial y_{+}}{\partial x}\right|_{x=h}$. Then, with the help of the interface condition (3), we have the transfer matrix M, that satisfies the relation

$$\begin{bmatrix}A_{+}\\ B_{+}\end{bmatrix}=M\begin{bmatrix}A_{-}\\ B_{-}\end{bmatrix}, \tag{13}$$

where

$$M:=\begin{bmatrix}M_{11} & M_{12}\\ M_{21} & M_{22}\end{bmatrix}=N^{-1}TD$$

such that

$$N := \begin{bmatrix} e^{+}(h,\eta) & \hat{e}^{+}(h,\eta) \\ e_{x}^{+}(h,\eta) & \hat{e}_{x}^{+}(h,\eta) \end{bmatrix}$$

and

$$D := \begin{bmatrix} \varphi(h,\eta) & \psi(h,\eta) \\ \varphi_{x}(h,\eta) & \psi_{x}(h,\eta) \end{bmatrix}.$$

As $\det N = -2i\eta$, it is easy to reach

$$\begin{aligned} M_{11}(\eta) &= \frac{i}{2\eta}\left\{\hat{e}_{x}^{+}(h,\eta)\left[\theta_{1}\varphi(h,\eta)+\theta_{2}\varphi_{x}(h,\eta)\right]-\hat{e}^{+}(h,\eta)\left[\theta_{3}\varphi(h,\eta)+\theta_{4}\varphi_{x}(h,\eta)\right]\right\}, \\ M_{12}(\eta) &= \frac{i}{2\eta}\left\{\hat{e}_{x}^{+}(h,\eta)\left[\theta_{1}\psi(h,\eta)+\theta_{2}\psi_{x}(h,\eta)\right]-\hat{e}^{+}(h,\eta)\left[\theta_{3}\psi(h,\eta)+\theta_{4}\psi_{x}(h,\eta)\right]\right\}, \\ M_{21}(\eta) &= \frac{i}{2\eta}\left\{-e_{x}^{+}(h,\eta)\left[\theta_{1}\varphi(h,\eta)+\theta_{2}\varphi_{x}(h,\eta)\right]+e^{+}(h,\eta)\left[\theta_{3}\varphi(h,\eta)+\theta_{4}\varphi_{x}(h,\eta)\right]\right\}, \\ M_{22}(\eta) &= \frac{i}{2\eta}\left\{-e_{x}^{+}(h,\eta)\left[\theta_{1}\psi(h,\eta)+\theta_{2}\psi_{x}(h,\eta)\right]+e^{+}(h,\eta)\left[\theta_{3}\psi(h,\eta)+\theta_{4}\psi_{x}(h,\eta)\right]\right\}. \end{aligned} \tag{14}$$

Let us think about any two solutions of equation (1) which are stated as

$$E^{+}(x,\eta) = \begin{cases} A_{-}^{+}\varphi(x,\eta)+B_{-}^{+}\psi(x,\eta), & 0 \le x < h \\ A_{+}^{+}e^{+}(x,\eta)+B_{+}^{+}\hat{e}^{+}(x,\eta), & h < x < \infty \end{cases} \tag{15}$$

and

$$F^{+}(x,\eta) = \begin{cases} A_{-}^{-}\varphi(x,\eta)+B_{-}^{-}\psi(x,\eta), & 0 \le x < h \\ A_{+}^{-}e^{+}(x,\eta)+B_{+}^{-}\hat{e}^{+}(x,\eta), & h < x < \infty, \end{cases} \tag{16}$$

where $A_{\pm}^{\pm}$ and $B_{\pm}^{\pm}$ are complex coefficients. If $E^{+}(x,\eta)$ and $F^{+}(x,\eta)$ are associated with the Jost solution of impulsive boundary value problem (1)-(3) and the boundary condition (2), respectively, then we obtain

$$B_{+}^{+}=0, \quad A_{+}^{+}=1, \quad A_{-}^{-}=k_{1}, \quad B_{-}^{-}=0, \tag{17}$$

where k_1 is a nonzero real constant. Additionally, by the interface condition (3) and (13), we have

$$A_{-}^{+}=\frac{M_{22}(\eta)}{\det M}, \quad B_{-}^{+}=-\frac{M_{21}(\eta)}{\det M}, \quad A_{+}^{-}=k_{1}M_{11}(\eta), \quad B_{+}^{-}=k_{1}M_{21}(\eta) \tag{18}$$

uniquely for the solutions $E^{+}(x,\eta)$ and $F^{+}(x,\eta)$. Obviously, when these coefficients (17) and (18) are substituted into (15) and (16), we obtain the new representations

$$E^{+}(x,\eta) = \begin{cases} \dfrac{M_{22}(\eta)}{\det M}\varphi(x,\eta)-\dfrac{M_{21}(\eta)}{\det M}\psi(x,\eta), & 0 \le x < h \\ e^{+}(x,\eta), & h < x < \infty \end{cases} \tag{19}$$

and

$$F^{+}(x,\eta) = \begin{cases} k_{1}\varphi(x,\eta), & 0 \le x < h \\ k_{1}M_{11}(\eta)e^{+}(x,\eta)+k_{1}M_{21}(\eta)\hat{e}^{+}(x,\eta), & h < x < \infty \end{cases} \tag{20}$$

for $\eta \in \overline{\mathbb{C}}_{+}$.

Now, using (19) and (20), we can present the next lemma.

Lemma 3.1. *The Wronskian of the solutions $E^{+}(x,\eta)$ and $F^{+}(x,\eta)$ fulfilles the subsequent equality for all $\eta \in \overline{\mathbb{C}}_{+}\setminus\{0\}$,*

$$W[E^{+},F^{+}](x,\eta)=\begin{cases} -\dfrac{k_1 M_{21}(\eta)}{\det M}, & x\in[0,h)\\ -2i\eta k_1 M_{21}(\eta), & x\in(h,\infty).\end{cases}$$

Besides, we shall denote the solutions of equation (1) in the interval $[0,h)$ and (h,∞) by $\tilde{y}_{-}$ and $\tilde{y}_{+}$, respectively, that is

$$\begin{cases} \tilde{y}_{-}(x):=\tilde{y}(x), & 0\le x<h\\ \tilde{y}_{+}(x):=\tilde{y}(x), & x>h.\end{cases}$$

Employing linearly independent solutions of (1) in the intervals $[0,h)$ and (h,∞), we can state the general solution of (1) for $\eta\in\overline{\mathbb{C}}_{-}$ as

$$\begin{cases} \tilde{y}_{-}(x,\eta)=\tilde{A}_{-}\varphi(x,\eta)+\tilde{B}_{-}\psi(x,\eta), & 0\le x<h\\ \tilde{y}_{+}(x,\eta)=\tilde{A}_{+}e^{-}(x,\eta)+\tilde{B}_{+}\hat{e}^{-}(x,\eta), & x>h,\end{cases}$$

where $\tilde{A}_{\pm}$ and $\tilde{B}_{\pm}$ are η dependent constant coefficients. By the help of (4), (5), (8) and (12), we get $\tilde{y}_{-}(h,\eta)$, $\tilde{y}_{+}(h,\eta)$, $\left.\frac{\partial\tilde{y}_{-}}{\partial x}\right|_{x=h}$ and $\left.\frac{\partial\tilde{y}_{+}}{\partial x}\right|_{x=h}$. Afterward, based on (3), we have the transfer matrix $\tilde{M}$, that satisfies the relation

$$\begin{bmatrix}\tilde{A}_{+}\\ \tilde{B}_{+}\end{bmatrix}=\tilde{M}\begin{bmatrix}\tilde{A}_{-}\\ \tilde{B}_{-}\end{bmatrix}, \tag{21}$$

where

$$\tilde{M}:=\begin{bmatrix}\tilde{M}_{11} & \tilde{M}_{12}\\ \tilde{M}_{21} & \tilde{M}_{22}\end{bmatrix}=\tilde{N}^{-1}TD$$

such that

$$\tilde{N}:=\begin{bmatrix}e^{-}(h,\eta) & \hat{e}^{-}(h,\eta)\\ e_x^{-}(h,\eta) & \hat{e}_x^{-}(h,\eta)\end{bmatrix}.$$

As $\det\tilde{N}=2i\eta$, it is easy to obtain

$$\tilde{M}_{11}(\eta)=-\frac{i}{2\eta}\left\{\hat{e}_x^{-}(h,\eta)\left[\theta_1\varphi(h,\eta)+\theta_2\varphi_x(h,\eta)\right]-\hat{e}^{-}(h,\eta)\left[\theta_3\varphi(h,\eta)+\theta_4\varphi_x(h,\eta)\right]\right\},$$

$$\tilde{M}_{12}(\eta)=-\frac{i}{2\eta}\left\{\hat{e}_x^{-}(h,\eta)\left[\theta_1\psi(h,\eta)+\theta_2\psi_x(h,\eta)\right]-\hat{e}^{-}(h,\eta)\left[\theta_3\psi(h,\eta)+\theta_4\psi_x(h,\eta)\right]\right\},$$

$$\tilde{M}_{21}(\eta)=-\frac{i}{2\eta}\left\{-e_x^{-}(h,\eta)\left[\theta_1\varphi(h,\eta)+\theta_2\varphi_x(h,\eta)\right]+e^{-}(h,\eta)\left[\theta_3\varphi(h,\eta)+\theta_4\varphi_x(h,\eta)\right]\right\}, \tag{22}$$

$$\tilde{M}_{22}(\eta)=-\frac{i}{2\eta}\left\{-e_x^{-}(h,\eta)\left[\theta_1\psi(h,\eta)+\theta_2\psi_x(h,\eta)\right]+e^{-}(h,\eta)\left[\theta_3\psi(h,\eta)+\theta_4\psi_x(h,\eta)\right]\right\}.$$

Let us present any two solutions of equation (1) which are stated as

$$E^{-}(x,\eta)=\begin{cases} \tilde{A}_{-}^{+}\varphi(x,\eta)+\tilde{B}_{-}^{+}\psi(x,\eta), & 0\le x<h\\ \tilde{A}_{+}^{+}e^{-}(x,\eta)+\tilde{B}_{+}^{+}\hat{e}^{-}(x,\eta), & h<x<\infty\end{cases} \tag{23}$$

and

$$F^{-}(x,\eta)=\begin{cases}\tilde{A}_{-}^{-}\varphi(x,\eta)+\tilde{B}_{-}^{-}\psi(x,\eta), & 0\le x<h\\ \tilde{A}_{+}^{-}e^{-}(x,\eta)+\tilde{B}_{+}^{-}\hat{e}^{-}(x,\eta), & h<x<\infty,\end{cases} \tag{24}$$

where $\tilde{A}_{\pm}^{\pm}$ and $\tilde{B}_{\pm}^{\pm}$ are complex coefficients. Assuming that $E^{-}(x,\eta)$ and $F^{-}(x,\eta)$ correspond to the Jost solution of impulsive boundary value problem (1)-(3) and the boundary condition (2), respectively, then we obtain

$$\tilde{B}_{+}^{+}=0,\quad \tilde{A}_{+}^{+}=1,\quad \tilde{A}_{-}^{-}=k_2,\quad \tilde{B}_{-}^{-}=0. \tag{25}$$

where k_2 is a nonzero real constant. Additionally, by the interface condition (3) and (21), we have

$$\tilde{A}_{-}^{+}=\frac{\tilde{M}_{22}(\eta)}{\det\tilde{M}},\quad \tilde{B}_{-}^{+}=-\frac{\tilde{M}_{21}(\eta)}{\det\tilde{M}},\quad \tilde{A}_{+}^{-}=k_2\tilde{M}_{11}(\eta),\quad \tilde{B}_{+}^{-}=k_2\tilde{M}_{21}(\eta) \tag{26}$$

uniquely for the solutions $E^{-}(x,\eta)$ and $F^{-}(x,\eta)$. Obviously, when these coefficients (25) and (26) are substituted into (23) and (24), we obtain the new representations

$$E^{-}(x,\eta)=\begin{cases}\dfrac{\tilde{M}_{22}(\eta)}{\det\tilde{M}}\varphi(x,\eta)-\dfrac{\tilde{M}_{21}(\eta)}{\det\tilde{M}}\psi(x,\eta) & 0\le x<h\\ e^{-}(x,\eta), & h<x<\infty\end{cases} \tag{27}$$

and

$$F^{-}(x,\eta)=\begin{cases}k_2\varphi(x,\eta), & 0\le x<h\\ k_2\tilde{M}_{11}(\eta)e^{-}(x,\eta)+k_2\tilde{M}_{21}(\eta)\hat{e}^{-}(x,\eta), & h<x<\infty.\end{cases} \tag{28}$$

for $\eta\in\overline{\mathbb{C}}_{-}$.

Now, using (27) and (28), we can present the next lemma.

Lemma 3.2. *For all $\eta\in\overline{\mathbb{C}}_{-}\setminus\{0\}$, the following equality is satisfied:*

$$W[E^{-},F^{-}](x,\eta)=\begin{cases}-\frac{k_2\tilde{M}_{21}(\eta)}{\det\tilde{M}}, & x\in[0,h)\\ 2i\eta k_2\tilde{M}_{21}(\eta), & x\in(h,\infty).\end{cases}$$

4. Analysis of eigenvalues and spectral singularities

Let us present the resolvent sets of $\mathcal{L}$ as

$$\rho_1(\eta)=\{\eta:\eta\in\mathbb{C}_{+},W[E^{+},F^{+}](\eta)\neq 0\},\quad \rho_2(\eta)=\{\eta:\eta\in\mathbb{C}_{-},W[E^{-},F^{-}](\eta)\neq 0\}.$$

By applying standard methods [18], it is evident that

$$\rho(\mathcal{L}(\eta))=\rho_1(\eta)\cup\rho_2(\eta).$$

Then, we can express the resolvent operator of $\mathcal{L}$ as stated in the next theorem.

Theorem 4.1. *The resolvent operator of $\mathcal{L}$ is defined as*

$$R_\eta(\mathcal{L})\mu(x)=\int_0^\infty G(x,t;\eta)\mu(t)dt$$

for $\eta\in\rho(\mathcal{L}(\eta))$ and $\mu\in L^2[0,\infty)$, where the Green's function for $\mathcal{L}$ is provided by

$$G(x,t;\eta)=\begin{cases}G^{+}(x,t;\eta), & \eta\in\rho_1(\eta)\\ G^{-}(x,t;\eta), & \eta\in\rho_2(\eta),\end{cases}$$

where

$$G^{+}(x,t;\eta)=\begin{cases}-\dfrac{F^{+}(t,\eta)E^{+}(x,\eta)}{W[E^{+},F^{+}](x,\eta)}, & 0\le t\le x,\quad t,x\neq h\\ -\dfrac{E^{+}(t,\eta)F^{+}(x,\eta)}{W[E^{+},F^{+}](x,\eta)}, & x\le t<\infty,\quad t,x\neq h\end{cases}$$

and

$$G^{-}(x,t;\eta)=\begin{cases}-\dfrac{F^{-}(t,\eta)E^{-}(x,\eta)}{W[E^{-},F^{-}](x,\eta)}, & 0\le t\le x,\quad t,x\neq h\\ -\dfrac{E^{-}(t,\eta)F^{-}(x,\eta)}{W[E^{-},F^{-}](x,\eta)}, & x\le t<\infty,\quad t,x\neq h.\end{cases}$$

Using conventional techniques [18], we can now define the sets of eigenvalues and spectral singularities for the impulsive operator $\mathcal{L}$ as follows:

$$\sigma_d(\mathcal{L})=\{\eta:\eta\in\mathbb{C}_+ \text{ and } M_{21}(\eta)=0\}\cup\left\{\eta:\eta\in\mathbb{C}_- \text{ and } \tilde{M}_{21}(\eta)=0\right\},$$

$$\sigma_{ss}(\mathcal{L})=\{\eta:\eta\in\mathbb{R}^*, M_{21}(\eta)=0\}\cup\left\{\eta:\eta\in\mathbb{R}^*,\tilde{M}_{21}(\eta)=0\right\},$$

where $\mathbb{R}^*:=\mathbb{R}\setminus\{0\}$. It is worthy of note that by utilizing (14), (22), and $W[e^+,e^-](x,\eta)=-2i\eta$, for $\eta\in\mathbb{R}^*$, we can directly obtain that

$$\{\eta:\eta\in\mathbb{R}^*, M_{21}(\eta)=0\}\cap\left\{\eta:\eta\in\mathbb{R}^*,\tilde{M}_{21}(\eta)=0\right\}=\emptyset.$$

In order to examine the quantitative characteristics of sets $\sigma_{ss}(\mathcal{L})$ and $\sigma_d(\mathcal{L})$, it is essential to investigate the quantitative features of the zeros of M_{21} and $\tilde{M}_{21}$ in $\overline{\mathbb{C}}_+$ and $\overline{\mathbb{C}}_-$, respectively. To achieve this, some essential sets may be defined as follows:

$$K_1=\{\eta:\eta\in\mathbb{C}_+, M_{21}(\eta)=0\},\quad \tilde{K}_1=\left\{\eta:\eta\in\mathbb{C}_-,\tilde{M}_{21}(\eta)=0\right\},$$

$$K_2=\{\eta:\eta\in\mathbb{R}^*, M_{21}(\eta)=0\},\quad \tilde{K}_2=\left\{\eta:\eta\in\mathbb{R}^*,\tilde{M}_{21}(\eta)=0\right\}.$$

Consequently, the sets $\sigma_d(\mathcal{L})$ and $\sigma_{ss}(\mathcal{L})$ can be redefined as follows:

$$\sigma_d(\mathcal{L})=\left\{\eta:\eta\in\mathbb{C}_+\cup\mathbb{C}_-,\ \eta\in K_1\cup\tilde{K}_1\right\},\quad \sigma_{ss}(\mathcal{L})=\left\{\eta:\eta\in\mathbb{R}^*,\eta\in K_2\cup\tilde{K}_2\right\}.$$

Theorem 4.2. *Based on condition* (6), *the function M_{21} fulfills the following asymptotic equations.*
(1) If $\theta_2\neq 0$, then

$$\eta M_{21}(\eta)=-\frac{i\theta_2}{4}e^{i\alpha(h)}\left[i\eta+O(1)\right],\ \eta\in\overline{\mathbb{C}}_+,\ |\eta|\to\infty.$$

(2) If $\theta_2=0$, then

$$\eta M_{21}(\eta)=\frac{\theta_1}{2}e^{i\alpha(h)}\left[1+O(\frac{1}{\eta})\right],\ \eta\in\overline{\mathbb{C}}_+,\ |\eta|\to\infty.$$

Proof. The function M_{21} defined by equation (14) can be arranged as

$$\begin{aligned}\eta M_{21}(\eta) \;=\; & \frac{i}{2}\left\{-\theta_1\left[e_x^+(h,\eta)e^{-i\eta h}\right]\left[\varphi(h,\eta)e^{i\eta h}\right]-\theta_2\left[e_x^+(h,\eta)e^{-i\eta h}\right]\left[\varphi_x(h,\eta)e^{i\eta h}\right]\right.\\ & \left.+\theta_3\left[e^+(h,\eta)e^{-i\eta h}\right]\left[\varphi(h,\eta)e^{i\eta h}\right]+\theta_4\left[e^+(h,\eta)e^{-i\eta h}\right]\left[\varphi_x(h,\eta)e^{i\eta h}\right]\right\}.\end{aligned}$$

Using the following asymptotic

$$e_x^+(x,\eta) = e^{i\alpha(x)}e^{i\eta x}[i\eta + O(1)], \quad x \in [0,\infty),\ \eta \in \overline{\mathbb{C}}_+,\ \left|\eta\right| \to \infty$$

and (11), the last equation can be rewritten as

$$\begin{aligned}\eta M_{21}(\eta) \;=\; & \frac{ie^{i\alpha(h)}}{2}\Bigg\{-\theta_1\,[i\eta+O(1)]\,\frac{1}{\eta}\left[\frac{e^{2i\eta h}}{2i}-\frac{1}{2i}+e^{i\eta h}\int_0^h \sin\eta(h-t)[-Q^2(t)+2\eta Q(t)]\varphi(t,\eta)dt\right]\\ & -\theta_2\,[i\eta+O(1)]\left[\frac{e^{2i\eta h}}{2}+\frac{1}{2}+e^{i\eta h}\int_0^h \cos\eta(h-t)[-Q^2(t)+2\eta Q(t)]\varphi(t,\eta)dt\right]\\ & +\theta_3\,[1+o(1)]\left[\frac{e^{2i\eta h}}{2i\eta}-\frac{1}{2i\eta}+e^{i\eta h}\int_0^h \frac{\sin\eta(h-t)}{\eta}[-Q^2(t)+2\eta Q(t)]\varphi(t,\eta)dt\right]\\ & +\theta_4\,[1+o(1)]\left[\frac{e^{2i\eta h}}{2}+\frac{1}{2}+e^{i\eta h}\int_0^h \cos\eta(h-t)[-Q^2(t)+2\eta Q(t)]\varphi(t,\eta)dt\right]\Bigg\}\end{aligned}$$

for $\eta \in \overline{\mathbb{C}}_+$ and $\left|\eta\right| \to \infty$. Hence, if $\theta_2 \neq 0$, the asymptotic form is obtained as

$$\eta M_{21}(\eta) = -\frac{i\theta_2}{4}e^{i\alpha(h)}\,[i\eta + O(1)]$$

for $\eta \in \overline{\mathbb{C}}_+$ and $\left|\eta\right| \to \infty$. If $\theta_2 = 0$, it is obtained as

$$\eta M_{21}(\eta) = \frac{\theta_1}{2}e^{i\alpha(h)}\left[1 + O(\frac{1}{\eta})\right]$$

for $\eta \in \overline{\mathbb{C}}_+$ and $\left|\eta\right| \to \infty$.

Here, we would like to emphasize an important point: since $\det T \neq 0$, the complex numbers θ_1 and θ_2 cannot be zero simultaneously. This completes the proof. □

Theorem 4.3. *Based on condition* (6), *the function* $\tilde{M}_{21}$ *fulfills the following asymptotic equations.*
(1) If $\theta_2 \neq 0$, *then*

$$\eta\tilde{M}_{21}(\eta) = \frac{i\theta_2}{4}e^{-i\alpha(h)}\,[-i\eta + O(1)]\,,\ \eta \in \overline{\mathbb{C}}_-,\ |\eta| \to \infty.$$

(2) If $\theta_2 = 0$, *then*

$$\eta\tilde{M}_{21}(\eta) = \frac{\theta_1}{2}e^{-i\alpha(h)}\left[1 + O(\frac{1}{\eta})\right],\ \eta \in \overline{\mathbb{C}}_-,\ |\eta| \to \infty.$$

Proof. The proof is obtained similarly to the proof of the previous theorem, using the asymptotic equality

$$e_x^-(x,\eta) = e^{-i\alpha(x)}e^{-i\eta x}[-i\eta + O(1)], \quad x \in [0,\infty),\ \eta \in \overline{\mathbb{C}}_-,\ \left|\eta\right| \to \infty.$$

□

Lemma 4.4. *Based on condition* (6),

i) the sets of K_1 *and* $\tilde{K}_1$ *are bounded with at most countably many elements, and their limit points are restricted to a bounded subinterval on the real axis,*

ii) the sets of K_2 *and* $\tilde{K}_2$ *are compact and their linear Lebesgue measure are zero.*

Proof. From Theorem 4.2, it can be concluded that for sufficiently large $\eta \in \overline{\mathbb{C}}_+$, $M_{21}(\eta) \neq 0$. Therefore, the zeros of $M_{21}(\eta)$ in $\overline{\mathbb{C}}_+$ are confined to a limited region, which establishes the boundedness of the sets K_1 and K_2. With a similar perspective, Theorem 4.3 demonstrates that $\tilde{M}_{21}(\eta)$ cannot be zero for sufficiently large $\eta \in \overline{\mathbb{C}}_-$. Thus, the boundedness of sets $\tilde{K}_1$ and $\tilde{K}_2$ is a direct consequence of Theorem 4.3. Additionally, since $M_{21}(\eta)$ and $\tilde{M}_{21}(\eta)$ are analytic in $\mathbb{C}_+$ and $\mathbb{C}_-$, respectively, the sets K_1 and $\tilde{K}_1$ consist of at most countably many elements, with their limit points residing within a restricted subinterval of $\mathbb{R}$. By utilizing the uniqueness theorem for analytic functions, we also establish that K_2 and $\tilde{K}_2$ are closed sets and their linear Lebesgue measure are zero. □

Theorem 4.5. *Based on condition* (6),

i) the set of eigenvalues of $\mathcal{L}$ are bounded with at most countably many elements, and their limit points are restricted to a bounded subinterval on the real axis,

ii) the set of spectral singularities of $\mathcal{L}$ are compact and their linear Lebesgue measure are zero.

In the rest of the paper, our focus will be solely on the zeros of $M_{21}(\eta)$ in $\overline{\mathbb{C}}_+$. A similar approach can also be applied to analyze the zeros of $\tilde{M}_{21}(\eta)$ in $\overline{\mathbb{C}}_-$.

An additional condition must be placed on the functions Q and Q' to guarantee that the sets $\sigma_d(\mathcal{L})$ and $\sigma_{ss}(\mathcal{L})$ contain a finite number of elements.

Theorem 4.6. *If*

$$\lim_{x\to\infty} Q(x) = 0, \quad |Q'(x)| \leq C\exp(-\epsilon x), \tag{29}$$

hold for all $\epsilon > 0$ and arbitrary positive constant C, then the eigenvalues and spectral singularities of the operator $\mathcal{L}$ are finite in number, with each having a finite multiplicity.

Proof. By the help of the conditions (29), we obtain that

$$|Q(x)| \leq C\exp(-\epsilon x). \tag{30}$$

By applying inequalities (9), (10) and (30), it has been demonstrated that

$$\left|P^+(h,t)\right| \leq c_1 \exp\left(-t\frac{\epsilon}{4}\right), \quad \left|P_x^+(h,t)\right| \leq c_2 \exp\left(-t\frac{\epsilon}{4}\right)$$

which indicate that the function M_{21} can be analytically extended from the real axis to the region $Im\eta > -\epsilon/4$ on the lower half-plane. Consequently, $\sigma_d(\mathcal{L})$ and $\sigma_{ss}(\mathcal{L})$ do not have any limit points on the real axis. Furthermore, utilizing Theorem 4.5, it is established that $\sigma_d(\mathcal{L})$ and $\sigma_{ss}(\mathcal{L})$ are bounded sets containing a finite number of elements. Finally, based on the properties of analytic functions [9], it is observed that the zeros of M_{21} in the set $\overline{\mathbb{C}}_+$ have finite multiplicities. In a similar manner, it can be demonstrated that if condition (29) is satisfied, $\tilde{M}_{21}(\eta)$ in $\overline{\mathbb{C}}_-$ has a finite number of zeros, each with a finite multiplicity. □

Now, let us define the set of all limit points of K_1 as K_3, all limit points of K_2 as K_4 and the set of all zeros of M_{21} with infinite multiplicity in $\overline{\mathbb{C}}_+$ as K_5. Utilizing the existence and uniqueness principles for analytic functions [9], we conclude that

$$K_3 \subset K_2, \quad K_4 \subset K_2, \quad K_5 \subset K_2.$$

Given the continuity of all orders derivatives of M_{21} extending to the real line, it is obtained that

$$K_3 \subset K_5, \quad K_4 \subset K_5.$$

By applying a condition less restrictive than (29), the conclusion of Theorem 4.6 is fulfilled in the theorem below.

Theorem 4.7. *Provided that*

$$\lim_{x\to\infty} Q(x) = 0, \quad |Q'(x)| \le C\exp(-\epsilon x^{\delta}) \tag{31}$$

hold for arbitrary constant C, some $\epsilon > 0$ *and* $\frac{1}{2} \le \delta < 1$*, then*

$$K_5 = \emptyset.$$

Proof. Since the function M_{21}, which is analytic on the real axis under condition (31), does not have to remain analytic in the lower half-plane , it is not possible to establish the finiteness of eigenvalues and spectral singularities as done in Theorem 4.6. In another approach, based on (14), the equation

$$\eta M_{21}(\eta) = \frac{i}{2}\left[-\theta_1 R_1(\eta)R_2(\eta) - \theta_2 R_1(\eta)R_3(\eta) + \theta_3 R_4(\eta)R_2(\eta) + \theta_4 R_4(\eta)R_3(\eta)\right] \tag{32}$$

can be expressed, where

$$\begin{aligned}
R_1(\eta) &= \left[i\alpha'(h) + i\eta\right] e^{i\alpha(h)} - P^+(h,h) + \int_h^{\infty} P_x^+(h,t)e^{i\eta(t-h)}dt,\\
R_2(\eta) &= e^{i\eta h}\frac{\sin\eta h}{\eta} + e^{i\eta h}\int_0^h \frac{\sin\eta(h-t)}{\eta}[-Q^2(t) + 2\eta Q(t)]\varphi(t,\eta)dt,\\
R_3(\eta) &= e^{i\eta h}\cos\eta h + e^{i\eta h}\int_0^h \cos\eta(h-t)[-Q^2(t) + 2\eta Q(t)]\varphi(t,\eta)dt,\\
R_4(\eta) &= e^{i\alpha(h)} + \int_h^{\infty} P^+(h,t)e^{i\eta(t-h)}dt.
\end{aligned}$$

Given that the function M_{21} is analytic in the open upper half-plane and all of its derivatives are continuous in the closed upper half-plane, the inequality

$$\left|\frac{d^m}{d\eta^m}R_i(\eta)\right| \le c_3\int_h^{\infty} t^m \exp\left(-\frac{\epsilon}{2}\left(\frac{t}{2}\right)^{\delta}\right)dt, \quad i = 1,4, \tag{33}$$

can be derived assuming (31), where $\eta \in \mathbb{C}_+$, $|\eta| < S$ and $m = 2,3,4,....$

Considering that R_2 and R_3 have continuous derivatives of all orders with respect to the variable η,

$$\left|\frac{d^m}{d\eta^m}R_i(\eta)\right| \le c_4, \quad i = 2,3 \tag{34}$$

can be obtained for $\eta \in \mathbb{C}_+$, $|\eta| < S$, $m = 1,2,3,,...$ and positive constant c_4 using Granoul inequality. By the help of (32)-(34), the inequality

$$\begin{aligned}
\left|\frac{d^m}{d\eta^m}(\eta M_{21})\right| \le \left(\frac{1}{2}\right)^m &\left\{\sum_{s=0}^{m}\binom{m}{s}\left|\frac{d^{m-s}}{d\eta^{m-s}}R_1(\eta)\right|\left[|\theta_1|^m\left|\frac{d^s}{d\eta^s}R_2(\eta)\right| + |\theta_2|^m\left|\frac{d^s}{d\eta^s}R_3(\eta)\right|\right]\right.\\
&\left.+\sum_{s=0}^{m}\binom{m}{s}\left|\frac{d^{m-s}}{d\eta^{m-s}}R_4(\eta)\right|\left[|\theta_3|^m\left|\frac{d^s}{d\eta^s}R_2(\eta)\right| + |\theta_4|^m\left|\frac{d^s}{d\eta^s}R_3(\eta)\right|\right]\right\}\\
&\le c_5\gamma\int_0^{\infty} t^m \exp\left[-\frac{\epsilon}{2}\left(\frac{t}{2}\right)^{\delta}\right]dt
\end{aligned} \tag{35}$$

can be found for $\gamma := [|\theta_1|^m + |\theta_2|^m + |\theta_3|^m + |\theta_4|^m]$ and $m = 2,3,4,...$.

At this point, using (35), we can express

$$\left|\frac{d^m}{d\eta^m}(\eta M_{21})\right| \leq U_m$$

for $m = 2, 3, ...$ and $\eta \in \mathbb{C}_+$, $|\eta| < S$, where

$$U_m = c_5\gamma \int_0^\infty t^m \exp\left[-\frac{\epsilon}{2}\left(\frac{t}{2}\right)^\delta\right] dt.$$

Given that M_{21} cannot be zero, Pavlov's Theorem [21] leads to the conclusion that

$$\int_0^z \ln J(t) d\mu(K_5, t) > -\infty, \tag{36}$$

where $J(t) = \inf\left\{\frac{U_m t^m}{m!} : m = 2, 3, ...\right\}$ and $\mu(K_5, t)$ denotes the linear Lebesgue measure of the $t-$neighborhood surrounding K_5. On the other side, the inequality

$$U_m \leq c_6\gamma b^m m^{m(1-\delta)/\delta} m!$$

is satisfied through the use of the Gamma function, where $c_6, b, c_5, \epsilon, \delta$ are constants dependent on each other. As a result, it follows that

$$J(t) \leq c_6\gamma \exp\left\{-\frac{1-\delta}{\delta} e^{-1} b^{-\delta/(1-\delta)} t^{-\delta/(1-\delta)}\right\}. \tag{37}$$

By applying equations (36) and (37), it can be deduced that

$$\int_0^z t^{-\delta/(1-\delta)} d\mu(K_5, t) < \infty. \tag{38}$$

Given that $\delta/(1-\delta) \geq 1$, it is concluded from inequality (38) that $\mu(K_5, t) = 0$, meaning $K_5 = \emptyset$. □

5. Conclusion

In this study, we have investigated the spectral properties of the impulsive Klein-Gordon s-wave equation, particularly focusing on the associated impulsive differential operator. By employing the transfer matrix and the resolvent operator, we demonstrated how the zeros of the transfer matrix components are related to the poles of the resolvent operator, providing a novel approach to describe the sets of eigenvalues and spectral singularities. Our analysis confirmed that, under specific conditions, the operator exhibits a finite number of eigenvalues and spectral singularities with finite multiplicities.

This work not only contributes to the understanding of the spectral behavior of impulsive operators but also offers new insights into the application of transfer matrix techniques in spectral theory. The results presented here pave the way for future studies on the spectral analysis of more complex impulsive systems. Further research could explore the broader implications of these results, extending the methodology first to impulsive Klein-Gordon operators on the whole real axis, and then to other types of differential operators.

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