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Signless Laplacian spectral analysis of a class of graph joins

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Abstract. A graph is said to be determined by its signless Laplacian spectrum (abbreviated as DQS) if no other non-isomorphic graph shares the same signless Laplacian spectrum. In this paper, we establish the following results:

(1) Every graph of the form $K_1 \vee (C_s \cup qK_2)$, where $q \geq 1$, $s \geq 3$, and the number of vertices is at least 16, is DQS;

(2) Every graph of the form $K_1 \vee (C_{s_1} \cup C_{s_2} \cup \cdots \cup C_{s_t} \cup qK_2)$, where $t \geq 2$, $q \geq 1$, $s_i \geq 3$, and the number of vertices is at least 52, is DQS.

Here, K_n and C_n denote the complete graph and the cycle of order n, respectively, while $\cup$ and $\vee$ represent the disjoint union and the join of graphs. Moreover, the signless Laplacian spectrum of the graphs under consideration is computed explicitly.

1. Introduction

Let $G = (V, E)$ be a finite simple undirected graph. The number of vertices in G is called the *order* of G, denoted by $n(G)$ (or simply n). The number of edges is referred to as the *size* of G, denoted by $m(G)$ (or m).

We use the notation K_n and C_n to denote, respectively, the complete graph and the cycle on n vertices. For two graphs G and H, their *disjoint union* is denoted by $G \cup H$. The disjoint union of q copies of a graph G is denoted by qG. Furthermore, the *join* of G and H, denoted by $G \vee H$, is the graph obtained from $G \cup H$ by adding all edges between every vertex of G and every vertex of H.

Let $D(G)$ and $A(G)$ be the diagonal matrix of vertex degree sequence and the adjacency matrix of G, respectively. The *signless Laplacian matrix* of G is $Q(G) = A(G) + D(G)$. Its eigenvalues are called *signless Laplacian eigenvalues*, and they form the *signless Laplacian spectrum* of G. For brevity, the foregoing notions will be referred to as the *Q-eigenvalues* and the *Q-spectrum*, respectively. Two graphs are said to be *Q-cospectral* if they share the same Q-spectrum. In this context, a graph G is said to be *determined by its signless*

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Laplacian spectrum (abbreviated as *DQS*) if no other non-isomorphic graph has the same Q-spectrum. When the context is clear, the prefix Q- will be omitted.

Identifying graphs that are, or are not, determined by their spectrum is one of the oldest and most extensively studied problems in spectral graph theory. Its origins can be traced back to the 1950s, when Günthard and Primas first investigated the question in the context of chemical applications [10].

Based on a combination of theoretical insights, exhaustive searches of graphs of small order, and comparisons of the spectra of the signless Laplacian, Laplacian, and adjacency matrices, Cvetković and Simić concluded that cospectrality occurs least frequently with respect to the Q-spectrum [4, 5]. This observation led them to the informal conclusion that spectral graph theory founded on the signless Laplacian may be more effective than those based on the other two matrices, which have traditionally received much greater attention in the literature.

Determining whether a graph is identified by its spectrum remains a challenge problem, even for graphs with seemingly simple structures. For foundational results and general developments, we refer the reader to [7, 8], while comprehensive treatments of DQS graphs can be found in [4, 5]. Specific families of DQS graphs have been investigated in [6, 13, 19, 21, 22, 24] and references therein. For a thorough survey and further discussion on spectral determination, see [3–5].

Motivated by the results of [6], which provide a complete classification of graphs whose components are paths and cycles according to whether or not they are DQS, we have undertaken an ambitious project aimed at investigating the determination by the Q-spectrum of cones over graphs whose components consist of cycles, edges, and isolated vertices. It turns out that, in its general formulation, this problem is highly non-trivial and demands a detailed and delicate spectral analysis. The case in which edge components are excluded has already been resolved in [22], while in the present paper we advance this line of inquiry by examining the complementary situation in which isolated vertices are omitted. Our broader goal is to progressively identify structural features that determine when the Q-spectrum is a complete invariant for cones over such decomposable graphs, thereby enriching the general understanding of spectral graph determination.

A particularly interesting phenomenon arises in this context. Even seemingly minor alterations in the configuration of the underlying graph can produce markedly different spectral behaviours. For example, the absence of a specific component type or the restriction to cycles of certain lengths may significantly affect the uniqueness of the Q-spectrum. Depending on these structural changes, the resulting cone may either be uniquely determined by its Q-spectrum or, conversely, give rise to entire families of non-isomorphic yet Q-cospectral graphs. This diversity of outcomes highlights the intricate relationship between combinatorial composition and spectral properties, and further demonstrates that the interaction among different component types within the base graph plays a decisive role in shaping the spectral identity of the cone. In contrast, the results presented in this paper establish families of DQS graphs without exception.

We prove the following theorems.

Theorem 1.1. *Every graph* $K_1 \vee (C_s \cup qK_2)$, *with* $q \geq 1$, $s \geq 3$, *and at least* 16 *vertices, is DQS.*

Theorem 1.2. *Every graph* $K_1 \vee (C_{s_1} \cup C_{s_2} \cup \cdots \cup C_{s_t} \cup qK_2)$, *with* $t \geq 2$, $q \geq 1$, $s_i \geq 3$ $(1 \leq i \leq t)$, *and at least* 52 *vertices, is DQS.*

The class of cones described in the above two theorems is illustrated in Figure 1. It can be observed that Theorem 1.2 constitutes a natural extension of the results established in [12, 23], which address the DQS problem for joins between an isolated vertex and collections of vertex disjoint cycles (i.e., multiwheel graphs). Moreover, it extends the findings on the Q-spectral determination of friendship graphs presented in [19]. In addition, Theorems 1.1 and 1.2 are closely related to the results in [2, 8, 14, 16, 17, 20, 22, 24, 25], which investigate similar graph joins or graph products. It should also be noted that Theorem 1.2 represents only a partial generalization of Theorem 1.1, owing to the additional assumption on the order of the graph. Last but not least, the threshold values constraining the number of vertices that appear in the formulations of both statements stem from more sophisticated spectral conditions, which are established in the forthcoming Lemma 2.6 (for $n \geq 16$ in the former theorem) and Lemma 5.2 (for $n \geq 52$ in the latter theorem).

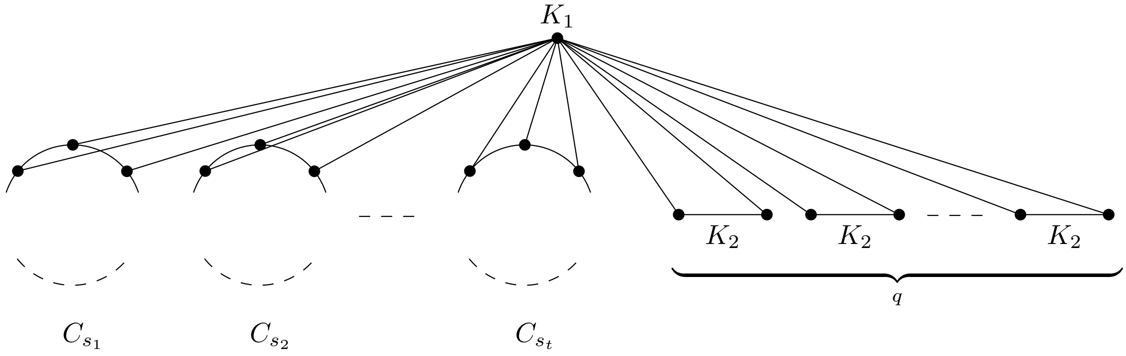

Figure 1: The graph $K_1 \vee (C_{s_1} \cup C_{s_2} \cup \cdots \cup C_{s_t} \cup qK_2)$.

The proofs are carried out in a more general setting, wherein the Q-spectra of the graphs under consideration are explicitly computed. In particular, several auxiliary results are established in a form that also accommodates the case where the cycles degenerate into two parallel edges. In this way, the corresponding results naturally extend to the setting of particular multigraphs.

The remainder of the paper is organized as follows. Section 2 introduces additional terminology and notation, as well as several known results. In Section 3, we compute the Q-spectrum of the graphs under consideration and present auxiliary results concerning relationships among Q-eigenvalues. The proofs of Theorems 1.1 and 1.2 are provided in Sections 4 and 5, respectively.

2. Preliminaries

We write $N_G(v)$ and $d_G(v)$ to denote the set of neighbours of a vertex v and the degree of the same vertex in a graph G, respectively. For a vertex subset $X \subset V$, $G[X]$ denotes the subgraph induced by X. The graphs obtained by deleting and edge e and a vertex v of G are denoted by $G - e$ and $G - v$, respectively. The Q-eigenvalues of graph G of order n are denoted by

$$\kappa_1(G) \geq \kappa_2(G) \geq \cdots \geq \kappa_n(G).$$

Since $Q(G)$ is positive semidefinite, it holds $\kappa_n(G) \geq 0$. Moreover, the equality occurs if and only if G has a bipartite component [18, Theorem 1.18]. The Q-spectrum of G is denoted by $S_Q(G)$; of course, it is considered as a multiset.

Henceforth, $\mathrm{mul}_G(\kappa)$ denotes the multiplicity of an eigenvalue κ in $S_Q(G)$, while $r^{(q)}$ denotes either q copies of a real number r or a vector of length q with all entries equal to r, depending on the context.

For convenience, we assume that the vertices of a graph under consideration are labelled in such a way that the corresponding degrees are arranged non-increasingly, i.e., $d_1 \geq d_2 \geq \cdots \geq d_n$ and, correspondingly, if the vertices of G are $v_1, v_2, \ldots, v_n$, then we always assume that the degree of v_i is $d_i, 1 \leq i \leq n$. In this context, $n_q(G)$ denotes the number of vertices of degree q (for short, q-vertices) in $V(G) \setminus \{v_1\}$, that is,

$$n_q(G) = |\{u \,:\, u \in V(G) \setminus \{v_1\} \text{ and } d_G(u) = q\}|,$$

where v_1 is a vertex attaining the maximum degree. Unless otherwise stated, the graph argument in the preceding notations will be omitted when no ambiguity arises.

The discussion now continues with a selection of results from linear algebra, some of which are well known. The first two results are drawn from [1, 9, 18]. Adapting the previous notation, let $\lambda_i(B)$, $1 \leq i \leq n$, denote the ith largest eigenvalue of an $n \times n$ real symmetric matrix B. Under this notation, we have $\kappa_i(G) = \lambda_i(Q(G))$.

Lemma 2.1. *Let B be an $n \times n$ real symmetric matrix. Then its largest eigenvalue satisfies*

$$\lambda_1(B) = \max_{\|x\|=1} \mathbf{x}^\intercal B\mathbf{x},$$

where $\mathbf{x} \in \mathbb{R}^n$ and $\|\cdot\|$ denotes the Euclidean norm. If, in particular, $\mathbf{x} = (x(u_1), x(u_2), \dots, x(u_n))^\intercal$ is a real unit vector defined on the vertex set of a graph G, then

$$\kappa_1(G) \geq \mathbf{x}^\intercal Q(G)\mathbf{x} = \sum_{uv \in E(G)} (x(u) + x(v))^2,$$

with equality if and only if $\mathbf{x}$ is an eigenvector associated with $\kappa_1(G)$.

Suppose now that the columns of B are indexed by $X = \{1, 2, \dots, n\}$. For a partition $X_1, X_2, \dots, X_k$, we set

$$B = \begin{bmatrix} B_{1,1} & \dots & B_{1,k} \\ \vdots & \ddots & \vdots \\ B_{k,1} & \dots & B_{k,k} \end{bmatrix}$$

where $B_{i,j}$ denotes the block of B formed by the rows in X_i and the columns in X_j. If $q_{i,j}$ denotes the average row sum in $B_{i,j}$, then the matrix $N = [q_{i,j}]$ is a *quotient matrix* of B. If, for every i, j, $B_{i,j}$ has a constant row sum, then the partition is called *equitable*, and N is refined to *equitable quotient matrix* of B.

Lemma 2.2. *Let B be a non-negative irreducible real symmetric matrix, and N an equitable quotient matrix of B. If λ is an eigenvalue of N, then λ is also an eigenvalue of B. Moreover, the largest eigenvalues of B and N coincide.*

The remainder of this section is devoted to specific results concerning Q-eigenvalues. We begin with a particular lower bound on κ_i.

Lemma 2.3. [15] *For a graph G, let $X = \{u_1, u_2, \dots, u_k\} \subset V(G)$ and $H \cong G[X]$. If, for $1 \leq j \leq k$, we have $0 \leq q_j \leq d_G(u_j)$, then the inequality*

$$\kappa_i(G) \geq \lambda_i\Big(\operatorname{diag}(q_1, q_2, \dots, q_k) + A(H)\Big)$$

holds for $1 \leq i \leq k$.

We also need the following interlacing between the eigenvalues of G and an edge-deleted subgraph. It is a 'signless Laplacian' counterpart to the standard Cauchy interlacing related to eigenvalues of the adjacency matrix.

Lemma 2.4. [11] *If G is a graph of order n ($n \geq 3$) and $e \in E(G)$, then*

$$\kappa_1(G) \geq \kappa_1(G - e) \geq \kappa_2(G) \geq \kappa_2(G - e) \geq \cdots \geq \kappa_n(G) \geq \kappa_n(G - e) \geq 0.$$

The following lemmas describe the relationship between the Q-eigenvalues and the structure of a graph, particularly in relation to its vertex degrees.

Lemma 2.5. [23] *If G is a graph of order $n \geq 12$ satisfying $\kappa_1 > n > 5 \geq \kappa_2 \geq \kappa_n > 0$, then G is necessarily connected, along with $d_1 \geq n - 3$ and $d_2 \leq 4$.*

The next result assumes almost the same chain of inequalities and establishes certain properties of a putative cospectral mate.

Lemma 2.6. [21] *Let G be a graph of order $n \geq 16$ satisfying $\kappa_1(G) > n > 5 > \kappa_2(G) \geq \kappa_n(G) > 0$. If H and G are Q-cospectral, then H is necessarily connected, with $d_2(H) \leq 4$ and $d_1(H) = d_1(G) \in \{n-1, n-2\}$.*

We proceed with an interplay between κ_1 and the largest vertex degree.

Lemma 2.7. [21] *Let G be a connected graph of order n, with $d_2 \le 4$. If either $d_1 \ge 8 > d_n \ge 2$ or $d_1 \ge 11 > 1 = d_n$, then $\kappa_1 \le d_1 + 3$.*

Finally, we consider the spectral moments. For a non-negative integer q, the sum

$$T_q(G) = \sum_{i=1}^{n} \kappa_i^q(G)$$

is the *qth spectral moment* of the signless Laplacian, or, equivalently, the *qth Q-spectral moment* of G.

For a graph G, we write $\varsigma_G(F)$ to denote the number of subgraphs isomorphic to a prescribed graph F.

Lemma 2.8. [3] *If G is a graph of order n and size m, then*

$$T_1(G) = \sum_{i=1}^{n} d_i = 2m, \quad T_2(G) = \sum_{i=1}^{n} d_i^2 + 2m, \quad \text{and} \quad T_3(G) = 6\varsigma_G(C_3) + \sum_{i=1}^{n} d_i^3 + 3\sum_{i=1}^{n} d_i^2.$$

All lemmas presented in this section constitute, to a greater or lesser extent, standard spectral tools employed in the subsequent sections. Most of them are adapted to the context of the Q-spectrum. Lemma 2.1 including a particular case involving κ_1 is applied only in the proof of Lemma 3.8 , while Lemma 2.3 regarding κ_2 is also applied only once, in the proof of Lemma 4.2. The remaining results are referred to more frequently throughout the paper.

3. Q-eigenvalues of certain cones

This section is concerned with the Q-eigenvalues of certain cones including, but not limited to, the following cases:

$$K_1 \vee (C_{s_1} \cup C_{s_2} \cup \cdots \cup C_{s_t} \cup qK_2), \quad K_1 \vee (P_3 \cup C_{s_1} \cup C_{s_2} \cup \cdots \cup C_{s_t} \cup qK_2), \quad \text{and} \quad K_1 \vee (P_l \cup G_1),$$

where P_l stands for the l-vertex path and G_1 is an arbitrary graph.

3.1. Q-spectrum of $K_1 \vee (C_{s_1} \cup C_{s_2} \cup \cdots \cup C_{s_t} \cup qK_2)$

This subsection focuses on the spectrum of the specific family described in the title.

Lemma 3.1. *Let n be the order of $G \cong K_1 \vee (C_{s_1} \cup C_{s_2} \cup \cdots \cup C_{s_t} \cup qK_2)$, with $t, q \ge 1$ and $s_i \ge 3$ $(1 \le i \le t)$. Then*

$$S_Q(G) = \left\{ r_1, r_2, r_3, 1^{(q)}, 3^{(q-1)}, 5^{(t-1)}, 3 + 2\cos\frac{2j\pi}{s_i} \ : \ 1 \le j \le s_i - 1, 1 \le i \le t \right\},$$

where r_1, r_2 and r_3 are the roots of $x^3 - (n+7)x^2 + (7n+8)x - 12n + 4q + 12$. Additionally, they satisfy $r_1 > n > 5 > r_2 > 3 > r_3 > 1$.

Proof. We suppose that $V(G) = \{u_i \ : \ 1 \le i \le n\}$, where $d_G(u_n) = n - 1$, $d_G(u_i) = 2$ for $1 \le i \le 2q$, and $u_{2j}u_{2j-1} \in E(G)$ for $1 \le j \le q$. In what follows, we construct the eigenvectors for certain eigenvalues.

We first deal with the eigenvalue 5. Let $\boldsymbol{\psi}_j = (\psi_j(u_1), \psi_j(u_2), \ldots, \psi_j(u_n))^\intercal$ for $1 \le j \le t - 1$, where

$$\psi_j(u_i) = \begin{cases} -s_{j+1}, & \text{for } u_i \in V(C_{s_1}), \\ s_1, & \text{for } u_i \in V(C_{s_{j+1}}), \\ 0, & \text{otherwise.} \end{cases}$$

It is straightforward to verify that this collection forms a set of linearly independent eigenvectors corresponding to the eigenvalue 5 of $Q(G)$. Consequently, $\mathrm{mul}_G(5) \ge t - 1$.

Secondly, we deal with the eigenvalues 1 and 3. As before,

$$\xi_1 = \left(1, -1, 0^{(n-2)}\right)^\intercal, \xi_2 = \left(0, 0, 1, -1, 0^{(n-4)}\right)^\intercal, \dots, \xi_q = \left(0^{(2q-2)}, 1, -1, 0^{(n-2q)}\right)^\intercal.$$

are linearly independent eigenvectors. Thus, $\mathrm{mul}_G(1) \geq q$. Similarly

$$\zeta_1 = (1, 1, -1, -1, 0^{(n-4)})^\intercal, \zeta_2 = (0^{(2)}, 1, 1, -1, -1, 0^{(n-6)})^\intercal, \dots, \zeta_{q-1} = (0^{(2q-4)}, 1, 1, -1, -1, 0^{(n-2q)})^\intercal.$$

are associated with 3, when $q \geq 2$. Thus, $\mathrm{mul}_G(3) \geq q-1$.

For convenience, we denote $V(C_{s_i}) = \{v_{i1}, v_{i2}, \dots, v_{is_i}\}, 1 \leq i \leq t$. Suppose that $A(C_{s_i})\boldsymbol{\omega} = \varrho\boldsymbol{\omega}$ ($\boldsymbol{\omega} \neq \mathbf{0}$), i.e., $\boldsymbol{\omega}$ is a unit eigenvector for $A(C_{s_i})$ corresponding to the eigenvalue ϱ. It is well known that $\varrho = 2\cos(2k\pi/s_i)$, $0 \leq k \leq s_i - 1$. Besides, it is easy to prove that if $\varrho \neq 2$ (i.e. $k \neq 0$), and then $\boldsymbol{\omega} = (\omega(v_{i1}), \omega(v_{i2}), \dots, \omega(v_{is_i}))^\intercal$ satisfies $\sum_{j=1}^{s_i} \omega(v_{ij}) = 0$. Therefore, by excluding $\varrho = 2$, we construct an eigenvector $\bar{\boldsymbol{\omega}} = (\bar{\omega}(u_1), \bar{\omega}(u_2), \dots, \bar{\omega}(u_n))^\intercal$ for $3 + \varrho$ by setting

$$\bar{\omega}(u) = \begin{cases} \omega(u), & \text{for } u \in V(C_{s_i}), \\ 0, & \text{otherwise.} \end{cases}$$

To verify this, one may observe that $I_{s_i} + Q(C_{s_i}) = 3I_{s_i} + A(C_{s_i})$ is a principal submatrix of $Q(G)$. Therefore, $3 + 2\cos(2k\pi/s_i)$, $1 \leq k \leq s_i - 1$, are the eigenvalues of $Q(G)$ for every i ($1 \leq i \leq t$).

It remains to deal with the Q-eigenvalues denoted by r_1, r_2 and r_3 in the statement formulation. Suppose $Q(G)\boldsymbol{\varphi} = \kappa\boldsymbol{\varphi}$ ($\boldsymbol{\varphi} \neq \mathbf{0}$), where $\boldsymbol{\varphi} = (\varphi(u_1), \varphi(u_2), \dots, \varphi(u_n))^\intercal$. Then for the vertex u_n with degree $n-1$, we have

$$(\kappa - n + 1)\varphi(u_n) = \sum_{i=1}^{n-1} \varphi(u_i). \tag{1}$$

For the $2q$ vertices $u_1, u_2, \dots, u_{2q}$ of degree 2, the equalities

$$\begin{cases} (\kappa - 2)\varphi(u_{2i-1}) & = & \varphi(u_{2i}) + \varphi(u_n), \\ (\kappa - 2)\varphi(u_{2i}) & = & \varphi(u_{2i-1}) + \varphi(u_n) \end{cases}$$

hold for $1 \leq i \leq q$. From the previous equalities, we deduce $(\kappa - 1)\varphi(u_{2i-1}) = (\kappa - 1)\varphi(u_{2i})$. Hence, if $\kappa \neq 1$, then $\varphi(u_{2i-1}) = \varphi(u_{2i})$, $1 \leq i \leq q$. Furthermore, if $\kappa \neq 3$, then we have

$$\varphi(u_j) = \frac{\varphi(u_n)}{\kappa - 3}, \quad 1 \leq j \leq 2q. \tag{2}$$

For the vertices in $V(C_{s_i}) = \{v_{i1}, v_{i2}, \dots, v_{i,s_i}\}$, $1 \leq i \leq t$, we have

$$\begin{cases} (\kappa - 3)\varphi(v_{i1}) & = & \varphi(u_n) + \varphi(v_{i,s_i}) + \varphi(v_{i2}), \\ (\kappa - 3)\varphi(v_{i2}) & = & \varphi(u_n) + \varphi(v_{i1}) + \varphi(v_{i3}), \\ & \vdots & \\ (\kappa - 3)\varphi(v_{i,s_i}) & = & \varphi(u_n) + \varphi(v_{i,s_i-1}) + \varphi(v_{i1}). \end{cases}$$

By setting $\varphi(v_{i1}) = \varphi(v_{i2}) = \cdots = \varphi(v_{i,s_i})$, for $\kappa \neq 5$, we obtain

$$\varphi(v_{ij}) = \frac{\varphi(u_n)}{\kappa - 5}, \quad 1 \leq j \leq s_i, \ 1 \leq i \leq t. \tag{3}$$

Combining (1), (2) and (3), we find

$$(\kappa - n + 1)\varphi(u_n) = 2q\frac{\varphi(u_n)}{\kappa - 3} + (n - 2q - 1)\frac{\varphi(u_n)}{\kappa - 5}.$$

For $\varphi(u_n) \neq 0$, the latter equality implies

$$\kappa^3 - (n+7)\kappa^2 + (7n+8)\kappa - 12n + 4q + 12 = 0.$$

It is verified directly that the corresponding roots, r_1, r_2 and r_3, satisfy $r_1 > n > 5 > r_2 > 3 > r_3 > 1$. Therefore,

$$\boldsymbol{\varphi}_i = \left((r_i - 5)^{(2q)}, (r_i - 3)^{(n-2q-1)}, (r_i - 3)(r_i - 5)\right)^\intercal$$

is an eigenvector of $Q(G)$ corresponding to the eigenvalue r_i for $1 \leq i \leq 3$.

An additional scenario must be considered: In certain cases, some of the computed eigenvalues may coincide, raising the question of whether their corresponding eigenvectors remain linearly independent. From the already proved chain of inequalities, we obtain that r_1, r_2 and r_3 are mutually distinct and do not belong to $\{1, 3, 5\}$. In addition, $1 \leq 3 + 2\cos\frac{2k\pi}{s_i} < 5$, for $1 \leq k \leq s_i - 1$ and $1 \leq i \leq t$. In this context, if the equality

$$3 + 2\cos\frac{2k\pi}{s_i} = r_j$$

holds for some choice of indices i, j and k, then the corresponding eigenvectors are linearly independent by construction. The same conclusion applies if the left-hand side equals either 1 or 3. This completes the argument and the entire proof. □

Remark 3.2. *Note that, in the previous lemma,* $3 + 2\cos(2j\pi/s_i) = 1$ *occurs if and only if* s_i *is even and* $j = s_i/2$. *Thus,* $\mathrm{mul}_G(1) = q + c_e(C)$, *where* $c_e(C)$ *denotes the number of even cycles among* C_{s_i}. *Besides, we also have*

$$n < r_1 = \kappa_1(G) < n+2,\ \kappa_n(G) = 1 < r_3,\ \kappa_2(G) < 5\ (\textit{when } t = 1),\ \textit{and } \kappa_2(G) = 5\ (\textit{when } t \geq 2).$$

Finally, for any fixed order n, *the largest eigenvalue* $\kappa_1(G)$ *depends on* q *but is independent of both the number of cycles and their lengths. This conclusion follows from the preceding proof and will also be corroborated by Corollary 3.5 in the next subsection.*

We provide an illustration of the previous lemma.

Example 3.3. *For the Q-spectrum of* $K_1 \vee (C_3 \cup 2K_2)$, *a direct algebraic computation leads to* $\{8.56, 4.44, 3, 2^{(3)}, 1^{(2)}\}$, *where* 8.56, 4.44 *and* 2 *are the roots of* $x^3 - 15x^2 + 64x - 76$.

3.2. The largest and least Q-eigenvalue of certain related cones

Next we will prove some properties of the largest and the least Q-eigenvalue of some prescribed cones. We first extend the context to include the possibility of cycles C_2, which are regarded as digons consisting of two parallel edges connecting the same pair of vertices. In this setting, the degree of a vertex is defined as the number of edges incident to it, and the entries of the adjacency matrix represent the multiplicity of the corresponding edges.

Lemma 3.4. *Let* $G^* \cong K_1 \vee (C_{s_1} \cup C_{s_2} \cup \cdots \cup C_{s_t} \cup qK_2)$, $t, q \geq 1$ *and* $s_i \geq 2$ $(1 \leq i \leq t)$. *If* n *is the order of* G^*, *then* $\kappa_1(G^*)$ *is equal to the largest root of* $x^3 - (n+7)x^2 + (7n+8)x - 12n + 4q + 12$.

Proof. The characteristic polynomial of the equitable quotient matrix

$$N = \begin{bmatrix} n-1 & n-1-2q & 2q \\ 1 & 5 & 0 \\ 1 & 0 & 3 \end{bmatrix}.$$

is $x^3 - (n+7)x^2 + (7n+8)x - 12n + 4q + 12$. By employing Lemma 2.2, we arrive at the desired result. □

Combining this result with Lemma 3.1, Remark 3.2 and Lemma 3.4, we immediately obtain an interesting corollary.

Corollary 3.5. *Given a graph* $G \cong K_1 \vee (C_{s_1} \cup C_{s_2} \cup \cdots \cup C_{s_t} \cup qK_2)$ *and a multigraph* $G^* \cong K_1 \vee (C_{s'_1} \cup C_{s'_2} \cup \cdots \cup C_{s'_r} \cup qK_2)$*, where* $t, r, q \geq 1$*,* $s_i \geq 3$ $(1 \leq i \leq t)$ *and* $s'_i \geq 2$ $(1 \leq i \leq r)$*. If* $n(G) = n(G^*)$*, then* $\kappa_1(G) = \kappa_1(G^*)$*.*

We now turn to the second join product, continuing within the broader context.

Lemma 3.6. *Let* $G^* \cong K_1 \vee (P_3 \cup C_{s_1} \cup C_{s_1} \cup \cdots \cup C_{s_t} \cup qK_2)$*, where* $t, q \geq 0$ *and* $s_i \geq 2$ $(1 \leq i \leq t)$*.Then the least Q-eigenvalue* $\kappa_n(G^*)$ *is less than* 1*.*

Proof. For $t, q \geq 1$, the characteristic polynomial of the equitable quotient matrix

$$N = \begin{bmatrix} n-1 & n-2q-4 & 2 & 1 & 2q \\ 1 & 5 & 0 & 0 & 0 \\ 1 & 0 & 2 & 1 & 0 \\ 1 & 0 & 2 & 3 & 0 \\ 1 & 0 & 0 & 0 & 3 \end{bmatrix}$$

is

$$g(x) = x^5 - (n+12)x^4 + (12n+47)x^3 + (4q-51n-52)x^2 + (88n-20q-48)x + 16q - 48n + 72.$$

It follows that $g(1) = 8 > 0$ and $g(0) = 16q - 48n + 72 < -32n + 72 < 0$. By the intermediate value theorem, we therefore deduce the existence of an eigenvalue of N lying in the interval $(0, 1)$. Consequently, by Lemma 2.2, we obtain $\kappa_n(G^*) < 1$.

Cases $q = 0$ or $t = 0$ simplify the previous computation, and lead to the same conclusion. □

In the remainder of this section, we present a lemma from [21] and establish an extension concerning the largest Q-eigenvalue. A detailed explanation of their role in the subsequent sections is provided at the end of this section. The following setting is used. Let

$$G \cong K_1 \vee (P_l \cup G_1) \tag{4}$$

be a graph with $V(G) = \{u_1, u_2, \ldots, u_n\}$, where $d_G(u_n) = n-1$. The vertices of P_l are denoted by $u_1, u_2, \ldots, u_l$, in the natural order. Also, G_1 denotes any graph. Let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)^\intercal$ be the Perron eigenvector (positive unit eigenvector associated with the largest eigenvalue) of G such that α_i corresponds to vertex u_i, $1 \leq i \leq n$.

Lemma 3.7. [21] *Under the above notation, the following statements hold:*

(i) If $l \geq 2$*, then* $\alpha_i = \alpha_{l+1-i}$ *holds for* $1 \leq i \leq \left\lfloor \frac{l}{2} \right\rfloor$*;*

(ii) If $l \geq 6$*,* $\alpha_i \neq \alpha_{i-2}$ *holds for* $3 \leq i \leq \lfloor \frac{l}{2} \rfloor$*;*

(iii) If $l \geq 4$*,* $\alpha_i \neq \alpha_{i+1}$ *holds for* $1 \leq i \leq \lfloor \frac{l}{2} \rfloor - 1$*;*

(iv) If $l \geq 3$*, then* $\alpha_1 < \alpha_2$*.*

Here is the announced extension.

Lemma 3.8. *In addition to the previous lemma:*

(i) If $l \geq 5$*, then*

$$\kappa_1(G) < \kappa_1(K_1 \vee (C_s \cup P_{l-s} \cup G_1))$$

holds for $3 \leq s \leq l-2$*;*

(ii) If $l = 4$*, then*

$$\kappa_1(G) < \kappa_1(K_1 \vee (C_2 \cup P_2 \cup G_1)),$$

where C_2 *is the digon.*

Proof. We first prove (ii). Denote $G^* \cong K_1 \vee (C_2 \cup P_2 \cup G_1)$. It is easy to see that $G^* \cong G - u_1u_2 - u_3u_4 + u_1u_4 + u_2u_3$, where G is given by (4). Note that $\kappa_1(G^*) \geq \boldsymbol{\alpha}^\intercal Q(G^*)\boldsymbol{\alpha}$ by Lemma 2.1 and $\kappa_1(G) = \boldsymbol{\alpha}^\intercal Q(G)\boldsymbol{\alpha}$ by the same result. Now, by employing Lemma 3.7 and performing a straightforward computation, we obtain

$$\kappa_1(G^*) - \kappa_1(G) \geq \boldsymbol{\alpha}^\intercal Q(G^*)\boldsymbol{\alpha} - \boldsymbol{\alpha}^\intercal Q(G)\boldsymbol{\alpha} = 2(\alpha_3 - \alpha_1)(\alpha_2 - \alpha_4) = 2(\alpha_2 - \alpha_1)^2 > 0,$$

as desired.

(i): Suppose that $\widetilde{G} \cong K_1 \vee (C_s \cup P_{l-s} \cup G_1)$. In this case, we restrict our attention to the subcase where l is even, as the alternative can be proved by analogous arguments.

Suppose $l = 2a$ with $a \geq 3$. By Lemma 3.7, we have $\alpha_{a-j} = \alpha_{a+j+1}$, $0 \leq j \leq a-1$. We next consider two separate situations, according to whether s is odd or even.

If $s = 2k+1$ ($k \geq 1$), then $a \geq k+2$ follows from $l - s \geq 2$. Obviously,

$$\widetilde{G} \cong G - u_{a-k+1}u_{a-k} - u_{a+k+1}u_{a+k+2} + u_{a-k+1}u_{a+k+1} + u_{a-k}u_{a+k+2}.$$

Note that $\alpha_{a-k} = \alpha_{a+1+k}$, $\alpha_{a+1-k} = \alpha_{a+k}$ and $\alpha_{a-k-1} = \alpha_{a+k+2}$. As before, Lemma 2.1 leads to

$$\begin{aligned}
\kappa_1(\widetilde{G}) - \kappa_1(G) &\geq \boldsymbol{\alpha}^\intercal Q(\widetilde{G})\boldsymbol{\alpha} - \boldsymbol{\alpha}^\intercal Q(G)\boldsymbol{\alpha} \\
&= 2\alpha_{a+1+k}\alpha_{a-k+1} + 2\alpha_{a+k+2}\alpha_{a-k} - 2\alpha_{a-k}\alpha_{a-k+1} - 2\alpha_{a+k+1}\alpha_{a+k+2} \\
&= 2\alpha_{a-k}\alpha_{a-k+1} + 2\alpha_{a-k-1}\alpha_{a-k} - 2\alpha_{a-k}\alpha_{a-k+1} - 2\alpha_{a-k}\alpha_{a-k-1} \\
&= 0.
\end{aligned}$$

If the equality holds, then $\boldsymbol{\alpha}$ is the Perron eigenvector of $\kappa_1(\widetilde{G})$. Moreover,

$$3\alpha_{a-k} + \alpha_{a-k-1} + \alpha_{a+k+2} = 3\alpha_{a-k} + \alpha_{a-k-1} + \alpha_{a-k+1},$$

which yields $\alpha_{a-k+1} = \alpha_{a+k+2} = \alpha_{a-k-1}$. However, this contradicts the statement of Lemma 3.7(ii).

On the other hand, if $s = 2k$ ($k \geq 2$), then $a \geq k+1$ deduces from $l - s \geq 2$. Moreover,

$$\widetilde{G} \cong G - u_{a-k+1}u_{a-k} - u_{a+k}u_{a+k+1} + u_{a-k+1}u_{a+k} + u_{a-k}u_{a+k+1}.$$

This, in combination with Lemma 3.7, gives

$$\begin{aligned}
\kappa_1(\widetilde{G}) - \kappa_1(G) &\geq \boldsymbol{\alpha}^\intercal Q(\widetilde{G})\boldsymbol{\alpha} - \boldsymbol{\alpha}^\intercal Q(G)\boldsymbol{\alpha} \\
&= 2\alpha_{a+1-k}\alpha_{a+k} + 2\alpha_{a-k}\alpha_{a+k+1} - 2\alpha_{a-k}\alpha_{a-k+1} - 2\alpha_{a+k}\alpha_{a+k+1} \\
&= 2\alpha_{a+1-k}^2 + 2\alpha_{a-k}^2 - 4\alpha_{a-k}\alpha_{a-k+1} \\
&= 2(\alpha_{a+1-k} - \alpha_{a-k})^2 \\
&> 0,
\end{aligned}$$

because $\alpha_{a+1-k} \neq \alpha_{a-k}$, and the proof is completed. □

Remark 3.9. *It is straightforward to verify that if G_1 is a multigraph, then Lemmas 3.7 and 3.8 remain valid, since the matrices $Q(G)$, $Q(\widetilde{G})$, and $Q(G^*)$ continue to be non-negative, irreducible, real, and symmetric.*

We briefly clarify the role of the above lemmas in the proofs of Theorems 1.1 and 1.2. Specifically, Lemma 3.1 provides an explicit computation of the spectra of the graphs considered in this paper, which is particularly useful for addressing their spectral determination. Moreover, the remaining lemmas serve as intermediate tools for establishing inequalities involving the largest and least Q-eigenvalues, thereby demonstrating that certain graphs cannot be Q-cospectral with the graph under consideration.

4. Proof of Theorem 1.1

Let G be as in the formulation of Theorem 1.1. Let further H be a simple graph that is Q-cospectral with G. So, $n(H) = n(G) = n$. For brevity, denote $n_i(H)$ and $d_j(H)$ by $\widetilde{n}_i$ and $\widetilde{d}_j$, respectively, where $0 \leq i \leq n-1$ and $1 \leq j \leq n$.

Due to Lemma 3.1 and Remark 3.2, the following ordering of eigenvalues is deduced:

$$1 = \kappa_n(H) < 3 < \kappa_2(H) < 5 < n < \kappa_1(H).$$

We recall the reader that the parameter $\varsigma(F)$ is defined prior Lemma 2.8.

Lemma 4.1. *If $n \geq 16$, then G and H share the same vertex degrees and $\varsigma_H(C_3) = \varsigma_G(C_3)$.*

Proof. From Lemma 2.6, we know that H is connected with $\widetilde{d}_1 = n-1$ and $\widetilde{d}_2 \leq 4$ when $n \geq 16$. By inserting $n_3 = s$, $n_2 = 2q$, $n_1 = n_4 = 0$ and $d_1 = \widetilde{d}_1 = n-1$ in Lemma 2.8, we arrive at

$$\begin{cases} \widetilde{n}_1 + \widetilde{n}_2 + \widetilde{n}_3 + \widetilde{n}_4 & = & 2q + s, \\ \widetilde{n}_1 + 2\widetilde{n}_2 + 3\widetilde{n}_3 + 4\widetilde{n}_4 & = & 4q + 3s, \\ \widetilde{n}_1 + 4\widetilde{n}_2 + 9\widetilde{n}_3 + 16\widetilde{n}_4 & = & 8q + 9s. \end{cases} \tag{5}$$

From the first two equalities, we have

$$-\widetilde{n}_1 + \widetilde{n}_3 + 2\widetilde{n}_4 = s. \tag{6}$$

Besides, from the first and the last equality, we deduce

$$-3\widetilde{n}_1 + 5\widetilde{n}_3 + 12\widetilde{n}_4 = 5s. \tag{7}$$

By combining equalities (6) and (7), we immediately obtain

$$\widetilde{n}_1 + \widetilde{n}_4 = 0.$$

Further by (6), we have $\widetilde{n}_3 = s$. Moreover, the first equality of (5) yields $\widetilde{n}_2 = 2q$. Therefore, G and H share vertex degrees.

Now, by Lemma 2.8 (precisely, the identity $T_3(H) = T_3(G)$), we arrive at $\varsigma_H(C_3) = \varsigma_G(C_3)$. Obviously, if $s \geq 4$, then $\varsigma_H(C_3) = \varsigma_G(C_3) = s+q$. Otherwise, that is, for $s = 3$, we have $\varsigma_H(C_3) = \varsigma_G(C_3) = s+1+q = q+4$. □

We now eliminate all but two structural possibilities for H.

Lemma 4.2. *For $n \geq 16$, the only structural possibilities for H are*

$$K_1 \vee (C_k \cup P_{l_1} \cup P_{l_2} \cup \cdots \cup P_{l_q}) \quad \textit{and} \quad K_1 \vee (P_{l_1} \cup P_{l_2} \cup \cdots \cup P_{l_q}),$$

where $3 \leq k \leq s$ and $l_1 \geq l_2 \geq \cdots \geq l_r \geq 3 > l_{r+1} = \cdots = l_q = 2$, with $0 \leq r \leq q$ in the former case and $1 \leq r \leq q$ in the latter.

Proof. By Lemma 4.1, $\widetilde{d}_1 = n-1$, so we may suppose $d_H(v_1) = n-1$. Besides, $\widetilde{d}_2 = 3$ and $\widetilde{n}_1 = 0$. These degree conditions lead to the conclusion that every component of $H - v_1$ is isomorphic to either C_k ($k \geq 3$) or P_l ($l \geq 2$). Moreover, from $\kappa_2(H) < 5$, we find that H cannot contain $K_1 \vee (C_{k_1} \cup C_{k_2})$ as a subgraph; otherwise, Lemmas 2.4 and 2.3 imply

$$\kappa_2(H) \geq \lambda_2(I_{k_1+k_2} + Q(C_{k_1} \cup C_{k_2})) = 5.$$

Hence, at most one component of $H - v_1$ is a cycle. Moreover, $\widetilde{n}_2 = 2q$ implies that there are exactly q disjoint paths of length at least 2 in $H - v_1$. Therefore, the lemma holds because H and G share vertex degrees. □

It remains to eliminate the possibilities listed in the formulation of the previous lemma. For convenience, we denote

$$\widetilde{H} \cong K_1 \vee (C_k \cup P_{l_1} \cup P_{l_2} \cup \cdots \cup P_{l_q}) \quad \text{and} \quad H^* \cong K_1 \vee (P_{l_1} \cup P_{l_2} \cup \cdots \cup P_{l_q}).$$

Proof of Theorem 1.1. We set $n \geq 16$ (as in the statement of this theorem). We shall prove that the graph H, introduced at the beginning of this section, is isomorphic to G. For a contradiction, we suppose $H \ncong G$. Then by Lemma 4.2, $H \cong \widetilde{H}$ or $H \cong H^*$.

Case 1: $H \cong \widetilde{H}$. By Lemma 4.2, we have $3 \leq k < s$ and $l_1 \geq l_2 \geq \cdots \geq l_r \geq 3 > l_{r+1} = \cdots = l_q = 2$ $(1 \leq r \leq q)$.

Subcase 1.1: $l_r \geq 4$. By Corollary 3.5, Lemma 3.8 and Remark 3.9, it follows that

$$\kappa_1(\widetilde{H}) < \kappa_1\Big(K_1 \vee (C_k \cup C_{l_1-2} \cup C_{l_2-2} \cup \cdots \cup C_{l_r-2} \cup qK_2)\Big) = \kappa_1(G),$$

which is a contradiction.

Subcase 1.2: $l_r = 3$. Lemmas 2.4 and 3.6 imply

$$\kappa_n(\widetilde{H}) \leq \kappa_n\Big(K_1 \vee (P_3 \cup C_k \cup C_{l_1} \cup \cdots C_{l_{r-1}} \cup (q-r)K_2)\Big) < 1 = \kappa_n(G),$$

violating the assumption on Q-cospectrality.

Case 2: $H \cong H^*$. Again, Lemma 4.2 gives $l_1 \geq l_2 \geq \cdots \geq l_r \geq 3 > l_{r+1} = \cdots = l_q = 2$ $(1 \leq r \leq q)$. As before, two subcases arise: $l_r \geq 4$ and $l_r = 3$. In the former situation, a similar argument yields

$$\kappa_1(H^*) < \kappa_1\Big(K_1 \vee (C_{l_1-2} \cup C_{l_2-2} \cup \cdots \cup C_{l_r-2} \cup qK_2)\Big) = \kappa_1(G).$$

In the latter one, it follows that

$$\kappa_n(H^*) \leq \kappa_n\Big(K_1 \vee (P_3 \cup C_{l_1} \cup \cdots \cup C_{l_{r-1}} \cup (q-r)K_2)\Big) < 1 = \kappa_n(G).$$

Both conclusions contradict the Q-cospectrality assumption, completing the proof. □

5. Proof of Theorem 1.2

Let G be as in the formulation of Theorem 1.2. In addition, recall that $q \geq 1$ and $t \geq 2$. Let further H be a simple graph that is Q-cospectral with G. By employing Lemma 3.1 and Remark 3.2, we deduce the following setting:

$$1 = \kappa_n(H) < \kappa_2(H) = 5 < n < \kappa_1(H).$$

Here we adopt the same notation for $n_i(H)$ and $d_j(H)$ as the previous section. From Lemma 2.5, we know that when $n \geq 12$, H is connected with $\widetilde{d_2} \leq 4$.

Before presenting our proof, we introduce several necessary lemmas. The first one is a known result concerning the multiplicity of 1 in the Q-spectrum of a graph containing vertices that share a common neighbourhood.

Lemma 5.1. [22] *Let F be a graph of order n $(n \geq 2)$ such that $N_F(u_1) = N_F(u_2) = \cdots = N_F(u_s) = \{u_{s+1}\}$, where $u_i \in V(F)$ $(1 \leq i \leq s+1)$ and $s \geq 1$. Then* $\mathrm{mul}_F(1) \geq s-1$.

By applying Lemma 5.1, we shall prove that, under certain conditions, the maximum vertex degrees of the graphs G and H must be equal.

Lemma 5.2. *If $n \geq 52$ or $q \geq 12$, then $\widetilde{d_1} = d_1 = n-1$.*

Proof. By Lemma 2.5, we have $\widetilde{d_1} \geq n-3 \geq 11$ when $n \geq 14$. By Lemma 2.7, we have $n < \kappa_1(H) \leq \widetilde{d_1}+3$, and thus $\widetilde{d_1} \geq n-2$. For a contradiction, we suppose that $\widetilde{d_1} = n-2$.

Lemma 2.8 leads to the system:

$$\begin{cases} 2\widetilde{n_1}+2\widetilde{n_4} &= 2n_1+2n_4+d_1(d_1-5)-\widetilde{d_1}(\widetilde{d_1}-5), \\ \widetilde{n_2}-3\widetilde{n_4} &= n_2-3n_4-d_1(d_1-4)+\widetilde{d_1}(\widetilde{d_1}-4), \\ 2\widetilde{n_3}+6\widetilde{n_4} &= 2n_3+6n_4+d_1(d_1-3)-\widetilde{d_1}(\widetilde{d_1}-3). \end{cases} \tag{8}$$

Since $d_1 = n-1$, $\widetilde{d_1} = n-2$, $n_2 = 2q$, $n_3 = n-1-2q$ and $n_1 = n_4 = 0$, from (8) we have

$$\begin{cases} \widetilde{n_1} &= n-4-\widetilde{n_4}, \\ \widetilde{n_2} &= 2q-2n+7+3\widetilde{n_4}, \\ \widetilde{n_3} &= 2n-2q-4-3\widetilde{n_4}. \end{cases} \tag{9}$$

Now, from (9) we find

$$\widetilde{n_2}+\widetilde{n_3} = 3.$$

Therefore, only four possible scenarios remain, each of which is considered separately.

Case 1: $\widetilde{n_2} = 0$ and $\widetilde{n_3} = 3$. Remark 3.2 implies $\mathrm{mul}_G(1) \leq q+(n-1-2q)/4 = (n+2q-1)/4$, while by (9), we get $\widetilde{n_4} = (2n-2q-7)/3$ and $\widetilde{n_1} = (n+2q-5)/3$. Note that $\widetilde{d_1} = n-2$ and $n \geq 2q+7$, since $t \geq 2$. If $n \geq 40$ (or $q \geq 9$), Lemma 5.1 leads to the impossible scenario: $\mathrm{mul}_H(1) \geq \widetilde{n_1}-2 > (n+2q-1)/4 \geq \mathrm{mul}_G(1)$.

Case 2: $\widetilde{n_2} = 1$ and $\widetilde{n_3} = 2$. Here, the system (9) yields $\widetilde{n_4} = (2n-2q-6)/3$ and $\widetilde{n_1} = (n+2q-6)/3$, whereas Lemma 5.1 implies $\mathrm{mul}_H(1) \geq \widetilde{n_1}-2 > (n+2q-1)/4 \geq \mathrm{mul}_G(1)$, whenever $n \geq 44$ (or $q \geq 10$).

Case 3: $\widetilde{n_2} = 2$ and $\widetilde{n_3} = 1$. From $\widetilde{n_4} = (2n-2q-5)/3$ and $\widetilde{n_1} = (n+2q-7)/3$, we arrive at $\mathrm{mul}_H(1) \geq \widetilde{n_1}-2 > (n+2q-1)/4 \geq \mathrm{mul}_G(1)$ for $n \geq 48$ (or $q \geq 11$).

Case 4: $\widetilde{n_2} = 3$ and $\widetilde{n_3} = 0$. Remark 3.2 gives $\mathrm{mul}_G(1) \leq (n+2q-1)/4$, and (9) gives $\widetilde{n_4} = (2n-2q-4)/3$ and $\widetilde{n_1} = (n+2q-8)/3$. Lemma 5.1 eliminates the possibility $n \geq 52$ (or $q \geq 12$).

In conclusion, if $n \geq 52$ or $q \geq 12$, then $\widetilde{d_1} = d_1$. □

The next two lemmas are formulated on the basis of a straightforward logical reasoning. The reader will recognize that they follow the line established in the previous section.

Lemma 5.3. *If $n \geq 52$ or $q \geq 12$, then G and H share the same vertex degrees.*

Proof. Lemma 5.2 ensures $\widetilde{d_1} = n-1$, whenever $n \geq 52$ or $q \geq 12$. By inserting $d_1 = \widetilde{d_1}$, $n_2 = 2q$, $n_3 = n-1-2q$ and $n_1 = n_4 = 0$ in (8), we arrive at

$$\widetilde{n_1} = \widetilde{n_4} = 0,\ \widetilde{n_3} = n-1-2q \ \text{ and } \widetilde{n_2} = 2q.$$

Therefore, G and H are as desired. □

Lemma 5.4. *For $n \geq 52$, the only structural possibilities for H are:*

$$K_1 \vee (C_{s'_1} \cup C_{s'_2} \cup \cdots \cup C_{s'_a} \cup P_{l_1} \cup P_{l_2} \cup \cdots \cup P_{l_q}) \quad \text{and} \quad K_1 \vee (P_{l_1} \cup P_{l_2} \cup \cdots \cup P_{l_q}),$$

where $s'_i \geq 3$, $l_1 \geq l_2 \geq \cdots \geq l_r \geq 3 > l_{r+1} = \cdots = l_q = 2$, with $0 \leq r \leq q$ in the former case and $1 \leq r \leq q$ in the latter.

Proof. Lemma 5.2 ensures $\widetilde{d_1} = n-1$ when $n \geq 52$, and we denote the corresponding vertex by v_1. In addition, Lemma 5.3 implies $\widetilde{d_2} = 3$ and $\widetilde{n_1} = 0$. These degree conditions lead to the conclusion that every component of $H - v_1$ is isomorphic to either C_s ($s \geq 3$) or P_l ($l \geq 2$). Moreover, $\widetilde{n_2} = 2q$ implies that there are exactly q disjoint paths of length at least 2 in $H - v_1$. Therefore, the statement of the lemma holds because H and G share vertex degrees. □

As before, we denote

$$\widetilde{H} \cong K_1 \vee (C_{s'_1} \cup C_{s'_2} \cup \cdots \cup C_{s'_a} \cup P_{l_1} \cup P_{l_2} \cup \cdots \cup P_{l_q}) \quad \text{and} \quad H^* \cong K_1 \vee (P_{l_1} \cup P_{l_2} \cup \cdots \cup P_{l_q}).$$

Proof of Theorem 1.2. We assume that $n \geq 52$ as in the statement of this theorem. Let H be as at the beginning of this section. Suppose that $H \ncong G$. Then by Lemma 5.4, $H \cong \widetilde{H}$ or $H \cong H^*$.

Case 1: $H \cong \widetilde{H}$. By Lemma 5.4, we have $s'_i \geq 3$ and $l_1 \geq l_2 \geq \cdots \geq l_r \geq 3 > l_{r+1} = \cdots = l_q = 2$ $(0 \leq r \leq q)$.

Subcase 1.1: $r = 0$. Then

$$H \cong \widetilde{H} \cong K_1 \vee (C_{s'_1} \cup C_{s'_2} \cup \cdots \cup C_{s'_a} \cup qK_2).$$

Firstly, $a = t$ since $\mathrm{mul}_H(5) = a - 1 = t - 1 = \mathrm{mul}_G(5)$ (by Lemma 3.1). Besides, we suppose $s_1 \geq s_2 \geq \cdots \geq s_t \geq 3$, $s'_1 \geq s'_2 \geq \cdots \geq s'_t \geq 3$. Observing that $\max\{3 + 2\cos\frac{2j\pi}{s_1} \ : \ 1 \leq j \leq s_1 - 1\} = 3 + 2\cos\frac{2\pi}{s_1}$, we get $s_1 = s'_1$. By excluding the Q-eigenvalues $3 + 2\cos\frac{2j\pi}{s_1}$, $1 \leq j \leq s_1 - 1$, from the common Q-spectrum given in Lemma 3.1, we obtain $s_2 = s'_2$. By repeating this iterative procedure, we arrive at $s_i = s'_i$, for all i, which means that H is isomorphic to G.

Subcase 1.2: $r \geq 1$ and $l_r \geq 4$. By employing Corollary 3.5, Lemma 3.8 and Remark 3.9, we obtain

$$\kappa_1(\widetilde{H}) < \kappa_1\Big(K_1 \vee (C_{s'_1} \cup C_{s'_2} \cup \cdots \cup C_{s'_a} \cup C_{l_1-2} \cup C_{l_2-2} \cup \cdots \cup C_{l_r-2} \cup qK_2)\Big) = \kappa_1(G),$$

which contradicts the assumption on Q-cospectrality.

Subcase 1.3: $r \geq 1$ and $l_r = 3$. By Lemmas 2.4 and 3.6,

$$\kappa_n(\widetilde{H}) \leq \kappa_n\Big(K_1 \vee (P_3 \cup C_{s'_1} \cup C_{s'_2} \cup \cdots \cup C_{s'_a} \cup C_{l_1} \cup \cdots \cup C_{l_{r-1}} \cup (q-r)K_2)\Big) < 1 = \kappa_n(G),$$

contradicting the same assumption.

Case 2: $H \cong H^*$. By Lemma 5.4, we have $l_1 \geq l_2 \geq \cdots \geq l_r \geq 3 > l_{r+1} = \cdots = l_q = 2$ $(1 \leq r \leq q)$, and there are two subcases that correspond to Subcases 1.2 and 1.3 of the previous part.

For $l_r \geq 4$, the same arguments lead to the impossible scenario $\kappa_1(H^*) < \kappa_1\Big(K_1 \vee (C_{l_1-2} \cup C_{l_2-2} \cup \cdots \cup C_{l_r-2} \cup qK_2)\Big) = \kappa_1(G)$.

Similarly, $l_r = 3$ reveals $\kappa_n(H^*) \leq \kappa_n\Big(K_1 \vee (P_3 \cup C_{l_1} \cup \cdots \cup C_{l_{r-1}} \cup (q-r)K_2)\Big) < 1 = \kappa_n(G)$, which is impossible. The proof is completed. □

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