Filomat 40:3 (2026), 1161–1173
https://doi.org/10.2298/FIL2603161T

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Boole-type inequalities and improved error estimates for quadrature formulae via twice differentiable convex functions

Muhammad Toseef[a], Artion Kashuri[b,*], Iram Javed[c]

[a] *School of Science, Nanjing University of Posts and Telecommunications, Nanjing, Jiangsu, 210023 China*
[b] *Department of Mathematical Engineering, Polytechnic University of Tirana, 1001 Tirana, Albania*
[c] *School of Mathematical Sciences, Beihang University, Beijing 100191, China*

Abstract. New inequalities presented by investigating quadrature formulae with their error bounds for twice differentiable convex functions. Being inspired by the central role of quadrature formulae for estimating definite integrals, this work proposes new Boole's identity for twice differentiable functions. By using the newly established identity, Boole's type inequalities for second-times differentiable convex functions are proved. By applying the power mean and Hölder's inequalities, the present research improved the error estimates for Boole's formula. The theoretical advancements are validated through numerical examples involving exponential and polynomial functions, demonstrating the efficacy of the proposed inequalities. Additionally, applications to the q-digamma function and modified Bessel functions underscore the versatility of the results. This work not only refines existing quadrature error analyses but also expands the applicability of convex function theory in numerical integration, contributing to both inequality theory and computational mathematics.

1. Introduction

Hermite and Hadamard established the first integral inequality for convex functions [1]. Hermite–Hadamard inequality, also known as the Hermite–Hadamard integral inequality for convex functions, it is stated as:

If $\Phi : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function on the interval I on the real numbers $z_1, z_2 \in I$, then we have

$$\Phi\left(\frac{z_1 + z_2}{2}\right) \leq \frac{1}{z_2 - z_1} \int_{z_1}^{z_2} \Phi(\xi)\, d\xi \leq \frac{\Phi(z_1) + \Phi(z_2)}{2}. \tag{1}$$

If inequality (1) holds in reverse, then Φ is concave.

2020 *Mathematics Subject Classification*. Primary 26D07; Secondary 26D10, 26D15, 26A33, 26A51.
Keywords. Boole's type inequalities; Convex functions; Hölder's inequalities; Power-mean inequality; Quadrature formulae; q-Digamma function; Modified Bessel functions.
Received: 19 April 2025; Accepted: 22 November 2025
Communicated by Miodrag Spalević
* Corresponding author: Artion Kashuri
Email addresses: toseefrana95@gmail.com (Muhammad Toseef), a.kashuri@fimif.edu.al (Artion Kashuri), iram.javed.412@gmail.com (Iram Javed)
ORCID iDs: https://orcid.org/0009-0004-1878-7136 (Muhammad Toseef), https://orcid.org/0000-0003-0115-3079 (Artion Kashuri), https://orcid.org/0000-0002-5212-4422 (Iram Javed)

In 1915, the term "numerical integration" debuted in the publication titled "A Course in Interpolation and Numerical Integration for the Mathematical Laboratory" by David Gibb [2]. Over recent decades, numerical integration has evolved into a pivotal tool in scientific computing, engineering, and data analysis. Advanced techniques such as adaptive quadrature algorithms, numerical integration with error estimation, and high-dimensional integration methods have been developed to tackle increasingly intricate problems. The term "quadrature", rooted in historical mathematical terminology, denotes area computation in numerical integration. A broad spectrum of quadrature rules can be derived using various interpolating polynomials. Among these, one of the simplest methods involves using a constant interpolating function (zero-degree polynomial). This is called the midpoint rule or rectangle rule. The interpolating function may be a straight line (a linear polynomial). This is known as the trapezoidal rule. We get Simpson's rule if the interpolating polynomial is of second degree. It is named Simpson's rule due to Mathematician Thomas Simpson (1710-1761). The most basic of Simpson's rule is Simpson's 1/3 rule. Another Simpson rule is the Second Simpson's or Simpson's 3/8 rule. It requires one more function evaluation with lower error bounds but does not improve the order of the error. In numerical integration with lower error bounds, the Third Simpson's or Simpson's 2/45 rule, also called Boole's rule, is named after George Boole.

In 1998, Dragomir and Agarwal [3] extend the idea of inequalities and find some error bounds of trapezoidal formula for differentiable convex functions. The principal findings of the paper summarized as follows:

Lemma 1.1. *[3] Let $\Phi : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° (the set of all interior points of I), and $z_1, z_2 \in I^{\circ}$ with $z_1 < z_2$, also $\Phi' \in L^1[z_1, z_2]$, then the following equality holds:*

$$\frac{\Phi(z_2) + \Phi(z_1)}{2} - \frac{1}{z_2 - z_1}\int_{z_1}^{z_2} \Phi(\xi)\, d\xi = \frac{(z_2 - z_1)}{2}\int_0^1 (1 - 2t)\,\Phi'(tz_2 + (1 - t)z_1)\, dt.$$

Theorem 1.2. *[3] Let $\Phi : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, and $z_1, z_2 \in I^{\circ}$ with $z_1 < z_2$, also $|\Phi'|$ is convex on $[z_1, z_2]$, then the following inequality holds:*

$$\left|\frac{\Phi(z_1) + \Phi(z_2)}{2} - \frac{1}{z_2 - z_1}\int_{z_1}^{z_2} \Phi(\xi)\, d\xi\right| \le \frac{(z_2 - z_1)}{8}\left[|\Phi'(z_1)| + |\Phi'(z_2)|\right].$$

After studying Dragomir's article, Kirmaci provided an error term of the midpoint rule in the same pattern in 2004 [4]. In 2010, Sarikaya *et al.* proved Simpson-type inequalities for differentiable convex functions and found their error bounds [5]. We need the convexity of first-time differentiable functions to estimate the error term of the above-described methods, which is a more significant class of functions than that of bounded two-times differentiable functions. That is why this is a considerable achievement in inequality theory.

The study of classical inequalities applied to integral operators linked with different fractional integrals has experienced a significant upsurge in recent years. Gronwall, Chebyshev, Hermite-Hadamard, Ostrowski, Hardy, Jensen, Milne, and Hölder inequality are a few examples of these investigated inequalities [6–8].

Kilbas et al. [10] give theory and applications of fractional differential equations. Sarikaya *et al.* proposed a novel integral identity for fractional integrals and utilized it to establish new generalized inequalities of Hermite-Hadamard's type [10]. Budak *et al.* [11] have established two novel identities involving quantum integrals. By utilizing these identities, they observed new inequalities related to Simpson's 1/3 formula for convex mappings. Hezenci *et al.* have proved a new identity for twice differentiable functions, they derived several fractional Simpson type inequalities for functions whose second derivatives in absolute value are convex [12]. By employing quantum differentiable (α, m)-convex functions, Soontharanon *et al.* have provided a generalized formulation of quantum Simpson's and quantum Newton's formula type inequalities [13]. In [14], Budak *et al.* have presented results for varied function classes; it explores Milne-type inequalities for fractional integrals, offering theoretical insights supported by specific examples and graphical representations. Ali *et al.* [15] have obtained error bounds for Milne's formula in fractional and classical calculus, with particular applications to differentiable convex functions. Additionally, they provide mathematical examples to support the validity of the newly determined bounds for Milne's formula. Numerous inequalities via fractional integrals one may refer to [16–22] and the references therein.

Simpson's quadrature formulas are among the widely recognized and extensively utilized methods in this domain. This mathematical tool is particularly useful in estimating the error associated with quadrature formulas.

Theorem 1.3. *Let $\Phi : [z_1, z_2] \to \mathbb{R}$ be a four times differentiable and continuous function on (z_1, z_2), and let $\left\|\Phi^{(4)}\right\|_\infty := \sup\limits_{\xi\in(z_1,z_2)} \left|\Phi^{(4)}(\xi)\right| < \infty$. Then, the following inequality can be declared as follows:*

$$\left|\frac{1}{6}\left[\Phi(z_1) + 4\Phi\left(\frac{z_1 + z_2}{2}\right) + \Phi(z_2)\right] - \frac{1}{z_2 - z_1}\int_{z_1}^{z_2} \Phi(\xi)d\xi\right| \leq \frac{1}{2880}\left\|\Phi^{(4)}\right\|_\infty (z_2 - z_1)^4.$$

Dragomir *et al.* [23] have found novel Simpson-type inequalities. Furthermore, they discuss their applications to quadrature formulas. In 2013, Sarikaya *et al.* [24] reported H-H-type inequalities for the first time via Riemann-Liouville fractional integrals. Budak and Karagözoğlu have investigated Milne-type inequalities in the term of Riemann-Liouville fractional integrals. Also, they discussed several function classes [25]. In [26], Sarikaya *et al.* conducted the first study of Simpson-type inequalities through the lens of convexity principles. Also, they discuss their applications in the context of special means for real numbers. By leveraging fractional integrals and *s*-convexity, Chen and Huang have formulated generalized results, which are obtained in [27]. In [28], authors have extended these inequalities by evolving *k*-fractional and Riemann-Liouville fractional integrals.

The following inequality is well known in the literature as Boole's rule inequality. This rule offers a fifth-order (quartic) approximation, surpassing the accuracy of Simpson's rule, by approximating the integral over five points with a polynomial of degree four.

Theorem 1.4. *Let suppose that $\Phi : [z_1, z_2] \to \mathbb{R}$, is a six times continuously differentiable function on (z_1, z_2) and $\|\Phi^{(6)}\| := \sup_{\xi\in(z_1,z_2)} \left|\Phi^{(6)}(\xi)\right| < \infty$. Then the subsequently inequality holds:*

$$\left|\frac{1}{90}\left[7\Phi(z_1) + 32\Phi\left(\frac{3z_1 + z_2}{4}\right) + 12\Phi\left(\frac{z_1 + z_2}{2}\right) + 32\Phi\left(\frac{z_1 + 3z_2}{4}\right) + 7\Phi(z_2)\right] - \frac{1}{z_2 - z_1}\int_{z_1}^{z_2} \Phi(\xi)d\xi\right|$$
$$\leq \frac{1}{1935360}\left\|\Phi^{(6)}\right\|_\infty (z_2 - z_1)^6.$$

Krukowski [29] has investigated a logical evolution of Simpson's rule, which is Boole's rule. By leveraging computer software, novel error bounds for Boole's rule have been demonstrated. For error bounds in *n*-point Newton-Cotes quadrature, which often rely on the n^{th} derivative of the integrand function, posing challenges for highly differentiable functions, Jamei [30] has defined a specific linear kernel to tackle this issue. By leveraging these kernels, he presented new error bounds for all closed and open type Newton-Cotes quadrature rules. The advantage of these error bounds is that they do not rely on the norms of the integrand function, ensuring broad applicability and coverage of existing literature. It is widely recognized that inequalities have been instrumental in advancing nearly all areas of pure and applied sciences see [31–33].

Studying twice differentiable convex functions allows for a detailed analysis of how errors behave as the function changes or as the interval size changes. This can help in adaptive quadrature methods, where the partition of the interval is adjusted based on the function's behavior. Some important articles that inspired me for twice differentiable convex functions [34–38]. Boole's rule, a fifth-order quadrature formula, offers superior accuracy by approximating integrals using quartic polynomials. Despite its potential, its error analysis has historically required sixth-order derivatives, limiting its applicability. Recent studies, such as those by Krukowski and Jamei, have sought to relax these constraints using kernel-based techniques.

Building on these foundations, this paper introduces a new Boole-type identity for twice differentiable functions for higher-order derivatives. By using the convexity of the second derivative, we derive sharper error bounds through inequalities that integrate power mean and Hölder's techniques. This approach not only simplifies error estimation but also broadens the scope of Boole's rule to a wider class of functions.

The remainder of this work is structured as follows: Section 2 presents the main results, including the novel Boole-type identity and associated inequalities. In Section 3, numerical examples validate the theoretical findings while Section 4, presents applications to special functions illustrate their practical utility. The paper concludes with a discussion of future research directions in adaptive quadrature and generalized convexity frameworks in the last Section 5.

2. Main Results

This section utilizes twice differentiable convex mappings to derive a sequence of Boole-type inequalities. In order to accomplish this, we first provide proof of that specific identity. Consequently, we derive three distinct inequalities by employing the modulus of this identity. This strategy not only presents new results but also enhances the advancement of the field.

Lemma 2.1. *Assume that $\Phi : [z_1, z_2] \subset \mathbb{R} \to \mathbb{R}$ is a twice differentiable function on (z_1, z_2). If $\Phi'' \in L[z_1, z_2]$ (the set of all Lebesgue integrable functions on the closed interval $[z_1, z_2]$), then, we have the subsequent inequality:*

$$\begin{aligned}
&\frac{1}{90}\left[\frac{(z_2 - z_1)}{16}\left(17\Phi'(z_2) - 21\Phi'\left(\frac{z_1 + 3z_2}{4}\right) + 21\Phi'\left(\frac{z_1 + z_2}{2}\right) - 17\Phi'\left(\frac{3z_1 + z_2}{4}\right)\right)\right. \\
&\qquad \left. +7\Phi(z_2) + 32\Phi\left(\frac{z_1 + 3z_2}{4}\right) + 12\Phi\left(\frac{z_1 + z_2}{2}\right) + 32\Phi\left(\frac{3z_1 + z_2}{4}\right) + 7\Phi(z_1)\right] - \frac{1}{z_2 - z_1}\int_{z_1}^{z_2} \Phi(\xi)\, d\xi \\
&= \frac{(z_2 - z_1)^2}{64}\sum_{j=0}^{3} S_j,
\end{aligned} \tag{2}$$

where

$$\begin{aligned}
S_0 &:= \int_0^1 \frac{t}{2}\left(\frac{28}{45} - t\right)\Phi''\left(\frac{4-t}{4}z_1 + \frac{t}{4}z_2\right)dt, \\
S_1 &:= \int_0^1 \frac{t}{2}\left(\frac{66}{45} - t\right)\Phi''\left(\frac{3-t}{4}z_1 + \frac{1+t}{4}z_2\right)dt, \\
S_2 &:= \int_0^1 \frac{t}{2}\left(\frac{24}{45} - t\right)\Phi''\left(\frac{2-t}{4}z_1 + \frac{2+t}{4}z_2\right)dt, \\
S_3 &:= \int_0^1 \frac{t}{2}\left(\frac{62}{45} - t\right)\Phi''\left(\frac{1-t}{4}z_1 + \frac{3+t}{4}z_2\right)dt.
\end{aligned}$$

Proof. Taking right hand side of (2), we have

$$\frac{(z_2 - z_1)^2}{64}\sum_{j=0}^{3} S_j = \frac{(z_2 - z_1)^2}{64}\left[S_0 + S_1 + S_2 + S_3\right].$$

Using the integration by parts, it yields

$$\begin{aligned}
S_0 &= \frac{(z_2 - z_1)^2}{64}\int_0^1 \frac{t}{2}\left(\frac{28}{45} - t\right)\Phi''\left(\frac{4-t}{4}z_1 + \frac{t}{4}z_2\right)dt \\
&= \frac{(z_2 - z_1)}{16}\left(\frac{28t}{90} - \frac{t^2}{2}\right)\Phi'\left(\frac{4-t}{4}z_1 + \frac{t}{4}z_2\right)\Bigg|_0^1 - \frac{(z_2 - z_1)}{16}\int_0^1 \left(\frac{28}{90} - t\right)\Phi'\left(\frac{4-t}{4}z_1 + \frac{t}{4}z_2\right)dt
\end{aligned} \tag{3}$$

$$= \frac{(z_2 - z_1)}{16}\left(-\frac{17}{90}\right)\Phi'\left(\frac{3z_1 + z_2}{4}\right) + \frac{1}{4}\left(\left(\frac{62}{90}\right)\Phi\left(\frac{3z_1 + z_2}{4}\right) + \frac{28}{90}\Phi(z_1) - \int_0^1 \Phi\left(\frac{4-t}{4}z_1 + \frac{t}{4}z_2\right)dt\right).$$

In a similar manner, we acquire

$$S_1 = \frac{(z_2 - z_1)^2}{64}\int_0^1 \frac{t}{2}\left(\frac{66}{45} - t\right)\Phi''\left(\frac{4-t}{4}z_1 + \frac{t}{4}z_2\right)dt \tag{4}$$

$$= \frac{(z_2 - z_1)}{16}\left(\frac{21}{90}\right)\Phi'\left(\frac{z_1 + z_2}{2}\right) + \frac{1}{4}\left(\left(\frac{24}{90}\right)\Phi\left(\frac{z_1 + z_2}{2}\right) + \frac{66}{90}\Phi\left(\frac{3z_1 + z_2}{4}\right) - \int_0^1 \Phi\left(\frac{3-t}{4}z_1 + \frac{1+t}{4}z_2\right)dt\right),$$

$$S_2 = \frac{(z_2 - z_1)^2}{64}\int_0^1 \frac{t}{2}\left(\frac{24}{45} - t\right)\Phi''\left(\frac{2-t}{4}z_1 + \frac{2+t}{4}z_2\right)dt \tag{5}$$

$$= \frac{(z_2 - z_1)}{16}\left(\frac{21}{90}\right)\Phi'\left(\frac{z_1 + 3z_2}{4}\right) + \frac{1}{4}\left(\left(\frac{66}{90}\right)\Phi\left(\frac{z_1 + 3z_2}{4}\right) + \frac{24}{90}\Phi\left(\frac{z_1 + z_2}{2}\right) - \int_0^1 \Phi\left(\frac{2-t}{4}z_1 + \frac{2+t}{4}z_2\right)dt\right),$$

and

$$S_3 = \frac{(z_2 - z_1)^2}{64}\int_0^1 \frac{t}{2}\left(\frac{62}{45} - t\right)\Phi''\left(\frac{1-t}{4}z_1 + \frac{3+t}{4}z_2\right)dt \tag{6}$$

$$= \frac{(z_2 - z_1)}{16}\left(\frac{21}{90}\right)\Phi'(z_2) + \frac{1}{4}\left(\left(\frac{28}{90}\right)\Phi(z_2) + \frac{62}{90}\Phi\left(\frac{z_1 + 3z_2}{4}\right) - \int_0^1 \left(\frac{1-t}{4}z_1 + \frac{3+t}{4}z_2\right)dt\right).$$

Adding (3)-(6) and multiply $\frac{(z_2-z_1)^2}{64}$, then by change of variable we get desired result. Hence, proof of Lemma 2.1 is completed. □

Theorem 2.2. *Assume that $\Phi : [z_1, z_2] \subset \mathbb{R} \to \mathbb{R}$ is a twice differentiable function on (z_1, z_2), such that $\Phi'' \in L[z_1, z_2]$. If $|\Phi''|$ is convex on $[z_1, z_2]$, then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{90}\left[\frac{(z_2 - z_1)}{16}\left(17\Phi'(z_2) - 21\Phi'\left(\frac{z_1 + 3z_2}{4}\right) + 21\Phi'\left(\frac{z_1 + z_2}{2}\right) - 17\Phi'\left(\frac{3z_1 + z_2}{4}\right)\right)\right.\right.\\
&\left.\left.\quad +7\Phi(z_2) + 32\Phi\left(\frac{z_1 + 3z_2}{4}\right) + 12\Phi\left(\frac{z_1 + z_2}{2}\right) + 32\Phi\left(\frac{3z_1 + z_2}{4}\right) + 7\Phi(z_1)\right] - \frac{1}{z_2 - z_1}\int_{z_1}^{z_2}\Phi(\xi)\,d\xi\right|\\
&\le \frac{(z_2 - z_1)^2}{64}\left[\frac{4916701}{24603750}|\Phi''(z_1)| + \frac{7081469}{24603750}|\Phi''(z_2)|\right].
\end{aligned}$$

Proof. Taking absolute property using Lemma 2.1, we have

$$\begin{aligned}
&\left|\frac{1}{90}\left[\frac{(z_2 - z_1)}{16}\left(17\Phi'(z_2) - 21\Phi'\left(\frac{z_1 + 3z_2}{4}\right) + 21\Phi'\left(\frac{z_1 + z_2}{2}\right) - 17\Phi'\left(\frac{3z_1 + z_2}{4}\right)\right)\right.\right.\\
&\left.\left.\quad +7\Phi(z_2) + 32\Phi\left(\frac{z_1 + 3z_2}{4}\right) + 12\Phi\left(\frac{z_1 + z_2}{2}\right) + 32\Phi\left(\frac{3z_1 + z_2}{4}\right) + 7\Phi(z_1)\right] - \frac{1}{z_2 - z_1}\int_{z_1}^{z_2}\Phi(\xi)\,d\xi\right|\\
&\le \frac{(z_2 - z_1)^2}{64}\left\{\int_0^1 \left|\frac{t}{2}\left(\frac{28}{45} - t\right)\right|\left|\Phi''\left(\frac{4-t}{4}z_1 + \frac{t}{4}z_2\right)\right|dt + \int_0^1 \left|\frac{t}{2}\left(\frac{66}{45} - t\right)\right|\left|\Phi''\left(\frac{3-t}{4}z_1 + \frac{1+t}{4}z_2\right)\right|dt\right.\\
&\left.\quad + \int_0^1 \left|\frac{t}{2}\left(\frac{24}{45} - t\right)\right|\left|\Phi''\left(\frac{2-t}{4}z_1 + \frac{2+t}{4}z_2\right)\right|dt + \int_0^1 \left|\frac{t}{2}\left(\frac{62}{45} - t\right)\right|\left|\Phi''\left(\frac{1-t}{4}z_1 + \frac{3+t}{4}z_2\right)\right|dt\right\}.
\end{aligned} \tag{7}$$

By using convexity of $|\Phi''|$, we attain

$$\left|\frac{1}{90}\left[\frac{(z_2 - z_1)}{16}\left(17\Phi'(z_2) - 21\Phi'\left(\frac{z_1 + 3z_2}{4}\right) + 21\Phi'\left(\frac{z_1 + z_2}{2}\right) - 17\Phi'\left(\frac{3z_1 + z_2}{4}\right)\right)\right.\right.$$

$$+7\Phi(z_2)+32\Phi\left(\frac{z_1+3z_2}{4}\right)+12\Phi\left(\frac{z_1+z_2}{2}\right)+32\Phi\left(\frac{3z_1+z_2}{4}\right)+7\Phi(z_1)\Bigg]-\frac{1}{z_2-z_1}\int_{z_1}^{z_2}\Phi(\xi)\,d\xi\Bigg|$$

$$\leq\frac{(z_2-z_1)^2}{64}\left\{\int_0^1\left|\frac{t}{2}\left(\frac{28}{45}-t\right)\right|\left[\left(\frac{4-t}{4}\right)|\Phi''(z_1)|+\frac{t}{4}|\Phi''(z_2)|\right]dt\right.$$
$$+\int_0^1\left|\frac{t}{2}\left(\frac{66}{45}-t\right)\right|\left[\left(\frac{3-t}{4}\right)|\Phi''(z_1)|+\left(\frac{1+t}{4}\right)|\Phi''(z_2)|\right]dt$$
$$+\int_0^1\left|\frac{t}{2}\left(\frac{24}{45}-t\right)\right|\left[\left(\frac{2-t}{4}\right)|\Phi''(z_1)|+\left(\frac{2+t}{4}\right)|\Phi''(z_2)|\right]dt$$
$$\left.+\int_0^1\left|\frac{t}{2}\left(\frac{62}{45}-t\right)\right|\left[\left(\frac{1-t}{4}\right)|\Phi''(z_1)|+\left(\frac{3+t}{4}\right)|\Phi''(z_2)|\right]dt\right\}$$
$$=\frac{(z_2-z_1)^2}{64}\left\{|\Phi''(z_1)|\left(\int_0^1\left|\frac{t}{2}\left(\frac{28}{45}-t\right)\right|\left(\frac{4-t}{4}\right)dt+\int_0^1\left|\frac{t}{2}\left(\frac{66}{45}-t\right)\right|\left(\frac{3-t}{4}\right)dt\right.\right.$$
$$\left.+\int_0^1\left|\frac{t}{2}\left(\frac{24}{45}-t\right)\right|\left(\frac{2-t}{4}\right)dt+\int_0^1\left|\frac{t}{2}\left(\frac{62}{45}-t\right)\right|\left(\frac{1-t}{4}\right)dt\right)$$
$$+\left(|\Phi''(z_2)|\left(\int_0^1\left|\frac{t}{2}\left(\frac{28}{45}-t\right)\right|\left(\frac{t}{4}\right)dt+\int_0^1\left|\frac{t}{2}\left(\frac{66}{45}-t\right)\right|\left(\frac{1+t}{4}\right)dt\right.\right.$$
$$\left.\left.\left.+\int_0^1\left|\frac{t}{2}\left(\frac{24}{45}-t\right)\right|\left(\frac{2+t}{4}\right)dt+\int_0^1\left|\frac{t}{2}\left(\frac{62}{45}-t\right)\right|\left(\frac{3+t}{4}\right)dt\right)\right\}$$
$$=\frac{(z_2-z_1)^2}{64}\left[\frac{4916701}{24603750}|\Phi''(z_1)|+\frac{7081469}{24603750}|\Phi''(z_2)|\right].$$

Hence, the proof of Theorem 2.2 is completed. □

Theorem 2.3. *Assume that $\Phi:[z_1,z_2]\subset\mathbb{R}\to\mathbb{R}$ is a twice differentiable function on (z_1,z_2), such that $\Phi''\in L[z_1,z_2]$. If $|\Phi''|^q$ is convex on $[z_1,z_2]$ with $q\geq 1$, then the following inequality holds:*

$$\Bigg|\frac{1}{90}\Bigg[\frac{(z_2-z_1)}{16}\left(17\Phi'(z_2)-21\Phi'\left(\frac{z_1+3z_2}{4}\right)+21\Phi'\left(\frac{z_1+z_2}{2}\right)-17\Phi'\left(\frac{3z_1+z_2}{4}\right)\right)$$
$$+7\Phi(z_2)+32\Phi\left(\frac{z_1+3z_2}{4}\right)+12\Phi\left(\frac{z_1+z_2}{2}\right)+32\Phi\left(\frac{3z_1+z_2}{4}\right)+7\Phi(z_1)\Bigg]-\frac{1}{z_2-z_1}\int_{z_1}^{z_2}\Phi(\xi)\,d\xi\Bigg|$$
$$\leq\frac{(z_2-z_1)^2}{64}\Bigg[\left(\frac{28027}{546750}\right)^{1-\frac{1}{q}}\left\{\left(\frac{16854253}{393660000}\right)|\Phi''(z_1)|^q+\left(\frac{3325187}{393660000}\right)|\Phi''(z_2)|^q\right\}^{\frac{1}{q}}$$
$$+\left(\frac{1}{5}\right)^{1-\frac{1}{q}}\left\{\left(\frac{173}{1440}\right)|\Phi''(z_1)|^q+\left(\frac{23}{288}\right)|\Phi''(z_2)|^q\right\}^{\frac{1}{q}}$$
$$+\left(\frac{1187}{20250}\right)^{1-\frac{1}{q}}\left\{\left(\frac{90373}{4860000}\right)|\Phi''(z_1)|^q+\left(\frac{194507}{4860000}\right)|\Phi''(z_2)|^q\right\}^{\frac{1}{q}}$$
$$+\left(\frac{8}{45}\right)^{1-\frac{1}{q}}\left\{\left(\frac{79}{4320}\right)|\Phi''(z_1)|^q+\left(\frac{689}{4320}\right)|\Phi''(z_2)|^q\right\}^{\frac{1}{q}}\Bigg].$$

Proof. When we first apply (7) to the power-mean inequality using absolute property, it becomes

$$\Bigg|\frac{1}{90}\Bigg[\frac{(z_2-z_1)}{16}\left(17\Phi'(z_2)-21\Phi'\left(\frac{z_1+3z_2}{4}\right)+21\Phi'\left(\frac{z_1+z_2}{2}\right)-17\Phi'\left(\frac{3z_1+z_2}{4}\right)\right)$$
$$+7\Phi(z_2)+32\Phi\left(\frac{z_1+3z_2}{4}\right)+12\Phi\left(\frac{z_1+z_2}{2}\right)+32\Phi\left(\frac{3z_1+z_2}{4}\right)+7\Phi(z_1)\Bigg]-\frac{1}{z_2-z_1}\int_{z_1}^{z_2}\Phi(\xi)\,d\xi\Bigg|$$

$$\leq \frac{(z_2-z_1)^2}{64}\left[\left(\int_0^1\left|\frac{t}{2}\left(\frac{28}{45}-t\right)\right|dt\right)^{1-\frac{1}{q}}\left(\int_0^1\left|\frac{t}{2}\left(\frac{28}{45}-t\right)\right|\left|\Phi''\left(\frac{4-t}{4}z_1+\frac{t}{4}z_2\right)\right|^q dt\right)^{\frac{1}{q}}\right.$$
$$+\left(\int_0^1\left|\frac{t}{2}\left(\frac{66}{45}-t\right)\right|dt\right)^{1-\frac{1}{q}}\left(\int_0^1\left|\frac{t}{2}\left(\frac{66}{45}-t\right)\right|\left|\Phi''\left(\frac{3-t}{4}z_1+\frac{1+t}{4}z_2\right)\right|^q dt\right)^{\frac{1}{q}}$$
$$+\left(\int_0^1\left|\frac{t}{2}\left(\frac{24}{45}-t\right)\right|dt\right)^{1-\frac{1}{q}}\left(\left|\int_0^1\frac{t}{2}\left(\frac{24}{45}-t\right)\right|\left|\Phi''\left(\frac{2-t}{4}z_1+\frac{2+t}{4}z_2\right)\right|^q dt\right)^{\frac{1}{q}}$$
$$\left.+\left(\int_0^1\left|\frac{t}{2}\left(\frac{62}{45}-t\right)\right|dt\right)^{1-\frac{1}{q}}\left(\int_0^1\left|\frac{t}{2}\left(\frac{62}{45}-t\right)\right|\left|\Phi''\left(\frac{1-t}{4}z_1+\frac{3+t}{4}z_2\right)\right|^q dt\right)^{\frac{1}{q}}\right].$$

As $|\Phi''|^q$ is convex on the interval $[z_1, z_2]$, we attain

$$\left|\frac{1}{90}\left[\frac{(z_2-z_1)}{16}\left(17\Phi'(z_2)-21\Phi'\left(\frac{z_1+3z_2}{4}\right)+21\Phi'\left(\frac{z_1+z_2}{2}\right)-17\Phi'\left(\frac{3z_1+z_2}{4}\right)\right)\right.\right.$$
$$\left.\left.+7\Phi(z_2)+32\Phi\left(\frac{z_1+3z_2}{4}\right)+12\Phi\left(\frac{z_1+z_2}{2}\right)+32\Phi\left(\frac{3z_1+z_2}{4}\right)+7\Phi(z_1)\right]-\frac{1}{z_2-z_1}\int_{z_1}^{z_2}\Phi(\xi)\,d\xi\right|$$
$$\leq \frac{(z_2-z_1)^2}{64}\left[\left(\int_0^1\left|\frac{t}{2}\left(\frac{28}{45}-t\right)\right|dt\right)^{1-\frac{1}{q}}\left\{\int_0^1\left|\frac{t}{2}\left(\frac{28}{45}-t\right)\right|\left(\left(\frac{4-t}{4}\right)|\Phi''(z_1)|^q+\frac{t}{4}|\Phi''(z_2)|^q\right)dt\right\}^{\frac{1}{q}}\right.$$
$$+\left(\int_0^1\left|\frac{t}{2}\left(\frac{66}{45}-t\right)\right|dt\right)^{1-\frac{1}{q}}\left\{\int_0^1\left|\frac{t}{2}\left(\frac{66}{45}-t\right)\right|\left(\left(\frac{3-t}{4}\right)|\Phi''(z_1)|^q+\left(\frac{1+t}{4}\right)|\Phi''(z_2)|^q\right)dt\right\}^{\frac{1}{q}}$$
$$+\left\{\left(\int_0^1\left|\frac{t}{2}\left(\frac{24}{45}-t\right)\right|dt\right)^{1-\frac{1}{q}}\int_0^1\left|\frac{t}{2}\left(\frac{24}{45}-t\right)\right|\left(\left(\frac{2-t}{4}\right)|\Phi''(z_1)|^q+\left(\frac{2+t}{4}\right)|\Phi''(z_2)|^q\right)dt\right\}^{\frac{1}{q}}$$
$$\left.+\left(\int_0^1\left|\frac{t}{2}\left(\frac{62}{45}-t\right)\right|dt\right)^{1-\frac{1}{q}}\left\{\int_0^1\left|\frac{t}{2}\left(\frac{62}{45}-t\right)\right|\left(\left(\frac{1-t}{4}\right)|\Phi''(z_1)|^q+\left(\frac{3+t}{4}\right)|\Phi''(z_2)|^q\right)dt\right\}^{\frac{1}{q}}\right]$$
$$=\frac{(z_2-z_1)^2}{64}\left[\left(\frac{28027}{546750}\right)^{1-\frac{1}{q}}\left\{\left(\frac{16854253}{393660000}\right)|\Phi''(z_1)|^q+\left(\frac{3325187}{393660000}\right)|\Phi''(z_2)|^q\right\}^{\frac{1}{q}}\right.$$
$$+\left(\frac{1}{5}\right)^{1-\frac{1}{q}}\left\{\left(\frac{173}{1440}\right)|\Phi''(z_1)|^q+\left(\frac{23}{288}\right)|\Phi''(z_2)|^q\right\}^{\frac{1}{q}}$$
$$+\left(\frac{1187}{20250}\right)^{1-\frac{1}{q}}\left\{\left(\frac{90373}{4860000}\right)|\Phi''(z_1)|^q+\left(\frac{194507}{4860000}\right)|\Phi''(z_2)|^q\right\}^{\frac{1}{q}}$$
$$\left.+\left(\frac{8}{45}\right)^{1-\frac{1}{q}}\left\{\left(\frac{79}{4320}\right)|\Phi''(z_1)|^q+\left(\frac{689}{4320}\right)|\Phi''(z_2)|^q\right\}^{\frac{1}{q}}\right].$$

Hence, the proof of Theorem 2.3 is completed. □

Theorem 2.4. *Assume that $\Phi : [z_1, z_2] \subset \mathbb{R} \to \mathbb{R}$ is a twice differentiable function on (z_1, z_2), such that $\Phi'' \in L[z_1, z_2]$. If $|\Phi''|^q$ is convex on $[z_1, z_2]$, then for $p > 1$, the following inequality holds:*

$$\left|\frac{1}{90}\left[\frac{(z_2-z_1)}{16}\left(17\Phi'(z_2)-21\Phi'\left(\frac{z_1+3z_2}{4}\right)+21\Phi'\left(\frac{z_1+z_2}{2}\right)-17\Phi'\left(\frac{3z_1+z_2}{4}\right)\right)\right.\right.$$
$$\left.\left.+7\Phi(z_2)+32\Phi\left(\frac{z_1+3z_2}{4}\right)+12\Phi\left(\frac{z_1+z_2}{2}\right)+32\Phi\left(\frac{3z_1+z_2}{4}\right)+7\Phi(z_1)\right]-\frac{1}{z_2-z_1}\int_{z_1}^{z_2}\Phi(\xi)\,d\xi\right|$$

$$\leq \frac{(z_2-z_1)^2}{64}\left[\Omega_1^p\left(\frac{7|\Phi''(z_1)|^q+|\Phi''(z_2)|^q}{8}\right)^{\frac{1}{q}}+\Omega_2^p\left(\frac{5|\Phi''(z_1)|^q+3|\Phi''(z_2)|^q}{8}\right)^{\frac{1}{q}}\right.$$
$$\left.+\Omega_3^p\left(\frac{3|\Phi''(z_1)|^q+5|\Phi''(z_2)|^q}{8}\right)^{\frac{1}{q}}+\Omega_4^p\left(\frac{|\Phi''(z_1)|^q+7|\Phi''(z_2)|^q}{8}\right)^{\frac{1}{q}}\right],$$

where $\frac{1}{p}+\frac{1}{q}=1$ *and*

$$\Omega_1^p := \left(\int_0^1\left|\frac{t}{2}\left(\frac{28}{45}-t\right)\right|^p dt\right)^{\frac{1}{p}},$$
$$\Omega_2^p := \left(\int_0^1\left|\frac{t}{2}\left(\frac{66}{45}-t\right)\right|^p dt\right)^{\frac{1}{p}},$$
$$\Omega_3^p := \left(\int_0^1\left|\frac{t}{2}\left(\frac{24}{45}-t\right)\right|^p dt\right)^{\frac{1}{p}},$$
$$\Omega_4^p := \left(\int_0^1\left|\frac{t}{2}\left(\frac{62}{45}-t\right)\right|^p dt\right)^{\frac{1}{p}}.$$

Proof. By utilizing Hölder's inequality in (7) using absolute property, it yields

$$\left|\frac{1}{90}\left[\frac{(z_2-z_1)}{16}\left(17\Phi'(z_2)-21\Phi'\left(\frac{z_1+3z_2}{4}\right)+21\Phi'\left(\frac{z_1+z_2}{2}\right)-17\Phi'\left(\frac{3z_1+z_2}{4}\right)\right)\right.\right.$$
$$\left.\left.+7\Phi(z_2)+32\Phi\left(\frac{z_1+3z_2}{4}\right)+12\Phi\left(\frac{z_1+z_2}{2}\right)+32\Phi\left(\frac{3z_1+z_2}{4}\right)+7\Phi(z_1)\right]-\frac{1}{z_2-z_1}\int_{z_1}^{z_2}\Phi(\xi)\,d\xi\right|$$
$$\leq \frac{(z_2-z_1)^2}{64}\left\{\left(\int_0^1\left|\frac{t}{2}\left(\frac{28}{45}-t\right)\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|\Phi''\left(\frac{4-t}{4}z_1+\frac{t}{4}z_2\right)\right|^q dt\right)^{\frac{1}{q}}\right.$$
$$+\left(\int_0^1\left|\frac{t}{2}\left(\frac{66}{45}-t\right)\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|\Phi''\left(\frac{3-t}{4}z_1+\frac{1+t}{4}z_2\right)\right|^q dt\right)^{\frac{1}{q}}$$
$$+\left(\int_0^1\left|\frac{t}{2}\left(\frac{24}{45}-t\right)\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|\Phi''\left(\frac{2-t}{4}z_1+\frac{2+t}{4}z_2\right)\right|^q dt\right)^{\frac{1}{q}}$$
$$\left.+\left(\int_0^1\left|\frac{t}{2}\left(\frac{62}{45}-t\right)\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|\Phi''\left(\frac{1-t}{4}z_1+\frac{3+t}{4}z_2\right)\right|^q dt\right)^{\frac{1}{q}}\right\}.$$

Since $|\Phi''|^q$ is convex on the interval $[z_1, z_2]$, it becomes

$$\left|\frac{1}{90}\left[\frac{(z_2-z_1)}{16}\left(17\Phi'(z_2)-21\Phi'\left(\frac{z_1+3z_2}{4}\right)+21\Phi'\left(\frac{z_1+z_2}{2}\right)-17\Phi'\left(\frac{3z_1+z_2}{4}\right)\right)\right.\right.$$
$$\left.\left.+7\Phi(z_2)+32\Phi\left(\frac{z_1+3z_2}{4}\right)+12\Phi\left(\frac{z_1+z_2}{2}\right)+32\Phi\left(\frac{3z_1+z_2}{4}\right)+7\Phi(z_1)\right]-\frac{1}{z_2-z_1}\int_{z_1}^{z_2}\Phi(\xi)\,d\xi\right|$$
$$\leq \frac{(z_2-z_1)^2}{64}\left[\left(\int_0^1\left|\frac{t}{2}\left(\frac{28}{45}-t\right)\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left[\left(\frac{4-t}{4}\right)|\Phi''(z_1)|^q+\frac{t}{4}|\Phi''(z_2)|^q\right]dt\right)^{\frac{1}{q}}\right.$$
$$+\left(\int_0^1\left|\frac{t}{2}\left(\frac{66}{45}-t\right)\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left[\left(\frac{3-t}{4}\right)|\Phi''(z_1)|^q+\frac{1+t}{4}|\Phi''(z_2)|^q\right]dt\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1\left|\frac{t}{2}\left(\frac{24}{45}-t\right)\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left[\left(\frac{2-t}{4}\right)|\Phi''(z_1)|^q+\frac{2+t}{4}|\Phi''(z_2)|^q\right]dt\right)^{\frac{1}{q}}$$
$$+\left(\int_0^1\left|\frac{t}{2}\left(\frac{62}{45}-t\right)\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left[\left(\frac{1-t}{4}\right)|\Phi''(z_1)|^q+\frac{3+t}{4}|\Phi''(z_2)|^q\right]dt\right)^{\frac{1}{q}}\Bigg]$$
$$=\frac{(z_2-z_1)^2}{64}\left[\left(\int_0^1\left|\frac{t}{2}\left(\frac{28}{45}-t\right)\right|^p dt\right)^{\frac{1}{p}}\left(\frac{7|\Phi''(z_1)|^q+|\Phi''(z_2)|^q}{8}\right)^{\frac{1}{q}}\right.$$
$$+\left(\int_0^1\left|\frac{t}{2}\left(\frac{66}{45}-t\right)\right|^p dt\right)^{\frac{1}{p}}\left(\frac{5|\Phi''(z_1)|^q+3|\Phi''(z_2)|^q}{8}\right)^{\frac{1}{q}}$$
$$+\left(\int_0^1\left|\frac{t}{2}\left(\frac{24}{45}-t\right)\right|^p dt\right)^{\frac{1}{p}}\left(\frac{3|\Phi''(z_1)|^q+5|\Phi''(z_2)|^q}{8}\right)^{\frac{1}{q}}$$
$$\left.+\left(\int_0^1\left|\frac{t}{2}\left(\frac{62}{45}-t\right)\right|^p dt\right)^{\frac{1}{p}}\left(\frac{|\Phi''(z_1)|^q+7|\Phi''(z_2)|^q}{8}\right)^{\frac{1}{q}}\right].$$

Hence, the proof of Theorem 2.4 is completed. □

3. Computational Analysis

In this section, numerical examples are given to check the validity of newly established results.

Example 3.1. *If $\Phi(z)=e^{2z}$ is twice differentiable convex function on (z_1,z_2), where $z_1,z_2\in\mathbb{R}$, then we have*

Left Inequality :

$$\left|\frac{1}{90}\left[\frac{(z_2-z_1)}{16}\left(17\Phi'(z_2)-21\Phi'\left(\frac{z_1+3z_2}{4}\right)+21\Phi'\left(\frac{z_1+z_2}{2}\right)-17\Phi'\left(\frac{3z_1+z_2}{4}\right)\right)\right.\right.$$
$$\left.\left.+7\Phi(z_2)+32\Phi\left(\frac{z_1+3z_2}{4}\right)+12\Phi\left(\frac{z_1+z_2}{2}\right)+32\Phi\left(\frac{3z_1+z_2}{4}\right)+7\Phi(z_1)\right]-\frac{1}{z_2-z_1}\int_{z_1}^{z_2}\Phi(\xi)\,d\xi\right|. \tag{8}$$

Right Inequality :

$$\frac{(z_2-z_1)^2}{64}\left[\frac{4916701}{24603750}|\Phi''(z_1)|+\frac{7081469}{24603750}|\Phi''(z_2)|\right]. \tag{9}$$

Inequalities (8) and (9) of Theorem 2.2 are valid, see Figure 1-(a).

Example 3.2. *If $\Phi(z)=z^5$ is twice differentiable convex function on (z_1,z_2), where $z_1,z_2\in\mathbb{R}$, then we have*

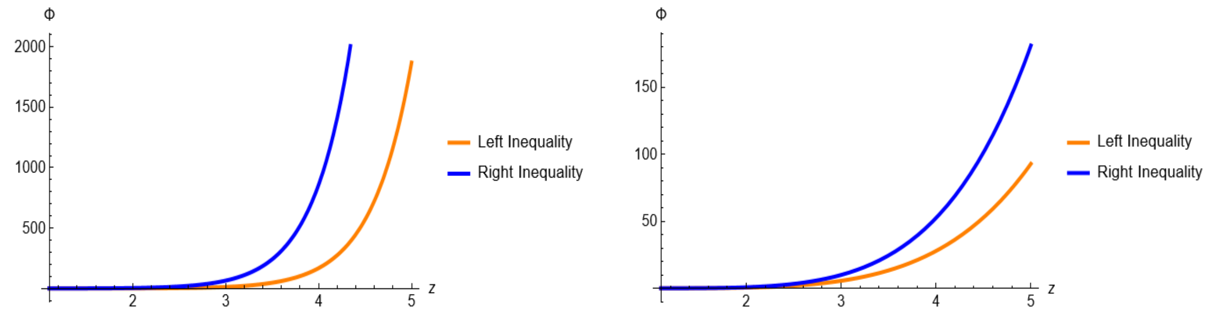
Left Inequality :

$$\left|\frac{1}{90}\left[\frac{(z_2-z_1)}{16}\left(17\Phi'(z_2)-21\Phi'\left(\frac{z_1+3z_2}{4}\right)+21\Phi'\left(\frac{z_1+z_2}{2}\right)-17\Phi'\left(\frac{3z_1+z_2}{4}\right)\right)\right.\right.$$
$$\left.\left.+7\Phi(z_2)+32\Phi\left(\frac{z_1+3z_2}{4}\right)+12\Phi\left(\frac{z_1+z_2}{2}\right)+32\Phi\left(\frac{3z_1+z_2}{4}\right)+7\Phi(z_1)\right]-\frac{1}{z_2-z_1}\int_{z_1}^{z_2}\Phi(\xi)\,d\xi\right|. \tag{10}$$

Right Inequality :

$$\frac{(z_2-z_1)^2}{64}\left[\frac{4916701}{24603750}|\Phi''(z_1)|+\frac{7081469}{24603750}|\Phi''(z_2)|\right]. \tag{11}$$

Inequalities (10) and (11) of Theorem 2.2 are valid, see Figure 1-(b).



(a) $\Phi(z) = e^{2z}$ (b) $\Phi(z) = z^5$

Figure 1: Comparative analysis of inequalities of Theorem 2.2

.

$\Phi(z)$	First time differentiable Boole's	Twice differentiable Boole's
$z_1 = 1, z_2 = 2$	$\frac{239(z_2-z_1)}{6480}\left[\lvert\Phi'(z_1)\rvert + \lvert\Phi'(z_2)\rvert\right]$	$\frac{(z_2-z_1)^2}{64}\left[\frac{4916701}{24603750}\lvert\Phi''(z_1)\rvert + \frac{7081469}{24603750}\lvert\Phi''(z_2)\rvert\right]$
z^6	7.30278	2.25233
z^5	3.13503	0.78200
e^z	0.372786	0.0417177

Table 1: Comparison of newly established error bounds in Theorem 2.2 and proved in [39] .

Remark 3.3. *Results proved in this paper gives better error estimates than proved in [39], clearly seen from Table 1.*

4. Applications

In this section, applications are given to show the effectiveness of newly established results.

4.1. *q-Digamma function*

The $\varphi_{\mathbf{q}}$-digamma function, defined as the logarithmic derivative of the **q**-gamma function, plays a significant role in **q**-calculus. Its monotonicity and complete monotonicity properties have been extensively studied, leading to notable inequalities (see [40, 41]).

Assume the **q**−analogue of the digamma function φ for $0 < \mathbf{q} < 1$ and $\xi > 0$, is the **q**−digamma function $\varphi_{\mathbf{q}}$, (see [42]) is given as:

$$\begin{aligned}\varphi_{\mathbf{q}}(\xi) &= -\ln(1-\mathbf{q}) + \ln\mathbf{q}\sum_{k=0}^{\infty}\frac{\mathbf{q}^{k+\xi}}{1-\mathbf{q}^{k+\xi}}\\ &= -\ln(1-\mathbf{q}) + \ln\mathbf{q}\sum_{k=1}^{\infty}\frac{\mathbf{q}^{k\xi}}{1-\mathbf{q}^{k}}.\end{aligned}$$

For $\mathbf{q} > 1$ and $\xi > 0$, **q**−digamma function $\varphi_{\mathbf{q}}$ can be given as:

$$\begin{aligned}\varphi_{\mathbf{q}}(\xi) &= -\ln(\mathbf{q}-1) + \ln\mathbf{q}\left[\xi - \frac{1}{2} - \sum_{k=0}^{\infty}\frac{\mathbf{q}^{-(k+\xi)}}{1-\mathbf{q}^{-(k+\xi)}}\right]\\ &= -\ln(\mathbf{q}-1) + \ln\mathbf{q}\left[\xi - \frac{1}{2} - \sum_{k=1}^{\infty}\frac{\mathbf{q}^{-k\xi}}{1-\mathbf{q}^{-k}}\right].\end{aligned}$$

Proposition 4.1. *Let $\varphi_{\mathbf{q}} : [z_1, z_2] \subset (0, \infty) \to \mathbb{R}$ be the $\mathbf{q}$-digamma function, where $\varphi''_{\mathbf{q}} \in L[z_1, z_2]$, then the following inequality holds:*

$$\left|\frac{1}{90}\left[\frac{(z_2 - z_1)}{16}\left(17\varphi'_q(z_2) - 21\varphi'_q\left(\frac{z_1+3z_2}{4}\right) + 21\varphi'_q\left(\frac{z_1+z_2}{2}\right) - 17\varphi'_q\left(\frac{3z_1+z_2}{4}\right)\right)\right.\right.$$
$$\left.\left. +7\varphi_q(z_2) + 32\varphi_q\left(\frac{z_1+3z_2}{4}\right) + 12\varphi_q\left(\frac{z_1+z_2}{2}\right) + 32\varphi_q\left(\frac{3z_1+z_2}{4}\right) + 7\varphi_q(z_1)\right] - \frac{1}{z_2 - z_1}\int_{z_1}^{z_2}\varphi_q(\xi)\,d\xi\right|$$
$$\leq \frac{(z_2 - z_1)^2}{64}\left[\frac{4916701}{24603750}\left|\varphi''_q(z_1)\right| + \frac{7081469}{24603750}\left|\varphi''_q(z_2)\right|\right].$$

Proof. *Substitute $\Phi(\xi) \to \varphi_q(\xi)$ in Theorem 2.2. The result is obtained immediately by $\varphi_q(\xi)$ that is a completely monotone function on $(0, \infty)$ for all $\xi > 0$ and consequently $\Phi''(\xi) := \varphi''_q(\xi)$ is convex on the same interval $(0, \infty)$.* □

4.2. Modified Bessel function

We revisit the series representation of the first kind modified Bessel function $\mathfrak{I}_\tau$ (see [43]).

$$\mathfrak{I}_\tau(\eta) = \Sigma_{\mathfrak{n}\geq 0}\frac{\left(\frac{\eta}{2}\right)^{\tau+2\mathfrak{n}}}{\mathfrak{n}!\Gamma(\tau+\mathfrak{n}+1)}.$$

where $\eta > 0$ and $\tau > -1$, while the second kind modified Bessel function $\hbar_\tau$ (see [43]) is usually defined as:

$$\hbar_\tau(\eta) = \frac{\pi}{2}\,\frac{\mathfrak{I}_{-\tau}(\eta) - \mathfrak{I}_\tau(\eta)}{\sin \tau\pi}.$$

Consider the function $\mathfrak{B}_\tau(\eta) : \mathbb{R} \to [1, \infty)$ defined by

$$\mathfrak{B}_\tau(\eta) = 2^\tau \Gamma(\tau+1)\,\eta^{-\tau}\hbar_\tau(\eta),$$

where $\Gamma(\cdot)$ is the gamma function.

The first order derivative formula of $\mathfrak{B}_\tau(\eta)$ is given by [43]:

$$\mathfrak{B}'_\tau(\eta) = \frac{\eta}{2(\tau+1)}\mathfrak{B}_{\tau+1}(\eta) \tag{12}$$

and the second derivative can be easily calculated from (12) to be

$$\mathfrak{B}''_\tau(\eta) = \frac{\eta^2\mathfrak{B}_{\tau+2}(\eta)}{4(\tau+1)(\tau+2)} + \frac{\mathfrak{B}_{\tau+1}(\eta)}{2(\tau+1)}. \tag{13}$$

The third derivative of $\mathfrak{B}_\tau(\eta)$ is given by:

$$\mathfrak{B}'''_\tau(\eta) = \frac{3\eta\mathfrak{B}_{\tau+2}(\eta)}{4(\tau+1)(\tau+2)} + \frac{\eta^3\mathfrak{B}_{\tau+3}(\eta)}{8(\tau+1)(\tau+2)(\tau+3)}. \tag{14}$$

Proposition 4.2. *Suppose $\tau > -1$, then*

$$\left|\frac{1}{90}\left[\frac{(z_2 - z_1)}{16}\left(17\mathfrak{B}''_\tau(z_2) - 21\mathfrak{B}''_\tau\left(\frac{z_1+3z_2}{4}\right) + 21\mathfrak{B}''_\tau\left(\frac{z_1+z_2}{2}\right) - 17\mathfrak{B}''_\tau\left(\frac{3z_1+z_2}{4}\right)\right)\right.\right.$$
$$\left.\left. +7\mathfrak{B}'_\tau(z_2) + 32\mathfrak{B}'_\tau\left(\frac{z_1+3z_2}{4}\right) + 12\mathfrak{B}'_\tau\left(\frac{z_1+z_2}{2}\right) + 32\mathfrak{B}'_\tau\left(\frac{3z_1+z_2}{4}\right) + 7\mathfrak{B}'_\tau(z_1)\right] - \frac{\mathfrak{B}(z_2) - \mathfrak{B}(z_1)}{z_2 - z_1}\right|$$
$$\leq \frac{(z_2 - z_1)^2}{64}\left[\frac{4916701}{24603750}\left|\mathfrak{B}'''_\tau(z_1)\right| + \frac{7081469}{24603750}\left|\mathfrak{B}'''_\tau(z_2)\right|\right].$$

Proof. *Applying Theorem 2.2 for the function $\Phi(\xi) := \mathfrak{B}'_\tau(\xi)$, with $\xi > 0$ and systematically integrating the identities* (12), (13), *and* (14), *we derive the required result.* □

5. Conclusion

In this study, we established a new Boole-type identity for twice differentiable functions and derived corresponding inequalities for convex functions. By employing the power mean and Hölder's inequalities, we enhanced the error bounds for Boole's quadrature formula, providing more precise estimates. Numerical examples validated the effectiveness of the proposed results, demonstrating their applicability in integral approximations. Applications to the **q**-digamma function and modified Bessel functions further demonstrated the versatility of our approach, bridging abstract inequality theory with practical computational problems. These contributions not only refine the analysis of Boole's quadrature but also underscore the efficacy of convexity in numerical integration. This work contributes to the theory of inequalities by refining classical quadrature estimates. Future research could extend these results to other convex classes, such as quasi-convex or harmonically convex functions, and explore adaptive quadrature algorithms that leverage the derived bounds. Additionally, integrating fractional calculus or quantum integrals may yield further generalizations, enhancing the applicability of Boole-type inequalities in diverse mathematical and engineering contexts. This work lays a foundation for such explorations, emphasizing the enduring relevance of convexity in numerical analysis.

References

[1] S. S. Dragomir, C. Pearce, *Selected topics on Hermite-Hadamard inequalities and applications*, RGMIA Monographs, Victoria University, 2000.

[2] D. Gibb, *A course in interpolation and numerical integration for the mathematical laboratory*, G. Bell & Sons, Limited, 1915.

[3] S. S. Dragomir, R. P. Agarwal, *Inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula*, Appl. Math. Lett., **11**(5) (1998), 91–95.

[4] U. S. Kirmaci, *Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula*, Appl. Math. Comput., **147**(1) (2004), 137–146.

[5] M. Z. Sarikaya, E. Set, M. E. Özdemir, *On new inequalities of Simpson's type for s-convex functions*, Comput. Math. Appl., **60** (2010), no. 8, 2191–2199.

[6] A. Abdeldaim, M. Yakout, *On some new integral inequalities of Gronwall-Bellman-Pachpatte type*, Appl. Math. Comput., **217**(20) (2011), 7887–7899.

[7] M. A. Ali, H. Budak, F. Michal, K. Sundas, *A new version of q-Hermite-Hadamard midpoint and trapezoid type inequalities for convex functions*, Math. Slovaca, **73**(2) (2023), 369–386.

[8] H. Kara, H. Budak, M. A. Ali, F. Hezenci, *On inequalities of Simpson's type for convex functions via generalized fractional integrals*, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat., **71**(3) (2022), 806–825.

[9] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, North-Holland Mathematics Studies, vol. 204, Elsevier Science B.V., Amsterdam, 2006.

[10] M. Z. Sarikaya, H. Budak, *Hermite-Hadamard type inequalities for twice differentiable mappings via fractional integrals*, Filomat, **31**(7) (2017), 1957–1965.

[11] H. Budak, S. Erden and M. A. Ali, *Simpson and Newton type inequalities for convex functions via newly defined quantum integrals*, Math. Methods Appl. Sci., **44**(1) (2021), 378–390.

[12] F. Hezenci, H. Budak, H. Kara, *The new version of fractional Simpson-type inequalities for twice differentiable functions*, Adv. Difference Equ., **2021**(1) (2021), Article ID 460, 1–10.

[13] J. Soontharanon, M. A. Ali, H. Budak, K. Nonlaopon, Z. Abdullah, *Simpson and Newton type inequalities for (α, m)-convex functions via quantum calculus*, Symmetry, **14**(4) (2022), Article ID 736, 1–16.

[14] H. Budak, P. Kösem, H. Kara, *On new Milne-type inequalities for fractional integrals*, J. Inequal. Appl., **2023**(1) (2023), Article ID 17, 1–15.

[15] M. A. Ali, Z. Zhang, M. Fečkan, *On some error bounds for Milne's formula in fractional calculus*, Mathematics, **11**(1) (2022), Article ID 146, 1–13.

[16] A. Akkurt, M. E. Yildirim, H. Y. Yildirim, *On some integral inequalities for (k, h)-Riemann-Liouville fractional integral*, New Trends Math. Sci., **4**(2) (2016), 138–146.

[17] S. Iftikhar, P. Kumam, S. Erden, *Newton-type integral inequalities via local fractional integrals*, Fractals, **28**(3) (2020), Article ID 2050037, 1–13.

[18] H. Kara, S. Erden, H. Budak, *Hermite-Hadamard, trapezoid and midpoint type inequalities involving generalized fractional integrals for convex functions*, Sahand Commun. Math. Anal., **20**(2) (2023), 85–107.

[19] M. E. Ozdemir, A. A. Merve, H. Kavurmaci-Onalan, *Hermite-Hadamard type inequalities for s-convex and s-concave functions via fractional integrals*, Turkish J. Sci., **1**(1) (2016), 28–40.

[20] A. Saglam, M. Z. Sarikaya, H. Yildirim, *Some new inequalities of Hermite-Hadamard type*, Kyungpook Math. J., **50**(3) (2010), 399–410.

[21] X. You, M. A. Ali, H. Budak, H. Kara, D. Zhao, *Some parameterized Simpson-type inequalities for differentiable convex functions involving generalized fractional integrals*, Adv. Contin. Discrete Models, **2022**(1) (2022), Article ID 22, 1–19.

[22] D. Zhao, M. A. Ali, Z. Y. He, *Some Bullen-type inequalities for generalized fractional integrals*, Fractals, **31**(8) (2023), Article ID 2340060, 1–13.

[23] S. S. Dragomir, R. P. Agarwal, P. Cerone, *On Simpson's inequality and applications*, J. Inequal. Appl., **5**(6) (2000), 533–579.

[24] M. Z. Sarikaya, E. Set, H. Yaldiz, N. Başak, *Hermite-Hadamard inequalities for fractional integrals and related fractional inequalities*, Math. Comput. Modelling, **57**(9-10) (2013), 2403–2407.

[25] H. Budak, P. Karagözoğlu, *Fractional Milne-type inequalities*, Acta Math. Univ. Comenian., **93**(1) (2024), 1–15.

[26] M. Z. Sarikaya, E. Set, M. E. Özdemir, *On new inequalities of Simpson's type for convex functions*, RGMIA Res. Rep. Coll., **13**(2) (2010), Article ID 7, 1–11.

[27] J. Chen, X. Huang, *Some new inequalities of Simpson's type for s-convex functions via fractional integrals*, Filomat, **31**(15) (2017), 4989–4997.

[28] F. Ertuğral, M. Z. Sarikaya, *Simpson-type integral inequalities for generalized fractional integral*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM, **113**(4) (2019), 3115–3124.

[29] M. Krukowski, *New error bounds for Boole's rule*, (2018), arXiv preprint arXiv:1808.02803.

[30] M. M. Jamei, *Unified error bounds for all Newton-Cotes quadrature rules*, J. Numer. Math., **23**(1) (2015), 67–80.

[31] J. H. Mathews, K. D. Fink, *Numerical methods using MATLAB*, 4th ed., Pearson Prentice Hall, 2004.

[32] M. A. Al-Alaoui, *A class of numerical integration rules with first-order derivatives*, ACM SIGNUM Newsl., **31**(2) (1996), 25–44.

[33] R. A. Fadhil, *Numerical calculation for error formula for Boole's rule*, Adv. Model. Anal., **56**(2-4) (2019), 54–58.

[34] H. Kara, H. Budak, S. Etemad, S. Rezapour, H. Ahmad, M. K. A. Kaabar, *A study on the new class of inequalities of midpoint-type and trapezoidal-type based on twice differentiable functions with conformable operators*, J. Funct. Spaces, **2023** (2023), Article ID 4624604, 1–12.

[35] F. Hezenci, H. Budak, *Midpoint-type inequalities via twice-differentiable functions on tempered fractional integrals*, J. Inequal. Appl., **2023**(1) (2023), Article ID 150, 1–18.

[36] M. A. Ali, H. Kara, J. Tariboon, S. Asawasamrit, H. Budak, F. Hezenci, *Some new Simpson-formula-type inequalities for twice-differentiable convex functions via generalized fractional operators*, Symmetry, **13**(12) (2021), Article ID 2249, 1–18.

[37] M. Vivas-Cortez, T. Abdeljawad, P. O. Mohammed, Y. Rangel-Oliveros, *Simpson's integral inequalities for twice differentiable convex functions*, Math. Probl. Eng., **2020** (2020), Article ID 1936461, 1–9.

[38] X. You, F. Hezenci, H. Budak, H. Kara, *New Simpson-type integral inequalities for twice differentiable functions via generalized fractional integrals*, AIMS Math., **7**(3) (2022), 3959–3971.

[39] A. Shehzadi, H. Budak, W. Haider, H. Chen, *Error bounds of Boole's formula for different function cases*, Appl. Math. J. Chinese Univ., 2025. (Accepted).

[40] A. Salem, *Complete monotonicity properties of functions involving q-gamma and q-digamma functions*, Math. Inequal. Appl., **17**(3) (2014), 801–811.

[41] S. Jain, K. Mehrez, D. Baleanu, P. Agarwal, *Certain Hermite-Hadamard inequalities for logarithmically convex functions with applications*, Mathematics, **7**(2) (2019), Article ID 163, 1–14.

[42] N. Batir, *Monotonicity properties of q-digamma and q-trigamma functions*, J. Approx. Theory, **192** (2015), 336–346.

[43] G. N. Watson, *A treatise on the theory of Bessel functions*, 2nd ed., Cambridge University Press, 1995.