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Matrix representation of Fibonacci sequence of complex uncertain variables

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Abstract. The main goal of this paper is to introduce and establish some relationship between the notions of convergence of complex uncertain sequences based on a regular matrix of Fibonacci numbers.

1. Introduction

The uncertainty theory is a branch of mathematics that model belief degree. When no samples are available, we use the uncertainty theory to quantify the future. In this scenario, we will invite some domain experts to determine the belief degree to which each event will occur. The uncertainty theory is just a tool for dealing with the belief degrees. The uncertainty theory and uncertain measure defined and developed by Liu [10] to describe the subjective uncertain phenomena. On sequences, Liu [10] applied the uncertainty theory because the convergence of sequences is important in exploring the fundamental theory of mathematics. After Liu, many researchers like You [23], Liu and Ha [11], You and Yan [24, 25] have been studied the uncertainty theory on sequences. In real life, the concept of uncertainty is not only restricted to real quantities but also appears in complex quantities. Peng [14] presented the notion of complex uncertain variable which is defined from an uncertainty space to the set of complex numbers. Chen *et al.* [2] researched the idea of convergence of uncertain sequences considering complex uncertain variables. Many other researchers, including Tripathy and Nath [21], Tripathy and Dowari [20], Nath and Tripathy [12, 13], Saha *et al.* [17–19], Roy *et al.* [15], have done extensive theoretical work based on complex uncertain variable.

Kara and Basarir [8] introduced the Fibonacci sequence into sequence theory. The Fibonacci sequence is one of the most well-known and interesting number sequences and mathematicians continue to remain interested by it because this sequence is an important and valuable tool for widen their mathematical horizons. Fibonacci numbers have many interesting properties and applications in the humanities, sciences and architecture. For example, the ratio of Fibonacci numbers converges to the golden section which is

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important in sciences and arts. Many mathematicians such as Basarir *et. al* [1] purposed regular matrix by using Fibonacci numbers and to investigate its matrix domain in the classical sequence spaces, Demiriz *et. al* [4] introduced almost convergent sequence space by using the Fibonacci difference matrix, Debnath and Saha [3], Saha and Tripathy [16], Das *et. al* [5], Yaying *et. al* [26], Kumar *et. al* [9] have done interesting work involving the Fibonacci matrix, which inspires the current study.

The main motivation of this article is to investigate the convergence properties of Fibonacci sequences in the context of complex uncertain variables. This research aims to provide new insights into the behavior of complex uncertain variables, which have gained significant attention in recent years due to their wide range of applications in fields such as finance, engineering, and economics. By exploring the convergence properties of Fibonacci sequences, this study aims to gain new insights into complex uncertain systems. Our main goal, as stated above, is to propose the various types of Fibonacci convergence of complex uncertain sequences. In addition, we have also attempted to form some relationships between them.

2. Preliminaries

We shall obtain some basic definitions and theorems of uncertainty theory in this section, which will be used in the work.

Definition 2.1. *[10] Let $\mathcal{L}$ be a $\sigma-$algebra on a nonempty set Γ. A set function $\mathcal{M}$ is called an uncertain measure if it satisfies the following axioms:*
Axiom 1. (Normality Axiom) $\mathcal{M}\{\Gamma\} = 1$;
Axiom 2. (Duality Axiom) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any $\Lambda \in \mathcal{L}$;
Axiom 3. (Subadditivity Axiom) For every countable sequence of $\left\{\Lambda_j\right\} \in \mathcal{L}$, we have

$$\mathcal{M}\left\{\bigcup_{j=1}^{\infty} \Lambda_j\right\} \leq \sum_{j=1}^{\infty} \mathcal{M}\left\{\Lambda_j\right\}.$$

The triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space and each element Λ in $\mathcal{L}$ is called an event. In order to obtain uncertainty measure of compound event, a product uncertain measure is defined by Liu [10] as follows:
Axiom 4. (Product Axiom) Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty space for $k = 1, 2, ...$ The product uncertainty measure $\mathcal{M}$ is an uncertain measure satisfying

$$\mathcal{M}\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{\Lambda_k\},$$ *where Λ_k are arbitrarily chosen events from $\mathcal{L}_k$ for $k = 1, 2, ...,$ respectively.*

Definition 2.2. *[14] A complex uncertain variable is a measurable function ζ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of complex numbers i.e., for any Borel set B of complex numbers, the set $\{\zeta \in B\} = \{\gamma \in \Gamma \mid \zeta(\gamma) \in B\}$ is an event.*

Definition 2.3. *[2] The complex uncertain sequence (ζ_n) is said to be convergent almost surely to ζ if there exists an event Λ with $\mathcal{M}\{\Lambda\} = 1$ such that*

$$\lim_{n\to\infty} \|\zeta_n(\gamma) - \zeta(\gamma)\| = 0,$$

for every $\gamma \in \Lambda$.

Definition 2.4. *[2] The complex uncertain sequence (ζ_n) is said to be convergent in measure to ζ if*

$$\lim_{n\to\infty} \mathcal{M}\{\|\zeta_n - \zeta\| \geq \varepsilon\} = 0,$$

for every $\varepsilon > 0$.

Definition 2.5. *[2] Let $\zeta, \zeta_1, \zeta_2, ...$ be complex uncertain variables with finite expected values. Then, the complex uncertain sequence (ζ_n) is said to be convergent in mean to ζ if*

$$\lim_{n\to\infty} E[\|\zeta_n - \zeta\|] = 0.$$

Definition 2.6. *[15] Let ζ and τ be complex uncertain sequence. Then, the $p-$distance between ζ and τ is defined by*

$$D_p(\zeta,\tau) = \ (E[||\zeta-\tau||^p])^{\frac{1}{p+1}},\ \ p>0.$$

Definition 2.7. *[15] Let $\mathcal{U}$ be the set of all complex uncertain variables having finite expected values. Then, the set $\mathcal{U}$ with the $p-$distance $\mathcal{D}_p$ is called a metric space of complex uncertain sequences and is denoted by $(\mathcal{U},\mathcal{D}_p)$.*

Definition 2.8. *[15] Let $\zeta,\zeta_1,\zeta_2,...$ be complex uncertain variables defined on metric space $(\mathcal{U},\mathcal{D}_p)$. Then, the sequence of a complex uncertain variable (ζ_n) is said to be $p-$distance convergent to ζ if*

$$\lim_{n\to\infty}\mathcal{D}_p(\zeta_n,\zeta)=0.$$

Lemma 2.9. *(Roy et al. [15], Theorem 3.1) Let ζ,τ,θ be complex uncertain variables and let $D_p(*,*)$ be the $p-$distance. Then*
(a)$D_p(\zeta,\tau)\geq 0$; (non-negativity)
(b)$D_p(\zeta,\tau)=0$; (identification)
(c)$D_p(\zeta,\tau)=D_p(\tau,\zeta)$; (symmetry)
(d)$D_p(\zeta,\theta)\leq D_p(\zeta,\tau)+D_p(\tau,\theta)$ (triangle inequality)

Lemma 2.10. *(Roy et al. [15], Theorem 3.2) Let $(\mathcal{U},\mathcal{D}_p)$ be a metric space of complex uncertain variables. For any complex uncertain variables $\zeta,\tau,\theta\in\mathcal{U}$ and $\lambda\in\mathcal{R}$, then*
(i)$D_p(\zeta+\tau,\theta+\tau)=D_p(\zeta,\theta)$; (translation invariant)
(ii)$D_p(\lambda\zeta,\lambda\theta)=|\lambda|^{\frac{p}{p+1}}D_p(\zeta,\theta)$

The Fibonacci numbers are discussed in several articles and books [6, 7]. The Fibonacci numbers are the sequence of numbers $\{f_n\}_{n=1}^{\infty}$ that are defined by the linear recurrence equations,

$$f_0=0 \text{ and } f_1=1,\ f_n=f_{n-1}+f_{n-2}.$$

The Fibonacci numbers have a number of basic features, which are listed below [7, 22]

$$\sum_{k=1}^{n}f_n=f_{n+2}-1;\ n\geq 1,$$
$$\sum_{k=1}^{n}f_k^2=f_nf_{n+1};\ n\geq 1.$$

Kara and Basarir [8] define the Fibonacci matrix $F=(f_{nk})_{n,k=1}^{\infty}$ by

$$f_{nk}=\begin{cases}\dfrac{f_k^2}{f_nf_{n+1}}, & (1\leq k\leq n);\\ 0, & (k>n),\end{cases}$$

that is $F=\begin{bmatrix}1&0&0&0&0&...\\ \frac{1}{2}&\frac{1}{2}&0&0&0&...\\ \frac{1}{6}&\frac{1}{6}&\frac{4}{6}&0&0&...\\ \frac{1}{15}&\frac{1}{15}&\frac{4}{15}&\frac{9}{15}&0&...\\ \frac{1}{40}&\frac{1}{40}&\frac{4}{40}&\frac{9}{40}&\frac{25}{40}&...\\ \vdots&\vdots&\vdots&\vdots&\vdots&\ddots\end{bmatrix}$.

So, the matrix F is a triangle. Kara and Basarir [8] defined Fibonacci sequence space

$$X(F)=\{x=(x_k):\frac{1}{f_nf_{n+1}}\sum_{k=1}^{n}f_k^2x_k\in X\},$$

for $X = \ell_\infty$, c, c_0 and ℓ_p that means the sets of all bounded, convergent, null sequences and $p-$absolutely convergent series.
Here, we extend the concepts of Fibonacci convergence in the setting of sequences of complex uncertain variables.

Definition 2.11. *A complex uncertain sequence (ζ_n) is said to be Fibonacci convergent almost surely to ζ if there exists an event Λ with $\mathcal{M}\{\Lambda\} = 1$ such that*

$$\lim_{n\to\infty} \left\| \frac{1}{f_n f_{n+1}} \sum_{k=1}^{n} f_k^2(\zeta_k(\gamma) - \zeta(\gamma)) \right\| = 0,$$

for every $\gamma \in \Lambda$.

Example 2.12. *Consider the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2, ...\}$ with*

$$M\{\Lambda\} = \begin{cases} \sup\limits_{\gamma_n \in \Lambda} \dfrac{n}{2n+1}, & if \sup\limits_{\gamma_n \in \Lambda} \dfrac{n}{2n+1} < \dfrac{1}{2}, \\ 1 - \sup\limits_{\gamma_n \in \Lambda^c} \dfrac{n}{2n+1}, & if \sup\limits_{\gamma_n \in \Lambda^c} \dfrac{n}{2n+1} < \dfrac{1}{2}, \\ \dfrac{1}{2}, & otherwise \end{cases}$$

and the complex uncertain variables be defined by

$$\zeta_n\{\gamma\} = \begin{cases} in, & if\ \gamma = \gamma_n, \\ 0, & otherwise \end{cases}$$

for $n = 1, 2, ...$ and $\zeta \equiv 0$. Then the sequence (ζ_n) Fibonacci convergent almost surely to ζ.

Definition 2.13. *A complex uncertain sequence (ζ_n) is said to be Fibonacci convergent in measure to ζ if*

$$\lim_{n\to\infty} \mathcal{M}\left\{ \left\| \frac{1}{f_n f_{n+1}} \sum_{k=1}^{n} f_k^2(\zeta_k - \zeta) \right\| \geq \varepsilon \right\} = 0,$$

for every $\varepsilon > 0$.

Example 2.14. *Consider the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2, ...\}$ with*

$$M\{\Lambda\} = \begin{cases} \sup\limits_{\gamma_n \in \Lambda} \dfrac{1}{n+1}, & if \sup\limits_{\gamma_n \in \Lambda} \dfrac{1}{n+1} < \dfrac{1}{2}, \\ 1 - \sup\limits_{\gamma_n \in \Lambda^c} \dfrac{1}{n+1}, & if \sup\limits_{\gamma_n \in \Lambda^c} \dfrac{1}{n+1} < \dfrac{1}{2}, \\ \dfrac{1}{2}, & otherwise \end{cases}$$

and the complex uncertain variables be defined by

$$\zeta_n(\gamma) = \begin{cases} (n+1)i, & if\ \gamma = \gamma_n, \\ 0, & otherwise \end{cases}$$

for $n = 1, 2, ...$ and $\zeta \equiv 0$. For some small $\epsilon > 0$ and $n \geq 2$, we have

$$\begin{aligned} \mathcal{M}\left\{ \left\| \frac{1}{f_n f_{n+1}} \sum_{k=1}^{n} f_k^2(\zeta_k(\gamma) - \zeta(\gamma)) \right\| \geq \varepsilon \right\} &= \mathcal{M}\left\{ \left\| \frac{1}{f_n f_{n+1}} \sum_{k=1}^{n} f_k^2(\zeta_k - \zeta) \right\| \geqslant \varepsilon \right\} \\ &= \mathcal{M}\{\gamma_n\} \\ &= 0. \end{aligned}$$

Thus, the sequence (ζ_n) is Fibonacci convergent in measure to ζ

Definition 2.15. *A complex uncertain sequence* (ζ_n) *is said to be Fibonacci convergent in mean to* ζ *if*

$$\lim_{n\to\infty} E\left[\left\|\frac{1}{f_n f_{n+1}}\sum_{k=1}^{n} f_k^2(\zeta_k-\zeta)\right\|\right]=0.$$

Definition 2.16. *A complex uncertain sequence* (ζ_n) *is said to be Fibonacci* $p-$*distance convergent to* ζ *if*

$$\lim_{n\to\infty}\left(E\left[\left\|\frac{1}{f_n f_{n+1}}\sum_{k=1}^{n} f_k^2(\zeta_k-\zeta)\right\|^p\right]\right)^{\dfrac{1}{p+1}}=0,\ p>0.$$

.

3. Some relationships among the several form of Fibonacci convergence

In this section, we establish relationships among the several forms of Fibonacci convergence in the setting of complex uncertain variables.

Theorem 3.1. *Let* (ζ_n) *be a sequence of complex uncertain variables. Then,* (ζ_n) *is convergent almost surely to* ζ *if and only if* (ζ_n) *is Fibonacci convergent almost surely to* ζ*.*

Proof. We have

$$\begin{aligned}
&\left\|\frac{1}{f_n f_{n+1}}\sum_{k=1}^{n} f_k^2\zeta_k(\gamma)-\zeta(\gamma)\right\|\\
&=\left\|\frac{1}{f_n f_{n+1}}(f_1^2\zeta_1(\gamma)+f_2^2\zeta_2(\gamma)+....+f_n^2\zeta_n(\gamma)-\sum_{k=1}^{n} f_k^2\zeta(\gamma))\right\|\\
&\leq\left\|\frac{f_1^2}{f_n f_{n+1}}(\zeta_1(\gamma)-\zeta(\gamma))\right\|+\left\|\frac{f_2^2}{f_n f_{n+1}}(\zeta_2(\gamma)-\zeta(\gamma))\right\|+...+\left\|\frac{f_n^2}{f_n f_{n+1}}(\zeta_n(\gamma)-\zeta(\gamma))\right\|\\
&=\frac{1}{f_n f_{n+1}}\sum_{k=1}^{n} f_k^2\|\zeta_k(\gamma)-\zeta(\gamma)\|.
\end{aligned}$$

Since, the sequence is convergent almost surely, so,

$$\left\|\frac{1}{f_n f_{n+1}}\sum_{k=1}^{n} f_k^2(\zeta_k(\gamma)-\zeta(\gamma))\right\|\to 0,$$

as $n\longrightarrow 0$.
Hence, (ζ_n) is Fibonacci convergent almost surely to ζ.
Conversely let (ζ_n) be Fibonacci convergent almost surely to ζ.

We denote $\mathcal{F}_n(\zeta) = \dfrac{1}{f_n f_{n+1}} \sum_{i=0}^{n} f_i^2 \zeta_i$ for simplicity.

$$\begin{aligned}
\zeta_n - \mathcal{F}_n(\zeta) &= \frac{1}{f_n f_{n+1}} \sum_{i=0}^{n} f_i^2 (\zeta_n - \zeta_i) \\
&= \frac{1}{f_n f_{n+1}} \sum_{i=0}^{n-1} f_i^2 (\zeta_n - \zeta_i) \\
&= \frac{1}{f_n f_{n+1}} \sum_{i=0}^{n-1} f_i^2 \sum_{k=i+1}^{n} (\zeta_k - \zeta_{k-1}) \\
&= \frac{1}{f_n f_{n+1}} \sum_{k=1}^{n} (\zeta_k - \zeta_{k-1}) \sum_{i=0}^{k-1} f_i^2 \\
&= \frac{1}{f_n f_{n+1}} \sum_{k=1}^{n} f_k f_{k-1} (\zeta_k - \zeta_{k-1}).
\end{aligned}$$

Therefore,

$$\zeta_n - \mathcal{F}_n(\zeta) = \alpha_n(\zeta), \text{ (say)}. \tag{1}$$

Again,

$$\begin{aligned}
\alpha_n(\zeta) &= \frac{1}{f_n f_{n+1}} \sum_{k=1}^{n} f_k f_{k-1} (\zeta_k - \zeta_{k-1}) \\
&= \frac{1}{f_n f_{n+1}} \left[\sum_{k=1}^{n} f_k f_{k-1} \zeta_k - \sum_{k=1}^{n} f_k f_{k-1} \zeta_{k-1} \right] \\
&= \frac{1}{f_n f_{n+1}} \left[\sum_{k=0}^{n} f_k f_{k-1} \zeta_k - \sum_{k=0}^{n-1} f_k f_{k+1} \zeta_k \right] \\
&= \frac{1}{f_n f_{n+1}} \left[f_n f_{n-1} \zeta_n + \sum_{k=1}^{n-1} f_k \left(f_{k-1} - f_{k+1} \right) \zeta_k \right] \\
&= \frac{1}{f_n f_{n+1}} \left[f_n f_{n-1} \zeta_n - \sum_{k=1}^{n-1} f_k^2 \zeta_k \right] \\
&= \frac{f_n f_{n-1}}{f_n f_{n+1}} \left[\zeta_n - \mathcal{F}_{n-1} \right] \\
&= \frac{f_{n-1}}{f_{n+1}} \left[\zeta_n - \mathcal{F}_{n-1} \right].
\end{aligned}$$

Therefore,

$$\begin{aligned}
\alpha_n(\zeta) &= \frac{f_{n-1}}{f_{n+1}} \left[\zeta_n - \mathcal{F}_{n-1} \right] \\
\Rightarrow \quad \frac{f_{n+1}}{f_{n-1}} \alpha_n(\zeta) &= \zeta_n - \mathcal{F}_{n-1} \\
\Rightarrow \alpha_n(\zeta) \left(\frac{f_{n+1} - f_{n-1}}{f_{n-1}} \right) &= \mathcal{F}_n - \mathcal{F}_{n-1} \\
\Rightarrow \quad \alpha_n(\zeta) &= \frac{f_{n-1}}{f_n} \left(\mathcal{F}_n - \mathcal{F}_{n-1} \right).
\end{aligned} \tag{2}$$

Since, the sequence (ζ_n) is Fibonacci convergent almost surely to ζ. Then,

$$\begin{aligned}\lim_{n\to\infty}\|\mathcal{F}_n-\mathcal{F}_{n-1}\| &= \lim_{n\to\infty}\|\mathcal{F}_n-\zeta+\zeta-\mathcal{F}_{n-1}\|\\ &\leq \lim_{n\to\infty}\left\|\mathcal{F}_n-\zeta\right\|+\lim_{n\to\infty}\left\|\zeta-\mathcal{F}_{n-1}\right\|\\ &\leq 0.\end{aligned}$$

So, $\lim_{n\to\infty}\|\mathcal{F}_n-\mathcal{F}_{n-1}\|=0$. It follows from (2) that
$\lim_{n\to\infty}\|\alpha_n(\zeta)\|=0$ and from (1) that
$\lim_{n\to\infty}\|\zeta_n-\mathcal{F}_n\|=0$.
Note that

$$\|\zeta_n-\zeta\|-\|\mathcal{F}_n-\zeta\|\leq\|\zeta_n-\mathcal{F}_n\|.$$

Therefore, $\lim_{n\to\infty}\|\zeta_n-\zeta\|=0$. Hence, the sequence (ζ_n) is convergent almost surely to ζ. □

Theorem 3.2. *Let (ζ_n) be a sequence of complex uncertain variables. Then, (ζ_n) is convergent in measure to ζ if and only if (ζ_n) is Fibonacci convergent in measure to ζ.*

Proof. Let $\varepsilon>0$ be given, then we have

$$\begin{aligned}&\mathcal{M}\left\{\left\|\frac{1}{f_nf_{n+1}}\sum_{k=1}^{n}f_k^2\zeta_k\right\|\geq\varepsilon\right\}\\ &=\mathcal{M}\left\{\left\|\frac{1}{f_nf_{n+1}}(f_1^2\zeta_1+f_2^2\zeta_2+...+f_n^2\zeta_n)\right\|\geq\varepsilon\right\}\\ &\leq\mathcal{M}\left\{\left\|\frac{1}{f_nf_{n+1}}f_1^2\zeta_1\right\|\geq\frac{\varepsilon}{n}\right\}+\mathcal{M}\left\{\left\|\frac{1}{f_nf_{n+1}}f_2^2\zeta_2\right\|\geq\frac{\varepsilon}{n}\right\}+...+\mathcal{M}\left\{\left\|\frac{1}{f_nf_{n+1}}f_n^2\zeta_n\right\|\geq\frac{\varepsilon}{n}\right\}\\ &\leq\mathcal{M}\left\{\|\zeta_1\|\geq\frac{f_nf_{n+1}\varepsilon}{f_1^2n}\right\}+\mathcal{M}\left\{\|\zeta_2\|\geq\frac{f_nf_{n+1}\varepsilon}{f_2^2n}\right\}+...+\mathcal{M}\left\{\|\zeta_n\|\geq\frac{f_nf_{n+1}\varepsilon}{f_n^2n}\right\}.\end{aligned}$$

Setting $\delta=min\left\{\frac{f_nf_{n+1}\varepsilon}{f_1^2n},\frac{f_nf_{n+1}\varepsilon}{f_2^2n},...,\frac{f_nf_{n+1}\varepsilon}{f_n^2n}\right\}$.
Thus, we have

$$\mathcal{M}\left\{\left\|\frac{1}{f_nf_{n+1}}\sum_{k=1}^{n}f_k^2(\zeta_k-\zeta)\right\|\geq\varepsilon\right\}\leq\sum_{k=1}^{n}\mathcal{M}\{\|\zeta_k-\zeta\|\geq\delta\}.$$

Since, the sequence (ζ_n) is convergent in mesaure to ζ, so from the above inequality we obtain
$\lim_{n\to\infty}\mathcal{M}\left\{\left\|\frac{1}{f_nf_{n+1}}\sum_{k=1}^{n}f_k^2(\zeta_k-\zeta)\right\|\geq\varepsilon\right\}=0$. As a result, the complex uncertain sequence is Fibonacci convergent in measure to ζ.
Conversely let the sequence (ζ_n) be Fibonacci convergent in measure to ζ. Then, for any small $\varepsilon>0$, we can deduce from the concept of convergence in measure of complex uncertain sequence that

$$\lim_{n\to\infty}\mathcal{M}\{\|\mathcal{F}_n-\zeta\|\geq\varepsilon\}=0. \qquad (3)$$

Note that

$$\|\mathcal{F}_n-\mathcal{F}_{n-1}\|\leq\|\mathcal{F}_n-\zeta\|+\|\zeta-\mathcal{F}_{n-1}\|.$$

We obtain using the subadditivity axiom of uncertain measure and taking as $n \to \infty$ that

$$
\begin{aligned}
0 &\leq \lim_{n\to\infty} \mathcal{M}\{\|\mathcal{F}_n - \mathcal{F}_{n-1}\| \geq \varepsilon\} \\
&\leq \lim_{n\to\infty} \mathcal{M}\{\|\mathcal{F}_n - \zeta\| \geq \varepsilon\} + \lim_{n\to\infty} \mathcal{M}\{\|\mathcal{F}_{n-1} - \zeta\| \geq \varepsilon\} \\
&= 0.
\end{aligned}
$$

Therefore,

$$\lim_{n\to\infty} \mathcal{M}\{\|\mathcal{F}_n - \mathcal{F}_{n-1}\| \geq \varepsilon\} = 0. \tag{4}$$

Thus, from (2) we have that

$$\{\|\alpha_n(\zeta)\| \geq \varepsilon\} = \left\{\left\|\frac{f_{n-1}}{f_n}(\mathcal{F}_n - \mathcal{F}_{n-1})\right\| \geq \varepsilon\right\}.$$

We obtain using the subadditivity axiom of uncertain measure and taking as $n \to \infty$ that

$$\lim_{n\to\infty} \mathcal{M}\{\|\alpha_n(\zeta)\| \geq \varepsilon\} = \lim_{n\to\infty} \mathcal{M}\left\{\left\|\frac{f_{n-1}}{f_n}(\mathcal{F}_n - \mathcal{F}_{n-1})\right\| \geq \varepsilon\right\}. \tag{5}$$

Combining equation (4) and (5), we obtain

$$\lim_{n\to\infty} \mathcal{M}\{\|\alpha_n(\zeta)\| \geq \varepsilon\} = 0. \tag{6}$$

It follows from (1) that

$$\zeta_n - \mathcal{F}_n(\zeta) = \alpha_n(\zeta).$$

Hence, we have

$$\lim_{n\to\infty} \mathcal{M}\{\|\zeta_n - \zeta\| \geq \varepsilon\} - \lim_{n\to\infty} \mathcal{M}\{\|\mathcal{F}_n(\zeta) - \zeta\| \geq \varepsilon\} \leq \lim_{n\to\infty} \mathcal{M}\{\|\alpha_n(\zeta)\| \geq \varepsilon\}$$

It follows from (3) and (6) that
$\lim_{n\to\infty} \mathcal{M}\{\|\zeta_n - \zeta\| \geq \varepsilon\} = 0$. Thus, the sequence (ζ_n) is convergent in measure to ζ. □

Theorem 3.3. *Let (ζ_n) be a sequence of complex uncertain variables. Then, (ζ_n) is convergent in $p-$distance to ζ if and only if (ζ_n) is Fibonacci convergent in $p-$distance to ζ.*

Proof. We have

$$
\begin{aligned}
&E\left[\left\|\frac{1}{f_n f_{n+1}}\sum_{k=1}^{n} f_k^2(\zeta_k - \zeta)\right\|^p\right] \\
&= \int_o^\infty \mathcal{M}\left\{\left\|\frac{1}{f_n f_{n+1}}\sum_{k=1}^{n} f_k^2(\zeta_k - \zeta)\right\|^p \geq x\right\} dx \\
&= \int_o^\infty \mathcal{M}\left\{\left\|\frac{1}{f_n f_{n+1}}\sum_{k=1}^{n} f_k^2(\zeta_k - \zeta)\right\| \geq x^{\frac{1}{p}}\right\} dx \\
&\leq \int_o^\infty \mathcal{M}\left\{\frac{1}{f_n f_{n+1}}\sum_{k=1}^{n} f_k^2 \|\zeta_k - \zeta\| \geq x^{\frac{1}{p}}\right\} dx \\
&\leq \int_o^\infty \mathcal{M}\left\{\|\zeta_k - \zeta\| \geq x^{\frac{1}{p}}\right\} dx \\
&= E[\|\zeta_k - \zeta\|^p].
\end{aligned}
$$

Hence, the first part of the theorem is proved.
Conversely let the sequence (ζ_n) be Fibonacci convergent in $p-$distance to ζ. Then, we can deduce from the concept of Fibonacci convergence in $p-$distance of complex uncertain sequence that

$$\lim_{n\to\infty} \mathcal{D}_p(\mathcal{F}_n(\zeta), \zeta) = 0. \tag{7}$$

Combining equation (1) and (2), we obtain

$$\zeta_n - \mathcal{F}_n(\zeta) = \frac{f_{n-1}}{f_n}(\mathcal{F}_n - \mathcal{F}_{n-1}) \tag{8}$$

Then, it follows from lemma 2.9 and lemma 2.10 that

$$\mathcal{D}_p(\zeta_n, \mathcal{F}_n(\zeta)) = \left(\frac{f_{n-1}}{f_n}\right)^{\frac{p}{1+p}} \mathcal{D}_p\left(\mathcal{F}_n(\zeta), \mathcal{F}_{n-1}(\zeta)\right) \tag{9}$$

$$\mathcal{D}_p(\mathcal{F}_n(\zeta), \mathcal{F}_{n-1}(\zeta)) \leq \mathcal{D}_p(\mathcal{F}_n(\zeta), \zeta) + \mathcal{D}_p(\mathcal{F}_{n-1}(\zeta), \zeta) \tag{10}$$

Since the sequence (ζ_n) is Fibonacci convergent in $p-$distance to ζ. Then, Taking as $n \to \infty$ we obtain

$$\lim_{n\to\infty} \mathcal{D}_p(\mathcal{F}_n(\zeta), \mathcal{F}_{n-1}(\zeta)) = 0$$

It follows from (9) that

$$\lim_{n\to\infty} \mathcal{D}_p(\zeta_n, \mathcal{F}_n(\zeta)) = 0.$$

We obtain using the triangular inequality that

$$\mathcal{D}_p(\zeta_n, \zeta) \leq \mathcal{D}_p(\zeta_n, \mathcal{F}_n(\zeta)) + \mathcal{D}_p(\mathcal{F}_n(\zeta), \zeta)$$

Therefore, $\lim_{n\to\infty} \mathcal{D}_p(\zeta_n, \zeta) = 0$. Hence, the sequence (ζ_n) is $p-$distance convergent to ζ. □

Theorem 3.4. *Let (ζ_n) be a sequence of complex uncertain variables. Then, (ζ_n) is convergent in mean to ζ if and only if (ζ_n) is Fibonacci convergent in mean to ζ.*

Proof. We have

$$\begin{aligned}
&E\left[\left\|\frac{1}{f_n f_{n+1}}\sum_{k=1}^{n} f_k^2(\zeta_k - \zeta)\right\|\right] \\
&= \int_o^{\infty} \mathcal{M}\left\{\left\|\frac{1}{f_n f_{n+1}}\sum_{k=1}^{n} f_k^2(\zeta_k - \zeta)\right\| \geq x\right\} dx \\
&\leq \int_o^{\infty} \mathcal{M}\left\{\frac{1}{f_n f_{n+1}}\sum_{k=1}^{n} f_k^2 \,||\zeta_k - \zeta|| \geq x\right\} dx \\
&\leq \int_o^{\infty} \mathcal{M}\left\{||\zeta_k - \zeta|| \geq x\right\} dx \\
&= E[||\zeta_k - \zeta||].
\end{aligned}$$

Hence, the first part of the theorem is proved.
Conversely let the sequence (ζ_n) be Fibonacci convergent in mean to ζ. Then, we can deduce from the concept of Fibonacci convergence in mean of complex uncertain sequence that

$$\lim_{n\to\infty} E\left\{||\mathcal{F}_n - \zeta|| \geq \varepsilon\right\} = 0. \tag{11}$$

Note that

$$\|\mathcal{F}_n - \mathcal{F}_{n-1}\| \leq \|\mathcal{F}_n - \zeta\| + \|\zeta - \mathcal{F}_{n-1}\|.$$

Taking as $n \to \infty$, we obtain

$$\begin{aligned} 0 &\leq \lim_{n\to\infty} E\{\|\mathcal{F}_n - \mathcal{F}_{n-1}\|\} \\ &\leq \lim_{n\to\infty} E\{\|\mathcal{F}_n - \zeta\|\} + \lim_{n\to\infty} E\{\|\mathcal{F}_{n-1} - \zeta\|\} \\ &= 0. \end{aligned}$$

Therefore,

$$\lim_{n\to\infty} E\{\|\mathcal{F}_n - \mathcal{F}_{n-1}\|\} = 0. \tag{12}$$

Thus, from (2) we have that

$$\{\|\alpha_n(\zeta)\|\} = \left\{\left\|\frac{f_{n-1}}{f_n}(\mathcal{F}_n - \mathcal{F}_{n-1})\right\|\right\}.$$

Taking as $n \to \infty$ we obtain

$$\lim_{n\to\infty} E\{\|\alpha_n(\zeta)\|\} = \lim_{n\to\infty} E\left\{\left\|\frac{f_{n-1}}{f_n}(\mathcal{F}_n - \mathcal{F}_{n-1})\right\|\right\}. \tag{13}$$

Combining equation (12) and (13), we obtain

$$\lim_{n\to\infty} E\{\|\alpha_n(\zeta)\|\} = 0. \tag{14}$$

It follows from (1) that

$$\zeta_n - \mathcal{F}_n(\zeta) = \alpha_n(\zeta).$$

Hence, we have

$$\lim_{n\to\infty} E\{\|\zeta_n - \zeta\|\} - \lim_{n\to\infty} E\{\|\mathcal{F}_n(\zeta) - \zeta\|\} \leq \lim_{n\to\infty} E\{\|\alpha_n(\zeta)\|\}$$

It follows from (11) and (14) that
$\lim_{n\to\infty} E\{\|\zeta_n - \zeta\|\} = 0$. Thus, the sequence (ζ_n) is convergent in mean to ζ. □

Theorem 3.5. *If the complex uncertains sequence (ζ_n) is Fibonacci convergent in mean to ζ, then (ζ_n) is Fibonacci convergent in measure to ζ.*

Proof. We have from the Markov inequality that for any given $\varepsilon > 0$,

$$\mathcal{M}\left\{\left\|\frac{1}{f_n f_{n+1}}\sum_{k=1}^{n} f_k^2(\zeta_k - \zeta)\right\| \geq \varepsilon\right\} \leq \frac{E\left[\left\|\frac{1}{f_n f_{n+1}}\sum_{k=1}^{n} f_k^2(\zeta_k - \zeta)\right\|\right]}{\varepsilon} \longrightarrow 0,$$

as $n \longrightarrow \infty$. Thus the complex uncertain sequence (ζ_n) is Fibonacci convergent in measure to ζ. □

Theorem 3.6. *Assume the complex uncertain sequence (ζ_n) with real part (ξ_n) and imaginary part (η_n), for n=1,2,... Then the complex uncertain sequence (ζ_n) is Fibonacci convergent in measure to ζ if and only if the uncertain sequences (ξ_n) and (η_n) are Fibonacci convergent in measure to ξ and η, respectively.*

Proof. From the definition of Fibonacci convergence in measure, for any $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathcal{M}\left\{\left\|\frac{1}{f_n f_{n+1}}\sum_{k=1}^{n} f_k^2(\xi_k - \xi)\right\| \geq \frac{\varepsilon}{\sqrt{2}}\right\} = 0$$

and

$$\lim_{n\to\infty} \mathcal{M}\left\{\left\|\frac{1}{f_n f_{n+1}}\sum_{k=1}^{n} f_k^2(\eta_k - \eta)\right\| \geq \frac{\varepsilon}{\sqrt{2}}\right\} = 0.$$

Note that

$$\begin{aligned}
&\left\|\frac{1}{f_n f_{n+1}}\sum_{k=1}^{n} f_k^2(\zeta_k - \zeta)\right\| \\
&\quad = \sqrt{\left\|\frac{1}{f_n f_{n+1}}\sum_{k=1}^{n} f_k^2(\xi_k - \xi)\right\|^2 + \left\|\frac{1}{f_n f_{n+1}}\sum_{k=1}^{n} f_k^2(\eta_k - \eta)\right\|^2}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\left\{\left\|\frac{1}{f_n f_{n+1}}\sum_{k=1}^{n} f_k^2(\zeta_k - \zeta)\right\| \geq \varepsilon\right\} \\
&\subset \left\{\left\|\frac{1}{f_n f_{n+1}}\sum_{k=1}^{n} f_k^2(\xi_k - \xi)\right\| \geq \frac{\varepsilon}{\sqrt{2}}\right\} \\
&\qquad \bigcup \left\{\left\|\frac{1}{f_n f_{n+1}}\sum_{k=1}^{n} f_k^2(\eta_k - \eta)\right\| \geq \frac{\varepsilon}{\sqrt{2}}\right\}.
\end{aligned}$$

Now, using subadditivity axiom of uncertain measure, we have

$$\begin{aligned}
0 &\leq \lim_{n\to\infty} \mathcal{M}\left\{\left\|\frac{1}{f_n f_{n+1}}\sum_{k=1}^{n} f_k^2(\zeta_k - \zeta)\right\| \geq \varepsilon\right\} \\
&\leq \lim_{n\to\infty} \mathcal{M}\left\{\left\|\frac{1}{f_n f_{n+1}}\sum_{k=1}^{n} f_k^2(\xi_k - \xi)\right\| \geq \frac{\varepsilon}{\sqrt{2}}\right\} \\
&\quad + \lim_{n\to\infty} \mathcal{M}\left\{\left\|\frac{1}{f_n f_{n+1}}\sum_{k=1}^{n} f_k^2(\eta_k - \eta)\right\| \geq \frac{\varepsilon}{\sqrt{2}}\right\} \\
&= 0.
\end{aligned}$$

So, $\lim_{n\to\infty} \mathcal{M}\left\{\left\|\frac{1}{f_n f_{n+1}}\sum_{k=1}^{n} f_k^2(\zeta_k - \zeta)\right\| \geq \varepsilon\right\} = 0$. Hence, the complex uncertain sequence (ζ_n) is Fibonacci convergent in measure to ζ.
Conversely, let the complex uncertain sequence (ζ_n) be Fibonacci convergence in measure to ζ. Then,

$$\lim_{n\to\infty} \mathcal{M}\left\{\left\|\frac{1}{f_n f_{n+1}}\sum_{k=1}^{n} f_k^2(\zeta_k - \zeta)\right\| \geq \varepsilon\right\} = 0.$$

Note that,

$$\left\{\left\|\frac{1}{f_n f_{n+1}}\sum_{k=1}^{n} f_k^2(\xi_k-\xi)\right\|\geq\varepsilon\right\}$$
$$\subset\left\{\left\|\frac{1}{f_n f_{n+1}}\sum_{k=1}^{n} f_k^2(\zeta_k-\zeta)\right\|\geq\varepsilon\right\}$$

and

$$\left\{\left\|\frac{1}{f_n f_{n+1}}\sum_{k=1}^{n} f_k^2(\eta_k-\eta)\right\|\geq\varepsilon\right\}$$
$$\subset\left\{\left\|\frac{1}{f_n f_{n+1}}\sum_{k=1}^{n} f_k^2(\zeta_k-\zeta)\right\|\geq\varepsilon\right\}.$$

Using subaddivity axiom of uncertain measure and taking as $n\longrightarrow\infty$, we obtain

$$\lim_{n\to\infty}\mathcal{M}\left\{\left\|\frac{1}{f_n f_{n+1}}\sum_{k=1}^{n} f_k^2(\xi_k-\xi)\right\|\geq\varepsilon\right\}=0$$

and

$$\lim_{n\to\infty}\mathcal{M}\left\{\left\|\frac{1}{f_n f_{n+1}}\sum_{k=1}^{n} f_k^2(\eta_k-\eta)\right\|\geq\varepsilon\right\}=0.$$

Hence, the uncertain sequences (ξ_n) and (η_n) are Fibonacci convergent in measure to ξ and η respectively. □

Corollary 3.7. *Assume the complex uncertain sequence (ζ_n) with real part (ξ_n) and imaginary part (η_n), for n=1,2,... If the complex uncertain sequence (ζ_n) is Fibonacci convergent in mean to ζ then the uncertain sequences (ξ_n) and (η_n) are Fibonacci convergent in measure to ξ and η, respectively.*

Proof. Combining Theorem 3.5 and Theorem 3.6, the above claim is obvious.

Theorem 3.8. *If the complex uncertain sequence (ζ_n) Fibonacci converges in $p-$ distance to ζ, then the sequence also Fibonacci converges in measure to ζ.*

Proof. It follows from the Markov inequality that

$$\mathcal{M}\left\{\left\|\frac{1}{f_n f_{n+1}}\sum_{k=1}^{n} f_k^2(\zeta_k-\zeta)\right\|\geq\varepsilon\right\}\leq\frac{E\left[\left\|\frac{1}{f_n f_{n+1}}\sum_{k=1}^{n} f_k^2(\zeta_k-\zeta)\right\|^p\right]}{\varepsilon^p}\longrightarrow 0,$$

as $n\longrightarrow\infty$. Thus, the complex uncertain sequence (ζ_n) Fibonacci converges in measure to ζ. □

4. Conclusion

This article initiates the concept of convergence of complex uncertain sequences based on a regular matrix of Fibonacci numbers. This study can be generalized by introducing other matrices in the same environment.

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