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More on (generalized) B-Fredholm theory in general Banach algebras

Ronalda Benjamin[a,*], **Jamiu Aliyu**[b]

[a]*Department of Mathematical Sciences, Mathematics Division, Stellenbosch University, Private Bag X1, Matieland 7602, South Africa*
[b]*University of Manitoba, Canada*

Abstract. We further develop (generalized) B-Fredholm theory relative to a fixed Banach algebra homomorphism.

1. Introduction

In [6], the interplay between Harte's Fredholm theory and Drazin invertibility (resp. Koliha-Drazin invertibility), which gives rise to B-Fredholm theory (resp. generalized B-Fredholm theory or, in short, GB-Fredholm theory) in general Banach algebras is studied relative to a fixed Banach algebra homomorphism.

We recall that the well-known Atkinson's Theorem gives a necessary and sufficient condition for a bounded linear operator T on a Banach space X to be Fredholm, which is that the coset of T is invertible in the Calkin algebra. This result inspired Harte's work on Fredholm theory relative to a Banach algebra homomorphism in [9], which has witnessed considerable development over the years. In [3, Theorem 3.4], Berkani and Sarih proved an Atkinson-type theorem for B-Fredholm operators, which led to the introduction of B-Fredholm theory (and an extension thereof, namely generalized B-Fredholm theory) relative to fixed Banach algebra homomorphism in [6]. The purpose of this paper, which is a continuation of [6], is to provide further (spectral) properties of the B-Fredholm, B-Weyl and B-Browder elements, which are studied in B-Fredholm theory, as well as the generalized B-Fredholm, generalized B-Weyl and generalized B-Browder elements, explored in GB-Fredholm theory (and the spectra that are derived from these classes of elements). We arrange our article in the following way. In Section 2, we list some necessary concepts, notations and results that are essential in the rest of the paper. In particular, in Subsection 2.1, we gather some concepts and results from the theory of Drazin and Koliha-Drazin invertible elements and review, in Subsection 2.2, some concepts and results from Fredholm theory relative to a fixed Banach algebra homomorphism. The basic information on B-Fredholm theory and GB-Fredholm theory, as developed in [6], is gathered in Section 3, where sufficient examples are also supplied to demonstrate certain statements. In Section 4, we state and prove further inclusion and algebraic properties of (generalized) B-Fredholm, (generalized) B-Weyl and (generalized) B-Browder elements relative to a fixed Banach algebra homomorphism. In particular,

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* Corresponding author: Ronalda Benjamin
Email addresses: ronalda@sun.ac.za (Ronalda Benjamin), aliyuj@myumanitoba.za (Jamiu Aliyu)
ORCID iDs: https://orcid.org/0000-0001-5016-9670 (Ronalda Benjamin)

we establish that the set of (generalized) B-Weyl elements (resp. (generalized) B-Browder elements) forms an upper semi-regularity relative to Banach algebra homomorphisms with commutative domains. Section 5 further investigates the spectra that are studied in B-Fredholm theory and GB-Fredholm theory. In particular, we state one-way spectral mapping theorems for the (generalized) B-Weyl and (generalized) B-Browder spectra and demonstrate that, if a Banach algebra homomorphism does not have the strong Riesz property, then the connected hulls of the (generalized) B-Fredholm, (generalized) B-Weyl and (generalized) B-Browder spectra may not coincide. Finally, in Section 6, we give a necessary condition for an element to be (generalized) B-Weyl (resp. (generalized) B-Browder) and specify some open questions.

2. Preliminaries

Throughout this paper, unless otherwise stated, A will denote a complex unital Banach algebra with unit 1_A. For $a \in A$, its ordinary spectrum will be denoted by

$$\sigma(a) := \{\lambda \in \mathbb{C} : \lambda 1_A - a \notin A^{-1}\},$$

where A^{-1} indicates the set of invertible elements of A. If any confusion occurs, we will write $\sigma_A(a)$ to emphasize the Banach algebra under discussion. The isolated (resp. accumulation) points of $\sigma(a)$ will be denoted by $\operatorname{iso}\sigma(a)$ (resp. $\operatorname{acc}\sigma(a)$) and, for $\lambda \notin \operatorname{acc}\sigma(a)$, the spectral idempotent associated with λ by $p(a, \lambda)$. If A is just a unital algebra, then the (Jacobson) radical and the sets of idempotent, nilpotent and quasinilpotent elements will be indicated by $\text{Rad}(A)$, $\text{Idem}(A)$, $\text{Nil}(A)$ and $\text{QN}(A)$, respectively. We further recall that A is said to be *semisimple* if its radical consists only of the zero element of A. Also recall that, for a Banach algebra element a, $a \in \text{QN}(A)$ if and only if $\sigma(a) = \{0\}$.

In [12, Definition 1.2], Kordula and Müller defined a *regularity* of a Banach algebra A as a subset R with the following two properties: (i) if $a \in A$ and $n \in \mathbb{N}$, then $a \in R$ if and only if $a^n \in R$ and, (ii), if $a, b \in A$ are *relatively prime* (i.e. there exist elements $c, d \in A$ such that $\{a, b, c, d\}$ is a commuting set and $ac + bd = 1_A$), then $a, b \in R$ if and only if $ab \in R$. In [17], Müller split the axioms of a regularity into two parts, and introduced the notions of a lower semi-regularity and an upper semi-regularity:

Definition 2.1. *([17], Definition 1) Let A be a Banach algebra. A non-empty subset R of A is said to be a lower semi-regularity if the following properties are satisfied:*
(i) *If $a \in A$ and $a^n \in R$ for some $n \in \mathbb{N}$, then $a \in R$.*
(ii) *If $a, b \in A$ are relatively prime and $ab \in R$, then $a, b \in R$.*

Definition 2.2. *([17], Definition 10) Let A be a Banach algebra. A non-empty subset R of A is said to be an upper semi-regularity if the following properties are satisfied:*
(i) *If $a \in R$, then $a^n \in R$ for all $n \in \mathbb{N}$.*
(ii) *If $a, b \in A$ are relatively prime and $a, b \in R$, then $ab \in R$.*
(iii) *R contains a neighbourhood of the unit 1_A.*

Consequently, a non-empty subset of a Banach algebra is a regularity if and only if it is both an upper and a lower semi-regularity. Finally, by $\mathcal{F}(X)$ and $\mathcal{K}(X)$ we denote, respectively, the ideal of finite-rank operators and the closed ideal of compact operators of the Banach algebra $\mathcal{L}(X)$ of all bounded linear operators on a Banach space X.

2.1. Drazin and Koliha-Drazin invertible elements

We continue with a dicussion on Drazin and Koliha-Drazin invertible elements, which are explored in the intensively studied subject of generalized inverses. Following [7], an element a of a unital algebra A is said to be *Drazin invertible* if there exists an element $b \in A$ such that $ab = ba$, $b = bab$ and $a - aba \in \text{Nil}(A)$. In [11], this concept was further generalized by Koliha who introduced a *Koliha-Drazin invertible* element $a \in A$ as one for which there exists an element $b \in A$ such that $ab = ba$, $b = bab$ and $a - aba \in \text{QN}(A)$. It is known that, if such b exists, then it unique, and we call it the *Drazin inverse* (resp. *Koliha-Drazin inverse*) of

a and indicate it by a^D (resp. a^{KD}). The sets of all Drazin and Koliha-Drazin invertible elements of A will be denoted by A^D and A^{KD}, respectively, which according to [3, Theorem 2.3] and [13, Theorem 1.2] form regularities. Evidently, $A^{-1} \subseteq A^D \subseteq A^{KD}$, where these inclusions are generally strict. In fact, we note that the closures of these sets coincide in general:

Proposition 2.3. *If A is a Banach algebra, then $A^{KD} \subseteq \overline{A^{-1}}$. In particular,*

$$\overline{A^{-1}} = \overline{A^D} = \overline{A^{KD}}.$$

Proof. Suppose that $a \in A^{KD}$. In view of [11, Theorem 3.1], $0 \notin \operatorname{acc} \sigma(a)$. If $0 \notin \sigma(a)$, then $a \in A^{-1} \subseteq \overline{A^{-1}}$, and in the case where $0 \in \operatorname{iso} \sigma(a)$, we let $\epsilon_a > 0$ be such that $B(0, \epsilon_a) \cap \sigma(a) \setminus \{0\} = \emptyset$. Since for every non-zero $\lambda \in B(0, \epsilon_a)$, $\lambda \notin \sigma(a)$, i.e. $a - \lambda 1_A \in A^{-1}$, we have for sufficiently large n that $a - \frac{1}{n} 1_A \in A^{-1}$. From the fact that the sequence $(a - \frac{1}{n} 1_A)$ of invertible elements converges to a, we obtain that $a \in \overline{A^{-1}}$, as desired. □

Using Example 4.3 from [10], we illustrate next that the inclusion $A^{KD} \subseteq \overline{A^{-1}}$ in Proposition 2.3 is generally strict.

Example 2.4. *Consider the Banach algebra $\mathcal{L}(\ell^2 \oplus \ell^2)$ of all bounded linear operators on the direct sum of the Banach space*

$$\ell^2 := \{(x_1, x_2, \dots) : \sum_{n=1}^{\infty} |x_i|^2 < \infty\}$$

of square summable sequences. Then $T : \ell^2 \oplus \ell^2 \to \ell^2 \oplus \ell^2$, defined by

$$T((x_1, x_2, \dots), (y_1, y_2, \dots)) = ((0, x_1, x_2, \dots), (y_2, y_3, \dots))$$

for all $(x_1, x_2, \dots), (y_1, y_2, \dots) \in \ell^2$ belongs to $\overline{(\mathcal{L}(\ell^2 \oplus \ell^2))^{-1}} \backslash \mathcal{L}(\ell^2 \oplus \ell^2)^{KD}$.

Proof. If $U, V : \ell^2 \to \ell^2$ define, respectively, the forward shift and backward shift operators on ℓ^2, i.e.

$$U(x_1, x_2, \dots) = (0, x_1, x_2, \dots)$$

and

$$V(x_1, x_2, \dots) = (x_2, x_3, \dots)$$

for all $(x_1, x_2, \dots) \in \ell^2$, then T can be represented as a 2×2 operator matrix

$$T := \begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix}.$$

Since $\sigma(T) = \sigma(U) \cup \sigma(V) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$, $0 \in \operatorname{acc} \sigma(T)$, so that from [11, Theorem 3.1] it follows that $T \notin \mathcal{L}(\ell^2 \oplus \ell^2)^{KD}$. From [10, Example 4.3] we have that T is a sum of an invertible operator T_1 and a finite-rank operator T_2, i.e. $T = T_1 + T_2$, and hence $\sigma(TT_1^{-1}) = \{1\} + \sigma(T_2 T_1^{-1})$. Since $T_2 T_1^{-1}$ is compact, 1 will be the only possible accumulation point of $\sigma(TT_1^{-1})$. We can therefore find a sequence (λ_n) in $\mathbb{C} \setminus \sigma(TT_1^{-1})$ that converges to 0 and, since the sequence $(\lambda_n - TT_1^{-1})(-T_1)$ in $\mathcal{L}(\ell^2 \oplus \ell^2)^{-1}$ converges to T, we conclude that $T \in \overline{\mathcal{L}(\ell^2 \oplus \ell^2)^{-1}}$. This proves that $T \in \overline{\mathcal{L}(\ell^2 \oplus \ell^2)^{-1}} \setminus \mathcal{L}(\ell^2 \oplus \ell^2)^{KD}$. □

Finally, we mention that $A^D = A^{KD}$ whenever A is a commutative semisimple Banach algebra [16, Lemma 6(a) and Corollary 4] or A is finite-dimensional. In the latter case, these sets coincide with the whole A.

Now using the regularities A^D and A^{KD}, the *Drazin spectrum*

$$\sigma_D(a) := \{\lambda \in \mathbb{C} : \lambda 1_A - a \notin A^D\}$$

of $a \in A$ and *Koliha-Drazin spectrum*

$$\sigma_{KD}(a) := \{\lambda \in \mathbb{C} : \lambda 1_A - a \notin A^{KD}\}$$

of $a \in A$, in view of [12, Theorem 1.4], both satisfy the spectral mapping theorem for functions which are analytic on a neighbourhood of the spectrum and non-constant on each component of this neighbourhood. In fact (in view of [13, Proposition 1.5]),

$$\begin{array}{ccccc} & & \mathrm{acc}\,\sigma(a) \cup (\mathrm{iso}\,\sigma(a) \backslash \Pi(a)) & & \\ & & \shortparallel & & \\ \sigma_{KD}(a) & \subseteq & \sigma_D(a) & \subseteq & \sigma(a), \\ \shortparallel & & & & \\ \mathrm{acc}\,\sigma(a) & & & & \end{array}$$

where $\Pi(a)$ denotes the set of poles of the resolvent of a. These identities confirm that the Drazin and Koliha-Drazin spectra are compact subsets of the complex plane (though A^D and A^{KD} are not open sets in general - see [11, Example 8.4]) that may be empty.

For a more detailed account on Drazin and Koliha-Drazin invertible elements in general Banach algebras, the reader may consult [11] and [16].

2.2. *Fredholm theory in general Banach algebras*

By a *unital homomorphism* (resp. *Banach algebra homomorphism*) we mean a linear operator T between unital algebras (resp. Banach algebras) A and B that satisfies $T(ab) = TaTb$ and $T1_A = 1_B$. It is easy to show that such mappings satisfy the inclusions $T(A^{-1}) \subseteq B^{-1}, T(A^D) \subseteq B^D$ and $T(A^{KD}) \subseteq B^{KD}$, so that

$$\sigma(Ta) \subseteq \sigma(a),\ \sigma_D(Ta) \subseteq \sigma_D(a)\ \text{ and }\ \sigma_{KD}(Ta) \subseteq \sigma_{KD}(a)$$

for all $a \in A$. A Banach algebra homomorphism T is said to have the *Riesz property* if $\mathrm{acc}\,\sigma(a) \subseteq \{0\}$ for all $a \in \mathcal{N}(T)$, where $\mathcal{N}(T) := \{a \in A : Ta = 0_B\}$ denotes the *null space* of T. In addition, T is said to have the *strong Riesz property* if $\mathrm{acc}\,\sigma(a) \subseteq \eta\sigma(Ta)$ for all $a \in A$, where ηK denotes the *connected hull* of a compact set $K \subseteq \mathbb{C}$ and is given by the complement of the unique unbounded component of $\mathbb{C} \setminus K$. Evidently, every Banach algebra homomorphism with the strong Riesz property has the Riesz property.

Relative to a fixed Banach algebra homomorphism, Harte studied in [9] natural generalizations of the notions of Fredholm, Weyl and Browder operators acting on a Banach space:

Definition 2.5. *([9], p.431) Let $T : A \to B$ be a Banach algebra homomorphism. Relative to T, an element $a \in A$ is called:*

- *Fredholm if $Ta \in B^{-1}$.*
- *Weyl if there exist elements $b \in A^{-1}$ and $c \in \mathcal{N}(T)$ such that $a = b + c$.*
- *Browder if there exist commuting elements $b \in A^{-1}$ and $c \in \mathcal{N}(T)$ such that $a = b + c$.*

The sets of all Fredholm, Weyl and Browder elements (relative to a Banach algebra homomorphism $T : A \to B$) will be denoted by $\mathcal{F}_T$, $\mathcal{W}_T$, and $\mathcal{B}_T$, respectively. Clearly, if A is commutative, then $\mathcal{B}_T = \mathcal{W}_T$, and, in general,

$$A^{-1} \subseteq \mathcal{B}_T \subseteq \mathcal{W}_T \subseteq \mathcal{F}_T.$$

As an immediate consequence of Proposition 2.3 and [14, Corollary 3.8] we have:

Proposition 2.6. *Let $T : A \to B$ be a Banach algebra homomorphism with the Riesz property. Then*

$$\overline{A^{KD}} = \overline{A^D} = \overline{A^{-1}} = \overline{\mathcal{B}_T} = \overline{\mathcal{W}_T}.$$

Harte also showed that the inclusion $A^{KD} \cap \mathcal{F}_T \subseteq \mathcal{B}_T$ generally holds, which is an immediate corollary of the following lemma:

Lemma 2.7. *Let $T : A \to B$ be a Banach algebra homomorphism and $a \in A^{KD} \cap \mathcal{F}_T$. Then $p(a, 0) \in \mathcal{N}(T)$.*
Hence, if $a \in A^{KD} \setminus A^{-1}$ and $p(a, 0) \notin \mathcal{N}(T)$, then $a \notin \mathcal{F}_T$, i.e. $Ta \notin B^{-1}$.

Proof. Let $a \in A^{KD} \cap \mathcal{F}_T$, i.e. $0 \notin \operatorname{acc} \sigma(a)$ and $Ta \in B^{-1}$. When considering the nontrivial case, i.e. $0 \in \operatorname{iso} \sigma(a)$, we have from [11, Theorem 4.2] that $p(a, 0) = 1_A - a^{KD}a$, and hence $Tp(a, 0) = 1_B - T(a^{KD})Ta = 1_B - (Ta)^{KD}Ta$. Since $Ta \in B^{-1}$ by assumption, it follows that $(Ta)^{KD} = (Ta)^{-1}$, from which we conclude that $Tp(a, 0) = 1_B - 1_B = 0_B$, i.e. $p(a, 0) \in \mathcal{N}(T)$. □

Consequently, for $a \in A^{KD} \cap \mathcal{F}_T$, we have that

$$a = \underbrace{a - p(a, 0)}_{A^{-1}} + \underbrace{p(a, 0)}_{\in \mathcal{N}(T)}.$$

Harte continued to show that, if T has the Riesz property, then the sets $A^{KD} \cap \mathcal{F}_T$ and $\mathcal{B}_T$ coincide (see [9, Theorem 1]). Furthermore, it is known that $\mathcal{B}_T$ (when T has the Riesz property) and $\mathcal{F}_T$ form regularities, while $\mathcal{W}_T$ forms an upper semi-regularity. The spectra obtain from the sets of Fredholm, Weyl and Browder elements are defined next:

Definition 2.8. *([9], p.433-434) Let $T : A \to B$ be a Banach algebra homomorphism. Relative to T,*

- *the Fredholm spectrum of $a \in A$, denoted by $\gamma_T(a)$, is the set*

$$\{\lambda \in \mathbb{C} : \lambda 1_A - a \notin \mathcal{F}_T\}.$$

- *the Weyl spectrum of $a \in A$, denoted by $\omega_T(a)$, is the set*

$$\{\lambda \in \mathbb{C} : \lambda 1_A - a \notin \mathcal{W}_T\}.$$

- *the Browder spectrum of $a \in A$, denoted by $\beta_T(a)$, is the set*

$$\{\lambda \in \mathbb{C} : \lambda 1_A - a \notin \mathcal{B}_T\}.$$

These are all non-empty compact subsets of the complex plane as can be seen from the following inclusions and identities [9]:

$$\begin{array}{ccccccc} & & \bigcap\limits_{c \in N(T)} \sigma(a+c) & & & & \\ & & \shortparallel & & & & \\ \tau_T(a) & \subseteq & \omega_T(a) & \subseteq & \beta_T(a) & \subseteq & \sigma(a). \\ \shortparallel & & & & \shortparallel & & \\ \sigma(Ta) & & & & \bigcap\limits_{\substack{c \in N(T) \\ ac=ca}} \sigma(a+c) & & \end{array}$$

Though the Fredholm, Weyl and Browder spectra do not generally coincide, in [19], Živković-Zlatanović and Harte showed that their connected hulls do:

Theorem 2.9. *([19], Corollary 2.2) Let $T : A \to B$ be a Banach algebra homomorphism with the strong Riesz property and $a \in A$. Then*

$$\eta\sigma(Ta) = \eta\omega_T(a) = \eta\beta_T(a).$$

In particular, if A is finite-dimensional, then $\sigma(Ta) = \omega_T(a) = \beta_T(a)$, i.e. $\mathcal{B}_T = \mathcal{W}_T = \mathcal{F}_T$.

We conclude this subsection with the spectral mapping properties of the Fredholm, Weyl and Browder spectra that are pointed out next:

Theorem 2.10. *([9], p.434, (2.4) and Theorem 2) Let $T : A \to B$ be a Banach algebra homomorphism. If $a \in A$, then*
(i) $\sigma(T(f(a))) = f(\sigma(Ta))$,
(ii) $\omega_T(f(a)) \subseteq f(\omega_T(a))$,
(iii) $\beta_T(f(a)) \subseteq f(\beta_T(a))$
for every function $f : U \mapsto \mathbb{C}$ which is analytic on a neighbourhood U of $\sigma(a)$ and non-constant on each component of U.
Moreover, if T has the Riesz property, then the reverse inclusion in (iii) also holds, i.e. $\beta_T(f(a)) = f(\beta_T(a))$.

For more on Fredholm theory in general Banach algebras, which has been widely studied by several authors, see for instance [9], [14], [15] and [21].

3. B-Fredholm theory and generalized B-Fredholm theory

In [2], Berkani introduced the notion of a B-Fredholm operator acting on a Banach space X, which extends the concept of a Fredholm operator on X. We recall that an operator $T \in \mathcal{L}(X)$ is said to be *B-Fredholm* if there exists a non-negative integer n such that the range $R(T^n)$ of T^n is closed and the restriction of T to $R(T^n)$ is a Fredholm operator on $R(T^n)$. In [3, Theorem 3.4], it was established that an operator $T \in \mathcal{L}(X)$ is B-Fredholm if and only if $T + \mathcal{F}(X)$ is Drazin invertible in $\mathcal{L}(X)/\mathcal{F}(X)$, i.e., relative to the canonical homomorphism $\pi : \mathcal{L}(X) \to \mathcal{L}(X)/\mathcal{F}(X)$, $\pi T \in (\mathcal{L}(X)/\mathcal{F}(X))^D$. Motivated by this Atkinson-type theorem for B-Fredhom operators, the authors of [6] described a B-Fredholm element of a general unital algebra A (w.r.t. a unital homomorphism $T : A \to B$) as an element whose image under T is Drazin invertible in B. They also introduced (generalized) B-Browder and (generalized) B-Weyl elements, which we define next.

Definition 3.1. *([6], Definition 2.3) Let $T : A \to B$ be a unital homomorphism. Relative to T, an element $a \in A$ is called:*

- *B-Fredholm if $Ta \in B^D$.*
- *B-Weyl if there exist elements $b \in A^D$ and $c \in \mathcal{N}(T)$ such that $a = b + c$.*
- *B-Browder if there exist commuting elements $b \in A^D$ and $c \in \mathcal{N}(T)$ such that $a = b + c$.*

The sets of Drazin invertible elements in Definition 3.1 are now replaced by the sets of Koliha-Drazin invertible elements:

Definition 3.2. *([6], Definition 2.3) Let $T : A \to B$ be a unital homomorphism. Relative to T, an element $a \in A$ is called:*

- *generalized B-Fredholm (or GB-Fredholm) if $Ta \in B^{KD}$.*
- *generalized B-Weyl (or GB-Weyl) if there exist elements $b \in A^{KD}$ and $c \in \mathcal{N}(T)$ such that $a = b + c$.*
- *generalized B-Browder (or GB-Browder) if there exist commuting elements $b \in A^{KD}$ and $c \in \mathcal{N}(T)$ such that $a = b + c$.*

The sets of all B-Fredholm, B-Weyl and B-Browder elements (relative to T) will be denoted by $\mathcal{F}_T^{(B)}$, $\mathcal{W}_T^{(B)}$ and $\mathcal{B}_T^{(B)}$, respectively, while the sets of all GB-Fredholm, GB-Weyl and GB-Browder elements (relative to T) will be indicated by $\mathcal{F}_T^{(GB)}$, $\mathcal{W}_T^{(GB)}$ and $\mathcal{B}_T^{(GB)}$, respectively. Evidently,

$$\mathcal{F}_T \subseteq \mathcal{F}_T^{(B)} \subseteq \mathcal{F}_T^{(GB)},$$
$$\mathcal{W}_T \subseteq \mathcal{W}_T^{(B)} \subseteq \mathcal{W}_T^{(GB)}$$

and

$$\mathcal{B}_T \subseteq \mathcal{B}_T^{(B)} \subseteq \mathcal{B}_T^{(GB)}.$$

Remark 3.3. *Note that the first inclusions in the inclusion-schemes above are always strict (cf. [6, Theorem 3.4 ((iii), (iv) and (v))]) since any $a \in \mathcal{N}(T)$ is obviously the sum of commuting elements $0_A \in A^D$ and $a \in \mathcal{N}(T)$, i.e. $a \in \mathcal{B}_T^{(B)}(\subseteq \mathcal{W}_T^{(B)} \subseteq \mathcal{F}_T^{(B)})$ but do not belong to $\mathcal{F}_T(\supseteq \mathcal{W}_T \supseteq \mathcal{B}_T)$ as $Ta = 0_B \notin B^{-1}$. In particular, we have that $\mathcal{N}(T) \subseteq X$, where $X \in \{\mathcal{B}_T^{(B)}, \mathcal{W}_T^{(B)}, \mathcal{F}_T^{(B)}\}$ and $\mathcal{N}(T) \cap Y = \emptyset$, where $Y \in \{B_T, \mathcal{W}_T, \mathcal{F}_T\}$.*

The following inclusions are straightforward to show:

Proposition 3.4. *([6], Remark 2.5(iv)) Let $T : A \to B$ be a unital homomorphism. Then*

$$A^D \subseteq \mathcal{B}_T^{(B)} \subseteq \mathcal{W}_T^{(B)} \subseteq \mathcal{F}_T^{(B)}$$

and

$$A^{KD} \subseteq \mathcal{B}_T^{(GB)} \subseteq \mathcal{W}_T^{(GB)} \subseteq \mathcal{F}_T^{(GB)}.$$

Moreover, if A is a commutative semisimple Banach algebra, then $\mathcal{F}_T^{(B)} = \mathcal{F}_T^{(GB)}$, $\mathcal{W}_T^{(B)} = \mathcal{W}_T^{(GB)}$ and $\mathcal{B}_T^{(B)} = \mathcal{B}_T^{(GB)}$.

In [6], the main properties of the elements defined in Definitions 3.1 and 3.2 are studied. Adding to [6], we list the following examples of (generalized) B-Fredholm, (generalized) B-Weyl and (generalized) B-Browder elements:

Example 3.5. *([3], Theorem 3.4; [4], Corollary 4.4; [20], p.3599) Consider the canonical map $\pi : \mathcal{L}(X) \to \mathcal{L}(X)/\mathcal{F}(X)$ and let $T \in \mathcal{L}(X)$. Then:*

(i) $T \in \mathcal{F}_\pi^{(B)}$ if and only if T is a B-Fredholm operator on X.

(ii) $T \in \mathcal{W}_\pi^{(B)}$ if and only if T is a B-Weyl operator on X, i.e. a B-Fredholm operator of index 0 [4, Definition 1.1], where the index of a B-Fredholm operator T is defined as the index of the relevant (restriction) Fredholm operator.

(iii) $T \in \mathcal{B}_\pi^{(B)}$ if and only if $T \in \mathcal{L}(X)^D$.

We remark that the statement in (*iii*) was not established by the authors of [20] but was stated there in the given form. One way to see why the nontrivial implication in (*iii*) holds is by using the fact that the Drazin spectrum remains stable under perturbation of commuting finite-rank operators [5, Theorem 2.7]: Let $T \in \mathcal{B}_\pi^{(B)}$. Then there exist commuting operators $T_1 \in \mathcal{L}(X)^D$ and $T_2 \in \mathcal{F}(X)$ such that $T = T_1 + T_2$. Since $T - T_2$ and $T_2 \in \mathcal{F}(X)$ commute, we have from [5, Theorem 2.7] that $\sigma_D(T) = \sigma_D((T - T_2) + T_2) = \sigma_D(T - T_2)$. Since $T - T_2 \in \mathcal{L}(X)^D$, so that $0 \notin \sigma_D(T - T_2)$, it follows that $0 \notin \sigma_D(T)$, and hence $T \in \mathcal{L}(X)^D$.

Remark 3.6. *The concepts of GB-Fredholm, GB-Weyl and GB-Browder elements, unlike the notions of B-Fredholm, B-Weyl and B-Browder elements (that were introduced in the context of bounded linear operators on Banach spaces), were first defined and studied in general Banach algebras in [6].*

For the function space in Example 3.7 - an example of a commutative semisimple Banach algebra - we have the following result, which can be viewed as an analogue of [9, p.432]. By $\mathrm{Ran}(f)$, we denote the range of a complex-valued function f. Also, if X and Y are arbitrary compact Hausdorff spaces, then we refer to the Banach algebra homomorphism $T : C(X) \to C(Y)$ defined by $Tf = f \circ \theta$ for all $f \in C(X)$, where $\theta : Y \to X$ is a fixed continuous map, as the *Banach algebra homomorphism induced by composition with θ*.

Example 3.7. *Let X and Y be compact Hausdorff spaces and $T : C(X) \to C(Y)$ be the Banach algebra homomorphism induced by composition with a fixed continuous map $\theta : Y \to X$, i.e. $Tf = f \circ \theta$ for all $f \in C(X)$. Then:*

(i) $f \in \mathcal{F}_T^{(B)} = \mathcal{F}_T^{(GB)}$ if and only if $0 \notin \mathrm{acc}\,\mathrm{Ran}(Tf)$.

(ii) $f \in \mathcal{W}_T^{(B)} = \mathcal{B}_T^{(B)} = \mathcal{B}_T^{(GB)} = \mathcal{W}_T^{(GB)}$ if and only if $f_{|\theta(Y)}$ has a (continuous) Koliha-Drazin invertible extension g to X, i.e. $g : X \to \mathbb{C}$ satisfies $0 \notin \mathrm{acc}\,\mathrm{Ran}(g)$.

Proof. Observe that $\mathcal{N}(T) = \{f \in C(X) : f_{|\theta(Y)} = 0\}$.

Let $f \in C(X)$.

(*i*) Then $f \in \mathcal{F}_T^{(B)}$ if and only if $Tf \in C(Y)^D = C(Y)^{KD}$, which (by [11, Theorem 3.1]) is equivalent to $0 \notin \operatorname{acc} \sigma(Tf) = \operatorname{acc} \operatorname{Ran}(Tf)$.

(*ii*) If $f \in \mathcal{W}_T^{(B)}$, then there exist $g \in C(X)^D = C(X)^{KD}$ and $h \in \mathcal{N}(T)$ such that $f = g + h$. Hence $f_{|\theta(Y)} = g_{|\theta(Y)} + h_{|\theta(Y)} = g_{|\theta(Y)}$, where $0 \notin \operatorname{acc} \sigma(g) = \operatorname{acc} \operatorname{Ran}(g)$ follows from [11, Theorem 3.1] as $g \in C(X)^{KD}$. For the reverse implication, suppose that $f_{|\theta(Y)}$ has a continuous Koliha-Drazin invertible extension g to X. Then $f = g + (f - g)$, where $g \in C(X)^{KD} = C(X)^D$ and $(f - g)_{|\theta(Y)} = 0$, i.e. $f - g \in \mathcal{N}(T)$. Hence $f \in \mathcal{W}_T^{(B)}$. This completes the proof. □

We show next that the inclusions in Proposition 3.4 are strict in general.

Example 3.8. *Consider the compact Hausdorff spaces $X = [0, 3] = Y$ and the Banach algebra homomorphism $T : C(X) \to C(Y)$ induced by composition with the fixed continuous map $\theta : Y \to X$ defined by $\theta(z) = \frac{z+1}{2}$ for all $z \in Y$. Then $f \in C(X)$, given by $f(z) = z$ for all $z \in X$, belongs to $\mathcal{B}_T^{(B)} \setminus C(X)^D$ (resp. $\mathcal{B}_T^{(GB)} \setminus C(X)^{KD}$).*

Proof. Observe that $\theta(Y) = [\frac{1}{2}, 2]$ and define $g : X \to \mathbb{C}$ by

$$g(z) := \begin{cases} \frac{1}{2} & \text{if } 0 \le z \le \frac{1}{2} \\ z & \text{if } \frac{1}{2} < z < 2 \\ 2 & \text{if } 2 \le z \le 3. \end{cases}$$

Then $g_{|\theta(Y)} = f_{|\theta(Y)}$ and $\sigma(g) = \operatorname{Ran}(g) = \left[\frac{1}{2}, 2\right]$, so that $0 \notin \operatorname{acc} \sigma(g)$, i.e. $g \in C(X)^{KD}$. By Example 3.7(*ii*), $f \in \mathcal{B}_T^{(B)}$. Since $0 \in [0, 3] = \operatorname{Ran}(f) = \operatorname{acc} \sigma(f)$, it follows that $f \notin C(X)^{KD}$. This completes the proof. □

Example 3.9. *Consider the canonical map $\pi : \mathcal{L}(\ell^2 \oplus \ell^2) \to \mathcal{L}(\ell^2 \oplus \ell^2)/\mathcal{F}(\ell^2 \oplus \ell^2)$ on the Banach algebra $\mathcal{L}(\ell^2 \oplus \ell^2)$ of all bounded linear operators on the direct sum of ℓ^2. If $U, V : \ell^2 \to \ell^2$ define, respectively, the forward shift and backward shift operators on ℓ^2, then $T : \ell^2 \oplus \ell^2 \to \ell^2 \oplus \ell^2$ given by*

$$T = \begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix}$$

belongs to $\mathcal{W}_\pi^{(B)} \setminus \mathcal{B}_\pi^{(B)}$.

Proof. From [10, Example 4.3] we have that T is a sum of an invertible operator and a finite-rank operator, i.e. T is Weyl (and hence B-Weyl), but T is not Browder w.r.t. π. It then follows from [1, Remark 3.1], which states that a Drazin invertible Weyl operator is Browder, that $T \notin \mathcal{L}(\ell^2 \oplus \ell^2)^D$, so that from Example 3.5(*iii*) we have that T does not belong to $\mathcal{B}_\pi^{(B)}$. □

For the next example we recall the concept of a retraction and a result involving retractions, so let X be a topological space and Y be a subset of X. A continuous function $f : X \to Y$ is said to be a *retraction* of X onto Y if the restriction $f_{|Y}$ of f to Y is the identity function on Y. According to [18, Theorem 34.5, p.236], there is no retraction from $D := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \le 1\}$ onto $S' := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. We also need the following observation:

Lemma 3.10. *Let $X, Y, T : C(X) \to C(Y)$ and $\theta : Y \to X$ be as in Example 3.7. If X is connected, then there is no $f \in C(X)$ with the property that* $\operatorname{iso}(\sigma(f) \setminus \sigma(Tf)) \neq \emptyset$.

Proof. First observe that $\operatorname{Idem}(C(X)) = \{0_{C(X)}, 1_{C(X)}\}$. If $f \in C(X)$ is such that there exists $\lambda_0 \in \operatorname{iso}(\sigma(f) \setminus \sigma(Tf))$, then by Lemma 2.7, $p(f, \lambda_0)$ is a non-zero idempotent that belongs to $\mathcal{N}(T)$, i.e.

$$1_{C(X)} = p(f, \lambda_0) \in \mathcal{N}(T) = \{f \in C(X) : f_{|\theta(Y)} = 0\},$$

which is a contradiction. □

Example 3.11. *Consider the compact Hausdorff spaces* $X := \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$ *and* $Y := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ *and the Banach algebra homomorphism* $T : C(X) \to C(Y)$ *induced by composition with the identity map* $\theta : Y \to X$ *on* Y*, i.e.* $\theta(\lambda) = \lambda$ *for all* $\lambda \in Y$*. Then* $f \in C(X)$*, defined by* $f(z) = z$ *for all* $z \in X$*, belongs to* $\mathcal{F}_T^{(B)} \setminus \mathcal{W}_T^{(B)}$ *(resp.* $\mathcal{F}_T^{(GB)} \setminus \mathcal{W}_T^{(GB)}$*).*

Proof. Note that $\sigma(Tf) = \mathrm{Ran}(Tf) = Y$ and, since $0 \notin \mathrm{acc}\, \sigma(Tf)$, we have that $Tf \in C(Y)^{KD} = C(Y)^D$, i.e. $f \in \mathcal{F}_T^{(B)}$.

We prove next by contradiction that $f \notin \mathcal{W}_T^{(B)}$. Hence, in view of Example 3.7(*ii*), suppose that $f_{|\theta(Y)} = f_{|Y}$ has a continuous Koliha-Drazin invertible extension g to X, i.e. $g : X \to \mathbb{C}$ satisfies $0 \notin \mathrm{acc}\, \mathrm{Ran}(g)$. Observe that

$$\mathrm{Ran}(Tf) = \{f(\theta(y)) : y \in Y\} = \{g(\theta(y)) : y \in Y\} = \mathrm{Ran}(Tg),$$

so that $0 \notin Y = \mathrm{Ran}(Tf) = \mathrm{Ran}(Tg)$. Since, in view of Lemma 3.10, the condition $0 \in \mathrm{iso}\, (\sigma(g) \setminus \mathrm{Ran}(Tg))$ cannot hold, it follows that $0 \notin \sigma(g) = g(X)$.

Now define $h : g(X) \to Y$ by $h(z) = \frac{z}{|z|}$ for all $z \in g(X)$. Then h is a continuous function on $g(X)$ that satisfies $h(Y) \subseteq Y$. Furthermore, the composition $h \circ g$ of h and g is a continuous function with domain X and range Y that satisfies (for all $y \in Y$)

$$(h \circ g)(y) = h(g(y)) = \frac{g(y)}{|g(y)|} = \frac{f(y)}{|f(y)|} = \frac{y}{|y|} = y,$$

i.e. $h \circ g$ is the identity function on Y. Consequently, $h \circ g$ is a retraction of X onto Y, which is a contradiction by the remark preceding Lemma 3.10. This shows that $f_{|\theta(Y)}$ has no continuous Koliha-Drazin invertible extension to X, so that $f \notin \mathcal{W}_T^{(B)}$ according to Example 3.7(*ii*). □

Example 3.12. *Consider* π, U, V *and* T *as in Example 3.9. Then* T *belongs to* $\mathcal{W}_\pi^{(GB)} \setminus \mathcal{B}_\pi^{(GB)}$*.*

Proof. First note from Example 3.9 that $T \in \mathcal{W}_\pi^{(B)} \subseteq \mathcal{W}_\pi^{(GB)}$. Assume now, by way of contradiction, that $T \in \mathcal{B}_\pi^{(GB)}$, i.e. there exist commuting operators $T_1 \in \mathcal{L}(\ell^2 \oplus \ell^2)^{KD}$ and $T_2 \in \mathcal{F}(\ell^2 \oplus \ell^2)$ such that $T = T_1 + T_2$. Since T is not Browder (as pointed out in Example 3.9), T_1 cannot be invertible in $\mathcal{L}(\ell^2 \oplus \ell^2)$. Hence, $0 \in \mathrm{iso}\, \sigma(T_1)$. Since $T \in \mathcal{W}_\pi^{(B)}$, there exist $T' \in \mathcal{L}(\ell^2 \oplus \ell^2)^D$ and $T'' \in \mathcal{F}(\ell^2 \oplus \ell^2)$ such that $T = T' + T''$. Consequently, since $\mathcal{F}(\ell^2 \oplus \ell^2)$ is an ideal, $T - T_2 = T' + (T'' - T_2) \in \mathcal{L}(\ell^2 \oplus \ell^2)^D + \mathcal{F}(\ell^2 \oplus \ell^2)$, i.e. $T_1 = T - T_2 \in \mathcal{W}_\pi^{(B)}$. From [4, Theorem 4.2] it now follows that $T_1 \in \mathcal{L}(\ell^2 \oplus \ell^2)^D$, so that T belongs to $\mathcal{B}_\pi^{(B)}$. But this is a contradiction in view of Example 3.9. Hence, $T \notin \mathcal{B}_\pi^{(GB)}$. □

4. Further properties

4.1. Inclusion properties

We recall the inclusion results for the classes of elements studied in Fredholm theory, B-Fredholm theory and GB-Fredholm theory relative to a Banach algebra homomorphism $T : A \to B$:

$$A^{-1} \subseteq \mathcal{B}_T \subseteq \mathcal{W}_T \subseteq \mathcal{F}_T,$$

$$A^D \subseteq \mathcal{B}_T^{(B)} \subseteq \mathcal{W}_T^{(B)} \subseteq \mathcal{F}_T^{(B)}$$

and

$$A^{KD} \subseteq \mathcal{B}_T^{(GB)} \subseteq \mathcal{W}_T^{(GB)} \subseteq \mathcal{F}_T^{(GB)}.$$

In [6, Theorem 3.4(x) and Theorem 3.5(v)], the authors showed that the sets $\mathcal{W}_T^{(B)} \setminus \mathcal{W}_T$ and $\mathcal{W}_T^{(GB)} \setminus \mathcal{W}_T$ are contained in $\mathcal{F}_T^{(B)} \setminus \mathcal{F}_T$ and $\mathcal{F}_T^{(GB)} \setminus \mathcal{F}_T$, respectively. A natural question arising now is whether the inclusions

$$A^D \setminus A^{-1} \subseteq \mathcal{B}_T^{(B)} \setminus \mathcal{B}_T \subseteq \mathcal{W}_T^{(B)} \setminus \mathcal{W}_T, \tag{4.1}$$

$$A^{KD} \setminus A^{-1} \subseteq \mathcal{B}_T^{(GB)} \setminus \mathcal{B}_T \subseteq \mathcal{W}_T^{(GB)} \setminus \mathcal{W}_T, \tag{4.2}$$

and

$$A^{KD} \setminus A^{D} \subseteq \mathcal{B}_T^{(GB)} \setminus \mathcal{B}_T^{(B)} \subseteq \mathcal{W}_T^{(GB)} \setminus \mathcal{W}_T^{(B)} \subseteq \mathcal{F}_T^{(GB)} \setminus \mathcal{F}_T^{(B)}. \tag{4.3}$$

also hold. This we examine next.

First, we observe that the containments $A^D \setminus A^{-1} \subseteq \mathcal{B}_T^{(B)} \setminus \mathcal{B}_T$ and $A^{KD} \setminus A^{-1} \subseteq \mathcal{B}_T^{(GB)} \setminus \mathcal{B}_T$ do not hold in general:

Example 4.4. *Consider the Banach algebra homomorphism $T : M_2^u(\mathbb{C}) \to \mathbb{C}$ defined by*

$$T\left(\begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{21} \end{bmatrix}\right) = a_{11}.$$

If we let $A := M_2^u(\mathbb{C})$, then $a := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in A^D (= A^{KD}) \setminus A^{-1}$ and $a \in \mathcal{B}_T$.

Proof. Since a is an idempotent with $\sigma(Ta) = \{1\} \subsetneq \{0, 1\} = \sigma(a)$, we have that $a \in A^D \setminus A^{-1}$ and $a \in \mathcal{F}_T$. Since A is finite-dimensional, it follows from Theorem 2.9 that $a \in \mathcal{B}_T$. □

We point out next that, under the additional assumption that $a \notin \mathcal{F}_T$, the implications

$$a \in A^D \setminus A^{-1} \Longrightarrow a \in \mathcal{B}_T^{(B)} \setminus \mathcal{B}_T$$

and

$$a \in A^{KD} \setminus A^{-1} \Longrightarrow a \in \mathcal{B}_T^{(GB)} \setminus \mathcal{B}_T$$

will hold:

Proposition 4.5. *Let $T : A \to B$ be a Banach algebra homomorphism. Then:*
(i) $(A^D \setminus A^{-1}) \cap \mathcal{F}_T \subseteq \mathcal{B}_T \subseteq \mathcal{B}_T^{(B)}$.
(ii) $(A^{KD} \setminus A^{-1}) \cap \mathcal{F}_T \subseteq \mathcal{B}_T \subseteq \mathcal{B}_T^{(GB)}$.
(iii) $(A^D \setminus A^{-1}) \cap (A \setminus \mathcal{F}_T) \subseteq \mathcal{B}_T^{(B)} \setminus \mathcal{B}_T$.
(iv) $(A^{KD} \setminus A^{-1}) \cap (A \setminus \mathcal{F}_T) \subseteq \mathcal{B}_T^{(GB)} \setminus \mathcal{B}_T$.

In the following result, we demonstrate that the second inclusions in both (4.1) and (4.2) - which are not mentioned in [6] - also hold. That is, every B-Browder (resp. GB-Browder) element which is not Browder belongs to the set of all B-Weyl (resp. GB-Weyl) elements that are not Weyl.

Theorem 4.6. *Let $T : A \to B$ be a Banach algebra homomorphism. Then*

$$\mathcal{B}_T^{(B)} \setminus \mathcal{B}_T \subseteq \mathcal{W}_T^{(B)} \setminus \mathcal{W}_T \subseteq \mathcal{F}_T^{(B)} \setminus \mathcal{F}_T$$

and

$$\mathcal{B}_T^{(GB)} \setminus \mathcal{B}_T \subseteq \mathcal{W}_T^{(GB)} \setminus \mathcal{W}_T \subseteq \mathcal{F}_T^{(GB)} \setminus \mathcal{F}_T.$$

Proof. Since $\mathcal{B}_T^{(B)} \setminus \mathcal{B}_T \subseteq \mathcal{B}_T^{(B)} \subseteq \mathcal{W}_T^{(B)} \subseteq \mathcal{F}_T^{(B)}$, we are only left to show that $a \in \mathcal{B}_T^{(B)} \setminus \mathcal{B}_T$ implies that $a \notin \mathcal{W}_T$, so let $a \in \mathcal{B}_T^{(B)} \setminus \mathcal{B}_T$. Then there exist commuting elements $b \in A^D \setminus A^{-1}$ and $c \in \mathcal{N}(T)$ such that $a = b + c$. Let $p := p(b, 0)$ be the idempotent from [11, Lemma 2.4 and Theorem 3.1]. Since b and c commute, $b + p \in A^{-1}$ and $c - p$ will also commute, and since $a \notin \mathcal{B}_T$, it follows that $p \notin \mathcal{N}(T)$. By Lemma 2.7, $Ta = Tb \notin B^{-1}$, i.e. $a \notin \mathcal{F}_T (\supseteq \mathcal{W}_T)$. Hence $a \in \mathcal{W}_T^{(B)} \setminus \mathcal{W}_T$. The proof of the implication $a \in \mathcal{B}_T^{(GB)} \setminus \mathcal{B}_T \Rightarrow a \in \mathcal{W}_T^{(GB)} \setminus \mathcal{W}_T$ is essentially the same. Simply replace the sets from B-Fredholm theory with their GB-counterparts. Finally, recall that the final inclusions were established in [6, Theorem 3.4(x) and Theorem 3.5(v)]. □

In [6, Theorem 3.4(*vi*)], the authors proved the containment $\mathrm{Idem}(A) \setminus T^{-1}\{1\} \subseteq \mathcal{F}_T^{(B)} \setminus \mathcal{F}_T$. Our next result display a more complete list of inclusions. In particular, we establish that the set $\mathrm{Idem}(A) \setminus T^{-1}\{1\}$ is contained in both $A^D \setminus A^{-1}$ and $A \setminus \mathcal{F}_T$ and then apply Proposition 4.5(*iii*) and Theorem 4.6.

Proposition 4.7. *Let $T : A \to B$ be a Banach algebra homomorphism. Then*

$$\mathrm{Idem}(A) \setminus T^{-1}\{1\} \subseteq (A^D \setminus A^{-1}) \cap (A \setminus \mathcal{F}_T) \subseteq \mathcal{B}_T^{(B)} \setminus \mathcal{B}_T \subseteq \mathcal{W}_T^{(B)} \setminus \mathcal{W}_T \subseteq \mathcal{F}_T^{(B)} \setminus \mathcal{F}_T.$$

Proof. Let $a \in \mathrm{Idem}(A) \setminus T^{-1}\{1\}$. Since $\mathrm{Idem}(A) \subseteq A^D$, we are only left to show that $a \notin \mathcal{F}_T (\supseteq A^{-1})$. By assumption, $Ta \neq 1_B$ is an idempotent in B. Therefore, either $\sigma(Ta) = \{0\}$ or $\sigma(Ta) = \{0, 1\}$. From both cases it follows that $Ta \notin B^{-1}$, i.e. $a \notin \mathcal{F}_T$, and hence the inclusion $a \in (A^D \setminus A^{-1}) \cap (A \setminus \mathcal{F}_T)$ holds. □

The authors do not know whether the inclusions in (4.3) hold in general.

4.2. Algebraic properties

It is well-known that the sets of Fredholm and Weyl elements are generally closed under multiplication (and hence also non-zero scalar multiplication since $\lambda 1_A$ is invertible, and therefore Weyl and Fredholm, for all $0 \neq \lambda \in \mathbb{C}$). In the case where the homomorphism has the Riesz property, then it is known that the product of commuting Browder elements is again Browder. As we see next, the sets of (G)B-Fredholm, (G)B-Weyl and (G)B-Browder elements are closed under arbitrary scalar multiplication.

Lemma 4.8. *Let $T : A \to B$ be a Banach algebra homomorphism and $\lambda \in \mathbb{C}$. Then the following holds:*
(*i*) *If $a \in \mathcal{F}_T^{(B)}$, then $\lambda a \in \mathcal{F}_T^{(B)}$.*
(*ii*) *If $a \in \mathcal{B}_T^{(B)}$, then $\lambda a \in \mathcal{B}_T^{(B)}$.*
(*iii*) *If $a \in \mathcal{W}_T^{(B)}$, then resp. $\lambda a \in \mathcal{W}_T^{(B)}$.*

Proof. (*i*) Let $a \in \mathcal{F}_T^{(B)}$ and $\lambda \in \mathbb{C}$. If $\lambda = 0$, then $\lambda a = 0_A \in \mathcal{N}(T) \subseteq \mathcal{F}_T^{(B)}$ in view of Remark 3.3. If $\lambda \neq 0$, then $\lambda 1_A \in A^{-1} \subseteq A^D \subseteq \mathcal{F}_T^{(B)}$, so that from [6, Theorem 3.4(*vii*)] it follows that $\lambda a \in \mathcal{F}_T^{(B)}$.

(*ii*) We prove only the implication $a \in \mathcal{B}_T^{(B)} \Rightarrow \lambda a \in \mathcal{B}_T^{(B)}$ for all $\lambda \in \mathbb{C}$, as a similar reasoning applies in the case where $\mathcal{W}_T^{(B)}$ is considered. So let $a \in \mathcal{B}_T^{(B)}$ and $\lambda \in \mathbb{C}$. Then there exist commuting elements $b \in A^D$ and $c \in \mathcal{N}(T)$ such that $a = b + c$. If $\lambda = 0$, then again by Remark 3.3 we have that $\lambda a = 0_A \in \mathcal{N}(T) \subseteq \mathcal{B}_T^{(B)}$. In the case where $\lambda \neq 0$, we have that $\lambda b \in A^D$ (since it is the product of commuting elements b and $\lambda 1_A$ of A^D) - cf. [11, p.375]. Hence, $\lambda a = \lambda b + \lambda c \in \mathcal{B}_T^{(B)}$. □

Using the facts that the sets of Drazin and Koliha-Drazin invertible elements form regularities, one can easily show that the sets of B-Fredholm and GB-Fredholm elements form regularities as well. We prove next that the sets of (G)B-Weyl and (G)B-Browder elements form upper semi-regularities relative to Banach algebra homomorphisms with commutative domains.

Proposition 4.9. *If A is a commutative Banach algebra and $T : A \to B$ a Banach algebra homomorphism, then $\mathcal{W}_T^{(B)}$ $(= \mathcal{B}_T^{(B)})$ is an upper semi-regularity.*

Proof. We show that conditions (*ii*) and (*iii*) in Definition 2.2 are satisfied as condition (*i*) was already established in [6, Theorem 3.4(*viii*)] (resp. [6, Theorem 3.4(*ix*)]) for B-Weyl elements (resp. B-Browder elements). To prove (*iii*), simply use the facts that $1_A \in A^{-1}$ and A^{-1} (which is contained in $\mathcal{B}_T^{(B)} \subseteq \mathcal{W}_T^{(B)}$) is an open subset of A. To establish (*ii*), which is where the assumption that A is commutative is required, suppose that $a, b \in \mathcal{W}_T^{(B)}$ (are relatively prime). Then $a = c + d$ and $b = e + f$, where $c, e \in A^D$ and $d, f \in \mathcal{N}(T)$. Recalling the fact that the product of commuting Drazin invertible elements is again Drazin invertible, it follows that

$$ab = (c + d)(e + f) = ce + (cf + de + df) \in A^D + \mathcal{N}(T) = \mathcal{W}_T^{(B)} (= \mathcal{B}_T^{(B)}),$$

which completes the proof. □

By applying a similar reasoning as in the proof of Proposition 4.9, and utilizing Theorems 3.5[(*iii*) and (*iv*)] in [6] instead of Theorems 3.4[(*viii*) and (*ix*)], respectively, the following result is obtained:

Proposition 4.10. *If A is a commutative Banach algebra and $T : A \to B$ a Banach algebra homomorphism, then $\mathcal{W}_T^{(GB)}$ $(= \mathcal{B}_T^{(GB)})$ is an upper semi-regularity.*

Hence, relative to arbitrary Banach algebra homomorphisms with commutative domains, the classes of elements from B-Fredholm theory and GB-Fredholm theory are all closed under finite products:

Proposition 4.11. *If A is a commutative Banach algebra and $T : A \to B$ a Banach algebra homomorphism, then:*
(i) $\mathcal{B}_T^{(B)}\mathcal{B}_T^{(B)} = \mathcal{W}_T^{(B)}\mathcal{W}_T^{(B)} \subseteq \mathcal{W}_T^{(B)} = \mathcal{B}_T^{(B)}$.
(ii) $\mathcal{F}_T^{(B)}\mathcal{F}_T^{(B)} \subseteq \mathcal{F}_T^{(B)}$
(iii) $\mathcal{B}_T^{(GB)}\mathcal{B}_T^{(GB)} = \mathcal{W}_T^{(GB)}\mathcal{W}_T^{(GB)} \subseteq \mathcal{W}_T^{(GB)} = \mathcal{B}_T^{(GB)}$.
(iv) $\mathcal{F}_T^{(GB)}\mathcal{F}_T^{(GB)} \subseteq \mathcal{F}_T^{(GB)}$

5. Connected hulls of corresponding spectra

The classes of elements from Definitions 3.1 and 3.2 give rise to six new spectra that are all compact subsets of the complex plane:

Definition 5.1. *([6], Definition 2.4) Let $T : A \to B$ be a Banach algebra homomorphism. Relative to T,*

- *the B-Fredholm spectrum of $a \in A$, denoted by $\gamma_T^{(B)}(a)$, is the set*

$$\{\lambda \in \mathbb{C} : \lambda 1_A - a \notin \mathcal{F}_T^{(B)}\}.$$

- *the B-Weyl spectrum of $a \in A$, denoted by $\omega_T^{(B)}(a)$, is the set*

$$\{\lambda \in \mathbb{C} : \lambda 1_A - a \notin \mathcal{W}_T^{(B)}\}.$$

- *the B-Browder spectrum of $a \in A$, denoted by $\beta_T^{(B)}(a)$, is the set*

$$\{\lambda \in \mathbb{C} : \lambda 1_A - a \notin \mathcal{B}_T^{(B)}\}.$$

- *the GB-Fredholm spectrum of $a \in A$, denoted by $\gamma_T^{(GB)}(a)$, is the set*

$$\{\lambda \in \mathbb{C} : \lambda 1_A - a \notin \mathcal{F}_T^{(GB)}\}.$$

- *the GB-Weyl spectrum of $a \in A$, denoted by $\omega_T^{(GB)}(a)$, is the set*

$$\{\lambda \in \mathbb{C} : \lambda 1_A - a \notin \mathcal{W}_T^{(GB)}\}.$$

- *the GB-Browder spectrum of $a \in A$, denoted by $\beta_T^{(GB)}(a)$, is the set*

$$\{\lambda \in \mathbb{C} : \lambda 1_A - a \notin \mathcal{B}_T^{(GB)}\}.$$

Then

$$\begin{array}{ccccccccc} & & \gamma_T^{(B)}(a) & \subseteq & \omega_T^{(B)}(a) & \subseteq & \beta_T^{(B)}(a) & \subseteq & \sigma_D(a) \\ & \rotatebox{45}{$\subsetneq$} & & \rotatebox{45}{$\subsetneq$} & & \rotatebox{45}{$\subsetneq$} & & \rotatebox{45}{$\subsetneq$} & \\ \gamma_T^{(GB)}(a) & \subseteq & \omega_T^{(GB)}(a) & \subseteq & \beta_T^{(GB)}(a) & \subseteq & \sigma_{KD}(a) & & \end{array}$$

and the B-Fredholm, B-Weyl and B-Browder spectra (resp. GB-Fredholm, GB-Weyl and GB-Browder spectra) of $a \in A$ can all be expressed in terms of the Drazin spectrum (resp. Koliha-Drazin spectrum):

$$\gamma_T^{(B)}(a) = \sigma_D(Ta);\ \ \omega_T^{(B)}(a) = \bigcap_{c \in \mathcal{N}(T)} \sigma_D(a+c);\ \ \beta_T^{(B)}(a) = \bigcap_{\substack{c \in \mathcal{N}(T)\\ ac=ca}} \sigma_D(a+c)$$

and

$$\gamma_T^{(GB)}(a) = \sigma_{KD}(Ta);\ \ \omega_T^{(GB)}(a) = \bigcap_{c \in \mathcal{N}(T)} \sigma_{KD}(a+c);\ \ \beta_T^{(GB)}(a) = \bigcap_{\substack{c \in \mathcal{N}(T)\\ ac=ca}} \sigma_{KD}(a+c).$$

In view of [17, Theorem 20], immediate consequences of Propositions 4.9 and 4.10 give one-way spectral mapping theorems for the (G)B-Weyl and (G)B-Browder spectra:

Theorem 5.2. *Let A be a commutative Banach algebra and $T : A \to B$ a Banach algebra homomorphism. If $a \in A$, then*

$$\beta_T^{(B)}(f(a)) = \omega_T^{(B)}(f(a)) \subseteq f(\omega_T^{(B)}(a)) = f(\beta_T^{(B)}(a))$$

for every function $f : U \mapsto \mathbb{C}$ which is analytic on a neighbourhood U of $\sigma(a)$ and non-constant on each component of U.

Theorem 5.3. *Let A be a commutative Banach algebra and $T : A \to B$ a Banach algebra homomorphism. If $a \in A$, then*

$$\beta_T^{(GB)}(f(a)) = \omega_T^{(GB)}(f(a)) \subseteq f(\omega_T^{(GB)}(a)) = f(\beta_T^{(GB)}(a))$$

for every function $f : U \mapsto \mathbb{C}$ which is analytic on a neighbourhood U of $\sigma(a)$ and non-constant on each component of U.

We recall Theorem 2.9 which states that the connected hulls of the Fredholm, Weyl and Browder spectra coincide relative to Banach algebra homomorphisms with the strong Riesz property. In $\mathcal{L}(X)$, relying on the relevant operator-theoretic definitions, the authors showed in [20, Theorem 3.8(2)] that the connected hulls of the B-Fredholm spectrum, B-Weyl spectrum and B-Browder spectrum (which coincides with the Drazin spectrum, cf. Example 3.5) of an operator $T \in \mathcal{L}(X)$ coincide. Even from Example 3.8 (where $\sigma_D(Tf) = \omega_T^{(B)}(f) = \beta_T^{(B)}(f) = \left[\frac{1}{2}, 2\right] \subsetneq [0,3] = \sigma_D(f)$) and Example 3.11 (where $\sigma_D(Tf) = Y \subsetneq X = \omega_T^{(B)}(f) = \beta_T^{(B)}(f) = \sigma_D(f)$) we observe that the connected hulls of the (G)B-Fredholm, (G)B-Weyl and (G)B-Browder spectra coincide. It is therefore natural to investigate the connections between the connected hulls of the B-Fredholm, B-Weyl and B-Browder spectra (resp. GB-Fredholm, GB-Weyl and GB-Browder spectra) of general Banach algebra elements. This question, as we see next, is only of interest in the case of infinite-dimensional Banach algebras:

Proposition 5.4. *If A is a finite-dimensional Banach algebra, $T : A \to B$ a Banach algebra homomorphism and $a \in A$, then $\sigma_{KD}(a) = \sigma_D(a) = \emptyset$, and hence*

$$\sigma_{KD}(Ta) = \omega_T^{(GB)}(a) = \beta_T^{(GB)}(a) = \sigma_{KD}(a) = \sigma_D(a) = \beta_T^{(B)}(a) = \omega_T^{(B)}(a) = \sigma_D(Ta).$$

Proof. Since $A^D = A^{KD} = A$, so that $\sigma_{KD}(a) = \sigma_D(a) = \emptyset$ for all $a \in A$, the list of identities holds. □

The authors do not know whether the connected hulls of the B-Fredholm, B-Weyl and B-Browder spectra (resp. GB-Fredholm, GB-Weyl and GB-Browder spectra) of an arbitrary Banach algebra element belonging to the domain of a general Banach algebra homomorphism with the strong Riesz property coincide, though some sufficient conditions for such to happen will be given in Theorem 5.6. Using an example from Harte [8], we demonstrate next that, if a Banach algebra homomorphism does not have the strong Riesz property, the relevant sets may not coincide.

Example 5.5. *Consider the disc algebra $\mathcal{A}(\mathbb{D})$ of all continuous complex-valued functions on the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| \leq 1\}$ which are analytic on the interior of $\mathbb{D}$ and the Banach algebra homomorphism $T : \mathcal{A}(\mathbb{D}) \to \mathcal{A}(\frac{1}{2}\mathbb{D})$ defined by $Tf = f_{|\frac{1}{2}\mathbb{D}}$ for all $f \in \mathcal{A}(\mathbb{D})$. For the identity function $g : \mathbb{D} \to \mathbb{C}$ on $\mathbb{D}$, defined by $g(z) = z$ for all $z \in \mathbb{D}$, we have that*

$$\sigma_{KD}(Tg) = \sigma_D(Tg) = \frac{1}{2}\mathbb{D} \subsetneq \mathbb{D} = \omega_T^{(B)}(g) = \beta_T^{(B)}(g) = \beta_T^{(GB)}(g) = \omega_T^{(GB)}(g).$$

Proof. Observe that $\sigma(g) = \mathbb{D} = \sigma_{KD}(g) = \sigma_D(g)$ and that

$$\sigma(Tg) = \mathrm{Ran}(g_{|\frac{1}{2}\mathbb{D}}) = \frac{1}{2}\mathbb{D} = \sigma_{KD}(Tg) = \sigma_D(Tg),$$

which confirms that T does not have the strong Riesz property. Since $\mathcal{N}(T) = \{0_{\mathcal{A}(\mathbb{D})}\}$ (which proves that T has the Riesz property), we have that

$$\beta_T^{(B)}(g) = \omega_T^{(B)}(g) = \bigcap_{h \in \mathcal{N}(T)} \sigma_D(g+h) = \sigma_D(g) = \mathbb{D},$$

and hence

$$\eta\sigma(Tg) = \frac{1}{2}\mathbb{D} \subsetneq \mathbb{D} = \eta\omega_T^{(B)}(g) = \eta\beta_T^{(B)}(g).$$

Since $\mathcal{A}(\mathbb{D})$ is also semisimple, the desired result follows. □

By a *group invertible* element of a unital algebra A we mean an element $a \in A$ for which there exists an element $b \in A$ satisfying $ab = ba, b = aba$ and $a = aba$. If A^g denotes the set of group invertible elements, then clearly $A^g \subseteq A^D$. The following result involves Banach algebras in which the sets of group and Drazin invertible elements coincide.

Theorem 5.6. *Let $T : A \to B$ be a Banach algebra homomorphism with the strong Riesz property. If $B^g = B^D$, then*

$$\eta\sigma_D(Ta) = \eta\omega_T^{(B)}(a) = \eta\beta_T^{(B)}(a)$$

and

$$\eta\sigma_{KD}(Ta) = \eta\omega_T^{(GB)}(a) = \eta\beta_T^{(GB)}(a)$$

for all elements $a \in A$ whose Fredholm spectra contain no isolated points.

Proof. Let $a \in A$ be such that $\sigma(Ta) = \mathrm{acc}\,\sigma(Ta)$. Then, obviously,

$$\sigma(Ta) = \sigma_D(Ta) = \sigma_{KD}(Ta).$$

First, we confirm the inclusion $A^{KD} \cap \mathcal{F}_T^{(B)} \subseteq \mathcal{B}_T^{(B)}$, so let $a \in A^{KD} \cap \mathcal{F}_T^{(B)}$. Since $B^g = B^D$, so that $Ta \in B^g$, it follows from [6, Theorem 3.4(*xii*)] that $a \in \mathcal{B}_T^{(B)}$, proving the desired inclusion. Consequently, by also using the fact that T has the strong Riesz property, we have that

$$\begin{aligned}\beta_T^{(B)}(a) &\subseteq \{\lambda \in \mathbb{C} : \lambda 1_A - a \notin A^{KD} \cap \mathcal{F}_T^{(B)}\}\\ &= \mathrm{acc}\,\sigma(a) \cup \sigma_D(Ta)\\ &\subseteq \mathrm{acc}\,\sigma(a)\\ &\subseteq \eta\sigma_D(Ta),\end{aligned}$$

from which we conclude that $\eta\sigma_D(Ta) = \eta\omega_T^{(B)}(a) = \eta\beta_T^{(B)}(a)$. For the second list of identities, we obtain from the containment $\beta_T^{(GB)}(a) \subseteq \sigma_{KD}(a)$ and the fact that T has the strong Riesz property that

$$\beta_T^{(GB)}(a) \subseteq \mathrm{acc}\,\sigma(a) \subseteq \eta\sigma(Ta) = \eta\sigma_{KD}(Ta),$$

so that

$$\eta\sigma_{KD}(Ta) = \eta\omega_T^{(GB)}(a) = \eta\beta_T^{(GB)}(a),$$

which completes the proof. □

Example 5.5, where $B := \mathcal{A}(\frac{1}{2}\mathbb{D})$ is a commutative semisimple Banach algebra (and hence $B^g = B^D$ according to [16, Corollary 4]), confirms the necessity of the condition that the Banach algebra homomorphism has the strong Riesz property even in Theorem 5.6.

6. Some open questions

Let $T : A \to B$ be a unital homomorphism. By $\mathrm{Nil}_T(A)$ (resp. $R_T(A)$) we denote the set of all T-nilpotent (resp. T-Riesz) elements of A, where an element $a \in A$ is called *T-nilpotent* (resp. *T-Riesz*) if $Ta \in \mathrm{Nil}(B)$ (resp. $Ta \in \mathrm{QN}(B)$). Clearly, $\mathrm{Nil}_T(A) \subseteq R_T(A)$. We also recall that T is said to have the *lifting property* if for each $q \in \mathrm{Idem}(A)$ there exists $p \in \mathrm{Idem}(A)$ such that $Tp = q$. In [6], relative to Banach algebra homomorphisms with the lifting property, the authors established necessary and sufficient conditions for an element to be (G)B-Fredholm. After observing that the lifting property assumption is only necessary to establish the sufficiency parts, we formulate their results as follow:

Theorem 6.1. *([6], Theorem 3.3(ii)) Let $T : A \to B$ be a Banach algebra homomorphism and $a \in A$. If there exists $p \in \mathrm{Idem}(A)$ such that*

$$a + p \in \mathcal{F}_T,\ pa(1_A - p), (1_A - p)ap \in \mathcal{N}(T) \text{ and } pap \in \mathrm{Nil}_T(A),$$

then $a \in \mathcal{F}_T^{(B)}$.
If, in addition, T has the lifting property, then the converse also holds.

Theorem 6.2. *([6], Theorem 3.3(i)) Let $T : A \to B$ be a Banach algebra homomorphism. If there exists $p \in \mathrm{Idem}(A)$ such that*

$$a + p \in \mathcal{F}_T,\ pa(1_A - p), (1_A - p)ap \in \mathcal{N}(T) \text{ and } pap \in R_T(A),$$

then $a \in \mathcal{F}_T^{(GB)}$.
If, in addition, T has the lifting property, then the converse also holds.

We further observed that the second part of Theorem 6.1 still holds when $\mathcal{F}_T$ and $\mathcal{F}_T^{(B)}$ are replaced by $\mathcal{W}_T$ and $\mathcal{W}_T^{(B)}$ (resp. $\mathcal{B}_T$ and $\mathcal{B}_T^{(B)}$) and the assumption that T has the lifting property is removed. Hence, the following results give a necessary condition for an element to be B-Weyl (resp. B-Browder) relative to an arbitrary Banach algebra homomorphism.

Proposition 6.3. *Let $T : A \to B$ be a Banach algebra homomorphism. If $a \in \mathcal{W}_T^{(B)}$, then there exists $p \in \mathrm{Idem}(A)$ such that*

$$a + p \in \mathcal{W}_T,\ pa(1_A - p), (1_A - p)ap \in \mathcal{N}(T) \text{ and } pap \in \mathrm{Nil}_T(A).$$

Proof. Let $a \in \mathcal{W}_T^{(B)}$. Then there exist $b \in A^D$ and $c \in \mathcal{N}(T)$ such that $a = b + c$. Let $p \in \mathrm{Idem}(A)$ be the idempotent from [16, Proposition 1]. Then

$$a + p = (b + p) + c \in A^{-1} + \mathcal{N}(T) = \mathcal{W}_T$$

and, since $bp = pb$, we obtain that $T(ap) = T(bp) = T(pbp) = T(pap)$ and $T(pa) = T(pb) = T(pbp) = T(pap)$, i.e. $pa(1_A - p), (1_A - p)ap \in \mathcal{N}(T)$. Finally, by using the fact that $bp \in \mathrm{Nil}(A)$, it follows that $T(pap) = T(pbp) = T(bp) \in \mathrm{Nil}(B)$, i.e. $pap \in \mathrm{Nil}_T(A)$, which completes the proof. □

Proposition 6.4. *Let $T : A \to B$ be a Banach algebra homomorphism. If $a \in \mathcal{B}_T^{(B)}$, then there exists $p \in \mathrm{Idem}(A)$ such that*

$$a + p \in \mathcal{B}_T,\ pa(1_A - p), (1_A - p)ap \in \mathcal{N}(T) \text{ and } pap \in \mathrm{Nil}_T(A).$$

Proof. Let $a \in \mathcal{B}_T^{(B)}$. Then there exist commuting elements $b \in A^D$ and $c \in \mathcal{N}(T)$ such that $a = b + c$. Let $p \in \mathrm{Idem}(A)$ be the idempotent from [16, Proposition 1]. As in the proof of Proposition 6.3, we have that $pa(1_A - p), (1_A - p)ap \in \mathcal{N}(T)$ and $pap \in \mathrm{Nil}_T(A)$. We show next that $a + p \in \mathcal{B}_T$. Now, if $b \in A^{-1}$, then $p = 0_A$ and $a \in \mathcal{B}_T$, so that $a + p = a \in \mathcal{B}_T$. Hence, suppose that $b \notin A^{-1}$, so that $0 \in \mathrm{iso}\,\sigma(b)$ and $p = p(b, 0)$ in view of [11, Theorem 3.1]. Since b commutes with c and the idempotent p belongs to the double commutant of b, it follows that $b + p \in A^{-1}$ and $c \in \mathcal{N}(T)$ commute, and hence $a + p = (b + p) + c \in \mathcal{B}_T$. This completes the proof. □

Remark 6.5. *The idempotent that works in Propositions 6.3 and 6.4 does not in general belong to the null space of T, so that the results above do not follow immediately.*

The analogues of Propositions 6.3 and 6.4 are given next. They provide necessary conditions for an element to be GB-Weyl and GB-Browder relative to arbitrary Banach algebra homomorphisms.

Proposition 6.6. *Let $T : A \to B$ be a Banach algebra homomorphism.*

(*i*) *If $a \in \mathcal{W}_T^{(GB)}$, then there exists $p \in \mathrm{Idem}(A)$ such that*

$$a + p \in \mathcal{W}_T,\ pa(1_A - p), (1_A - p)ap \in \mathcal{N}(T),\ \text{and } pap \in R_T(A).$$

(*ii*) *If $a \in \mathcal{B}_T^{(GB)}$, then there exists $p \in \mathrm{Idem}(A)$ such that*

$$a + p \in \mathcal{B}_T,\ pa(1_A - p), (1_A - p)ap \in \mathcal{N}(T),\ \text{and } pap \in R_T(A).$$

Proof. Simply replace [16, Proposition 1] in Propositions 6.3 and 6.4 by [11, Lemma 2.4]. □

Do the converses of Propositions 6.3, 6.4 and 6.6[(i) and (ii)] hold (under the assumption that T has the lifting property)?

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