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On q-statistical convergence and statistical solution of q-Cauchy problem

Gülizar Gülenay Zengin[a,c], Hemanta Kalita[b,*], Ekrem Savas[a], Tarini Kumar Dutta[c]

[a]*Department of Mathematics, Usak University, Usak, Turkey*
[b]*Mathematics Division, VIT Bhopal University, Bhopal-Indore Highway, Kothrikalan, Sehore, Madhya Pradesh 466114, India*
[c]*Department of Mathematics, Assam Don Bosco University, Sonapur, Assam, India*

Abstract. In this paper, we present the concept of q-statistical convergence for any sequence of real-valued functions. Several characteristics of q-statistical convergence for sequences of functions with real values are examined. Further, we introduce the notion of q-statistical convergence for sequences of Jackson integrable functions. Finally, we determine the q-statistical solution of q-differential equations that involve non-uniquely solvable Cauchy problems.

1. Introduction

Summability theory addresses the generalization of the limit concept associated with sequences or series, which is usually influenced by an auxiliary sequence of linear means derived from the specified sequences or series.Although the original sequence or series may exhibit divergence, it is essential that the linear mean sequence converges. It is known that Zigmund [33] first proposed the concept of statistical convergence in his well-known work "Trigonometric series" in 1935. The notion was formally established by Fast [2] and Steinhaus. The principle of convergence pertaining to an infinite series was first satisfactorily clarified by the French mathematician A.L Cauchy. Then, Bilalov et al. [5] presented the idea of the statistical convergence in Lebesgue spaces L^p. One can see [11–14] and references therein for several work of statistical convergence sequence of functions. One can see [22, 31] and their references for recent trends of statistical convergence and their related works. Illner et al. [10] put forth the notion of employing a statistical method for differential equations tied to a Cauchy problem that does not yield a unique solution.

Quantum calculus [9, 21, 30], referred to as q-calculus, represents a type of calculus that operates without the concept of limits. Recently, q-calculus has drawn the attention of numerous researchers due to its wide-ranging applications in Mathematics and Physics. Jackson systematically introduced and explored the q-derivative and q-integral [18, 19]. The creation and annihilation operator matrix elements are used by

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* Corresponding author: Hemanta Kalita
Email addresses: glnyzngn@gmail.com (Gülizar Gülenay Zengin), hemanta30kalita@gmail.com (Hemanta Kalita), ekremsavas@yahoo.com (Ekrem Savas), tarini.kumar.dutta@dbuniversity.ac.in (Tarini Kumar Dutta)
ORCID iDs: https://orcid.org/0009-0007-8825-0651 (Gülizar Gülenay Zengin), https://orcid.org/0000-0002-9798-6608 (Hemanta Kalita), https://orcid.org/0000-0003-2135-3094 (Ekrem Savas), https://orcid.org/0000-0002-9798-6608 (Tarini Kumar Dutta)

Kaniadakis et al. [20] to present a generalized exclusion-inclusion principle that is intrinsically related to the quantum q-algebra. The fractional q-integral with the parametric lower limit, Rajkovic et al. [27] use the fractional q-derivative of Caputo type as the integration. In particular, they introduce its applications to q-exponential functions: Mittag-Leffler function q-analogs. In broad approach: Zhang et al. [32] investigate the solution theory pertaining to the Caputo type nonlinear q-fractional differential equation. Ultimately, a successive approximation technique is introduced to determine the analytical approximate solution for this issue. The research conducted by Atici et al. [3] in this document serves as a bridge that connects the established fractional q-calculus present in the literature with the fractional q-calculus formulated within a time scale.

Lately, Bekar et al. [6] introduced the concept of q-density and q-statistical convergence. Mursaleen et al. [23] investigate q-statistical convergence, q-statistical limit points, and q-statistical cluster points. They provide a definition for q-statistical Cauchy and investigate its relationship with q-statistical convergence. Additionally, they introduce two concepts, namely q-strongly Cesàro summable sequences and statistically $C_1^{(q)}$-summable sequences, and illustrate their connection to q-statistical convergence. In their analysis, they consider q-statistical convergence to scrutinize a Korovkin-type approximation finding. Further, Mursaleen et al. [24] investigate q-statistical convergence in double sequences. They offer definitions for the statistical pre-Cauchy and q-analog of statistical Cauchy pertaining to double sequences. Additionally, they identified the necessary and sufficient criteria for a double sequence to exhibit distinct statistical limits. It is illustrated that a q-statistical convergent sequence qualifies as a q-statistical Cauchy sequence, and the opposite is confirmed. Mursaleen et al. [25] introduce a new category of Lupa s-Bernstein operators defined by the shape parameter λ and establish a Korovkin-type approximation theorem. Furthermore, they ascertain the rate of statistical convergence associated with these operators. Additionally, they determine the rate of statistical convergence for these operators. Moreover, authors provide various graphs and numerical illustrations demonstrating the convergence of the newly introduced operators and indicate that in certain scenarios, the errors are smaller than those of the conventional ones. In recent times, q-calculus has been employed in several summability approaches, encompassing both matrix and non-matrix frameworks, including q-Cèsaro matrices, q-Hausdorff summability and q-statistical convergence (see [1, 6, 7] and references therein for details).

Recently, Jena et al. [15] discuss statistical gauge integrable functions. In application, Korovkin-type aproximation theorem is proved. Jena et al. [29] defined statistical Riemann and Lebesgue integrable sequence of functions with Korovkin-type approximation theorems. Jena et al. [16] discuss approximation of Fourier series via a class of product deferred summability mean. Jena et al. [17] introduce equi-statistical convergence of distribution product via deferred Nörlund summability mean. Satapathy et al. [28] find a new class of Korovkin-type approximation theorem based on equi-statistical convergence of double sequence. Parida et al. [26] extend statistical Riemann summability and fuzzy approximation.

The idea of [5] encourage us to introduce q-statistical convergence of sequence of functions. Further, the methods presented in [10] inspired us to seek a q-statistical solution of non-uniquely solvable Cauchy problems in our settings.

The structure of the manuscript is as follows: In Section 1, we recall several definitions, results that are useful to our next section. Additionally, we discuss several properties of sequence of q-integrable functions. In Section 2, we discuss q-statistical convergence sequence of functions and several results related to the same. In Section 3, we present q-statistical convergence of sequence of Jackson integrable or q-integrable functions. We provide the necessary and sufficient condition for the q-statistical convergence of sequences in sense of Jackson integrable functions. In Section 4, we present the q-statistical solution for the non-uniquely solvable Cauchy problem within the framework of q-calculus.

2. Preliminaries

We recall several definitions, and theorems of q-calculus that will be use in our Sections. We denote $\mathbb{N}$ is natural number set and $\mathbb{R}$ is real number set.

Definition 2.1. *[18] Let $0 < q < 1$. The quantum number or q-number of $n \in \mathbb{N}$ is defined by*

$$[n]_{q} = [n] = \begin{cases} \dfrac{1-q^{n}}{1-q} & , n > 0 \\ 1 & , n = 0. \end{cases}$$

One may notice that when $q \to 1$ *then* $[n]_q = n$ *for* $n > 0$.
The q*-analog of binomial coefficient or* q*-binomial coefficient is defined by*

$$\begin{bmatrix} n \\ r \end{bmatrix}_{q} = \begin{bmatrix} n \\ r \end{bmatrix} \begin{cases} \dfrac{[n]_q!}{[n-r]_q![r]_q!} & , n \geq r \\ 0 & , n < r \end{cases}$$

where q*-factorial* $[n]_q!$ *of* n *is given by*

$$[n]_q! = [n]! = \begin{cases} 1 & , \ n = 0 \\ [n][n-1]\cdots[2][1] & , \ n > 0. \end{cases}$$

The q-differential of an arbitrary $\wp$ function is defined by $d_q\wp(x) = \wp(qx) - \wp(x)$. In particular let $d_q x = (q-1)x$. Then the q-derivative of $\wp$ defined by

$$D_q\wp(x) = \frac{d_q\wp(x)}{d_q x} = \frac{\wp(qx) - \wp(x)}{(q-1)x}$$

where $x \neq 0$ and $0 < q < 1$. Note that if $\wp$ is differentiable function, then

$$\lim_{q\to 1} D_q\wp(x) = \lim_{q\to 1} \frac{\wp(qx) - \wp(x)}{(q-1)x} = \frac{x\wp'(x)}{x} = \wp'(x) = \frac{d\wp(x)}{dx}.$$

One can see [4, 7, 8, 18, 20, 21, 31] and references therein for details of q-differential of an arbitrary $\wp$ functions and their related work. The q-analogue of $(a-b)^n$ is defined by

$$(a-b)_q^n = \begin{cases} 1 & , \ n = 0 \\ (a-b)(a-qb)\cdots(a-q^{n-1}b) & , \ n \geq 1 \end{cases}$$

for every $a, b \in \mathbb{R}$. In other saying

$$(a-b)_q^n = \prod_{i=1}^{n-1}(a - q^i b) \quad \textit{and} \quad (a-b)_q^0 = 1, \qquad n \in \mathbb{N}.$$

Recall the q-integral or Jackson integral as follows:

Suppose $\wp(x)$ is an arbitrary function. To construct its q-derivative $\Phi(x)$, recall the operator $M_q(\Phi(x)) = \Phi(qx)$, and

$$\begin{aligned} \frac{1}{(q-1)x}(M_q - 1)\Phi(x) &= \frac{\Phi(qx) - \Phi(x)}{(q-1)x} \\ &= \wp(x). \end{aligned}$$

Since the operator do not commute, we can formulate the q-derivative as

$$\begin{aligned} \Phi(x) &= \frac{1}{1 - M_q}((1-q)x\wp(x)) \\ &= (1-q)\sum_{j=0}^{\infty} M_q^j(x\wp(x)). \end{aligned}$$

2.1. Sequence of $\mathfrak{q}$*-integrable functions and their properties.*

In this Section, we discuss several properties of sequence of $\mathfrak{q}$-integrable sequence of functions. We start the Section with the following definition.

Definition 2.2. *Let* $\wp(x)$ *be an arbitrary function. We called* $\wp(x)$ *to be Jackson integrable (or* $\mathfrak{q}$*-integrable) if for* $\tau > 0$ *there exists* $\epsilon > 0$ *such that*

$$\left|(1-\mathfrak{q})x\sum_{j=0}^{\infty}\mathfrak{q}^j\wp(\mathfrak{q}^jx) - \int \wp(x)d_{\mathfrak{q}}x\right| < \epsilon$$

whenever $\left|(1-\mathfrak{q})x\sum_{j=0}^{\infty}\mathfrak{q}^j\wp(\mathfrak{q}^jx)\right| < \tau$.

It is not hard to find, if $\wp(x)$ is Jackson integrable then this is unique. Linearity and sub-additivity holds for Jackson integrable functions. In this manuscript, we are focusing on several properties of sequence of Jackson integrable functions. For the purpose of furthering our research, we articulate the following.

Definition 2.3. *Let* $\wp_n(x)$ *be a sequence of any functions converges to* $\wp(x)$ *in sense of Jackson integral if* $\lim_{n\to\infty}\wp_n(x) = \wp(x)$ *and* $(1-\mathfrak{q})x\sum_{j=0}^{\infty}\mathfrak{q}^j\wp_n(\mathfrak{q}^jx) = \int\wp(x)d_{\mathfrak{q}}x$ *as* $n\to\infty$.

Definition 2.4. *Let* $\wp_n(x)$ *be a sequence of Jackson integrable functions on* $[a.b] \subset \mathbb{R}$*. We placed a call* $\wp_n(x)$ *converges* $\mathfrak{q}$*-uniformly to the function* $\wp : [a,b] \to \mathbb{R}$ *on* $[a,b]$*, if for any* $\epsilon > 0$ *there exists* $k \in \mathbb{N}$ *such that* $n \geq k$ $\left|\wp_n(x) - \wp(x)\right| < \epsilon$ *for all* $x \in [a,b]$ *with* $(1-\mathfrak{q})x\sum_{j=0}^{\infty}\mathfrak{q}^j\wp_n(\mathfrak{q}^jx) = \int\wp(x)d_{\mathfrak{q}}x$ *as* $n\to\infty$.

Remark 2.5. *The convergence of any* $\mathfrak{q}$*-integrable function in sense of the Definition 2.4 implies the convergence of the Definition 2.3.*

Theorem 2.6. *The sequence* $\wp_n(x)$ *convergence* $\mathfrak{q}$*-uniformly in sense of Jackson integrable if and only if for every* $\epsilon > 0$ *there exists* $k \in \mathbb{N}$ *such that if* $m,n \geq k$ *with* $\left|H_m(x) - H_n(x)\right| < \epsilon$ *where* $H_n(x) = (1-\mathfrak{q})x\sum_{j=0}^{\infty}\mathfrak{q}^j\wp_n(\mathfrak{q}^jx)$.

Proof. Suppose that $\wp_n(x)$ sequence $\mathfrak{q}$-uniformly convergence to $\wp(x)$ in sense of Jackson integral on $[a,b]$. That is for given $\epsilon > 0$ and all $x \in [a,b]$, there exists $k \in \mathbb{N}$ such that for each $n \geq k$ $|H_n(x) - \int\wp(x)d_{\mathfrak{q}}x| < \frac{\epsilon}{2}$. Let $m \geq k$ a number and $N = \max(m,n)$, for every x and $N \geq k$

$$\begin{aligned}\left|H_n(x) - H_m(x)\right| &= \left|H_n(x) - \int\wp(x)d_{\mathfrak{q}}x + \int\wp(x)d_{\mathfrak{q}}x - H_m(x)\right| \\ &\leq \left|H_n(x) - \int\wp(x)d_{\mathfrak{q}}x\right| + \left|H_m(x) - \int\wp(x)d_{\mathfrak{q}}x\right| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

Conversely, suppose that for every $\epsilon > 0$ there exists $k \in \mathbb{N}$ such that if $m,n \geq k$ with $\left|H_m(x) - H_n(x)\right| < \epsilon$ where $H_n(x) = (1-\mathfrak{q})x\sum_{j=0}^{\infty}\mathfrak{q}^j\wp_n(\mathfrak{q}^jx)$ for all $x \in [a,b]$. Then by Cauchy criterion for series the equality $(1-\mathfrak{q})x\sum_{j=0}^{\infty}\mathfrak{q}^j\wp_n(\mathfrak{q}^jx) = \int\wp(x)d_{\mathfrak{q}}x$ as $n\to\infty$ exists for every x. Taking the limit of $|H_n(x) - H_m(x)|$ as $m\to\infty$, we have $\left|H_n(x) - \int\wp(x)d_{\mathfrak{q}}x\right| < \epsilon$ for all $x \in [a,b]$ and $n \geq k$. □

Definition 2.7. *Suppose $\wp_n(x)$ be a sequence of Jackson integrable functions on $[a.b]$. If for every $x \in [a, b]$ and $n \in \mathbb{N}$ there exists a $L \in \mathbb{R}$ such that $\left|(1-\mathfrak{q})x \sum_{j=0}^{\infty} \mathfrak{q}^j \wp_n(\mathfrak{q}^j x)\right| \leq L$ then $\wp_n(x)$ is said to be $\mathfrak{q}$-bounded.*

Definition 2.8. *$\wp_n(x)$ sequence of Jackson integrable functions on $[a.b]$ is called $\mathfrak{q}$-continuous if for every $\epsilon > 0$ and $n \in \mathbb{N}$ there exists a $\tau > 0$ such that for all $x, y \in [a, b]$, $|H_n(x) - H_n(y)| < \epsilon$ whenever $|\wp_n(x) - \wp_m(y)| < \tau$.*

Now we give results involving the relations between uniform convergence, bounded and continuity. Since proofs are directly follows from definitions, we have ommitted their proofs.

Theorem 2.9. *Suppose $\wp_n(x)$ be a sequence of Jackson integrable functions and $\wp_n \to \wp$ $\mathfrak{q}$-uniformly on $[a, b]$. If each $\wp_n$ is bounded on $[a, b]$ then the sequence $\wp_n$ is $\mathfrak{q}$-uniformly bounded on $[a, b]$ and $\wp$ is bounded on $[a, b]$.*

Theorem 2.10. *Let $\wp_n(x)$ be a sequence of Jackson integrable functions on $[a, b]$ converging uniformly to $\wp$ on $[a, b]$. If every $\wp_n$ is $\mathfrak{q}$-continuous on $[a, b]$ then $\wp$ is $\mathfrak{q}$-continuous on $[a, b]$.*

Theorem 2.11. *Let $\wp_n(x)$ be a sequence of Jackson integrable function on $[a, b]$. If $\wp_n$ converges uniformly to $\wp$ on $[a, b]$, then $\wp$ is Jackson integrable and $\lim_{n\to\infty} H_n(x) = \int_a^b \wp(x) d_{\mathfrak{q}}x$.*

Recall the notion of $\mathfrak{q}$-statistical convergence, which is linked to both density and statistical convergence.

Definition 2.12. *[6] Suppose $\mathcal{K} \subseteq \mathbb{N}$ and let $\mathcal{K}_n = \{j : j \leq n, j \in \mathcal{K}\}$. Then the natural density $\partial(\mathcal{K})$ of $\mathcal{K}$ is defined by*

$$\partial(\mathcal{K}) = \lim_{n\to\infty} \frac{|\mathcal{K}_n|}{n} = k$$

where the k is a real number and finite, $|\mathcal{K}_n|$ is the cardinality of $\mathcal{K}_n$.

A given sequence (x_n) is statistically convergent to $\mathcal{L}$, if for each $\epsilon > 0$

$$\mathcal{K}_\epsilon = \{j : j \in \mathbb{N}, |x_j - \mathcal{L}| \geq \epsilon\}$$

has zero natural density. Thus for each $\epsilon > 0$, we have

$$\partial(\mathcal{K}_\epsilon) = \lim_{n\to\infty} \frac{|\mathcal{K}_\epsilon|}{n} = 0$$

Here we write

$$stat\text{-}\lim_{n\to\infty}(x_n) = \mathcal{L}.$$

Definition 2.13. *[1] A sequence (x_n) is called $\mathfrak{q}$-statistically convergent to $\mathcal{L}$ number, if for every $\epsilon > 0$ and the $\mathfrak{q}$-density of the set $\mathcal{K}_\epsilon = \{k : k \in \mathbb{N}, |x_k - \mathcal{L}| \geq \epsilon\}$*

$$\partial_{\mathfrak{q}}(\mathcal{K}_\epsilon) = \lim_{n\to\infty} \frac{1}{[n]_{\mathfrak{q}}} \left| \{k \leq n : \mathfrak{q}^k |x_k - \mathcal{L}| \geq \epsilon\} \right| = 0$$

and we write

$$st_{\mathfrak{q}}\text{-}\lim_{n\to\infty}(x_n) = \mathcal{L}.$$

One should recall that the statistical convergence of a sequence corresponds to $\mathfrak{q}$-statistical convergence, yet the reverse is not valid. This indicates that $\mathfrak{q}$-statistical convergence encompasses a wider range than statistical convergence. Consequently, we were motivated to extend the framework of $\mathfrak{q}$-statistical convergence from sequences of real numbers to sequences of functions that demonstrate $\mathfrak{q}$-statistical convergence.

3. $\mathfrak{q}$-Statistical Convergence sequence of functions

In this section, we introduce the notion of $\mathfrak{q}$-statistical convergence for sequences of functions. We explore various properties and results pertaining to $\mathfrak{q}$-statistical convergence of function sequences. Additionally, we demonstrate that every sequence of functions that converges statistically is also $\mathfrak{q}$-statistically convergent; the opposite does not hold. Below, we provide a definition for the $\mathfrak{q}$-statistical convergence of sequences of functions.

Definition 3.1. *A sequence $(\wp_n)$ of function is called $\mathfrak{q}$-statistically convergent to a function $\wp$ if for every $\epsilon > 0$, $\mathfrak{q}$-density of the set $\mathcal{K}_\epsilon = \left\{k : k \in \mathbb{N} \text{ and } |\wp_k - \wp| \geq \epsilon\right\}$ is zero i.e. $\delta_{\mathfrak{q}}(\mathcal{K}_\epsilon) = \lim_{n\to\infty} \frac{1}{[n]_{\mathfrak{q}}} \left|\left\{k \leq n : \mathfrak{q}^k |\wp_k - \wp| \geq \epsilon\right\}\right| = 0.$*

We denote $\mathfrak{q}$-statistical convergence of $\wp_n \to \wp$ by $st_{\mathfrak{q}}\text{-}\lim_{n\to\infty} \wp_n(x) = \wp(x)$.

Lemma 3.2. *The Definition 3.1 is hold if and only if there exists a set $\mathcal{K} = \{k_1 < k_2 < \cdots < k_n < \cdots\} \subset \mathbb{N}$ with $\delta_{\mathfrak{q}}(\mathcal{K}) = 1$ and $\lim_{n\to\infty} \wp_{k_n} = \wp$.*

Proof. The proof is an analogous of [23, Theorem 1]. □

The following properties are directly follows from the Definition 3.1.

Theorem 3.3. *If $st_{\mathfrak{q}}\text{-}\lim_{n\to\infty} \wp_n(x) = \wp(x)$, $st_{\mathfrak{q}}\text{-}\lim_{n\to\infty} \Im_n(x) = \Im(x)$ and λ is any real number, then*

1. $st_{\mathfrak{q}}\text{-}\lim_{n\to\infty}(\wp_n(x) + \Im_n(x)) = \wp(x) + \Im(x)$.
2. $st_{\mathfrak{q}}\text{-}\lim_{n\to\infty}(\lambda\wp_n(x)) = \lambda\wp(x)$.

According to Theorem 3.3, the collection of all bounded $\mathfrak{q}$-statistically convergent sequences of real functions constitutes a linear subspace within the linear normed space $\mathcal{G}$ of all $\mathfrak{q}$-bounded sequences of real functions, where the norm is defined as $\|\wp\| = \sup_x |\wp(x)|$.

Theorem 3.4. *Let $\mathcal{G}_0$ the set of all $\mathfrak{q}$-statistically bounded convergence sequence of real functions. Then the set $\mathcal{G}_0$ is a closed linear subspace of the linear normed space $\mathcal{G}$.*

Proof. Let $\wp_n \in \mathcal{G}_0$ $(n = 1, 2, \cdots)$ and $\wp_n \to \wp \in \mathcal{G}$ in sense of $\mathfrak{q}$-statistically. In order to prove $\mathcal{G}_0$ be a closed set, it is enough to prove that $\wp_n \to \wp \in \mathcal{G}_0$. According to the assumption for each n there exists a real sequence of functions $\Im_n$ such that $\wp_n \to \Im_n$ hold for $n = 1, 2, \cdots$. That is if $\wp_n = \left\{\xi_k^n\right\}_{k=1}^\infty$ then $st_{\mathfrak{q}}\text{-}\xi_k^n = \Im_n$ for $n = 1, 2, \cdots$. In order to complete our proof we need to establish the following facts:

1. The sequence $(\wp_n)_{n=1}^\infty$ of real functions converges to a real function $\wp$;
2. $\wp_n \to \wp$ in sense of $\mathfrak{q}$-statistical.

For (1) : Since $(\wp_n)_{n=1}^\infty$ is convergent sequence of real functions from $\mathcal{G}$, for $\epsilon > 0$ there exists a $n_0 \in \mathbb{N}$ that for every $j, n \geq n_0$, we have

$$\|\wp_j - \wp_n\| < \frac{\epsilon}{3}. \tag{1}$$

By Lemma 3.2, there exist such sets $A_j, A_n, A_i, A_n \subset \mathbb{N}$ that

$$\lim_{k\to\infty} \wp_k^j = \wp_j, \text{ where } k \in A_j \tag{2}$$

and

$$\lim_{k\to\infty} \wp_k^n = \wp_n, \text{ where } k \in A_n. \tag{3}$$

Since $\mathfrak{q}$-density of $A_j \cap A_n$ is one, so it is clear that $A_j \cap A_n$ is a infinite set. We can choose such a $k \in A_j \cap A_n$ that have

$$|\wp_k^j - \wp_j| < \frac{\epsilon}{3} \text{ and } |\wp_k^n - \wp_n| < \frac{\epsilon}{3}. \tag{4}$$

The Eqn. 1 and Eqn. 4 gives for each $j, n > n_0$

$$|\wp_j - \wp_n| < \epsilon. \tag{5}$$

Clearly, $(\wp_j)$ is a Cauchy sequence of functions. So it is converges to a real function $\wp$ i.e. $\lim_{k\to\infty} \wp_k = \wp$.

For (2) : Let $\eta > 0$. It suffices to demonstrate the existence of a set $A \subset \mathbb{N}$ with $\delta_{\mathfrak{q}}(A) = 1$ and for each $k \in \mathbb{N}$, $|\wp_k - \wp| < \eta$. Since $\wp_j \to \wp$, there exists a number $p \in \mathbb{N}$ such that

$$\|\wp_p - \wp\| < \frac{\eta}{3}. \tag{6}$$

Next, the number p can be choosen in such a way that together with the Eqn. 5 also the inequality

$$|\wp_p - \wp| < \frac{\eta}{3} \tag{7}$$

holds. Since $\wp_p \to \Im_p$ as $\mathfrak{q}$-statistically, there exists a set $A \subset \mathbb{N}$ with $\delta_{\mathfrak{q}}(A) = 1$ and for each $k \in A$ we have

$$|\wp_k^p - \Im_p| < \frac{\eta}{3}. \tag{8}$$

Clearly, by Eqn. 6, Eqn. 7, and Eqn. 8 we can find for each $k \in A$, $|\wp_k - \Im_m| < \eta$. Hence the set $\mathcal{G}_0$ is a closed linear subspace of the linear normed space $\mathcal{G}$. □

Corollary 3.5. *The set $\mathcal{G}_0$ is a nowhere dense set in $\mathcal{G}$.*

Theorem 3.6. *Every statistical convergence sequence of functions are $\mathfrak{q}$-statistical convergence. The opposite might not be true.*

Proof. The proof is similar to [6, Theorem 3.2.2]. □

Example 3.7. *Let $(\wp_n)$ be a sequence of functions defined by $\wp_n = \frac{1+(-1)^{[log_2^n]}}{2}$ on the set $\mathcal{K} = \{k \in \mathbb{N} : \wp_n(x) = 1\}$. Clearly $\delta(\mathcal{K})$ does not exist. Hence the sequence of the function is not statistical convergence. In the other hand $\lim_{n\to\infty} \frac{1}{[n]_{\mathfrak{q}}} \left| \{k \leq n : \mathfrak{q}^k \left| \frac{1+(-1)^{[log_2^k]}}{2} - 0 \right| \} \right| = 0$. Thus $\wp_n$ is $\mathfrak{q}$-statistical convergence and converges to zero.*

Definition 3.8. *We say $(\wp_n)$ to be $\mathfrak{q}$-statistical uniformly converges to $\wp$ on $M \subseteq \mathbb{R}$ if*
$st_{\mathfrak{q}}\text{-}\lim_{n\to\infty} \frac{1}{[n]_{\mathfrak{q}}} \sup_{x\in M} \mathfrak{q}^k \left| \wp_n(x) - \wp(x) \right| = 0.$

It is clear that $\mathfrak{q}$-statistical uniformly convergence of $\wp_n$ implies $\mathfrak{q}$-statistical convergence of $\wp_n$. If the Definition 3.8 true, then the following theorem is true.

Theorem 3.9. *Let $(\wp_n) \subset C[a, b]$ and $\mathfrak{q}$-statistically uniformly convergent which is converges to $\wp$ on $[a, b]$, then $\wp$ is in $C[a, b]$ and $st_{\mathfrak{q}}\text{-}\lim_{n\to\infty} \int_a^b \wp_n(x) d_{\mathfrak{q}}x = \int_a^b \wp(x) d_{\mathfrak{q}}x$.*

Proof. Let $(\wp_n) \subset C[a, b]$ and $\mathfrak{q}$-statistically uniformly convergent and converges to $\wp$ on $M \subset \mathbb{R}$. Let $\Upsilon_n = \frac{1}{[n]_{\mathfrak{q}}} \sup_{x\in M} \mathfrak{q}^k \left| \wp_n(x) - \wp(x) \right|$ $\forall$ $n \in \mathbb{N}$ and $k \leq n$. Clearly, $st_{\mathfrak{q}}\text{-}\lim_{n\to\infty} \Upsilon_n = 0$ and there exists $\mathcal{K} \equiv (n_r)_{r\in\mathbb{N}} : n_1 < n_2 < \cdots$, $\delta_{\mathfrak{q}}(\mathcal{K}) = 1$ and $\lim_{r\to\infty} \Upsilon_{n_r} = 0$. Thus if $(\wp_n)$ is continuous on M and $\wp_n \to \wp$ which is $\mathfrak{q}$-statistically uniformly on M, then $\wp$ is also continuous on M. Moreover for $M = [a, b]$ we have $st_{\mathfrak{q}}\text{-}\lim_{n\to\infty} \int_a^b \wp_n(x) d_{\mathfrak{q}}x = \int_a^b \wp(x) d_{\mathfrak{q}}x$. □

4. $\mathfrak{q}$-statistical convergence of sequence of Jackson integrable functions

In this section, we discuss the $\mathfrak{q}$-statistical convergence of sequences of Jackson integrable functions, utilizing the definition of $\mathfrak{q}$-statistical convergence. In addition, we outline the necessary and sufficient condition for the $\mathfrak{q}$-statistical convergence of sequences regarding Jackson integrable functions.

Definition 4.1. *A sequence of function $\wp_n(x)$ is called $\mathfrak{q}$-statistically convergent to $\wp$ in the sense of $\mathfrak{q}$-integral or Jackson integral if for every $\epsilon > 0$, $\partial_{\mathfrak{q}}(\mathcal{K}_\epsilon) = 0$, where $\mathcal{K}_\epsilon = \left\{k : \left|(1-\mathfrak{q})x \sum_{j=0}^{\infty} \mathfrak{q}^j \wp_k(\mathfrak{q}^j x) - \int \wp(x) d_{\mathfrak{q}} x\right| \geq \epsilon\right\}$.*

We write

$$st_{\mathfrak{q}}\text{-} \lim_{n\to\infty} (\wp_n)(x) = \wp(x)$$

and

$$\partial_{\mathfrak{q}}(\mathcal{K}_\epsilon) = \lim_{n\to\infty} \frac{1}{[n]_{\mathfrak{q}}} \left| \left\{ k : \ \mathfrak{q}^k \left| (1-\mathfrak{q})x \sum_{j=0}^{\infty} \mathfrak{q}^j \wp_k(\mathfrak{q}^j x) - \int \wp(x) d_{\mathfrak{q}} x \right| \geq \epsilon \right\} \right|.$$

Example 4.2. *Let a sequence of functions $\{\phi_n\}_{n=1}^{\infty}$ is defined by $\phi_n = \frac{1-x^n}{n}$ for $x \in [0,1]$. The sequence $\mathfrak{q}$-statistically convergent to $\phi(x) = 0$ for $x \in [0,1]$ in sense of Jackson integral.*

Proof. Let $\epsilon > 0$

$$\begin{aligned}
\mathcal{K}_\epsilon &= \left\{k : \left|(1-\mathfrak{q}) \sum_{j=0}^{\infty} \mathfrak{q}^j \phi_k(\mathfrak{q}^j x) - \int \phi(x) d_{\mathfrak{q}} x\right| \geq \epsilon\right\} \\
&= \left\{k : \left|(1-\mathfrak{q}) \sum_{j=0}^{\infty} \mathfrak{q}^j \frac{1-(\mathfrak{q}^j x)^k}{k} - \int 0 d_{\mathfrak{q}} x\right| \geq \epsilon\right\} \\
&= \left\{k : \left|(1-\mathfrak{q}) \sum_{j=0}^{\infty} \mathfrak{q}^j \frac{1-(\mathfrak{q}^j x)^k}{k}\right| \geq \epsilon\right\}
\end{aligned}$$

for every $x \in [0,1]$ $\mathcal{K}_\epsilon$ is a null set. So,

$$\lim_{n\to\infty} \frac{1}{[n]_{\mathfrak{q}}} \left| \left\{ k : \ \mathfrak{q}^k \left| (1-\mathfrak{q}) \sum_{j=0}^{\infty} \mathfrak{q}^j \frac{1-(\mathfrak{q}^j x)^k}{k} \right| \geq \epsilon \right\} \right| = 0.$$

Therefore $\{\phi_n\}$ is $\mathfrak{q}$-statistically convergent to ϕ in sense of Jackson integral. □

Theorem 4.3. *Every $\mathfrak{q}$-integrable convergent sequence of functions is $\mathfrak{q}$-statistically convergent in sense of Jackson integral.*

Proof. Let $\wp_n(x)$ sequence of functions converges to $\wp(x)$ in sense of $\mathfrak{q}$-integral. Then we have $\lim \lim_{n\to\infty} (\wp_n)(x) = \wp(x)$. For any arbitrary $\epsilon > 0$, we write $\mathcal{K}_\epsilon = \left\{k : \left|(1-\mathfrak{q})x \sum_{j=0}^{\infty} \mathfrak{q}^j \wp_k(\mathfrak{q}^j x) - \int \wp(x) d_{\mathfrak{q}} x\right| \geq \epsilon\right\}$. Since $\mathfrak{q}$-integrable $\wp_n(x)$ convergent to $\wp(x)$,

$$\partial_{\mathfrak{q}}(\mathcal{K}_\epsilon) = \lim_{n\to\infty} \frac{1}{[n]_{\mathfrak{q}}} \left| \left\{ k : \ \mathfrak{q}^k \left| (1-\mathfrak{q})x \sum_{j=0}^{\infty} \mathfrak{q}^j \wp_k(\mathfrak{q}^j x) - \int \wp(x) d_{\mathfrak{q}} x \right| \geq \epsilon \right\} \right| = 0.$$

Therefore $(\wp_n)$ is $\mathfrak{q}$-statistically convergent to $\wp$ in sense of Jackson integral. □

We shall examine the uniqueness of $\mathfrak{q}$-statistically convergent in sense of Jackson integral in the following theorem.

Theorem 4.4. *If a sequence $(\wp_n)$ of functions is $\mathfrak{q}$-statistically convergent to $\wp$ in sense of $\mathfrak{q}$-integral, then $\wp$ is unique.*

Proof. Suppose that $\wp$ and $\flat$ are $\mathfrak{q}$-integrable functions and for each $\epsilon > 0$ the sequence $(\wp_n)$ be $\mathfrak{q}$-statistically convergent to $\wp$ and $\flat$ functions in sense of $\mathfrak{q}$-integral. Now we write

$$\begin{aligned}\left|\int \wp d_{\mathfrak{q}}x - \int \flat d_{\mathfrak{q}}x\right| &= \left|\int \wp d_{\mathfrak{q}}x + (1-\mathfrak{q})x\sum_{j=0}^{\infty} q^j \wp_k(\mathfrak{q}^j x) - (1-\mathfrak{q})x\sum_{j=0}^{\infty} q^j \wp_k(\mathfrak{q}^j x) - \int \flat d_{\mathfrak{q}}x\right| \\ &\leq \left|(1-\mathfrak{q})x\sum_{j=0}^{\infty} q^j \wp_k(\mathfrak{q}^j x) - \int \wp d_{\mathfrak{q}}x\right| + \left|(1-\mathfrak{q})x\sum_{j=0}^{\infty} q^j \wp_k(\mathfrak{q}^j x) - \int \flat d_{\mathfrak{q}}x\right| \\ &\to 0\end{aligned}$$

Since $st_{\mathfrak{q}}\text{-}\lim_{n\to\infty}(\wp_n)(x) = \wp(x)$ and $st_{\mathfrak{q}}\text{-}\lim_{n\to\infty}(\wp_n)(x) = \flat(x)$, we get $\wp = \flat$. □

Theorem 4.5. *Let $(\wp_n)$ and $(\flat_n)$ are sequence of functions that are $\mathfrak{q}$-statistically convergent to $\wp$ and $\flat$ functions respectively in sense of $\mathfrak{q}$-integral.*

1. $st_{\mathfrak{q}}\text{-}\lim_{n\to\infty}(c\wp_n)(x) = \wp(x)$, *where c is nonzero scalar.*
2. $st_{\mathfrak{q}}\text{-}\lim_{n\to\infty}(\wp_n + \flat_n)(x) = \wp(x) + \flat(x)$.

The proofs of linearity properties given above are directly follows from the definition.

Now we present the $\mathfrak{q}$-statistically Cauchy sequence for sequence of Jackson integrable functions that is closely relation with $\mathfrak{q}$-statistically convergent.

Definition 4.6. *A sequence $(\wp_n)$ of functions is called $\mathfrak{q}$-statistically Cauchy sequence in sense of Jackson integral if for any $\epsilon > 0$ there exists a N natural number such that the set $\partial_{\mathfrak{q}}(\{k : |H_K - H_N| \geq \epsilon\}) = 0$ where $H_K := (1-\mathfrak{q})x\sum_{j=0}^{\infty} \mathfrak{q}^j \wp_k(\mathfrak{q}^j x)$.*

Theorem 4.7. *Every sequence of $\mathfrak{q}$-statistically convergent functions in sense of Jackson integral is $\mathfrak{q}$-statistically Cauchy sequence.*

Proof. Let $\wp_n$ is $\mathfrak{q}$-statistically convergent to $\wp$ function in sense of Jackson integral. Thus for each $\epsilon > 0$, $\partial_{\mathfrak{q}}(\mathcal{K}_\epsilon) = 0$ where $\mathcal{K}_\epsilon = \left\{k : \left|H_K - \int \wp(x)d_{\mathfrak{q}}x\right| \geq \epsilon\right\}$ and $H_K := (1-\mathfrak{q})x\sum_{j=0}^{\infty} \mathfrak{q}^j \wp_k(\mathfrak{q}^j x)$. Now we choose a $n < k$ number such that $\partial_{\mathfrak{q}}(\mathcal{N}_\epsilon) = 0$ where $\mathcal{N}_\epsilon = \left\{n : \left|H_N - \int \wp(x)d_{\mathfrak{q}}x\right| \geq \epsilon\right\}$ and $H_N := (1-\mathfrak{q})x\sum_{j=0}^{\infty} \mathfrak{q}^j \wp_n(\mathfrak{q}^j x)$. Since the density has subadditive property

$$\partial_{\mathfrak{q}}(\mathcal{K}_\epsilon \cup \mathcal{N}_\epsilon) = \partial_{\mathfrak{q}}(\{k : |H_K - H_N| \geq \epsilon\}) \leq \partial_{\mathfrak{q}}(\mathcal{K}_\epsilon) + \partial_{\mathfrak{q}}(\mathcal{N}_\epsilon) = 0.$$

Therefore the sequence $\wp_n$ is $\mathfrak{q}$-statistically Cauchy sequence in sense of Jackson integral. □

We shall introduce $\mathfrak{q}$-statistically bounded sequence of Jackson integrable functions.

Definition 4.8. *A sequence $(\wp_n)$ of functions is said to be $\mathfrak{q}$-statistically bounded in sense of Jackson integral if there exist a $L > 0$ such that the set $\{k : |(1-\mathfrak{q})x\sum_{j=0}^{\infty} \mathfrak{q}^j \wp_k(\mathfrak{q}^j x)| \geq L\}$ has zero $\mathfrak{q}$-density.*

Theorem 4.9. *Every $\mathfrak{q}$-statistically convergent sequence of Jackson integrable functions is $\mathfrak{q}$-statistically bounded in sense of Jackson integral.*

Proof. Suppose that $(\wp_n)$ is a $\mathfrak{q}$-statistically convergent sequence of Jackson integrable functions. Let $\epsilon > 0$ and $(\wp_n)$ be unbounded. Then for the set $\mathcal{K}_\epsilon = \left\{k : \left|(1-\mathfrak{q})x \sum_{j=0}^{\infty} \mathfrak{q}^j \wp_k(\mathfrak{q}^j x) - \int \wp(x) d_{\mathfrak{q}} x\right| \geq \epsilon\right\}$,

$$\partial_{\mathfrak{q}}(\mathcal{K}_\epsilon) = \lim_{n\to\infty} \frac{1}{[n]_{\mathfrak{q}}} \left|\left\{k : \ \mathfrak{q}^k \left|(1-\mathfrak{q})x \sum_{j=0}^{\infty} \mathfrak{q}^j \wp_k(\mathfrak{q}^j x) - \int \wp(x) d_{\mathfrak{q}} x\right| \geq \epsilon\right\}\right| = 0.$$

This implies,

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{[n]_{\mathfrak{q}}} \left|\left\{k : \ \mathfrak{q}^k \left|(1-\mathfrak{q})x \sum_{j=0}^{\infty} \mathfrak{q}^j \wp_k(\mathfrak{q}^j x)\right| \geq \mathfrak{q}^k \left|\int \wp(x) d_{\mathfrak{q}} x\right| - \epsilon\right\}\right| \\
&\leq \lim_{n\to\infty} \frac{1}{[n]_{\mathfrak{q}}} \left|\left\{k : \ \mathfrak{q}^k \left|(1-\mathfrak{q})x \sum_{m=0}^{p} \mathfrak{q}^m \wp_k(\mathfrak{q}^m x)\right| \geq \mathfrak{q}^k \left|\int \wp(x) d_{\mathfrak{q}} x\right| - \epsilon\right\}\right| \\
&+ \lim_{n\to\infty} \frac{1}{[n]_{\mathfrak{q}}} \left|\left\{k : \ \mathfrak{q}^k \left|(1-\mathfrak{q})x \sum_{r\neq m}^{s} \mathfrak{q}^r \wp_k(\mathfrak{q}^r x)\right| \geq \mathfrak{q}^k \left|\int \wp(x) d_{\mathfrak{q}} x\right| - \epsilon\right\}\right|.
\end{aligned}$$

Therefore $\lim_{n\to\infty} \frac{1}{[n]_{\mathfrak{q}}} \left|\left\{k : \ \mathfrak{q}^k \left|(1-\mathfrak{q})x \sum_{j=0}^{\infty} \mathfrak{q}^j \wp_k(\mathfrak{q}^j x) - \int \wp(x) d_{\mathfrak{q}} x\right| \geq \epsilon\right\}\right| \neq 0$. This is a contradiction of $\mathfrak{q}$-statistically convergence. So, we conclude that $(\wp_n)$ is bounded sequence in sense of $\mathfrak{q}$-integral. □

Theorem 4.10. *A sequence $(\wp_n)$ of Jackson integrable functions is $\mathfrak{q}$-statistically convergent in sense of $\mathfrak{q}$-integral if and only if for every $\epsilon > 0$ the following condition is satisfied:*
If for the set

$$\mathcal{K}_\epsilon = \{k, m : \ |(1-\mathfrak{q})x \sum_{j=0}^{\infty} \mathfrak{q}^j \wp_k(\mathfrak{q}^j x) - (1-\mathfrak{q})x \sum_{j=0}^{\infty} \mathfrak{q}^j \wp_m(\mathfrak{q}^j x)| \geq \epsilon\}$$

has zero $\mathfrak{q}$-density that is $\partial_{\mathfrak{q}}(\mathcal{K}_\epsilon) = 0$ whenever $(\wp_m)$ is convergence subsequence of $(\wp_k)$.

Proof. Let $(\wp_n)$ sequence of Jackson integrable functions is $\mathfrak{q}$-statistically convergent to $\wp$ in sense $\mathfrak{q}$-integral. Then for every $\epsilon > 0$

$$\lim_{n\to\infty} \frac{1}{[n]_{\mathfrak{q}}} \left|\left\{k : \mathfrak{q}^k \left|H_K - \int \wp(x) d_{\mathfrak{q}} x\right| \geq \epsilon\right\}\right| = 0$$

where $H_K = (1-\mathfrak{q})x \sum_{j=0}^{\infty} \mathfrak{q}^j \wp_k(\mathfrak{q}^j x)$. Let H_M be a convergence subsequence of H_K then,

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{[n]_{\mathfrak{q}}} \left|\left\{k, m : \mathfrak{q}^k \left|H_K - H_M\right| \geq \epsilon\right\}\right| \\
&= \lim_{n\to\infty} \frac{1}{[n]_{\mathfrak{q}}} \left|\left\{k, m : \mathfrak{q}^k \left|H_K - \int \wp(x) d_{\mathfrak{q}} x + \int \wp(x) d_{\mathfrak{q}} x - H_M\right| \geq \epsilon\right\}\right| \\
&\leq \lim_{n\to\infty} \frac{1}{[n]_{\mathfrak{q}}} \left|\left\{k, m : \mathfrak{q}^k \left|H_K - \int \wp(x) d_{\mathfrak{q}} x\right| \geq \epsilon\right\}\right| + \lim_{n\to\infty} \frac{1}{[n]_{\mathfrak{q}}} \left|\left\{k, m : \mathfrak{q}^k \left|H_M - \int \wp(x) d_{\mathfrak{q}} x\right| \geq \epsilon\right\}\right| \\
&= \lim_{n\to\infty} \frac{1}{[n]_{\mathfrak{q}}} \left|\left\{k, m : \mathfrak{q}^k \left|H_M - \int \wp(x) d_{\mathfrak{q}} x\right| \geq \epsilon\right\}\right|.
\end{aligned}$$

Since $\lim_{m\to\infty} \wp_m = \wp$, $(\wp_m)$ is convergent to $\wp$. Hence $\lim_{n\to\infty} \frac{1}{[n]_{\mathfrak{q}}} \left|\left\{k : \mathfrak{q}^m \left|H_M - \int \wp(x) d_{\mathfrak{q}} x\right| \geq \epsilon\right\}\right| = 0$. Therefore, we conclude that $\lim_{n\to\infty} \frac{1}{[n]_{\mathfrak{q}}} \left|\left\{k, m : \mathfrak{q}^k \left|H_K - H_M\right| \geq \epsilon\right\}\right| = 0$.

Conversely, suppose $\lim_{n\to\infty}\frac{1}{[n]_{\mathfrak{q}}}\left|\left\{k,m : \mathfrak{q}^k\left|H_K - H_M\right| \geq \epsilon\right\}\right| = 0$ for $(\wp_m)$ convergence subsequence of $(\wp_k)$ such that $\lim \wp_m = \wp$. Now we get

$$\begin{aligned}
&\lim_{n\to\infty}\frac{1}{[n]_{\mathfrak{q}}}\left|\left\{k : \mathfrak{q}^k\left|H_K - \int \wp(x)d_{\mathfrak{q}}x\right| \geq \epsilon\right\}\right| \\
&= \lim_{n\to\infty}\frac{1}{[n]_{\mathfrak{q}}}\left|\left\{k,m : \mathfrak{q}^k\left|H_K - H_M + H_M - \int \wp(x)d_{\mathfrak{q}}x\right| \geq \epsilon\right\}\right| \\
&\leq \lim_{n\to\infty}\frac{1}{[n]_{\mathfrak{q}}}\left|\left\{k,m : \mathfrak{q}^k\left|H_K - H_M\right| \geq \epsilon\right\}\right| + \lim_{n\to\infty}\frac{1}{[n]_{\mathfrak{q}}}\left|\left\{k,m : \mathfrak{q}^k\left|H_M - \int \wp(x)d_{\mathfrak{q}}x\right| \geq \epsilon\right\}\right| \\
&= \lim_{n\to\infty}\frac{1}{[n]_{\mathfrak{q}}}\left|\left\{k,m : \mathfrak{q}^k\left|H_M - \int \wp(x)d_{\mathfrak{q}}x\right| \geq \epsilon\right\}\right|.
\end{aligned}$$

Since $\lim \wp_m = \wp$, $(\wp_m)$ is $\mathfrak{q}$-statistically convergent to $\wp$. Therefore $\lim_{n\to\infty}\frac{1}{[n]_{\mathfrak{q}}}\left|\left\{k,m : \mathfrak{q}^k\left|H_M - \int \wp(x)d_{\mathfrak{q}}x\right| \geq \epsilon\right\}\right| = 0$. Consequently, $\lim_{n\to\infty}\frac{1}{[n]_{\mathfrak{q}}}\left|\left\{k : \mathfrak{q}^k\left|H_K - \int \wp(x)d_{\mathfrak{q}}x\right| \geq \epsilon\right\}\right| = 0$. This conclusion that H_K is $\mathfrak{q}$-statistically convergent in sense of $\mathfrak{q}$-integral. □

5. $\mathfrak{q}$-statistical solution of non-uniquely solvable Cauchy problem

In this Section of the manuscript, we find $\mathfrak{q}$-statistical solution of non-uniquely solvable Cauchy problem in the settings of $\mathfrak{q}$-Calculus. In order to develop the solution concept, we put forward several assumptions regarding the fundamental initial value problems. We consider $\mathfrak{q}$-Cauchy problem

$$D_{\mathfrak{q}}\mathfrak{x} = \wp(\mathfrak{x}),\ \mathfrak{x}(0) = \mathfrak{x}_0 \tag{9}$$

allows for a globally unique solution for every $\mathfrak{x}_0 \in \mathbb{R}$. The function $\wp$ satisfies the global Lipschitz condition, and there exists a family of $\mathfrak{q}$-diffeomorphisms $S(t) : \mathbb{R} \to \mathbb{R}$, $t \in [0, \infty)$ such that $S(t)\mathfrak{x}_0$ represents the solution to Eqn. 9. Let $S(0) = d$ and let μ_0 represent any probability measure on $\mathbb{R}$. Let

$$\mu_t(A) = \mu_0(S(t)^{-1}A) \tag{10}$$

for all Borel sets A in $\mathbb{R}$. Then $\mu_y(S(t)A) = \mu_0(A)$ signifies that μ_t is the measure generated by μ_0 as influenced by $S(t)$.

Let w belong to $C_0^1([0,\infty) \times \mathbb{R})$ be an arbitrary test function. The use of the transformation theorem for integrals alongside $\mathfrak{q}$-differentiation produces

$$\begin{aligned}
d_{\mathfrak{q}}\int w(t,\mathfrak{x})d_{\mathfrak{q}}\mu_t(\mathfrak{x}) &= d_{\mathfrak{q}}\int w(t, S(t)\mathfrak{x})d_{\mathfrak{q}}\mu_0(\mathfrak{x}) \\
&= \int [w_t(t, S(t)\mathfrak{x}) + \wp(S(t)\mathfrak{x}).w_{\mathfrak{x}}(t, S(t)\mathfrak{x})]d_{\mathfrak{q}}\mu_0(\mathfrak{x}) \\
&= \int [w_t + \wp(\mathfrak{x})w_{\mathfrak{x}}(t,\mathfrak{x})d_{\mathfrak{q}}\mu_t(\mathfrak{x})
\end{aligned}$$

and integration from 0 to ∞, we get

$$\int_0^\infty \int [w_{\mathfrak{x}} + \wp(t)w_{\mathfrak{x}}]d_{\mathfrak{q}}\mu_t(\mathfrak{x})d_{\mathfrak{q}}t + \int w(0,\mathfrak{x})d_{\mathfrak{q}}\mu_0(\mathfrak{x}) = 0. \tag{11}$$

Remark 5.1. *It can be seen that $S(t)$ does not contain explicitly in the Eqn. 11. Hence Eqn. 11 differemt from the Eqn. 10, and makes sese even if Eqn. 9 is not uniquely solvable for all $\mathfrak{x}_0$.*

In analog approach of [10], we shall investigate the $\mathfrak{q}$-statistical solution of $D_{\mathfrak{q}}\mathfrak{x} = \wp(\mathfrak{x})$. In this situation if can claim a measure-valued mapping will be $\mathfrak{q}$-statistical solution of $D_{\mathfrak{q}}\mathfrak{x} = \wp(\mathfrak{x})$ then our $\mathfrak{q}$-statistical solution will be as follows.

Definition 5.2. *We define a measure-valued mapping* $[0,\infty) \to M_y$, $t \to u_t$, *as a* $\mathfrak{q}$-*statistical solution of* $D_{\mathfrak{q}}\mathfrak{x} = \wp(\mathfrak{x})$ *with the initial value* u_0 *if Equation 11 is satisfied for every* $w \in C_0^1([0,\infty)\times\mathbb{R})$, *where* M_t *represents the collection of probability measures on* $\mathbb{R}$.

In order to construct $\mathfrak{q}$-statistical solution, we consider the following axioms:

1. $\wp : \mathbb{R} \to \mathbb{R}^+$ is $\mathfrak{q}$-continuous;
2. $\wp : \mathbb{R} \setminus \{0\} \to \mathbb{R}^+$ is $\mathfrak{q}$-Lipschitz continuous;
3. The only stationary point of the equation $D_{\mathfrak{q}}\mathfrak{x} = \wp(\mathfrak{x})$ is 0;
4. The equation $D_{\mathfrak{q}}\mathfrak{x} = \wp(\mathfrak{x})$, $\mathfrak{x}(0) = \mathfrak{x}_0$ possesses a positive solution given by $\mathfrak{x}(t) = T^+_{t0}0$, $t \in (0,t^*)$ where $(0,t^*)$ represents the maximal existence interval in $\mathbb{R}^+$;
5. for each solution $T^-_{t0}a$ of $D_{\mathfrak{q}}\mathfrak{x} = \wp(\mathfrak{x})$, $\mathfrak{x}(0) = a < 0$, there exists a finite time $\tau(a)$ such that $T^-_{\tau(a)}a = 0$ and $T^-_{t0}a < 0$ for the interval $0 \le t \le \tau(a)$.

Furthermore, let us consider the mapping $\tau : (-\infty,0) \to (0,\infty)$ which is bijective, and assume that τ is $\mathfrak{q}$-differentiable. Let μ_0 be $\mathfrak{q}$-uniformly continuous with $\mathfrak{q}$-density $\delta_{\mathfrak{q}} \in \mathfrak{q}\text{-}L^1_+$, and let $\int \delta_{\mathfrak{q}}(\mathfrak{x})d_{\mathfrak{q}}\mathfrak{x} = 1$. In order to examine "not stopping in 0 solution of $D_{\mathfrak{q}}\mathfrak{x} = \wp(\mathfrak{x})$, $\mathfrak{x}(0) = a < 0$, We define $\delta_0(\mathfrak{x}) = u(\mathfrak{x}) + v(\mathfrak{x})$, where $u(\mathfrak{x}) = 0$ for all $\mathfrak{x} > 0$ and $v(\mathfrak{x}) = 0$ for all $\mathfrak{x} < 0$.Then,

$$\begin{aligned}\mathfrak{x}(t) &= T^-_{t0}a,\ t \le \tau(a)\\ &= T^+_{t-\tau(a),0}0,\ t > \tau(a).\end{aligned}$$

Next, we can find the $\mathfrak{q}$-uniformly continuous solution of (11) corresponds to

$$\begin{aligned}\delta(t,\mathfrak{x}) &= u(T^-_{0t}\mathfrak{x})D_{\mathfrak{q},\mathfrak{x}}(T^-_{0t}\mathfrak{x}),\ \mathfrak{x} < 0\\ &= u\Big(\tau^-(\sigma_t(s))\Big)D_{\mathfrak{q},\mathfrak{x}}\Big(\tau^-(\sigma_t(s))\Big),\ 0 < \mathfrak{x} < T^+_{t0}0\\ &= v\Big(T^+_{0t}\mathfrak{x}\Big)D_{\mathfrak{q},\mathfrak{x}}\Big(T^+_{0t}\mathfrak{x}\Big),\ T^+_{t0}0 < \mathfrak{x}.\end{aligned}$$

Let μ_0 denote an arbitrary probability measure representing initial values, and consider μ_0 as an aggregation of numerous identical particles, where $\int_A d_{\mathfrak{q}}\mu_0$ indicates the relative quantity of particles within the set A. Additionally, let all these particles evolve in accordance with the equation $D_{\mathfrak{q}}\mathfrak{x} = \wp(\mathfrak{x})$. Then $T^+_{t,0}$, $T^-_{t,0}$ produce the trajectories of particles that can linger at 0 for as long as they choose. The expression

$$\delta(t,\mathfrak{x}) = u\Big(\tau^-(\sigma_t(\mathfrak{x})))D_{\mathfrak{q},\mathfrak{x}}\Big(\tau^-(\sigma_t(\mathfrak{x}))\Big),\ 0 < a,\ T^+_{t0}0$$

is reinstated if no particle remains at 0 at any point in time. Additional $\mathfrak{q}$-statistical solutions can be derived when a particle arrives at time "s" at 0, remains there for a certain arbitrary "waiting time" t', and then departs from 0 at $s + t'$ on the trajectory $T^+_{t-s-t^*,0}0$. To express this heuristic observation within a $\mathfrak{q}$-stochastic context, let $\{P_t\}_{t\in[0,\infty)}$ denote a collection of $\mathfrak{q}$-substochastic measures defined on the interval $[0,\infty)$, such that

1. $suppP_t \subset [t,\infty)$ and
2. The mapping $t \to P_t(A)$ is measurable for every Borel set $A \subset [0,\infty)$. The term $P_t(A)$ can be understood as the likelihood that a particle arriving at time t exits within the interval A. Utilizing the "waiting measures" P_t, we define the transition probabilities $P(t,a,E)$ as follows:

$$P(t,a,E) = P_{\tau(\mathfrak{x})}\Big(\sigma_t\Big(E \cap (0,T^+_{t0}0]\Big)\Big) + \Big(1 - P_\tau(a)\Big([0,t)\Big)\Big)\delta_0(E) \text{ if } a < 0 \ \&\ t > \tau(a).$$

Let $H^*_t : M_1 \to M_1$ be the $\mathfrak{q}$-forward evolution in time of a measure μ_0, which is defined by the equation $H^*_t[\mu_0](E) = \int P(t,\mathfrak{x},E)d_{\mathfrak{q}}\mu_0(\mathfrak{x})$.

Next we claim our main result.

Theorem 5.3. *Let $SuppP_t \subset [t,\infty)$, and if $t \to P_t(A)$ is measurable for each Borel set $A \subset [0,\infty)$, then for any collection of substochastic measures $\{P_i\}$ on $[0,\infty)$, $\mu_t = H_t^*[\mu_0]$ constitutes a $\mathfrak{q}$-statistical solution of $D_{\mathfrak{q}}\mathfrak{x} = \wp(\mathfrak{x})$.*

Proof. Let $w \in C_0^1\Big([0,\infty)\times\mathbb{R}\Big)$. Then

$$\begin{aligned}\int_0^\infty\int_{-\infty}^\infty [w_t+\wp(\mathfrak{x})w_{\mathfrak{x}}]d_{\mathfrak{q}}\mu_t(\mathfrak{x})d_{\mathfrak{q}}t &= \int_0^\infty\int_{T_{t0}^+0}^\infty [w_t+\wp(\mathfrak{x})w_{\mathfrak{x}}]d_{\mathfrak{q}}\mu_t(\mathfrak{x})d_{\mathfrak{q}}t+\int_0^\infty\int_0^{T_{t0}^+0} [w_t+\wp(\mathfrak{x})w_{\mathfrak{x}}]d_{\mathfrak{q}}\mu_t(\mathfrak{x})d_{\mathfrak{q}}t\\ &\quad+\int_0^\infty\int_{-\infty}^0 [w_t+\wp(\mathfrak{x})w_{\mathfrak{x}}]d_{\mathfrak{q}}\mu_t(\mathfrak{x})d_{\mathfrak{q}}t\\ &= I+II+III.\end{aligned}$$

Given the involvement of not only $\mathfrak{q}$-uniform continuous measures, it is essential to clarify which boundaries correspond to which integrals. In "I" the inner integral is taken over (T_{t0}^+0,∞), in (II) over $[0,T_{t0}^+0]$, and in (III) over $(-\infty,0)$.

$$\begin{aligned}I &= \int_0^\infty\int_{-\infty}^\infty\int_{T_{t0}}^+ 0[w_t+\wp.w_{\mathfrak{x}}](t,a)P(t,\mathfrak{x},d_{\mathfrak{q}}a)d_{\mathfrak{q}}\mu_0(a)d_{\mathfrak{q}}(t)\\ &= \int_0^\infty\int_0^\infty [w_t+\wp.w_{\mathfrak{x}}](t,T_{t0}^+\mathfrak{x})d_{\mathfrak{q}}\mu_0(\mathfrak{x})d_{\mathfrak{q}}t\\ &= \int_0^\infty\int_0^\infty [w_t+\wp.w_{\mathfrak{x}}](t,T_{t0}^+\mathfrak{x})d_{\mathfrak{q}}\mu_0(\mathfrak{x})d_{\mathfrak{q}}t\\ &= \int_0^\infty\int_0^\infty D_{\mathfrak{q}}[w(t,T_{t0}^+\mathfrak{x})]d_{\mathfrak{q}}td_{\mathfrak{q}}\mu_0(\mathfrak{x})\\ &= -\int_0^\infty w(0,\mathfrak{x})d_{\mathfrak{q}}\mu_0(\mathfrak{x}).\end{aligned}$$

The calculation of (II) is

$$\begin{aligned}II &= \int_0^\infty\int_{-\infty}^\infty\int_0^{T_{t0}^+0} [w_t+\wp.w_{\mathfrak{x}}](t,a)P(t,\mathfrak{x})d_{\mathfrak{q}}a)d_{\mathfrak{q}}\mu_0(\mathfrak{x})d_{\mathfrak{q}}t\\ &= \int_0^\infty\int_0^\infty [w_t+\wp.w_{\mathfrak{x}}](t,T_{t0}^+\mathfrak{x})d_{\mathfrak{q}}\mu_0(\mathfrak{x})d_{\mathfrak{q}}t\\ &= \int_0^\infty\int_0^\infty D_{\mathfrak{q}}[w(t,T_{t0}^+\mathfrak{x})]d_{\mathfrak{q}}td_{\mathfrak{q}}\mu_0(\mathfrak{x})\\ &= -\int_0^\infty w(0,\mathfrak{x})d_{\mathfrak{q}}\mu_0(\mathfrak{x})\end{aligned}$$

The calculation of (III) is as follows:

$$\begin{aligned}III &= \int_0^\infty\int_{-\infty}^\infty\int_{-\infty}^0 [w_t+\wp w_t](t,a)P(t,\mathfrak{x},d_{\mathfrak{q}}a)d_{\mathfrak{q}}a)d_{\mathfrak{q}}\mu_0(\mathfrak{x})d_{\mathfrak{q}}t\\ &= \int_0^\infty\int_{-\infty}^0\int_{-\infty}^0 [w_t+\wp w_{\mathfrak{x}}](t,a)d_{\mathfrak{q}}\delta_{T_{t0}^-\mathfrak{x}}(a)d_{\mathfrak{q}}\mu_0(\mathfrak{x})d_{\mathfrak{q}}t\\ &= \int_0^\infty\int_{-\infty}^{\tau^-(t)} D_t[w(t,T_{t0}^-\mathfrak{x})]d_{\mathfrak{q}}td_{\mathfrak{q}}\mu_0(\mathfrak{x})\end{aligned}$$

$$= \int_{-\infty}^{0} [w(\tau(\mathfrak{x}), 0) - w(0, \mathfrak{x})] d_{\mathfrak{q}} \mu_0(\mathfrak{x}).$$

Adding up $(I) - (III)$ we can conclude the complete proof. □

Conclusion

In this manuscript, we introduce the concept of $\mathfrak{q}$-statistical convergence sequences of functions. Various important characteristics of these sequences are analyzed. We have established that every statistically convergent sequence of functions shows q-statistical convergence. Moreover, we explore the $\mathfrak{q}$-statistical convergence of sequences that consist of Jackson integrable functions, applying the definition of $\mathfrak{q}$-statistical convergence. We also present both the necessary and sufficient conditions for the q-statistical convergence of sequences pertaining to Jackson integrable functions. Additionally, we determine the q-statistical solution for the Cauchy problem that lacks unique solvability in the context of $\mathfrak{q}$-calculus. Theorem3.6 shows a generalization of [6, Theorem 3.2.2]. In future, we shall find relationship of $\mathfrak{q}$-statistical convergence sequences of Jackson integrable functions and $\mathfrak{q}$-statistical convergence sequences of gauge integrable functions. One can find a new class of Korovkin-type approximation theorem based on equi-statistical convergence of double $\mathfrak{q}$-sequences.

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